THEORETICAL PEARLS

Applications of Plotkin-terms: partitions and morphisms for closed terms

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Abstract

This theoretical pearl is about the closed term model of pure untyped lambda-terms modulo β -convertibility. A consequence of one of the results is that for arbitrary distinct combinators (closed lambda terms) M, M', N, N' there is a combinator H such that

$$HM = HM' \neq HN = HN'.$$

The general result, which comes from Statman [1998], is that uniformly r.e. partitions of the combinators, such that each “block” is closed under β -conversion, are of the form $\{H^{-1}\{M\}\}_{M \in \Lambda^\emptyset}$. This is proved by making use of the idea behind the so-called Plotkin-terms, originally devised to exhibit some global but non-uniform applicative behavior. For expository reasons we present the proof below. The following consequences are derived: a characterization of morphisms and a counter-example to the perpendicular lines lemma for β -conversion.

1. Introduction

We use notations from recursion theory and lambda calculus, see Rogers [1987] and Barendregt [1984].

NOTATION. (i) φ_e is the e -th partial recursive function of one argument.

(ii) $W_e = \text{dom}(\varphi_e) \subseteq \mathbb{N}$ is the r.e. set with index e .

(iii) Λ is the set of lambda-terms and $\Lambda^\emptyset$ is the set of closed-lambda terms (combinators).

(iv) $\mathcal{W}_e = \{M \in \Lambda^\emptyset \mid \#M \in W_e\} \subseteq \Lambda^\emptyset$; here $\#M$ is the code of the term M .

1.1. DEFINITION. (i) Inspired by Visser [1980] we define a *Visser-partition* (V-partition) of $\Lambda^\emptyset$ to be a family $\{\mathcal{W}_e\}_{e \in S}$ such that

- (1) $S \subseteq \mathbb{N}$ is an r.e. set
 - (2) $\forall e \in S \forall M, N (M \in \mathcal{W}_e \ \& \ N = M) \Rightarrow N \in \mathcal{W}_e$.
 - (3) $\mathcal{W}_e \cap \mathcal{W}_{e'} \neq \emptyset \Rightarrow \mathcal{W}_e = \mathcal{W}_{e'}$.
- (ii) A family $\{\mathcal{W}_e\}_{e \in S}$ is a *pseudo-V-partition* if it satisfies just 1 and 2.

1.2. DEFINITION. Let $\{\mathcal{W}_e\}_{e \in S}$ be a V-partition.

- 1. The partition is said to be *covering* if $\bigcup_{e \in S} \mathcal{W}_e = \Lambda^\emptyset$.
- 2. The partition is said to be *inhabited* if $\forall e \in S \ \mathcal{W}_e \neq \emptyset$.
- 3. A V-partition $\{\mathcal{W}_e\}_{e \in S'}$ is said to be (*extensionally*) *equivalent* with $\{\mathcal{W}_e\}$ if these families define the same collection of non-empty sets, i.e. if

$$\{\mathcal{W}_e \mid e \in S \ \& \ \mathcal{W}_e \neq \emptyset\} = \{\mathcal{W}_e \mid e \in S' \ \& \ \mathcal{W}_e \neq \emptyset\}.$$

1.3. EXAMPLE. Let H be some given combinator. Define

$$\mathcal{W}_{e(M,H)} = \{N \in \Lambda^\emptyset \mid HN = HM\},$$

Then $\{\mathcal{W}_e\}_{e \in S_H}$, with $S_H = \{e(M, H) \mid M \in \Lambda^\emptyset\}$, is an example of a covering and inhabited V-partition. We denote this V-partition by $\{\mathcal{W}_{e(M,H)}\}_{M \in \Lambda^\emptyset}$.

1.4. PROPOSITION. (i) *Every V-partition is effectively equivalent to an inhabited one.*

(ii) *Every V-partition can effectively be extended to a covering one.*

PROOF. (i) Given $\{\mathcal{W}_e\}_{e \in S}$ define $S' = \{e \in S \mid \mathcal{W}_e \neq \emptyset\}$. Then $\{\mathcal{W}_e\}_{e \in S'}$ is the required modified partition.

(ii) Given $\{\mathcal{W}_e\}_{e \in S}$ define

$$\mathcal{W}_{e(M)} = \{N \mid N = M \vee \exists e \in S \ M, N \in \mathcal{W}_e\}.$$

Then $\{\mathcal{W}_{e(M)}\}_{M \in \Lambda^\emptyset}$ is the required V-partition. ■

The main theorem comes in two version. The second more sharp version is needed for the construction of so called inevitably consistent equations, see Statman [1999].

1.5. THEOREM (Main theorem). (i) *Let $\{\mathcal{W}_e\}_{e \in S}$ be a V-partition. Then one can construct effectively a combinator H such that for all $M, N \in \Lambda^\emptyset$*

$$HM = HN \Leftrightarrow M = N \vee \exists e \in S \ M, N \in \mathcal{W}_e. \quad (*)$$

The construction of H is effective in the code of the underlying r.e. set S .

(ii) *Let $\{\mathcal{W}_e\}_{e \in S}$ be a pseudo-V-partition. Then one can construct effectively a combinator H such that if $\{\mathcal{W}_e\}_{e \in S}$ is an actual V-partition, then $(*)$ holds.*

The theorem will be proved in §2. It has several consequences. In order to state these we have to formulate the notion of morphism on $\Lambda^\emptyset$ and the so-called perpendicular lines lemma.

1.6. DEFINITION. Let $\varphi : \Lambda^\emptyset \rightarrow \Lambda^\emptyset$ be a map. Then φ is a *morphism* if

- 1. $\varphi(M) = \mathbf{Ec}_{f(\#M)}$, for some recursive function f .
- 2. $M = N \Rightarrow \varphi(M) = \varphi(N)$.

1.7. LEMMA. (i) Let F be a combinator and define $\varphi_H(M) \equiv HM$. Then φ_H is a morphism.

(ii) Let F, G be combinators such that for all $M \in \Lambda^\emptyset$ there exists a unique $N \in \Lambda^\emptyset$ with $FM = GN$. Then there is a map $\varphi_{F,G}$ such that $FM = G\varphi_{F,G}(M)$, for all M , which is a morphism.

PROOF. (i) For the coding $\#$ let app be the recursive function such that $\#(PQ) = \text{app}(\#P, \#Q)$. Define $f(m) = \text{app}(\#H, m)$. Then $\varphi_H(M) = \text{Ec}_{f(\#M)}$. It is obvious that φ_H preserves β -equality.

(ii) Let $R(m, n)$ be an r.e. relation. Then we have $R(m, n) \Leftrightarrow \exists z T(m, n, z)$, for some recursive T . Let $\langle n, z \rangle$ be a recursive pairing with recursive inverses $\langle n, z \rangle .0 = n, \langle n, z \rangle .1 = z$. Define (μ is the least number operator)

$$\iota_n.R(m, n) = (\mu p.T(m, p.0, p.1)).0.$$

Then $\exists n \in \mathbb{N} R(m, n) \Rightarrow R(m, \iota_n.R(m, n))$. In order to construct the morphism $\varphi_{F,G}$, define

$$f(m) = \iota_n.F(\text{Ec}_m) = G(\text{Ec}_n).$$

By the assumption (existence) f is total. Define $\varphi_{F,G}(M) = \text{Ec}_{f(\#M)}$. Now $f(\#M) = n \Rightarrow F(\text{Ec}_n) = G(\text{Ec}_n)$. Therefore $FM = G\varphi_{F,G}(M)$, for all M . The condition

$$M = M' \Rightarrow \varphi_{F,G}(M) = \varphi_{F,G}(M')$$

holds by the assumption (unicity). ■

One may wonder whether dropping the unicity condition in lemma 1.7 (ii) one may obtain a morphism by making a right uniformization. This is not the case.

1.8. PROPOSITION. There exists combinators F, G such that $\forall M \exists N FM = GN$ but without any morphism satisfying $\forall M FM = G\varphi(N)$.

PROOF. Let $\Delta = \Upsilon\Omega$ and define $F = \lambda x.\langle x, \Delta, \text{I} \rangle$ and $G = \lambda y.\langle Ey, y\Omega\Delta, y\text{I} \rangle$. Then, see Statman [1986],

$$FM =_\beta GN \Leftrightarrow (N =_\beta c_n \vee N =_\beta \text{I}) \ \& \ EN =_\beta M. \quad (1)$$

Any morphism φ such that $FM = G\varphi(M)$ would solve the convertibility problem recursively: one has by (1)

$$M = M' \Leftrightarrow \varphi(M) = \varphi(M'), \quad (2)$$

and since $\varphi(M), \varphi(M')$ have nf's by (1), the RHS of (2) is decidable. ■

1.9. PROPOSITION. Not every morphism is of the form φ_H .

PROOF. Let $F, G \in \Lambda^\emptyset$ be such that $F \circ G = \text{I}$. Then F, G determine a so-called *inner model* $\llbracket \cdot \rrbracket = \llbracket \cdot \rrbracket^{F,G}$ as follows.

$$\begin{aligned} \llbracket x \rrbracket &= x; \\ \llbracket PQ \rrbracket &= F\llbracket P \rrbracket \llbracket Q \rrbracket; \\ \llbracket \lambda x.P \rrbracket &= G(\lambda x.\llbracket P \rrbracket). \end{aligned}$$

Using the condition on F, G it can be proved that

$$M =_{\beta} N \Rightarrow \llbracket M \rrbracket = \llbracket N \rrbracket.$$

Therefore defining $\varphi(M) = \llbracket M \rrbracket$ we obtain a morphism.

Now take $F \equiv \lambda y. ul$, $G \equiv \lambda xy. yx$. Then indeed $F \circ G = \text{id}$ and for the resulting inner model one has $\llbracket I \rrbracket = \lambda y. y!$ and $\llbracket \Omega \rrbracket = (\lambda y. y(\lambda z. z!z))!(\lambda y. y(\lambda z. z!z))$.

Suppose towards a contradiction that the resulting φ is of the form φ_H . Then $H! = \lambda y. y!$, so H is solvable and hence has a hnf $\lambda x_1 \dots x_n. \cdot_i M_1 \dots M_m$. But $H\Omega = (\lambda y. y(\lambda z. z!z))!(\lambda y. y(\lambda z. z!z))$, which is unsolvable. Therefore the head-variable x_i is x_1 . But then $H\Omega = \lambda x_2 \dots x_n. \Omega M_1^* \dots M_m^*$ which is not of the correct form. ■

The following is a corollary to the main theorem.

1.10. COROLLARY. *Every morphism φ is of the form $\varphi_{F,G}$.*

PROOF. Let φ be a given morphism. Define

$$\mathcal{W}_{e(N)} = \{Z \mid \exists M \in \Lambda^\emptyset [\varphi(M) = N \ \& \ [Z = \langle \mathbf{c}_0, M \rangle \vee Z = \langle \mathbf{c}_1, N \rangle]]\}.$$

Then $\{\mathcal{W}_{e(N)}\}$ is a V-partition. By the main theorem there exists an H such that

$$\begin{aligned} H\langle \mathbf{c}_0, M \rangle = H\langle \mathbf{c}_1, N \rangle &\Leftrightarrow \langle \mathbf{c}_0, M \rangle = \langle \mathbf{c}_1, N \rangle \vee N = \varphi(M) \\ &\Leftrightarrow N = \varphi(M). \end{aligned}$$

Define

$$\begin{aligned} F &= \lambda m. H\langle \mathbf{c}_0, m \rangle; \\ G &= \lambda n. H\langle \mathbf{c}_1, n \rangle. \end{aligned}$$

Then $FM = GN \Leftrightarrow N = \varphi(M)$. Therefore $\varphi = \varphi_{F,G}$. ■

Note that for a given morphism φ one can define by

$$\mathcal{W}_{e(M, \varphi)} = \{N \in \Lambda^\emptyset \mid \varphi(M) = \varphi(N)\}.$$

This is an inhabited V-partition. It is not difficult to show that that each V-partition is equivalent to one of the form $\{\mathcal{W}_{e(M, \varphi)}\}$. Note that $\{\mathcal{W}_{e(M, H)}\} = \{\mathcal{W}_{e(M, \varphi_H)}\}$, see lemma 1.7. The following result shows that covering V-partitions are always of this more restricted form.

1.11. COROLLARY. *If $\{\mathcal{W}_e\}$ is a covering V-partition, then $\{\mathcal{W}_e\}$ is equivalent to $\{\mathcal{W}_{e(M, H)}\}_{M \in \Lambda^\emptyset}$ for some H , effectively found from $\{\mathcal{W}_e\}$.*

PROOF. Let H be the combinator constructed effectively from $\{\mathcal{W}_e\}$. We will show that $\mathcal{W}_{e(M, H)} = \{N \mid HN = HM\}$ is equivalent to $\{\mathcal{W}_e\}$. Claim. For $N \in \mathcal{W}_e$ one has $\mathcal{W}_e = \mathcal{W}_{e(M, H)}$. Indeed,

$$\begin{aligned} N \in \mathcal{W}_e &\Leftrightarrow M = N \vee M, N \in \mathcal{W}_e \\ &\Leftrightarrow HN = HM \\ &\Leftrightarrow N \in \mathcal{W}_{e(M, H)}. \end{aligned}$$

Therefore, noting that $M \in \mathcal{W}_{e(M,H)}$,

$$\{\mathcal{W}_e \mid M \in \Lambda^\emptyset, \mathcal{W}_e \neq \emptyset\} \subseteq \{\mathcal{W}_{e(M,H)} \mid \mathcal{W}_{e(M,H)} \neq \emptyset, M \in \Lambda^\emptyset\}.$$

The converse inclusion holds also, since every M belongs to some $\mathcal{W}_e$ and hence $\mathcal{W}_{e(M,H)} = \mathcal{W}_e$ for this e . ■

The following theorem states that if a combinator, seen as function of n arguments, is constant—modulo Böhm-tree equality—on n perpendicular lines, then it is constant everywhere.

1.12. THEOREM (Perpendicular lines lemma). *Let F be a combinator. Suppose that for $n \in \mathbb{N}$ there are combinators M_{ij} , $1 \leq i \neq j \leq n$, and $N_1, \dots, N_n$ such that for all combinators Z one has ($\cong$ denotes Böhm-tree equality, i.e. $M \cong N \Leftrightarrow BT(M) = BT(N)$)*

$$\begin{array}{ccccccc} F & Z & M_{12} & \dots & M_{1n-1} & M_{1n} & \cong N_1; \\ F & M_{21} & Z & \dots & M_{2n-1} & M_{2n} & \cong N_2; \\ & & & \dots & & & \\ & & & \dots & & & \\ F & M_{n1} & M_{n2} & \dots & M_{nn-1} & Z & \cong N_n. \end{array}$$

Then for all $P_1, \dots, P_n \in \Lambda^\emptyset$ one has

$$FP_1 \dots P_n \cong N_1 (\cong N_2 \cong \dots \cong N_n).$$

PROOF. This is the restriction to closed terms of a theorem in Barendregt [1984], theorem 14.4.12, having the same proof. ■

1.13. COROLLARY. *The perpendicular lines lemma is false for any $n > 1$, if $\cong$ is replaced by $=_\beta$.*

PROOF (For $n = 1$ the perpendicular lines lemma is trivially true for $=_\beta$). Let $n > 1$. For notational simplicity we assume $n = 2$ and give a counter example. Define

$$\begin{aligned} \mathcal{W}_{e_1} &= \{N \in \Lambda^\emptyset \mid N = \langle S, S \rangle\} \\ \mathcal{W}_{e_2} &= \{N \in \Lambda^\emptyset \mid \exists Z \in \Lambda^\emptyset [N = \langle I, Z \rangle \vee N = \langle Z, I \rangle]\} \end{aligned}$$

Then $\{\mathcal{W}_e\}_{e \in \{e_1, e_2\}}$ is a V-partition. Let H be the combinator obtained from this partition by the main theorem. Then for all $Z \in \Lambda^\emptyset$

$$H\langle S, S \rangle \neq H\langle I, Z \rangle = H\langle Z, I \rangle.$$

Now define $F \equiv \lambda xy. H\langle x, y \rangle$. Then for all $Z \in \Lambda^\emptyset$

$$FSS \neq FI Z = FZI.$$

This is indeed a counterexample. ■

We do believe the conjecture in Barendregt [1984], stating that the perpendicular line lemma with $\cong$ replaced by $=_\beta$ is correct for open terms.

2. Proof of the main theorem

In order to prove the main theorem 1.5, let a V -partition determined by S be fixed in this section. By proposition 1.4 it may be assumed that the partition is inhabited.

2.1. LEMMA. *Let $\{\mathcal{W}_e\}_{e \in S}$ be an inhabited V -partition.*

(i) *There exists a total recursive function $f = f_S$ such that*

$$\forall e \in S \ W_e = \{f((2e+1)2^n) \mid n \in \mathbb{N}\}.$$

(ii) *There exists a combinator E^S such that*

$$\forall e \in S \ \mathcal{W}_e = \{E^S c_{(2e+1)2^n} \mid n \in \mathbb{N}\}.$$

PROOF. (i) By elementary recursion theory there exists a recursive function h such that $W_e = \text{Range}(\varphi_{h(e)})$ and $\varphi_{h(e)}$ is total, for all $e \in S$. Observing that e, n are uniquely determined by $k = (2e+1)2^n$, define f by $f(0) = 0$, $f((2e+1)2^n) = \varphi_{h(e)}(n)$.

(ii) Take $E^S = E \circ F_S$, where F_S lambda defines f_S and $E c_{\#M} = M$ for all $M \in \Lambda^\emptyset$. ■

2.2. DEFINITION. (i) Define

$$\begin{aligned} \text{odd}(0) &= 0; \\ \text{odd}((2e+1)2^n) &= 2e+1. \end{aligned}$$

(ii) Define $M \sim N$ iff $M = N \vee M = E_m, N = E_n$ and $\text{odd}(m) = \text{odd}(n)$, for some m, n .

Notice that $M \sim N$ iff $M = N$ or $\exists e \in S, M, N \in \mathcal{W}_e$. Therefore we have to prove that there exists a combinator H such that

$$HM = HN \Leftrightarrow M \sim N.$$

The proof consists in constructing a combinator $H = H^S$ such that

1. $M \sim N \Rightarrow HM = HN$, proposition 2.4;
2. $HM = HN \Rightarrow M \sim N$, proposition 2.9.

The second part of the main theorem easily follows by inspecting the proof.

2.3. DEFINITION. (i) Define

$$\begin{aligned} T &\equiv \lambda xyz. xy(xyz); \\ A &\equiv \lambda f gxyz. fx(a(Ex))[f(S^+x)y(g(S^+x))z]; \\ B &\equiv \lambda f g x. f(Sx)(a(E(Tx))(g(S^+x))(gx). \end{aligned}$$

(ii) By the double fixed-point theorem there exists terms F, G such that

$$\begin{aligned} F &\rightarrow AFG; \\ G &\rightarrow BFG. \end{aligned}$$

To be explicit, write

$$\begin{aligned} D &\equiv (\lambda xy.y(xxy)); \\ Y &\equiv DD; \\ G &\equiv Y(\lambda u.B(Y(\lambda v.Auv))u); \\ F &\equiv Y(\lambda u.AuG). \end{aligned}$$

(iii) Finally define

$$H \equiv \lambda xa.Fc_1(ax)(Gc_1).$$

NOTATION. Write

$$\begin{aligned} F_k &\equiv Fc_k; \\ G_k &\equiv Gc_k; \\ E_k &\equiv Ec_k; \\ a_k &\equiv aE_k; \\ H_k[] &\equiv F_k[]G_k; \\ C_k[] &\equiv F_k a_k([]G_k). \end{aligned}$$

Note that by construction

$$\begin{aligned} F_k MN &\rightarrow F_k a_k(F_{k+1} M G_{k+1} N); \\ G_k &\rightarrow F_{k+1} a_{2k} G_{k+1} G_k. \end{aligned}$$

By reducing F , respectively G , it follows that

$$H_k[a_p] \equiv F_k a_p G_k \rightarrow C_k[H_{k+1}[a_p]] \quad (1)$$

$$H_k[a_k] \equiv F_k a_k G_k \rightarrow C_k[H_{k+1}[a_{2k}]] \quad (2)$$

2.4. PROPOSITION. $M \sim N \Rightarrow HM = HN$.

PROOF. By lemma 2.1 it suffices to show $HE_k = HE_{2k}$ for all k .

$$\begin{aligned} HE_k &= \lambda a.H_1[a_k] \\ &= \lambda a.C_1[C_2[\dots C_{k-1}[H_k[a_k]]\dots]], && \text{by (1),} \\ &= \lambda a.C_1[C_2[\dots C_{k-1}[C_k[H_k[a_{2k}]]\dots]], && \text{by (2),} \\ HE_{2k} &= \lambda a.H_1[a_{2k}] \\ &= \lambda a.C_1[C_2[\dots C_{k-1}[C_k[H_k[a_{2k}]]\dots]], && \text{by (1). } \blacksquare \end{aligned}$$

As a piece of art we exhibit in more detail the reduction flow (contracted redexes are underlined).

$$\begin{aligned}
& \frac{HE_k}{\lambda a. \underline{F_1 a_k G_1}} \\
& \lambda a. F_1 a_1 (\underline{F_2 a_2 G_2 G_1}) \\
& \lambda a. F_1 a_1 (F_2 a_2 (\underline{F_3 a_k G_3 G_2}) G_1) \\
& \dots \\
& \lambda a. F_1 a_1 (F_2 a_2 (F_3 a_3 (\dots (F_k a_k G_k G_{k-1}) \dots) G_2) G_1) \equiv \\
& \lambda a. F_1 a_1 (F_2 a_2 (F_3 a_3 (\dots (F_k a_k \underline{G_k} G_{k-1}) \dots) G_2) G_1) \\
& \lambda a. F_1 a_1 (F_2 a_2 (F_3 a_3 (\dots (F_k a_k (F_{k+1} a_{2k} G_{k+1} G_k) G_{k-1}) \dots) G_2) G_1)
\end{aligned}$$

And also

$$\begin{aligned}
& HE_{2k} \twoheadrightarrow \dots \twoheadrightarrow \\
& \lambda a. F_1 a_1 (F_2 a_2 (F_3 a_3 (\dots (F_k a_k (F_{k+1} a_{2k} G_{k+1} G_k) G_{k-1}) \dots) G_2) G_1)
\end{aligned}$$

For the converse implication we need the fine structure of the reduction.

2.5. DEFINITION. Define

$$\begin{aligned}
D_k^0[M] & \equiv F_k(aM) \equiv Y(\lambda u. AuG) \mathbf{c}_k(aM) \\
D_k^1[M] & \equiv (\lambda y. y(DDy))(\lambda u. AuG) \mathbf{c}_k(aM) \\
D_k^2[M] & \equiv (\lambda u. AuG) F_k(aM) \\
D_k^3[M] & \equiv AFG \mathbf{c}_k(aM) \\
D_k^4[M] & \equiv (\lambda gxyz. F_x(aE_x)(F_{S+x}y(g(S^+x)z))G) \mathbf{c}_k(aM) \\
D_k^5[M] & \equiv (\lambda xyz. F_x(aE_x)(F_{S+x}yG_{S+x}z)) \mathbf{c}_k(aM) \\
D_k^6[M] & \equiv (\lambda yz. F_k(aE_k)(F_{S+\mathbf{c}_k}yG_{S+\mathbf{c}_k}z))(aM) \\
D_k^7[M] & \equiv (\lambda z. F_k(aE_k)(F_{S+\mathbf{c}_k}(aM)G_{S+\mathbf{c}_k}z))
\end{aligned}$$

2.6. LEMMA. Let $F_k(aM)N$ head-reduce in $8p+q$ steps to W . Then

$$\begin{aligned}
W & \equiv D_k^q[M]N, & \text{if } p=0; \\
& \equiv D_k^q[E_k]((H_{k+1}[E_k])^{p-1}(H_{k+1}[M]N)), & \text{else.}
\end{aligned}$$

PROOF. Note that $F_k(aM)N \equiv D_k^0[M]N$. Moreover,

$$\begin{aligned}
D_k^q[M]N & \rightarrow_h D_k^{q+1}[M]N, & \text{for } q < 7; \\
D_k^7[M]N & \rightarrow_h D_k^0[E_k](H_{k+1}[M]N).
\end{aligned}$$

The rest is clear. At steps 16, 24 we obtain for example

$$\begin{aligned}
D_k^7[E_k](H_{k+1}[M]N) & \rightarrow_h D_k^0[E_k]((H_{k+1}[E_k])(H_{k+1}[M]G_k)). \\
D_k^7[E_k]((H_{k+1}[E_k])(H_{k+1}[M]G_k)) & \rightarrow_h D_k^0[E_k]((H_{k+1}[E_k])^2(H_{k+1}[M]G_k)). \blacksquare
\end{aligned}$$

Remember that a standard reduction $\sigma:M \rightarrow_s N$ always consists of a head-reduction followed by an internal reduction:

$$\sigma:M \twoheadrightarrow_h W \twoheadrightarrow_i N.$$

NOTATION. Write $M =_{s \leq n} N$ if there are standard reductions of length $\leq n$ from M respectively N to a common reduct Z . Similarly $M =_{i \leq n} N$ for internal standard reductions. Also the notations $=_{s < n}$ and $=_{i < n}$ will be used.

- 2.7. LEMMA. (i) $D_k^q[M]N =_{i \leq n} D_k^{q'}[M']N' \Rightarrow q = q' \ \& \ N =_{s \leq n} N'$.
(ii) $D_k^q[M]N =_{i \leq n} D_k^q[M']N' \ \& \ q < 7 \Rightarrow M =_{s \leq n} M'$.
(iii) $D_k^7[M]N =_{i \leq n} D_k^7[M']N' \Rightarrow H_{k+1}[M] =_{s \leq n} H_{k+1}[M']$.

PROOF. (i) Suppose $D_k^q[M]N =_{i \leq n} D_k^{q'}[M']N'$. Then By observing where the free variable a occurs one can conclude that $q = q'$. Since the reductions to a common reduct are internal, the positions of N, N' are not changed and hence $N =_{s \leq n} N'$.

(ii) Obvious from the definition of D_k^q .

(iii) In this case it follows that

$$D_k^0[E_k](H_{k+1}[M]z) =_{i \leq n} D_k^0[E_k](H_{k+1}[M']z).$$

The conclusion $H_{k+1}[M] =_{s \leq n} H_{k+1}[M']$ depends on the fact that there are the free variables z to mark the residuals. ■

2.8. LEMMA. Suppose $G_k =_{s \leq n} (H_{k+1}[E_k])^d(H_{k+1}[M]G_k)$. Then

$$H_{k+1}[E(T\mathbf{c}_k)] =_{s < n} H_{k+1}[M].$$

PROOF. By induction on d . If $d = 0$, then we have $G_k =_{s \leq n} H_{k+1}[M]G_k$. So there are standard reductions of these two terms to a common reduct. Observe that the head-reduction starting with G_k begins as follows.

$$\begin{aligned} G_k &\equiv Y(\lambda u. B(Y(\lambda v. Avu))u)\mathbf{c}_k \\ &\rightarrow_h (\lambda x. x(Yx))(\lambda u. B(Y(\lambda v. Avu))u)\mathbf{c}_k \\ &\rightarrow_h (\lambda u. B(Y(\lambda v. Avu))u)G\mathbf{c}_k \\ &\rightarrow_h BFG\mathbf{c}_k \\ &\rightarrow_h (\lambda gx. F(S^+k)(a(E^S(Tx)))(g(S^+k))(gx)G\mathbf{c}_k \\ &\rightarrow_h (\lambda x. F(S^+k)(a(E^S(Tx)))(G(S^+k))(Gx))\mathbf{c}_k \\ &\rightarrow_h F(S^+k)(a(E^S(T\mathbf{c}_k)))(G(S^+k))(G\mathbf{c}_k). \end{aligned}$$

The heads of these terms are not of order 0 except the last one. But $H_{k+1}[X]$ is always of order 0. Therefore the mentioned standard reduction of G_k goes at least to this last term $H_{k+1}[E^S(T\mathbf{c}_k)]G_k$. But then $H_{k+1}[E^S(T\mathbf{c}_k)] =_{s < n} H_{k+1}[M]$.

If $d > 0$, then start the same argument as above, but at the intermediate conclusion

$$H_{k+1}[E^S(T\mathbf{c}_k)]G_k =_{s < n} (H_{k+1}[E_k])^d(H_{k+1}[M]G_k),$$

one precedes by concluding that

$$G_k =_{s < n} H_{k+1}[E_k]^{d-1}(H_{k+1}[M]G_k)$$

and uses the induction hypothesis. ■

2.9. PROPOSITION. $H_k[M] = H_k[N] \Rightarrow M \sim N$.

PROOF. By the standardization theorem it suffices to show for all n that

$$\forall k \in \mathbb{N} [H_k[M] =_{s \leq n} H_k[N] \Rightarrow M \sim N].$$

This will be done by induction on n . From $H_k[M] =_{s \leq n} H_k[N]$ it follows that

$$\begin{array}{ccccc} H_k[M] & \twoheadrightarrow_h & W_M & \twoheadrightarrow_i & Z \\ H_k[N] & \twoheadrightarrow_h & W_N & \twoheadrightarrow_i & Z. \end{array}$$

for some W_M, W_N, Z .

Case 1. W_M, W_N are both reached after < 8 steps. Then by lemma 2.6 $W_M \equiv D_k^q[M]G_k, W_N \equiv D_k^{q'}[N]G_k$. By lemma 2.7(i) it follows that $q = q'$. If $q < 7$, then by 2.7(ii) one has $M = N$ so $M \sim N$. If $q = 7$, then by 2.7(iii) one has $H_{k+1}[M] =_{s < n} H_{k+1}[N]$ and by the induction hypothesis one has $M \sim N$.

Case 2. W_M is reached after $p \geq 8$ steps and W_N after $q < 8$ steps. Then $p = 8d + q$ and, keeping in mind lemma 2.7(i), it follows that $W_M \equiv D_k^q[M]G_k, W_N \equiv D_k^q[E_k]R, G_k =_{s < n} R$, where $R \equiv (H_{k+1}[E_k])^{d-1}(H_{k+1}[N]G_k)$. Then as in case 1 it follows that $M \sim E_k$. Moreover, by lemma 2.8 $H_{k+1}[E_{2k}] =_{s < n} H_{k+1}[N]$, so by the induction hypothesis $E_{2k} \sim N$. So $M \sim E_k \sim E_{2k} \sim N$.

Case 3. Both W_M, W_N are reached after ≥ 8 steps. Then

$$\begin{array}{lcl} W_M & \equiv & D_k^j[E_k]((H_{k+1}[E_k])^d(H_{k+1}[M]G_k)); \\ W_N & \equiv & D_k^j[E_k]((H_{k+1}[E_k])^{d'}(H_{k+1}[N]G_k)). \end{array}$$

If $d = d'$, then by lemma 2.7

$$(H_{k+1}[E_k])^d(H_{k+1}[M]G_k) =_{s < n} (H_{k+1}[E_k])^d(H_{k+1}[N]G_k),$$

so

$$H_{k+1}[M] =_{s < n} H_{k+1}[N],$$

since $H_{k+1}[X]$ is always of order 0. Therefore by the induction hypothesis $M \sim N$.

If on the other hand, say, $d < d'$, then (writing $d' = d + e$)

$$\begin{array}{lcl} W_M & \equiv & D_k^j[E_k]((H_{k+1}[E_k])^d(H_{k+1}[M]G_k)); \\ W_N & \equiv & D_k^j[E_k]((H_{k+1}[E_k])^d(H_{k+1}[E_k] \underline{((H_{k+1}[E_k])^{e-1}(H_{k+1}[N]G_k)}))). \end{array}$$

so

$$\begin{array}{ccc} H_{k+1}[M] & =_{s < n} & H_{k+1}[E_k] \\ G_k & =_{s < n} & (H_{k+1}[E_k])^{e-1}(H_{k+1}[N]G_k), \end{array}$$

since $H_{k+1}[X]$ is always of order 0. Therefore by lemma 2.8

$$H_{k+1}[E_{2k}] =_{s < n} H_{k+1}[N]$$

Therefore by the induction hypothesis twice we obtain $M \sim E_k \sim E_{2k} \sim N$. ■

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