Mon. Not. R. Astron. Soc. **000**, 1–4 (2012) Printed 6 June 2012 (MN LaTeX style file v2.2)

On the point mass approximation to calculate the gravitational wave signal from white dwarf binaries

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6 June 2012

ABSTRACT
Double white dwarf binaries in the Galaxy dominate the gravitational wave sky and would be detectable for an instrument such as LISA. Most studies have calculated the expected gravitational wave signal under the assumption that the binary white dwarf system can be represented by two point masses in orbit. We discuss the accuracy of this approximation for real astrophysical systems. For non-relativistic binaries in circular orbit the gravitational wave signal can easily be calculated. We show that for these systems the point mass approximation is completely justified when the individual stars are axisymmetric irrespective of their size. We find that the signal obtained from Smoothed-Particle Hydrodynamics simulations of tidally deformed, Roche-lobe filling white dwarfs, including one case when an accretion disc is present, is consistent with the point mass approximation. The difference is typically at the level of one per cent or less in realistic cases, yielding small errors in the inferred parameters of the binaries.

Key words: stars: white dwarfs – gravitational waves

1 INTRODUCTION

At low frequencies (mHz), millions of double white dwarf binaries in the Galaxy are expected to dominate the gravitational wave (GW) sky. At the lowest frequencies they form an unresolved foreground, while at frequencies above several mHz, thousands of sources would be individually detectable for an instrument such as the Laser Interferometer Space Antenna, LISA (Evans et al. 1987, Lipunov et al. 1987, Hils et al. 1990, Nelemans et al. 2001, 2004) or eLISA/NGO (Amaro-Seoane et al. 2012). These binaries come in two flavours: detached systems and semi-detached (mass-transferring) systems that are know as AM CVn systems (see Solheim 2010, Marsh 2011 for reviews). Several known binaries should be detected by LISA within the first weeks of operation and are known as verification binaries (Stroeer & Vecchio 2006, Roelofs et al. 2007, 2010, Brown et al. 2011). By measuring their gravitational wave amplitude and frequency (evolution), the type of astrophysical source and its parameters can be determined (e.g. Cutler & Flanagan 1994, Littenberg 2011, Blaut 2011). In all these calculations the gravitational wave signal was determined under the assumption that the binary white dwarf system can be represented by two point masses in orbit, even for the tidally deformed stars in semi-detached binaries. The goal of this study is to determine the accuracy of this assumption. In section 2 we will discuss our method of calculating the GW signal. In section 3 we will give an algebraic view on the assumption of using point masses and in section 4 we will calculate the GW signal from smoothed-particle hydrodynamics (SPH) simulations of AM CVn stars. In section 5 we discuss the conclusions of this study.

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2 GRAVITATIONAL WAVES FROM ARBITRARY SOURCES IN CIRCULAR MOTION

We calculate the GW wave signal from a collection of point particles with arbitrary coordinates and masses, all rotating about a fixed point with the same angular speed ω. Since the stars do not move at highly relativistic speeds we use linearised general relativity. In linearised general relativity the trace reversed metric for any non-relativistic, far away source is given by (e.g. Rindler 2001)

$$\bar{h}_{ij} = \frac{-2G}{c^4 R}\int \frac{d^2}{dt^2}\rho \boldsymbol{x_i x_j} dV = \frac{-2G}{c^4 R}\sum_\alpha \frac{d^2}{dt^2} m_\alpha \boldsymbol{x_{i\alpha} x_{j\alpha}}, \tag{1}$$

where R is the distance from observer to the source taken as an average over the distance to all source points. m_α and $\boldsymbol{x_{i\alpha}}$ are the mass and coordinates of points in the source. $\sum_\alpha m_\alpha \boldsymbol{x_{i\alpha} x_{j\alpha}}$ is the quadrupole moment of the source. In the transverse-traceless (TT) gauge for a wave travelling in the z-direction, the wave has only two degrees of freedom left. They manifest themselves as so called $+$ and $\times$ polarisations that can be measured by a detector.

The wave metric then becomes (Rindler 2001)

$$h_{ij} = \left(\begin{array}{ccc} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{array} \right). \quad (2)$$

h_+ and $h_\times$ can be obtained by (Price & Wang 2008)

$$h_+ = \frac{1}{2}(\overline{h}_{xx} - \overline{h}_{yy}) h_\times = \overline{h}_{xy}. \quad (3)$$

If we now take N particles and assign them masses m_α and positions in polar coordinated r_α and $\phi_\alpha = \omega t + \theta_\alpha$, we can derive a general expression for h_+ and $h_\times$. Also taking into account the inclination angle i for the relative orientation of the source, we get:

$$h_+ = \frac{-4\omega^2 G}{c^4 R}(1+\cos^2 i)\left(S_1 \cos 2\omega t - S_2 \sin 2\omega t\right) \quad (4)$$

$$h_\times = \frac{-4\omega^2 G}{c^4 R}\cos i \left(S_2 \cos 2\omega t - S_1 \sin 2\omega t\right), \quad (5)$$

where

$$S_1 \equiv \sum_\alpha m_\alpha r_\alpha^2 \cos 2\theta_\alpha \quad (6)$$

$$S_2 \equiv \sum_\alpha m_\alpha r_\alpha^2 \sin 2\theta_\alpha. \quad (7)$$

These expressions depend only on the given initial coordinates and can easily be calculated numerically. Because h_+ and $h_\times$ have a cosine and a sine term with different amplitudes it is useful to define the average strain amplitude as:

$$h \equiv \sqrt{\frac{1}{2}(h_{+max}^2 + h_{\times max}^2)}. \quad (8)$$

As we will later determine the accuracy of the point mass approximation, it is worth estimating the error we make by using the linearised theory for these systems. The first Post-Newtonian correction is proportional to $(v/c)^2$ (e.g. Misner, Thorne, & Wheeler 1973,Blanchet et al. 1995) which for double white dwarf systems with orbital period of several minutes is of order 10^{-5}.

3 PARALLEL AXIS THEOREM FOR GRAVITATIONAL WAVES

Using equations (4) and (5) we look at what we can say about the two point mass approximation algebraically. The parts of the equations that depend on the configuration of the system are S_1 and S_2. For these expressions something very similar to the parallel axis theorem for moment of inertia can be expressed. Consider a body in the xy-plane with its centre of mass at the origin. Its mass distribution can be described by point masses with coordinates x_α,y_α and mass m_α with respect to its own centre of mass. So S_1 and S_2 are obtained from Eqs. 6,7. We call these $S_{1,\mathrm{spinning}}$ and $S_{2,\mathrm{spinning}}$, because these are the S's that would arise if the body was spinning around its centre of mass. We now change the coordinates of our origin to another point in the plane such that all point masses get new coordinates $x'_\alpha = x_\alpha - x_{\mathrm{CM}}$ and $y'_\alpha = y_\alpha - y_{\mathrm{CM}}$. All coordinates denoted with subscript CM refer to the location of the body's centre of mass in the new frame and we assume the body is in circular motion around the new origin.

To do the translation we first transform to Cartesian coordinates, translate and then transform back to polar coordinates. For the transform we use

$$\frac{y_\alpha}{x_\alpha} = \left\{ \begin{array}{ll} \tan(\theta_\alpha) & \text{for} \quad x_\alpha \geqslant 0 \\ \tan(\theta_\alpha + \pi) & \text{for} \quad x_\alpha < 0 \end{array} \right\}. \quad (9)$$

The mirror symmetry, $\cos(2\theta) = \cos(2(\theta+\pi))$ and $\sin(2\theta) = \sin(2(\theta+\pi))$, relieves us from having to split the sum over particles into separate sums for positive and negative x_α, allowing the new S to be written as the old S plus an additional term. The resulting S's that determine the GW radiation are:

$$S_1 = S_{1,\mathrm{spinning}} + \sum_\alpha m_\alpha r_{\mathrm{CM}}^2 \cos 2\theta_{\mathrm{CM}} \quad (10)$$

$$S_2 = S_{2,\mathrm{spinning}} + \sum_\alpha m_\alpha r_{\mathrm{CM}}^2 \sin 2\theta_{\mathrm{CM}}, \quad (11)$$

i.e. a first term denoting the original $S_{1,\mathrm{spinning}}$ and $S_{2,\mathrm{spinning}}$ and a second displacement term that sums all mass in one point, so the change of coordinates has contributed only the effect of a point mass at the body's centre of mass.

So far we have considered the case in which there is one body rotating around an arbitrary point. In the case of a binary system, the procedure can be repeated for the second star and the $x_{\mathrm{CM}}, y_{\mathrm{CM}}$ for each star now refer to the distance of the centres of the two stars to the system barycentre.

We thus can conclude that, when determining h, every body with $S_{1,\mathrm{spinning}} = S_{2,\mathrm{spinning}} = 0$ can be considered as a point mass without implications to the result. These are bodies that do not radiate GW if they were only spinning around an axis through their own CM. In other words: bodies without a quadrupole moment, like axisymmetric spheres or disks. This result even holds when the body overlaps with the point it is rotating about, if that is physically possible. So only asymmetries in the stars that form a binary GW source may lead to deviations from the result as calculated using a source consisting only of two point masses.

4 NUMERICAL CALCULATION OF GRAVITATIONAL WAVES FROM SPH SIMULATIONS

For semi-detached binaries, we know they do not consist of spherical stars. The contribution of the accretion disk will depend on if it is circular or not and on how the mass is distributed over the disk. The contribution of the donor will also depend on the mass distribution over its shape. We therefore need to look at the GW signals of more realistic mass distributions.

4.1 The simulations

Roche lobe filling stars and accretion disks are not completely symmetric around their centres of mass. Using equations (4) and (5) we can calculate the contribution this has to the gravitational wave signal if we know their mass distributions. For this study we use a set of SPH simulations of double white dwarfs at the onset of mass transfer (taken from Dan et al. 2011, 2012), hereafter referred to as RocheSPH. We sample the different mass combinations as shown in Table 1. An example of one star of an SPH simulation is shown in Fig. 1 in such a way that the non-axisymmetric SPH particles are shown more prominently. In addition we have used an SPH simulation of an accretion disc in an ultra-compact binary, kindly provided to us by Prof. Matt Wood (based on Wood, Thomas, & Simpson 2009). The latter is a $M_{\mathrm{donor}}/M_{\mathrm{accretor}} = q = 1/10$ system with an accretion disk around the accretor formed by adding mass at

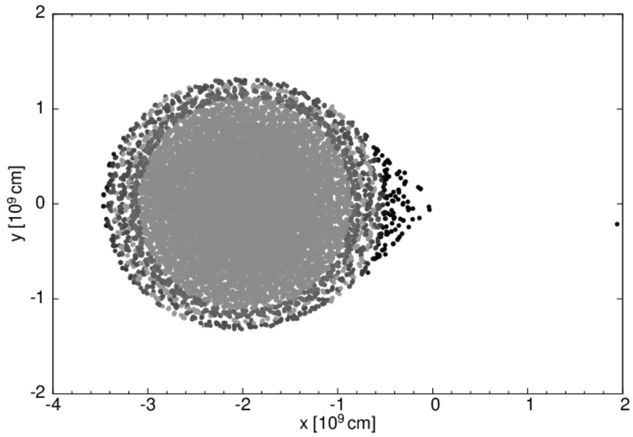

Figure 1. Projection of SPH particles onto the orbital plane of one of the stars in a double white dwarf system with $M_{\rm donor} = M_{\rm accretor} = 0.2M_{\odot}, q = 1$. The colour coding accentuates the asymmetry in the mass distribution that is responsible for the deviation of the gravitational wave signal from that of two point masses. For each ring around the center of mass of the star the ratio of the mass on the left (x coordinates smaller than $x_{\rm CM}$) to that on the right(x coordinates larger than $x_{\rm CM}$) is calculated. Light grey represents values of unity (smoothed for clarity) and black values below 0.5.

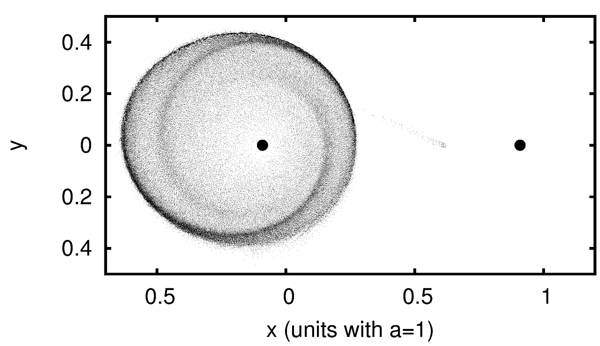

Figure 2. Projection of SPH particles onto the orbital plane of a double white dwarf system with $q = 1/10$, the donor and accretor are represented as point masses (the bigger dots) and an accretion disk is present around the accretor. The coordinates are in units of the separation a between the two stars. In this SPH simulation particles were continuously added at the inner Lagrange point of the Roche potential. These particles have formed the accretion disk around the accretor. This is a picture of the 390th orbit after the mass transfer started.

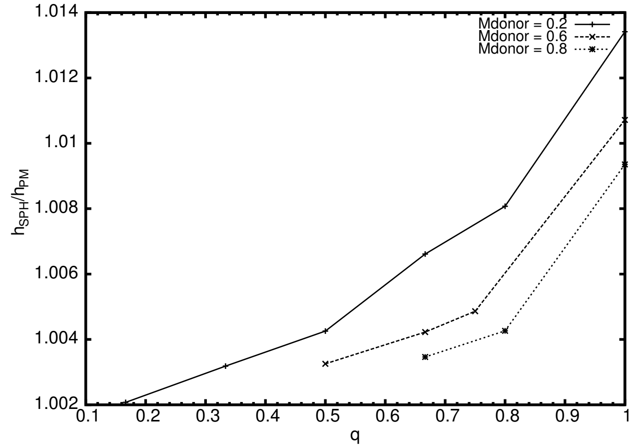

Figure 3. Ratio of the GW strain amplitude (equation 8) from the RocheSPH systems (Fig. 1) and corresponding the two point mass systems as a function of the mass ratio, for different donor masses. See also Table 1

Table 1. Ratio of GW strain amplitude of SPH systems and two point mass systems. The mass of the accretion disk in DiskSPH was taken to be $10^{-5}M_{\rm accretor}$.

SPH simulation	$M_{\rm donor}$ ($M_{\odot}$)	$M_{\rm accretor}$ ($M_{\odot}$)	q	$\frac{h_{\rm SPH}}{h_{\rm PM}}$
DiskSPH			0.1	1.000010
RocheSPH	0.2	0.2	1.0	1.013418
RocheSPH	0.2	0.25	0.8	1.008075
RocheSPH	0.2	0.3	0.667	1.006612
RocheSPH	0.2	0.4	0.5	1.004254
RocheSPH	0.2	0.6	0.33	1.003183
RocheSPH	0.2	1.2	0.167	1.002072
RocheSPH	0.6	0.6	1.0	1.010717
RocheSPH	0.6	0.8	0.75	1.004861
RocheSPH	0.6	0.9	0.667	1.004226
RocheSPH	0.6	1.2	0.5	1.003254
RocheSPH	0.8	0.8	1.0	1.009355
RocheSPH	0.8	1.0	0.8	1.004264
RocheSPH	0.8	1.2	0.667	1.003461

the inner Lagrange point of the Roche potential with the two stars treated as point masses (Fig. 2), from here onwards referred to as the DiskSPH. We assume the mass in the disk is 10^{-5} of the mass of the accretor.

All simulation results are snapshots at some point in the evolution of the binary. Because of this, we can not take all movement of the system into account, so we assumed that these mass distributions rotate with fixed angular speeds around their centres of mass. Now equations 4, 5 and 8 can be used to calculate the GW strain amplitude.

4.2 The results

The differences of GW strain amplitude for the DiskSPH and all RocheSPH systems compared to two point systems are listed in Table 1. The DiskSPH results shows a deviation at the 10^{-5} level, while the RocheSPH results, depending on the mass of the deformed donor star and the mass ratio q, range between 0.2 and 1.3 per cent. The results of the RocheSPH calculations as function of mass ratio q and donor mass are shown graphically in Fig. 3. The deviations are largest for equal mass systems, but more interesting, our results clearly show the fact that lower donor masses are more deformed than more massive donors at the same mass ratio.

To explore the influence of an accretion disk in somewhat more detail, we varied the mass of the accretion disk compared to the mass in the two stars (Fig. 4). As can be expected, the deviation from the point mass approximation scales linearly with the disk mass, which has to be unrealistically high (1 per cent of the total mass) in order to get near the effect of the deformed donor star discussed above. The deviation for a system with Roche-lobe filling donor plus an accretion disk is approximately the sum of the RocheSPH and DiskSPH result (see above) and thus for realistic disk masses is dominated by the deformation of the donor.

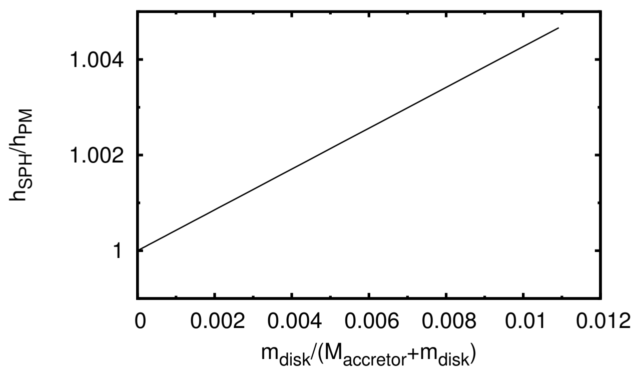

Figure 4. Ratio of the GW strain amplitude (equation 8) from the DiskSPH system (Fig. 2) and the two point mass system. The x-axis shows ratio of disk mass to the accretor plus disk mass. The donor is considered to be a point mass, so at $m_{\rm disk} = 0$ the system is equal to the two point system.

5 CONCLUSIONS

We have calculated the deviation of the gravitational wave signal of finite size and non-spherical white dwarfs to that of point masses, which is usually assumed. For any finite size, non-relativistic axisymmetric body in circular orbit, the result is exactly the same. For semi-detached white dwarf binaries, we find that an accretion disk of reasonable mass changes the gravitational wave signal at the level of 10^{-4} to 10^{-3}, small but still significantly larger than errors due to the neglect of post-Newtonian corrections in the calculation of the signal. Deformations due to filling the Roche lobe of semi-detached binaries increase for mass ratios closer to unity for fixed donor mass and are distinctly stronger at fixed mass ratio for lower mass donors and can in the most extreme cases be of order 1 per cent. This in principle will change the frequency evolution and the accuracy with which the parameters can be determined (e.g. Blanchet et al. 1995, Cutler & Flanagan 1994), so calculations in which accuracies of better than one per cent are needed should take the finite size into account. Also, the different strength of the gravitational wave angular momentum losses will affect the mass transfer rate and stability of the mass transfer in semi-detached systems (e.g. Marsh et al. 2004). However, the level of deviation we found shows that the calculations presented in the literature on the expected signals of (verification) binaries for LISA, are essentially unaffected: for monochromatic sources, the amplitude of the signal will be slightly different, but indistinguishable from sources with slightly higher (chirp) masses and/or smaller distances, properties that are much more uncertain than one per cent. For systems with measurable period derivatives, the degeneracy between (chirp) mass and distance is broken, but the deviations we found can still only be detected if there is independent extremely accurate measurement of the (chirp) mass of the binary. For a LISA-like detector such independent mass estimates, if available, are typically accurate at the 10 per cent level at best (e.g. Littenberg 2011). The detailed evolution (and possible merger of the system) will be affected as well, but the calculations of this phase (e.g. Dan et al. 2012) already take the finite size of the stars into account. We therefore conclude that in the majority of cases, the use of the point mass approximation is well justified.

ACKNOWLEDGMENTS

We thank Matt Wood for sharing the results of his SPH simulations with us and the anonymous referee for comments that greatly improved the paper.

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