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A Fock space representation for the quantum Lorentz gas

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A Fock space representation is given for the quantum Lorentz gas, i.e., for random Schrödinger operators of the form $H(\omega) = \mathbf{p}^2 + V_\omega = \mathbf{p}^2 + \sum \varphi(\mathbf{x} - \mathbf{x}_j(\omega))$, acting in $\mathcal{H} = L^2(\mathbb{R}^d)$, with Poisson distributed $\mathbf{x}_j$ s. An operator H is defined in $\mathcal{H} = \mathcal{H} \otimes \mathcal{P} = \mathcal{H} \otimes L^2(\Omega, P(d\omega)) = L^2(\Omega, P(d\omega); \mathcal{H})$ by the action of $H(\omega)$ on its fibers in a direct integral decomposition. The stationarity of the Poisson process allows a unitarily equivalent description in terms of a new family $\{H(\mathbf{k}) | \mathbf{k} \in \mathbb{R}^d\}$, where each $H(\mathbf{k})$ acts in $\mathcal{P}$ [A. Tip, *J. Math. Phys.* **35**, 113 (1994)]. The space $\mathcal{P}$ is then unitarily mapped upon the symmetric Fock space over $L^2(\mathbb{R}^d, \rho d\mathbf{x})$, with ρ the intensity of the Poisson process (the average number of points $\mathbf{x}_j$ per unit volume; the scatterer density), and the equivalent of $H(\mathbf{k})$ is determined. Averages now become vacuum expectation values and a further unitary transformation (removing ρ in $\rho d\mathbf{x}$) is made which leaves the former invariant. The resulting operator $H_F(\mathbf{k})$ has an interesting structure: On the n th Fock layer we encounter a single particle moving in the field of n scatterers and the randomness now appears in the coefficient $\sqrt{\rho}$ in a coupling term connecting neighboring Fock layers. We also give a simple direct self-adjointness proof for $H_F(\mathbf{k})$, based upon Nelson's commutator theorem. Restriction to a finite number of layers (a kind of low scatterer density approximation) still gives nontrivial results, as is demonstrated by considering an example. © 1995 American Institute of Physics.

I. INTRODUCTION

One of the earliest random systems ever considered is the Lorentz gas.¹ It pertains to the situation where a small number of light molecules are moving in a background of heavy ones. Then the observation is made that the position and velocity of a heavy molecule are only slightly affected by an encounter with a light one (the Born–Oppenheimer situation) and that the light molecules move nearly independently of each other. This leads to a picture where a single particle moves through an assembly of stationary scattering centers; the Lorentz gas. In statistical mechanics the usual procedure is to start out with a finite volume in which the scattering centers are homogeneously distributed, whereupon the thermodynamic limit is taken. This leads to a Poisson distribution of the scatterers and, in a quantum-mechanical description, to the quantum Lorentz gas. In the following we present a Fock space setting for the latter. The connection between Gaussian processes and Fock space is well known; that a similar situation exists for Poisson processes should not come as a surprise if one realizes that photon counting statistics (formulated in a quantum-electrodynamical Fock space setting) are often Poisson processes. In fact, a whole class of infinitely divisible processes can be generated within a Fock space context.^{2,3}

The Lorentz gas, be it in its classical or its quantum version, has always been an important testing ground in statistical mechanics. For example, the asymptotic expression for its diffusion coefficient as the scatterer density ρ tends to zero, calculated by means of statistical mechanical methods, already shows the characteristic logarithmic terms in ρ , commonly found in more general situations. Mathematical results concerning the density dependence of this and other physical quantities seem to be lacking. Indeed, since ρ appears as a parameter in the Poisson random

measure $P_\rho(d\mu)$ (see below), it is not easily accessible in asymptotic and perturbation considerations. In addition two Poisson measures with different ρ are mutually singular. In the Fock space representation, on the other hand, $\sqrt{\rho}$ enters as the coupling constant in a perturbation term in a Hamiltonian, and this may turn out to be an advantage in the study of the ρ dependence of physical quantities.

The quantum Lorentz gas is an example of a random family of Schrödinger operators with so-called topological disorder, i.e., its members are self-adjoint operators, acting in $\mathcal{H} = L^2(\mathbb{R}^d, \mu_L)$ of the type $(\mathbf{p}$ is the momentum operator, φ is the real-valued single center potential, $\varphi_{\mathbf{x}}^-(\mathbf{y}) := \varphi(\mathbf{x} - \mathbf{y})$, and μ_L is Lebesgue measure)

$$H(\omega) = \mathbf{p}^2 + V(\mathbf{x}, \omega) = \mathbf{p}^2 + \sum_j \varphi(\mathbf{x} - \mathbf{x}_j) = \mathbf{p}^2 + \int \mu(d\mathbf{y}) \varphi(\mathbf{x} - \mathbf{y}) = \mathbf{p}^2 + \langle \varphi_{\mathbf{x}}^-, \mu \rangle, \quad (1.1)$$

where $\omega = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots\}$ is distributed according to some given probability law $(\Omega, P(d\omega))$. In the quantum Lorentz case, the points $\mathbf{x}_j$ are Poisson distributed. In the fourth expression we have written the sum as an integral over a point measure $\mu = \mu_\omega$ (a sum of Dirac δ -measures, each with unit strength) and in the last as a linear functional. In the literature^{4,5} it is common practice to use μ , a point from a state space M , as the random variable, i.e., our probability space becomes $(M, P(d\mu))$. Recently one of us⁶ considered such systems from the point of view of direct integrals. There each $H(\omega)$ is considered as acting in the fiber $\mathcal{H}$ in a direct integral decomposition of

$$\mathcal{K} = \int_{\Omega}^{\oplus} P(d\omega) \mathcal{H} = L^2(\Omega, P; \mathcal{H}) = L^2(\Omega, P) \otimes \mathcal{H} = L^2(\Omega \times \mathbb{R}^d, P \otimes \mu_L), \quad (1.2)$$

the Hilbert space of $\mathcal{H}$ -valued functions square integrable over $P(d\omega)$, thus defining H , acting in $\mathcal{K}$:

$$H = \int_{\Omega}^{\oplus} P(d\omega) H(\omega). \quad (1.3)$$

If $P(d\omega)$ is translation invariant, the family of operators $\{U(\mathbf{a}) | \mathbf{a} \in \mathbb{R}^d\}$ on $L^2(\Omega, P)$, defined by

$$U(\mathbf{a})\psi(\omega) := \psi(\omega_{\mathbf{a}}), \quad \omega_{\mathbf{a}} = \{\mathbf{y} - \mathbf{a} | \mathbf{y} \in \omega\},$$

constitutes a d -parameter group of unitary operators, which is strongly continuous under quite general conditions.⁶ Thus $U(\mathbf{a})$ can be written as $\exp[i\mathbf{a} \cdot \mathbf{q}]$ with $\mathbf{q} = \{q_1, \dots, q_d\}$, a set of commuting self-adjoint operators. Next we define on $\mathcal{K}$ the unitary operator W according to

$$W\psi(\omega, \mathbf{x}) := \psi(\omega_{\mathbf{x}}, \mathbf{x}).$$

On a suitable core its generator reduces to $\mathbf{q} \cdot \mathbf{x} = \sum_{j=1}^d q_j \otimes x_j$. W transforms H into

$$H' := WHW^{-1} = (\mathbf{p} - \mathbf{q})^2 + \sum_j \varphi(-\mathbf{x}_j) = (\mathbf{p} - \mathbf{q})^2 + \langle \varphi^-, \mu \rangle.$$

In case $P(d\omega)$ is also invariant under reflections, a further unitary transformation, which leaves averages invariant, changes H' into

$$\hat{H} = (\mathbf{p} + \mathbf{q})^2 + \sum_j \varphi(\mathbf{x}_j) = (\mathbf{p} + \mathbf{q})^2 + \langle \varphi, \mu \rangle. \quad (1.4)$$

Since neither $\mathbf{x}$ nor $\mathbf{p}$ occurs in the second term any longer, we can decompose $\hat{H}$ into Fourier components as

$$\hat{H} = \int_{\mathbb{R}^d}^{\oplus} d\mathbf{k} \hat{H}(\mathbf{k}), \quad \hat{H}(\mathbf{k}) := (\mathbf{q} + \mathbf{k})^2 + \sum_j \varphi(\mathbf{x}_j) = (\mathbf{q} + \mathbf{k})^2 + \langle \varphi, \mu \rangle \quad (1.5)$$

with each $\hat{H}(\mathbf{k})$ acting in $L^2(\Omega, P(d\omega))$. In addition, physically relevant quantities, such as the expectation $\mathbb{E}[z - H(\omega)]^{-1}$ of the resolvent of $H(\omega)$, and the integrated density of state $N(E)$ can be given in terms of H . They are invariant under $H \rightarrow \hat{H}$, and hence can be expressed in terms of the $\hat{H}(\mathbf{k})$ s.

For the study of spectral and dynamical properties associated with H , the expression (1.4) is not yet the most convenient one. In particular, the infinite sum is a source of difficulties. For the case of a Poisson distribution we shall provide a representation in which only finite sums occur.

We recall that the Poisson distribution $\mathbb{P}_\rho$ with density $\rho > 0$ is determined by the requirement^{4,5} that for all $f \in C_c(\mathbb{R}^d)$, the compactly supported, continuous functions on $\mathbb{R}^d$,

$$\mathbb{E}_\rho(e^{f, \mu}) = \exp[\langle (e^f - 1), \rho \mu_L \rangle]. \quad (1.6)$$

Alternatively, we can characterize $\mathbb{P}_\rho$ by the independence of $\mu_\omega(B_1), \mu_\omega(B_2), \dots, \mu_\omega(B_n)$, for mutually disjoint Borel sets $B_1, \dots, B_n$, and their distribution, given by

$$\mathbb{P}(\mu_\omega(B) = k) = (k!)^{-1} [\rho \mu_L(B)]^k \exp[-\rho \mu_L(B)]. \quad (1.7)$$

In the Hilbert space $\mathcal{P}_\rho := L^2(\Omega, \mathbb{P}_\rho)$ we can now distinguish layers $\mathcal{P}_\rho(n)$, $n \in \mathbb{N}$, in the following way: $\mathcal{P}_\rho(0)$ consists of the constant functions, whereas $\mathcal{P}_\rho(1)$ contains all functions of the form $\omega \rightarrow \langle f, \mu_\omega \rangle - \langle f, \mu_L \rangle$ ($f \in L^1 \cap L^2(\mathbb{R}^d)$), the subtraction orthogonalizing $\mathcal{P}_\rho(1)$ to $\mathcal{P}_\rho(0)$. In this way we continue, i.e., $\mathcal{P}_\rho(n+1)$ is the space of polynomials of degree $n+1$ in the functions $\omega \rightarrow \langle f, \mu_\omega \rangle$, orthogonal to those of degree $\leq n$. It can then be shown that $\mathcal{P}_\rho(n)$ is isomorphic to $\mathcal{F}_\rho(n)$, the symmetric n th tensor power of $L^2(\mathbb{R}^d, \rho \mu_L)$. In this way we obtain

$$\mathcal{P}_\rho = \bigoplus_{n=0}^{\infty} \mathcal{P}_\rho(n) = \bigoplus_{n=0}^{\infty} (\mathcal{H}^{\otimes n})_{\text{sym}} := \mathcal{F}_\rho, \quad \mathcal{H} = L^2(\mathbb{R}^d, \rho \mu_L), \quad (1.8)$$

with norm squared given by

$$\|f\|^2 = \sum_{n=0}^{\infty} \frac{\rho^n}{n!} \|f^{(n)}\|^2, \quad f = f^{(0)} \oplus f^{(1)} \oplus f^{(2)} \oplus \dots, \quad f^{(n)} \in (\mathcal{H}^{\otimes n})_{\text{sym}}.$$

Averages in $\mathcal{P}_\rho$ become vacuum expectation values in $\mathcal{F}_\rho$. Let now the isomorphism (1.8) be followed by a scale transformation $\mathcal{F}_\rho \rightarrow \mathcal{F} := \mathcal{F}_1$. Denoting the annihilation and creation operators for $f \in L^2(\mathbb{R}^d, \mu_L)$ by $b(f)$, respectively $b(f)^*$, and the number operator for f by $V(f)$, an operator on $\mathcal{P}_\rho$ such as $\psi \rightarrow \mu_\omega(B)$. ψ is mapped into

$$\Phi(\chi_B) := V(\chi_B) + \sqrt{\rho} W(\chi_B) + \rho \mu_L(B) \cdot 1, \quad (1.9)$$

where $W(f) = b(f) + b(f)^*$. This operator has the same distribution in the vacuum state Ω_0 as $V(\chi_B)$ has in the coherent state $\exp[-\rho \mu_L(B)] \exp[\sqrt{\rho} b(\chi_B)^*] \Omega_0$, i.e., the Poisson distribution with mean $\rho \mu_L(B)$, in accordance with (1.7). In particular, the operator $\hat{H}(\mathbf{k})$ as defined in (1.5) takes the following form on the Fock space $\mathcal{F}$:

$$H_F(\mathbf{k}) = H_0(\mathbf{k}) + \sqrt{\rho} W(\varphi) + \rho \langle \varphi, \mu_L \rangle, \quad H_0(\mathbf{k}) = \bigoplus_{n=0}^{\infty} H_0^{(n)}(\mathbf{k}). \quad (1.10)$$

Here $H_0^{(0)}(\mathbf{k}) = k^2$ and, for $n > 0$,

$$H_0^{(n)}(\mathbf{k}) = (\mathbf{q}^{(n)} + \mathbf{k})^2 + \sum_{j=1}^n \varphi(\mathbf{x}_j) = (\mathbf{q}^{(n)} + \mathbf{k})^2 + V^{(n)}(\varphi), \quad \mathbf{q}^{(n)} = \sum_{j=1}^n \mathbf{q}_j, \quad \mathbf{q}_j = -i \partial_{\mathbf{x}_j}, \quad (1.11)$$

the latter being unitarily equivalent to

$$H_{\text{BO}}^{(n)} = (\mathbf{q}_1)^2 + \varphi(\mathbf{x}_1) + \sum_{j=2}^n \varphi(\mathbf{x}_1 + \mathbf{x}_j).$$

(Thus we deal with an n -potential Born–Oppenheimer Hamiltonian on the n th Fock layer.) As an example,

$$\mathbb{E}[z - H(\omega)]^{-1} = \int P(d\omega) [z - H(\omega)]^{-1} = \int d\mathbf{k} ([z - H_F(\mathbf{k})]^{-1} \Omega_0, \Omega_0). \quad (1.12)$$

These results are obtained in Sec. II, whereas in Sec. III we present a direct self-adjointness proof for $H(\mathbf{k})$, based upon Nelson's commutator theorem. The basic requirement is here that Σ_n , the bottom of the spectrum of $H_0^{(n)}$, obeys $\Sigma_n \geq -na$ with $a \geq 0$ n -independent. In this we recognize a simplified form of the stability of matter condition (if the bottom of the spectrum tends to $-\infty$ as the number of scatterers n increases, it does so at most linear in n). In Sec. IV we discuss our results and point out that in the approximation where only two Fock layers are retained, nontrivial results can already be obtained. The proofs of the theorems in Sec. II are given in the Appendix.

II. FOCK SPACE AND POISSON SPACE

In this section we discuss the unitary correspondence between Fock space and Poisson space alluded to in the Introduction. It was first discussed by Itô,⁷ and developed further by Ogura,⁸ Kabanov,⁹ Segall and Kailath,¹⁰ and, in particular, by Surgailis.¹¹ In this section we employ a notation, introduced by one of us (Maassen,¹² Lindsay and Maassen¹³), which simplifies many of the arguments considerably.

We start with a separable metric space S with its Borel σ -algebra of measurable subsets $\mathcal{B} = \mathcal{B}(S)$, on which a diffuse measure λ is given. As our main application we think of $S = \mathbb{R}^d$ and λ as Lebesgue measure. $C_c(S)$ will denote the set of compactly supported, continuous functions on S , $\mathcal{B}_b$ is the collection of bounded sets in $\mathcal{B}$, and $\mathcal{H}$ is the Hilbert space $L^2(S, \mathcal{B}, \lambda)$. By $\langle f, \nu \rangle$ we denote the integral $\int_S f d\nu$. In Secs. II A and II B below we discuss how a Fock space $\mathcal{F} = \mathcal{F}(S, \mathcal{B}, \lambda)$ and a Poisson space $\mathcal{P} = \mathcal{P}(S, \mathcal{B}, \lambda)$ are associated with the measure space $(S, \mathcal{B}, \lambda)$, and in the remaining subsections we collect the ingredients for the determination of $H(\mathbf{k})$.

A. Fock space

We introduce the space $\Gamma_n(S)$, $n \in \mathbb{N}$, and the space $\Gamma(S)$ of finite subsets of S by

$$\Gamma_n(S) := \{\omega \subset S \mid \#(\omega) = n\}, \quad \Gamma(S) := \bigcup_{n=0}^{\infty} \Gamma_n(S) = \{\omega \subset S \mid \#(\omega) < \infty\}. \quad (2.1)$$

Now let $S_\#^n$ denote the “off-diagonal” part of S^n , i.e., the set of all n -tuples of different points in S . Then $\Gamma_n(S)$ may be considered as the “symmetric part” of $S_\#^n$, obtained by identifying all n -tuples in S which only differ by a permutation, i.e., all n -tuples which have the same image under the map

$$j_n: S_\#^n \rightarrow \Gamma_n(S): (x_1, x_2, \dots, x_n) \rightarrow \{x_1, x_2, \dots, x_n\}.$$

Next let $\mathcal{B}_n$ denote the σ -algebra

$$\mathcal{B}_n := \{B \subset \Gamma_n(S) \mid j^{-1}(B) \in \mathcal{B}(S)^{\otimes n}\},$$

and define $\lambda_n: \mathcal{B}_n \rightarrow [0, \infty]$, by

$$\lambda_0(\{\emptyset\}) = 1, \quad \lambda_n(B) = [n!]^{-1} \lambda^{\otimes n}(j^{-1}(B)), \quad n \geq 1.$$

The sequence $\lambda_0, \lambda_1, \lambda_2, \lambda_3, \dots$ determines a measure e^λ on $\Gamma(S)$ by

$$e^\lambda\left(\bigcup_{n=0}^{\infty} B_n\right) = \sum_{n=0}^{\infty} \lambda_n(B_n), \quad B_n \in \mathcal{B}_n. \quad (2.2)$$

Finally we define the symmetric Fock space over (S, λ) by

$$\mathcal{F}(S, \lambda) := L^2(\Gamma(S), e^\lambda). \quad (2.3)$$

Note that

$$\mathcal{F}(S, \lambda) = \bigoplus_{n=0}^{\infty} L^2(\Gamma_n(S), \lambda_n) = \mathbb{C} \oplus L^2(S, \lambda) \oplus L^2(\Gamma_2(S), \lambda_2) \oplus \dots, \quad (2.4)$$

where $L^2(\Gamma_n(S), \lambda_n)$ is isomorphic to the n -fold symmetric tensor power of $\mathcal{H}$.

The present notation has the advantage that it suppresses factorials in many places. For instance, the well-known coherent vectors become straightforward products: For $f \in \mathcal{H}$ we define

$$\pi_f(\sigma) := \prod_{x \in \sigma} f(x), \quad \sigma \in \Gamma(S) \text{ nonempty}, \quad \pi_f(\emptyset) = 1. \quad (2.5)$$

The vectors $\{\pi_f \mid f \in \mathcal{H}\}$ are fundamental in $\mathcal{F}$ if $\mathcal{D}$ is dense in $\mathcal{H}$. Annihilation and creation operators $b(f)$ and $b(f)^*$ are given by

$$(b(f)\psi)(\sigma) = \int_S \lambda(dx) \overline{f(x)} \psi(\sigma \cup \{x\}), \quad (b(f)^*\psi)(\sigma) = \sum_{x \in \sigma} f(x) \psi(\sigma \setminus \{x\}). \quad (2.6)$$

Here σ is allowed to have any finite number of elements, including zero. The role of the vacuum vector (Ω_0 in Sec. I) is played by $\delta_\emptyset = \pi_0$; $\delta_\emptyset(\sigma) = 1$ if $\sigma = \emptyset$ and zero otherwise. Note that $b(f)\delta_\emptyset = 0$ and $b(f)^*\delta_\emptyset = 0 \oplus f \oplus 0 \oplus 0 \dots$, and also that $[b(f), b(g)^*] = (g, f)$. Furthermore, if N denotes the multiplication operator by the “particle number,” $(N\psi)(\sigma) := \#(\sigma)\psi(\sigma)$, and $\psi \in \mathcal{L}(N)$, then $\|b(f)\psi\|^2 \leq (N\psi, \psi)\|f\|^2 = \|N^{1/2}\psi\|^2\|f\|^2$ and $\|b(f)^*\psi\|^2 \leq \|(N+1)^{1/2}\psi\|^2\|f\|^2$. In particular, if $\{u_j \mid j \in \mathbb{N}\}$ is an orthonormal basis for $\mathcal{H}$, then $\sum_{j=0}^{\infty} \|b(u_j)\psi\|^2 = (N\psi, \psi)$. We finally note that

$$\begin{aligned}
(\pi_f, \pi_g) &= \int_{\Gamma(S)} e^{\lambda(d\sigma)} \pi_f(\sigma) \overline{\pi_g(\sigma)} \\
&= \sum_{n=0}^{\infty} \int_{\Gamma_n(S)} \lambda_n(d\sigma) \pi_{f\bar{g}}(\sigma) \\
&= 1 + \sum_{n=1}^{\infty} [n!]^{-1} \int_{S^n_{\#}} \lambda(dx_1) \cdots \lambda(dx_n) \pi_{f\bar{g}}(\{x_1, x_2, \dots, x_n\}) \\
&= \sum_{n=0}^{\infty} [n!]^{-1} \left(\int_S \lambda(dx) f(x) \overline{g(x)} \right)^n = \exp[(f, g)], \tag{2.7}
\end{aligned}$$

since $S^n \setminus S^n_{\#}$ has $\lambda^{\otimes n}$ -measure zero.

B. Poisson space

Let $\Omega(S)$ denote the collection of countable subsets of S without accumulation points. In the present context we prefer to use these countable subsets ω rather than the corresponding measures given by $\mu_{\omega} = \sum_{x \in \omega} \delta_x$, or, equivalently, $\mu_{\omega}(B) = \#(\omega \cap B)$, $B \in \mathcal{B}(S)$. For $B \in \mathcal{B}_b(S)$ and X a Borel subset of $\Gamma(B)$, we consider the cylinder set

$$\Sigma(B, X) := \{\omega \in \Omega(S) \mid \omega \cap B \in X\},$$

consisting of those configurations which, as seen through the “window” B , look like a configuration in X . The Poisson distribution P_{λ} with intensity λ is the probability measure on $\Omega(S)$ defined by

$$P_{\lambda}(\Sigma(B, X)) = e^{\lambda(X)} \exp[-\lambda(B)].$$

In particular

$$P_{\lambda}(\{\omega \in \Omega \mid \#(\omega \cap B) = n\}) = e^{\lambda(\Gamma_n(B))} \exp[-\lambda(B)] = [n!]^{-1} \lambda(B)^n \exp[-\lambda(B)]. \tag{2.8}$$

We define the Poisson space $\mathcal{P}(S, \lambda)$ by

$$\mathcal{P}(S, \lambda) := L^2(\Omega(S), \mathcal{F}, P_{\lambda}), \tag{2.9}$$

where $\mathcal{F}$ denotes the σ -algebra generated by the cylinder sets $\Sigma(B, X)$. Important vectors in $\mathcal{P}(S, \lambda)$ are again the product vectors, given by

$$\pi_f(\omega) := \prod_{x \in \omega} f(x), \tag{2.10}$$

defined for those $f \in C(S)$ which take the value 1 outside some bounded region.

C. Unitary equivalence

The decomposition in “orthogonal chaos” or “chaotic decomposition”¹⁴ of vectors in $\mathcal{P} = L^2(\Omega, P_{\lambda})$, which yields a natural equivalence with $\mathcal{F} = L^2(\Gamma, e^{\lambda})$, is based upon repeated stochastic integration with respect to the Poisson distribution. We can integrate functions $f \in C_c(S)$ with respect to the random measure $\mu: \omega \rightarrow \mu_{\omega}$ as follows:

$$\int_S f(x) \mu_\omega(dx) = \langle f, \mu_\omega \rangle = \sum_{x \in \omega} f(x).$$

However, repeated stochastic integration with respect to μ does not yield good orthogonality properties for the stochastic integrals; for the chaotic decomposition one rather considers the compensated Poisson distribution $\nu := \mu - \lambda$. A function $\vartheta \in \mathcal{P}$ is formally expanded as

$$\vartheta = \sum_{n=0}^{\infty} \int_{\Gamma_n(S)} \nu(dx_1) \nu(dx_2) \cdots \nu(dx_n) \psi(x_1, x_2, \dots, x_n). \quad (2.11)$$

In the notation of Sec. II A this can, again formally, be written as

$$\vartheta(\omega) = \int_{\Gamma(S)} e^{\mu_\omega - \lambda}(d\zeta) \psi(\zeta) = \sum_{\substack{\sigma \subset \omega \\ \sigma \text{ finite}}} \int_{\Gamma(S)} e^\lambda(d\eta) (-1)^{\#(\eta)} \psi(\sigma \cup \eta). \quad (2.12)$$

In particular, if we substitute $\psi = \pi_f$ in (2.12) we obtain

$$\vartheta = \exp \left[- \int f d\lambda \right] \pi_{f+1}. \quad (2.13)$$

The latter formula gives us an entrance to a precise formulation:

Theorem 2.1: There exists a unique unitary equivalence $U: \mathcal{F} \rightarrow \mathcal{P}$, which maps π_f to $\exp[-\int f d\lambda] \pi_{f+1}$ for all $f \in C_c(S)$. The image $U\psi$ of any $\psi \in \mathcal{F}$ is given by the right-hand side of (2.12) whenever the function $C: \Omega \rightarrow \mathbb{C}$,

$$C(\omega) := \sum_{\substack{\sigma \subset \omega \\ \sigma \text{ finite}}} \int_{\Gamma(S)} e^\lambda(d\eta) |\psi(\sigma \cup \eta)|,$$

is square integrable (P_λ). This includes the case where ψ has bounded support and lies in the domain of $(\sqrt{3})^N$.

Proof: Cf. the Appendix.

Note that $U\delta_\emptyset = 1$ and $E(U\psi) = \langle 1, U\psi \rangle = \langle U^{-1}1, \psi \rangle = \langle \delta_\emptyset, \psi \rangle = \psi(\emptyset)$ for $\psi \in \mathcal{F}$. Moreover, U maps the space Λ_B of functions in $\mathcal{F}$ with support B to the space Λ'_B of functions in $\mathcal{P}$ whose values depend only on $\omega \cap B$. Since the orthogonal projection onto Λ'_B of a function ϑ is the conditional expectation $E_B \vartheta$ of ϑ , given $\omega \cap B$, it follows that

$$U\pi_{\chi_B} U^{-1} = E_B, \quad B \in \mathcal{B}(S). \quad (2.14)$$

D. The multiplication formula

When two functions $\vartheta_1, \vartheta_2 \in \mathcal{P}$ have chaotic expansions $\vartheta_1 = U\psi_1$ and $\vartheta_2 = U\psi_2$, and their product is square integrable, then it, too, allows a unique chaotic decomposition, say $\vartheta_1 \cdot \vartheta_2 = U\psi$. We shall denote ψ by $\psi_1 \star \psi_2$, the Poisson product of ψ_1 and ψ_2 . This product was first described by Surgailis¹¹ and was given in the present notation without proof by Lindsay and Maassen¹³ and Meyer,¹⁴ among products of other processes with independent increments. Here we shall prove the Poisson formula for an infinite measure space and indicate a range of validity.

Theorem 2.2: The Poisson product of ψ_1 and ψ_2 in $\mathcal{F}(S, \lambda)$ is given by

$$(\psi_1 \star \psi_2)(\sigma) := \sum_{\substack{\alpha \cup \beta \cup \gamma = \sigma \\ \alpha, \beta, \gamma \text{ disjoint}}} \int_{\Gamma(S)} e^{\lambda(d\eta)} \psi_1(\alpha \cup \gamma \cup \eta) \psi_2(\alpha \cup \beta \cup \eta), \quad (2.15)$$

whenever the right-hand side converges absolutely and (2.15) with the integrand replaced by $|\psi_1(\alpha \cup \gamma \cup \eta) \psi_2(\alpha \cup \beta \cup \eta)|$ yields a square integrable function of σ . This is the case if ψ_1 and ψ_2 have a common support $A \in \mathcal{B}_b(S)$ and

$$\int_{\Gamma(S)} e^{\lambda(d\sigma)} 4^{\#(\sigma)} |\psi_j(\sigma)|^4 < \infty, \quad j=1,2. \quad (2.16)$$

Proof: Cf. the Appendix.

E. The potential $\langle \varphi, \mu_\omega \rangle$ in Fock space

Our main aim is to translate Eq. (1.5) into its Fock space equivalent. In particular we must find the equivalent $\Phi(\varphi)$ of multiplication by the function $\omega \rightarrow \langle \varphi, \mu_\omega \rangle = \sum_{x \in \omega} \varphi(x)$ as announced in (1.9). Now, this function has the chaotic decomposition

$$\langle \varphi, \mu \rangle = \langle \varphi, \lambda \rangle + \langle \varphi, \nu \rangle = \int_S \varphi \, d\lambda + \int_S \varphi \, d\nu = U(\langle \varphi, \lambda \rangle \oplus \varphi \oplus 0 \oplus \dots), \quad (2.17)$$

provided $\varphi \in L^1 \cap L^2(S)$. The following summary of the situation is a corollary to Theorems 2.1 and 2.2. By $V(\varphi)$ we shall denote the operator of multiplication by $\sigma \rightarrow \sum_{x \in \omega} \varphi(x)$. Note that this is a finite sum, unlike the one defining $\Phi(\varphi)$.

Corollary 2.3: Let $\varphi \in L^1 \cap L^2(S)$. Then the random variable

$$F(\omega) := \sum_{x \in \omega} \varphi(x)$$

is well defined and has mean $\int_S \varphi \, d\lambda$ and variance $\int_S |\varphi|^2 \, d\lambda$. Moreover, for all $\psi \in \mathcal{F}$ in the intersection of the domains of $b(|\varphi|)$, $b(|\varphi|)^*$, and $V(|\varphi|)$ (which contains the dense set of finite particle vectors with compact support on each Fock layer) one has

$$\Phi(\varphi)\psi = F \cdot U\psi = U(\langle \varphi, \lambda \rangle \cdot \mathbf{1}_{\mathcal{F}} + V(\varphi) + b(\varphi) + b(\varphi)^*)\psi. \quad (2.18)$$

Proof: See the Appendix.

F. The operator $H_F(\mathbf{k})$

We are now in a position to present the Fock space version $H_F(\mathbf{k})$ of $\hat{H}(\mathbf{k})$. For this it remains to determine the equivalent of the term $(\mathbf{q} + \mathbf{k})^2$ in (1.5). Here we have to be specific about the measure space (S, λ) . Thus $S = \mathbb{R}^d$ and $\lambda = \rho \cdot \mu_L$, where μ_L is Lebesgue measure (see Sec. I). It is readily seen that the generator of translations $\mathbf{q}$ in the Poisson context translates into the set of translation generators on the respective Fock layers. Thus, in terms of a direct sum over the latter, $\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{F}^{(n)}$,

$$\mathbf{q}_F := U^{-1} \mathbf{q} U = \bigoplus_{n=0}^{\infty} \mathbf{q}^{(n)}, \quad \mathbf{q}^{(0)} = 0, \quad \mathbf{q}^{(n)} = \sum_{j=1}^n \mathbf{q}_j, \quad h > 0, \quad (2.19)$$

where $\mathbf{q}_j = -i\partial_{x_j}$ on the C_c^∞ -functions in $\mathcal{F}^{(n)} = L^2(\mathbb{R}^{dn})_{\text{sym}}$. Collecting results we arrive at

$$H_F(\mathbf{k}) = H_0(\mathbf{k}) + b(\varphi) + b(\varphi)^* + \rho \langle \varphi, \mu_L \rangle = H_0(\mathbf{k}) + W(\varphi) + \rho \langle \varphi, \mu_L \rangle, \quad (2.20)$$

$$H_0(\mathbf{k}) = \bigoplus_{n=0}^{\infty} H_0^{(n)}(\mathbf{k}), \quad H_0^{(0)}(\mathbf{k}) = k^2, \quad H_0^{(n)}(\mathbf{k}) = (\mathbf{q}^{(n)} + \mathbf{k})^2 + \sum_{j=1}^n \varphi(\mathbf{x}_j), \quad n > 0.$$

So far we did not consider domain questions. However, as discussed in Ref. 6, there exists a core for H , such that $H = \mathbf{p}^2 + \langle \varphi_{\mathbf{x}}, \mu \rangle$ on the former. Cores being unitarily invariant, this translates to $\hat{H}$, $\hat{H}(\mathbf{k})$, and $H_F(\mathbf{k})$. In the next section, however, we shall present a direct proof of the essential self-adjointness of $H_F(\mathbf{k})$ on the C_c^∞ -finite particle vectors in $\mathcal{F}$.

F. A scale transformation

In the special case $S = \mathbb{R}^d$, $\lambda = \rho \cdot \mu_L$, it is convenient to translate our results from the Fock space $\mathcal{F}_\rho := \mathcal{F}(\mathbb{R}^d, \rho \cdot \mu_L)$ to the more conventional $\mathcal{F} := \mathcal{F}(\mathbb{R}^d, \mu_L)$. This is readily done by means of the unitary map

$$T_\rho: \mathcal{F}_\rho \rightarrow \mathcal{F}: \psi \mapsto \rho^{N/2} \psi.$$

Noting that $T_\rho b(f) T_\rho^{-1} = \sqrt{\rho} b(f)$, $T_\rho b(f)^* T_\rho^{-1} = \sqrt{\rho} b(f)^*$ and $T_\rho V(f) T_\rho^{-1} = V(f)$, we find that $H_F(\mathbf{k})$, acting in $\mathcal{F}$, takes the form $H'_F(\mathbf{k}) := T_\rho H_F(\mathbf{k}) T_\rho^{-1}$, given by

$$H'_F(\mathbf{k}) = H_0(\mathbf{k}) + \sqrt{\rho} \{b(\varphi, \mu_L) + b(\varphi, \mu_L)^*\} + \rho \langle \varphi, \mu_L \rangle = H_0(\mathbf{k}) + \sqrt{\rho} W + \rho \langle \varphi, \mu_L \rangle. \quad (2.21)$$

This is Eq. (1.10), where we have dropped the prime in $H'_F(\mathbf{k})$. Since the vacuum state is invariant under T_ρ , vacuum expectation values of functions of $H_F(\mathbf{k})$ directly translate into vacuum expectation values of the same functions of $H'_F(\mathbf{k})$. This is, however, no longer true for more general inner products. Note further that, apart from the trivial term $\rho \langle \varphi, \mu_L \rangle$, the dependence on ρ now enters through the coupling constant $\sqrt{\rho}$ in the term connecting neighboring Fock layers.

III. SELF-ADJOINTNESS OF $H_F(\mathbf{k})$

The expression (2.21) consists of three terms, the last one being a constant, which is finite for $\varphi \in L^1(\mathbb{R}^d)$. The second, $W(\varphi) = b(\varphi) + b(\varphi)^*$, is known to be essentially self-adjoint on the finite particle vectors $\mathcal{F}_0 \subset \mathcal{F}$, provided $\varphi \in L^2(\mathbb{R}^d)^{15}$ and this remains true on $\mathcal{E}$, the C_c^∞ -finite particle vectors. We now turn to the first term $H_0(\mathbf{k})$. Its vacuum component is trivial, so it remains to consider $H_0^{(n)}(\mathbf{k})$ for $n > 0$. We do so with $\mathcal{F}^{(n)}$ replaced by $L^2(\mathbb{R}^d)$. Then we can always make the restriction to $\mathcal{F}^{(n)}$. $H_0^{(n)}(\mathbf{k})$ is well defined as a symmetric operator for locally square integrable φ and, as observed in the introduction, is unitarily equivalent to

$$\tilde{H} = (\mathbf{q}_1)^2 + \varphi(\mathbf{x}_1) + \sum_{j=2}^n \varphi(\mathbf{x}_1 + \mathbf{x}_j). \quad (3.1)$$

The connecting unitary transformation consists of shifts and boosts and leaves $C_c^\infty(\mathbb{R}^{dn})$ invariant. Clearly $\tilde{H}$ allows a direct integral decomposition, the coordinates $\mathbf{x}_2, \dots, \mathbf{x}_n$, labelling the fibers.

Now, for fixed $\mathbf{x}_2, \dots, \mathbf{x}_n$, we are dealing with an ordinary one-particle Schrödinger operator $\tilde{H}(\mathbf{x}_2, \dots, \mathbf{x}_n)$. For a large class of φ $C_c^\infty(\mathbb{R}^d)$ is a core for this operator and φ is $((\mathbf{q}_1)^2, 0)$ -bounded, i.e., $(\mathbf{q}_1)^2$ -bounded with zero relative bound in either the operator or form sense. For $\varphi \in L^1 \cap L^2(\mathbb{R}^d)$, as we already assumed earlier in this section, this is the case for $d \leq 3$, whereas for $d > 3$, $\varphi \in L^p(\mathbb{R}^d)$, $p > 2$ for $d = 4$, and $p \geq d/2$ for $d \geq 5$ is a sufficient condition. Weaker conditions are allowed if $\varphi \geq 0$, but what is essential here is that there is an n -independent $a \geq 0$, such that

$$\tilde{H}(\mathbf{x}_2, \dots, \mathbf{x}_n) \geq -na, \quad (3.2)$$

which follows for $((\mathbf{q}_1)^2, 0)$ -bounded φ . Then (3.2) also holds for $\tilde{H}$ and, *a fortiori*, for $H_0^{(n)}(\mathbf{k})$. It now follows that $H_0(\mathbf{k})$ is essentially self-adjoint on $\mathcal{E}$ and is bounded from below by $-aN$, where N is the number operator. As observed earlier this is a simple form of the “stability of matter” condition.

Theorem 3.1: Let $\varphi \in L^1 \cap L^2(\mathbb{R}^d)$ and suppose in addition that $\mathcal{E}$ is a core for $H_0(\mathbf{k}) \geq -aN$, $a \geq 0$. Then $H(\mathbf{k})$ is essentially self-adjoint on $\mathcal{E}$.

Proof: The proof consists of a straightforward application of Nelson’s commutator theorem. We already have $H_0(\mathbf{k}) + aN \geq 0$. We also have, using the estimates from Sec. II, $\|Wf\| \leq 2\|\varphi\| \|(N+1)^{1/2}f\|$ and hence $\sqrt{\rho}W$ is $(H_0(\mathbf{k}) + cN, 1-\varepsilon)$ -bounded for sufficiently large c . Consequently $H(\mathbf{k}) + cN$ and $M := H(\mathbf{k}) + cN + 1$ are self-adjoint on $\mathcal{D}(H_0(\mathbf{k}) + cN)$ [note that $H_0(\mathbf{k}) + cN$ is essentially self-adjoint on $\mathcal{E}$]. We also note that there exists a $d > 0$ such that $W + dN \geq 0$. Thus, for $c \geq a + d + 1$, as a form on $\mathcal{E}$: $M = \{H_0(\mathbf{k}) + aN\} + \{\sqrt{\rho}W + dN\} + \{(c - a - d)N + 1\} \geq N + 1 \geq 1$, since all terms between parenthesis are non-negative. For $f \in \mathcal{E}$ we now have

$$\|H(\mathbf{k})f\| = \|(M - cN - 1)f\| \leq (2 + c)\|Mf\|$$

and

$$\begin{aligned} |(H(\mathbf{k})f, Mf) - (Mf, H(\mathbf{k})f)| &= c\sqrt{\rho}|([N, W]f, f)| = c\sqrt{\rho}|(\{b(\varphi) - b(\varphi)^*\}f, f)| \\ &= c\sqrt{\rho}|(\{\operatorname{Im} b(\varphi)\}f, f)| \leq 2c\sqrt{\rho}\|b(\varphi)\|^{1/2}\|f\|^2 \\ &\leq 2c\sqrt{\rho}\|\varphi\| \|(N+1)\|^{1/4}\|f\|^2 \leq \text{constant} \cdot \|M^{1/2}f\|^2. \end{aligned}$$

Thus Nelson’s theorem¹⁵ applies and we are finished. $\square$

IV. DISCUSSION

Our results about the Poisson–Fock correspondence immediately raise the question whether they can be generalized to other situations. Indeed a large class of processes can be generated within a Fock space context (Ref. 2, p. 239, Ref. 3, ch. 2-21, p. 152), but the simple correspondence $\langle \varphi, \mu \rangle \leftrightarrow \Phi(\varphi)$ will be lost in general.

Let us return to Eq. (1.9)

$$P(\#(\omega \cap B) = k) = (\delta_{k, V(\chi_B)} \psi_{\text{coh}}(\rho, B), \psi_{\text{coh}}(\rho, B))_{\mathcal{F}}, \quad (4.1)$$

which follows immediately from $\psi_{\text{coh}}(\rho, B) = \exp[-\rho|B|/2] \pi \chi_B$. Different statistics emerge if the coherent states in (4.1) are replaced by other ones. In quantum optics (based upon a Fock space formalism for the photon field) such situations are the object of intense research (see Ref. 16 for a survey).

In the preceding sections we have found a Fock space representation for the operator $\hat{H}(\mathbf{k})$ and have shown that averages can be presented in terms of $H_F(\mathbf{k})$, given by (1.10). The latter is actually acting in a fiber in a direct integral decomposition and we can reconstruct

$$\hat{H}_F(\mathbf{p}) = \int_{\mathbb{R}^d}^{\oplus} d\mathbf{k} H_F(\mathbf{k}), \quad (4.2)$$

acting in $\mathcal{H} \otimes \mathcal{F}$. We can retrace the step $\hat{H} \rightarrow H$ in the present Fock space setting, giving

$$H_F = \mathbf{p}^2 \otimes \mathbf{I}_{\mathcal{F}} + V(\varphi_{\mathbf{x}}^-) + \sqrt{\rho}\{b(\varphi_{\mathbf{x}}^-) + b(\varphi_{\mathbf{x}}^-)^*\} + \rho\langle \varphi, \mu_L \rangle. \quad (4.3)$$

The expression 1.10, the result of corollary 2.3, is an intriguing one. On each Fock layer n we encounter a particle moving in n potentials and the randomness manifests itself in the coupling of the layers. Vacuum expectation values are functions of ρ rather than its square root since we have to “ladder upwards and downwards” an equal number of steps if we start from and end up in the vacuum state. For $\rho=0$ vacuum expectation values are free particle quantities. However, as soon as we include a second layer ($n=1$), i.e., we cut $H_F(\mathbf{k})$ off beyond the second layer, $H_F(\mathbf{k}) \rightarrow H^{(1)} := H_0^{(1)} + W^{(1)}$, we already obtain nontrivial results. Upon such a restriction W becomes a bounded operator and we can expand:

$$\mathbb{E}[z - H^{(1)}]^{-1} = ([z - H^{(1)}]^{-1} \Omega_0, \Omega_0) = \sum_{j=0}^{\infty} ([z - H_0^{(1)}]^{-1} [\sqrt{\rho} W^{(1)} [z - H_0^{(1)}]^{-1}]^j \Omega_0, \Omega_0). \quad (4.4)$$

Here $W^{(1)}$ maps Ω_0 into $\varphi^{(1)} = (0 \oplus \varphi \oplus 0 \oplus \dots)$ and $\psi^{(1)} = (0 \oplus \psi \oplus 0 \oplus \dots)$ into $(\varphi, \psi)_1$, the subscript 1 referring to the inner product on the layer $n=1$, i.e., on $\mathcal{H} = L^2(\mathbb{R}^d)$. Working things out we find

$$\mathbb{E}[z - H^{(1)}]^{-1} = [z - k^2 - \rho \cdot \langle \mathbf{k} | t(z) | \mathbf{k} \rangle]^{-1}, \quad (4.5)$$

where t is the one-potential transition operator: $t(z) := \varphi + \varphi [z - \mathbf{p}^2 - \varphi]^{-1} \varphi$ on $\mathcal{H}$. Note that the square integrable function φ converts the L^∞ -plane wave $|\mathbf{k}\rangle$ into $\varphi|\mathbf{k}\rangle \mapsto (2\pi)^{-d/2} \cdot \exp[i\mathbf{k} \cdot \mathbf{x}] \varphi(\mathbf{x}) \in \mathcal{H}$. For a large class of φ s a limiting absorption principle applies with the result that $\langle \mathbf{k} | t(E \pm i\varepsilon) | \mathbf{k} \rangle$ have limits for $E > 0$ (and for $E < 0$, not an eigenvalue). Thus, in this situation, the mass operator $\Sigma(z, \mathbf{k}, \rho)$ (see Ref. 6 for its definition) becomes $\Sigma^{(1)}(z, \mathbf{k}, \rho) = \rho \cdot \langle \mathbf{k} | t(z) | \mathbf{k} \rangle$ and has the above limits. The integrated density of states, $N(E)$, can also be calculated in the same approximation. Adding more Fock layers, say up to order n , does not change the picture. Repeated use of the second resolvent equation with $\sqrt{\rho}W$ as the perturbation gives an expansion for the approximate mass operator in powers of ρ . Whether or not $\Sigma(z, \mathbf{k}, \rho)$ has an asymptotic expansion or is actually analytic in a neighborhood of $\rho=0$ seems to be an open problem.

A similar approach can be attempted on the density operator level. Without going into the complicated calculations, we note that the heuristic methods, commonly used in statistical mechanics, applied to the two-layer approximation give rise to a Boltzmann equation in three dimensions, thus leading to diffusive behavior. This raises the question whether corresponding rigorous results can be obtained for diffusion in the two-layer case.

Finally, for a special class of repulsive potentials φ , commutator estimates of the Mourre type¹⁷ can be derived (within the Fock space setting) with the result that $N(E)$ is absolutely continuous for $E > k\rho \langle \varphi, \mu_L \rangle$, where k is a dimension-dependent positive number.¹⁸

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APPENDIX: PROOFS OF THE RESULTS OF SECTION II

In this appendix we provide the proofs of Theorems 2.1 and 2.2 and Corollary 2.3. In view of the formal similarity between our proofs of Theorems 2.1 and 2.2 we shall treat them together. Let $\Pi \subset \mathcal{F}$ denote the linear span of the functions π_f ($f \in C_0(S)$) and $\mathcal{E} \subset \mathcal{P}$ the span of π_h ($h \in C(S)$, $h=1$ outside some bounded region). We define the unitary map $U: \mathcal{F} \rightarrow \mathcal{P}$ by continuous extension of the linear map $U_0: \Pi \rightarrow \mathcal{E}$, defined by

$$U_0 \pi_f = \exp \left[- \int_S f d\lambda \right] \pi_{f+1}. \quad (\text{A1})$$

This continuous extension is possible and uniquely defined since Π and $\mathcal{E}$ are dense subspaces of $\mathcal{F}$ and $\mathcal{P}$, respectively, and, for all $f, g \in C_c(S)$ [say with common support $A \in \mathcal{B}_b(S)$],

$$\begin{aligned} (U_0 \pi_f, U_0 \pi_g)_{\mathcal{P}} &= \exp \left[- \int_A d\lambda (f + \bar{g}) \right] (\pi_{f+1}, \overline{\pi_{g+1}})_{\mathcal{P}} \\ &= \exp \left[- \int_A d\lambda (f + \bar{g}) \right] \exp[-\lambda(A)] \int_{\Gamma(A)} d e^\lambda \pi_{f+1} \cdot \overline{\pi_{g+1}} \\ &= \exp \left[- \int_A d\lambda (f + \bar{g}) - \lambda(A) + \int_A d\lambda (f+1)(\bar{g}+1) \right] \\ &= \exp \left[\int_A d\lambda f \bar{g} \right] = (\pi_f, \pi_g)_{\mathcal{F}}. \end{aligned} \quad (\text{A2})$$

This unitary correspondence U enables us to introduce a binary operation $\star$ on a large domain $\mathcal{D} = \{(\psi_1, \psi_2) \in \mathcal{F} \times \mathcal{F} \mid U\psi_1 \cdot U\psi_2 \in \mathcal{P}\}$ as follows:

$$\psi_1 \star \psi_2 := U^{-1}(U\psi_1 \cdot U\psi_2). \quad (\text{A3})$$

Our aim is to show that the expressions

$$(\tilde{U}\psi)(\omega) := \sum_{\substack{\sigma \subseteq \omega \\ \sigma \text{ finite}}} \int_{\Gamma(S)} e^\lambda(d\eta) (-1)^{\#(\eta)} \psi(\sigma \cup \eta) \quad (\text{A4})$$

and

$$\psi_1 \star \psi_2(\sigma) := \sum_{\substack{\alpha \cup \beta \cup \gamma = \sigma \\ \alpha, \beta, \gamma \text{ disjoint}}} \int_{\Gamma(S)} e^\lambda(d\eta) \psi_1(\alpha \cup \gamma \cup \eta) \psi_2(\alpha \cup \beta \cup \eta) \quad (\text{A5})$$

define a map $\tilde{U}$ and an operation $\star$, which, on their respective domains, coincide with U and $\star$. As was noted in (2.13), $\tilde{U}\psi = U\psi$ holds for $\psi \in \Pi$, and a straightforward calculation shows that $\psi_1 \star \psi_2 = \psi_1 \star \psi_2$ for $\psi_1, \psi_2 \in \Pi$:

$$\begin{aligned} (\pi_f \star \pi_g)(\sigma) &= \sum_{\substack{\alpha \cup \beta \cup \gamma = \sigma \\ \alpha, \beta, \gamma \text{ disjoint}}} \int_{\Gamma(S)} e^\lambda(d\eta) \pi_f(\alpha \cup \gamma \cup \eta) \pi_g(\alpha \cup \beta \cup \eta) \\ &= \pi_{f+g+f_g}(\sigma) \cdot \exp \left[\int_A d\lambda f g \right], \end{aligned}$$

so that

$$U(\pi_f \star \pi_g) = \exp \left[- \int_A d\lambda (f + g) \right] \pi_{(f+1)(g+1)} = U\pi_f \cdot U\pi_g.$$

In order to extend these results we need the following estimates.

Proposition A.1: Let the measurable functions $\psi, \psi_1, \psi_2: S \rightarrow \mathbb{C}$ have support $A \in \mathcal{B}_b(S)$. Then (the value infinity is allowed on the right-hand sides)

$$\|\tilde{U}\psi\| \leq \exp[\lambda(A)] \|(\sqrt{3})^N \psi\|, \quad (\text{A6})$$

$$\|\psi_1 \star \psi_2\| \leq \exp[8\lambda(A)] \|(\sqrt{2})^N \psi_1\|_4 \|(\sqrt{2})^N \psi_2\|_4. \quad (\text{A7})$$

Remark: It has been noted^{11,19} that $\psi_1, \psi_2 \in \mathcal{F}_1$ does not imply that $\psi_1 \star \psi_2$ is square integrable. This is also borne out by the L^4 -norm in (A7).

For the proofs we need two lemmas.

Lemma A.2: For all $f \in L^1(\Gamma(S) \times \Gamma(S))$

$$\int_{\Gamma(S)} e^{\lambda(d\alpha)} \int_{\Gamma(S)} e^{\lambda(d\beta)} f(\alpha, \beta) = \int_{\Gamma(S)} e^{\lambda(d\sigma)} \sum_{\alpha \in \sigma} f(\alpha, \sigma \setminus \alpha).$$

Proof: See Ref. 20, Lemma 2.3.3, or Ref. 21, Prop. 3.1.

Lemma A.3: Let $\psi \in \text{Dom}(\sqrt{1+\varepsilon})^N$ for some $\varepsilon > 0$. Then, for almost all $\alpha \in \Gamma(S)$, the function

$$T_\alpha \psi: \beta \rightarrow \psi(\alpha \cup \beta)$$

is square integrable. Moreover

$$\int_{\Gamma(S)} e^{\lambda(d\alpha)} \varepsilon^{\#(\alpha)} \|T_\alpha \psi\|^2 = \|(\sqrt{1+\varepsilon})^N \psi\|^2. \quad (\text{A8})$$

Proof: By Lemma A.2 and Fubini's theorem we have

$$\begin{aligned} \int_{\Gamma(S)} e^{\lambda(d\alpha)} \varepsilon^{\#(\alpha)} \|T_\alpha \psi\|^2 &= \int_{\Gamma(S)} e^{\lambda(d\alpha)} \varepsilon^{\#(\alpha)} \int_{\Gamma(S)} e^{\lambda(d\beta)} |\psi(\alpha \cup \beta)|^2 \\ &= \int_{\Gamma(S)} e^{\lambda(d\sigma)} \sum_{\alpha \subset \sigma} |\psi(\sigma)|^2 \varepsilon^{\#(\alpha)} \\ &= \int_{\Gamma(S)} e^{\lambda(d\sigma)} (1+\varepsilon)^{\#(\sigma)} |\psi(\sigma)|^2 = \|(\sqrt{1+\varepsilon})^N \psi\|^2, \end{aligned}$$

and it follows that $T_\alpha \psi \in \mathcal{F}$ for almost all α . □

Proof of Proposition A.1: Let $\varepsilon_A: \Gamma(S) \rightarrow \{-1, 0, 1\}$ be defined through

$$\varepsilon_A(\sigma) = \begin{cases} (-1)^{\#(\sigma)} & \text{if } \sigma \subset A, \\ 0 & \text{otherwise} \end{cases}.$$

For $\psi \in \text{Dom}((\sqrt{3})^N)$, we have $T_\alpha \psi \in \mathcal{F}$ a.e. and according to the definition (A4) of $\tilde{U}$,

$$(\tilde{U}\psi)(\omega) := \sum_{\substack{\sigma \subset \omega \\ \sigma \text{ finite}}} (\varepsilon_A, T_\sigma \psi)_{\mathcal{F}}.$$

By the general inequality

$$\left| \sum_{j=1}^n z_j \right|^2 \leq n \sum_{j=1}^n |z_j|^2, \quad z_j \in \mathbb{C},$$

we have

$$\begin{aligned} |(\tilde{U}\psi)(\omega)|^2 &\leq 2^{*(\omega \cap A)} \sum_{\sigma \subset \omega \cap A} |(\varepsilon_A, T_\sigma \psi)|^2 \\ &\leq 2^{*(\omega \cap A)} \sum_{\sigma \subset \omega \cap A} \|\varepsilon_A\|^2 \|T_\sigma \psi\|^2 = 2^{*(\omega \cap A)} e^{\lambda(A)} \sum_{\sigma \subset \omega \cap A} \|T_\sigma \psi\|^2. \end{aligned}$$

By Lemmas A.2 and A.3 we deduce that

$$\begin{aligned} \|\tilde{U}\psi\|^2 &= \int_{\Omega} P_{\lambda}(d\omega) |(\tilde{U}\psi)(\omega)|^2 = e^{-\lambda(A)} \int_{\Gamma(A)} e^{\lambda(d\xi)} |(\tilde{U}\psi)(\xi)|^2 \leq \int_{\Gamma(A)} e^{\lambda(d\xi)} 2^{*(\xi)} \sum_{\sigma \subset \xi} \|T_\sigma \psi\|^2 \\ &= \int_{\Gamma(A)} e^{\lambda(d\tau)} \int_{\Gamma(A)} e^{\lambda(d\sigma)} 2^{*(\tau \cup \sigma)} \|T_\sigma \psi\|^2 \\ &= \left\{ \int_{\Gamma(A)} e^{\lambda(d\tau)} 2^{*(\tau)} \right\} \left\{ \int_{\Gamma(A)} e^{\lambda(d\sigma)} 2^{*(\sigma)} \|T_\sigma \psi\|^2 \right\} = e^{2\lambda(A)} \|(\sqrt{3})^N \psi\|^2, \end{aligned}$$

proving the first inequality (A.6).

Next, for $\psi_1, \psi_2 \in \mathcal{F}$, supported by $A \in \mathcal{B}_b(S)$, we may write the definition (A5) of $\psi_1 \star \psi_2$ as

$$\psi_1 \star \psi_2(\sigma) = \sum_{\substack{\alpha \cup \beta \cup \gamma = \sigma \\ \alpha, \beta, \gamma \text{ disjoint}}} (T_\alpha T_\gamma \psi_1, T_\beta T_\gamma \psi_2).$$

Hence

$$|\psi_1 \star \psi_2(\sigma)|^2 \leq 3^{*(\sigma)} \sum_{\substack{\alpha \cup \beta \cup \gamma = \sigma \\ \alpha, \beta, \gamma \text{ disjoint}}} |(T_\alpha T_\gamma \psi_1, T_\beta T_\gamma \psi_2)|^2,$$

so that, by Lemmas A.2 and A.3 and Schwartz's inequality,

$$\begin{aligned} \|\psi_1 \star \psi_2\|^2 &\leq \int_{\Gamma(A)} e^{\lambda(d\sigma)} |\psi_1 \star \psi_2(\sigma)|^2 \\ &\leq \int_{\Gamma(A)^3} e^{\lambda(d\alpha)} e^{\lambda(d\beta)} e^{\lambda(d\gamma)} 3^{*(\alpha) + *(\beta) + *(\gamma)} \|T_\alpha T_\gamma \psi_1\|^2 \cdot \|T_\beta T_\gamma \psi_2\|^2 \\ &= \int_{\Gamma(A)} e^{\lambda(d\gamma)} 3^{*(\gamma)} \left[\int_{\Gamma(A)} e^{\lambda(d\alpha)} 3^{*(\alpha)} \|T_\alpha T_\gamma \psi_1\|^2 \right] \left[\int_{\Gamma(A)} e^{\lambda(d\beta)} 3^{*(\beta)} \|T_\beta T_\gamma \psi_2\|^2 \right] \\ &= \int_{\Gamma(A)} e^{\lambda(d\gamma)} 3^{*(\gamma)} \left[\int_{\Gamma(A)} e^{\lambda(d\delta)} 4^{*(\delta)} |T_\gamma \psi_1(\delta)|^2 \right] \left[\int_{\Gamma(A)} e^{\lambda(d\varepsilon)} 4^{*(\varepsilon)} |T_\gamma \psi_2(\varepsilon)|^2 \right] \end{aligned}$$

$$\begin{aligned}
&= \int_{\Gamma(A)} e^{\lambda(d\delta)} e^{\lambda(d\varepsilon)} 4^{\#(\delta)+\#(\varepsilon)} (T_\delta |\psi_1|^2, 3^N T_\varepsilon |\psi_2|^2) \\
&\leq \left[\int_{\Gamma(A)} e^{\lambda(d\delta)} 4^{\#(\delta)} \|(\sqrt{3})^N T_\delta |\psi_1|^2\| \right] \left[\int_{\Gamma(A)} e^{\lambda(d\varepsilon)} 4^{\#(\varepsilon)} \|(\sqrt{3})^N T_\varepsilon |\psi_2|^2\| \right] \\
&=: \| |\psi_1| \|^2 \| |\psi_2| \|^2.
\end{aligned}$$

Now, again by Schwartz's inequality, the norm $\| \cdot \|$, defined above, satisfies the bound

$$\begin{aligned}
\| |\psi| \|^2 &\leq \left[\int_{\Gamma(A)} e^{\lambda(d\sigma)} (4^{\#(\sigma)})^2 \right]^{1/2} \left[\int_{\Gamma(A)^2} e^{\lambda(d\delta)} e^{\lambda(d\eta)} 3^{\#(\eta)} |\psi(\delta \cup \eta)|^4 \right]^{1/2} \\
&= (\exp[16\lambda(A)])^{1/2} \left(\int_{\Gamma(A)} e^{\lambda(d\tau)} \sum_{\eta \subset \tau} 3^{\#(\eta)} |\psi(\tau)|^4 \right)^{1/2} \\
&= \exp[8\lambda(A)] \left(\int_{\Gamma(A)} e^{\lambda(d\tau)} 4^{\#(\tau)} |\psi(\tau)|^4 \right)^{1/2} \\
&= \exp[8\lambda(A)] \|(\sqrt{2})^N \psi\|_4^2
\end{aligned}$$

and the estimate (A7) follows. $\square$

Completion of the proof of Theorem 2.1: First suppose that ψ , with support $A \in \mathcal{B}_b(S)$, is contained in $\mathcal{D}(\sqrt{3}^N)$ and let $\varphi \in \Pi$ have the same support. Then, since $\tilde{U}\varphi = U\varphi$,

$$\|\tilde{U}\psi - U\psi\| \leq \|\tilde{U}(\psi - \varphi)\| + \|U(\varphi - \psi)\| \leq \exp[\lambda(A)] \|(\sqrt{3})^N(\psi - \varphi)\| + \|\psi - \varphi\|.$$

Both terms can be made arbitrarily small by a suitable choice of φ and it follows that $\tilde{U}\psi = U\psi$. Next let $\psi \in \mathcal{F}$ be such that $C(\omega)$, as defined in the theorem, is square integrable. Define $\psi_{A,n}$ [$n \in \mathbb{N}$, $A \in \mathcal{B}_b(S)$] by

$$\psi_{A,n}(\sigma) = \begin{cases} \psi(\sigma) & \text{if } \sigma \subset A \text{ and } \#(\sigma) \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Then clearly $\psi_{A,n} \rightarrow \psi$ pointwise as $n \rightarrow \infty$ and $A \uparrow S$, but also, for almost all $\omega \in \Omega$,

$$\lim_{n \rightarrow \infty} \lim_{A \uparrow S} \tilde{U}\psi_{A,n}(\omega) = \tilde{U}\psi(\omega),$$

by the absolute convergence of the sum and integral in (A4) which define $\tilde{U}\psi$. Now, again for almost all ω ,

$$|\tilde{U}\psi_{A,n}(\omega) - \tilde{U}\psi(\omega)| \leq 2C(\omega),$$

so that, according to Lebesgue's dominated convergence theorem, $\|\tilde{U}\psi_{A,n} - \tilde{U}\psi\|$ can be made arbitrarily small. Since $\tilde{U}\psi_{A,n} = U\psi_{A,n}$, it follows that $\tilde{U}\psi = U\psi$ by a density argument.

Completion of the proof of Theorem 2.2: Let $\mathcal{D}_A$ denote the space

$$\mathcal{D}_A := \{ \psi \in \mathcal{F} \mid (\sqrt{2})^N \psi \in L^4(\Gamma(A), \lambda) \}.$$

On $\mathcal{D}_A$ the operation $\star$ is well defined by Proposition A.1. We claim that $U\mathcal{D}_A \subset L^4(\Omega, \mathbb{P}_\lambda)$, which implies that also the operation $\star$ is well defined on $\mathcal{D}_A$. Indeed, if $\varphi \in \Pi$ has support A , then

$$\|U\varphi\|_4 = \|U\bar{\varphi} \cdot U\varphi\|_2^{1/2} = \|U(\bar{\varphi} \star \varphi)\|_2^{1/2} = \|\bar{\varphi} \star \varphi\|^{1/2} \leq \exp[4\lambda(A)] \|(\sqrt{2})^N \varphi\|_4.$$

Observing that the space Π is invariant under the operators a^N , $a > 0$, we have for $\varphi \in \Pi$ with support A

$$\|U(\sqrt{2})^{-N} \varphi\|_4 \leq \exp[4\lambda(A)] \|\varphi\|_4.$$

With Π being dense in $\mathcal{S}$, the map $\varphi \rightarrow U(\sqrt{2})^{-N} \varphi$ extends to a bounded operator, $L^4(\Gamma(A), \lambda) \rightarrow L^4(\Omega, P_\lambda)$, i.e.,

$$U\mathcal{D}_A = \{U(\sqrt{2})^{-N} \varphi \mid \varphi \in L^4(\Gamma(A), \lambda)\} \subset L^4(\Omega, P_\lambda).$$

Next choose $\psi_1, \psi_2 \in \mathcal{D}_A$ and let $\varphi_1, \varphi_2 \in \Pi$ also be supported by A . Then, since $\star$ and $\bar{\star}$ coincide on $\Pi \times \Pi \subset \mathcal{D}_A \times \mathcal{D}_A$,

$$\begin{aligned} \|\psi_1 \bar{\star} \psi_2 - \psi_1 \star \psi_2\| &\leq \|\psi_1 \bar{\star} (\psi_2 - \varphi_2)\| + \|(\psi_1 - \varphi_1) \bar{\star} \varphi_2\| + \|(\varphi_1 - \psi_1) \star \varphi_2\| + \|\psi_1 \star (\varphi_2 - \psi_2)\| \\ &\leq 2 \exp[8\lambda(A)] \{ \|(\sqrt{2})^N \psi_1\|_4 \cdot \|(\sqrt{2})^N (\psi_2 - \varphi_2)\|_4 + \|(\sqrt{2})^N (\psi_1 - \varphi_1)\|_4 \\ &\quad \cdot \|(\sqrt{2})^N \varphi_2\|_4 \}, \end{aligned}$$

where we have used that, for $\psi, \xi \in \mathcal{D}_A$, $\|\psi \star \xi\| = \|U(\psi \star \xi)\| \leq \|U\psi \cdot U\xi\| \leq \|U\psi\|_4 \|U\xi\|_4$.

The right-hand side can be made arbitrarily small by a proper choice of φ_1 and φ_2 and it follows that $\psi_1 \bar{\star} \psi_2 = \psi_1 \star \psi_2$. Finally, let ψ_1 and ψ_2 be such that the function

$$G(\sigma) := \sum_{\substack{\alpha \cup \beta \cup \gamma = \sigma \\ \alpha, \beta, \gamma \text{ disjoint}}} \int_{\Gamma(S)} e^{\lambda(d\eta)} |\psi_1(\alpha \cup \gamma \cup \eta) \psi_2(\alpha \cup \beta \cup \eta)|$$

is square integrable, and let $\psi_{A,n}^{(j)} = \psi_j(\sigma)$ if $\sigma \subset A$ and $\#(\sigma) \leq n$ and let it be 0 otherwise ($j = 1, 2$). Let $A_1 \subset A_2 \subset A_3 \subset \dots$ be an increasing sequence of bounded measurable subsets of S such that $\bigcup_{n=1}^\infty A_n = S$. Then, since $G \in L^2(S)$,

$$L^2 - \lim_{n \rightarrow \infty} \psi_{A_n,n}^{(1)} \bar{\star} \psi_{A_n,n}^{(2)} = \psi_1 \bar{\star} \psi_2.$$

With $\psi_{A,n}^{(j)}$ being contained in $\mathcal{D}_A$, we now have

$$L^2 - \lim_{n \rightarrow \infty} U\psi_{A_n,n}^{(1)} \cdot U\psi_{A_n,n}^{(2)} = U(\psi_1 \bar{\star} \psi_2).$$

It follows that there exists an increasing sequence $\{n_k\}_{k=1}^\infty$ such that for P_λ -almost all $\omega \in \Omega$

$$\lim_{k \rightarrow \infty} U\psi_{A_{n_k},n_k}^{(1)}(\omega) \cdot U\psi_{A_{n_k},n_k}^{(2)}(\omega) = U(\psi_1 \bar{\star} \psi_2)(\omega).$$

Now, as $U\psi_{A_n,n}^{(j)} \rightarrow U\psi_j$, the sequence $\{n_k\}$ can be diluted, resulting in pointwise convergence for a.e. ω . Substituting and taking the limit we obtain for a.e. ω

$$U\psi_1(\omega) \cdot U\psi_2(\omega) = U(\psi_1 \bar{\star} \psi_2)(\omega),$$

i.e., $\psi_1 \bar{\star} \psi_2 = \psi_1 \star \psi_2$. □

Proof of Corollary 2.3: Let $f = |\varphi|$. Then $f \in L^1 \cap L^2(S)$ and is non-negative. Define $f_A := f \cdot \chi_A$ for $A \in \mathcal{B}_b(S)$. Then, by Theorem 2.1, the partial sums

$$F_A(\omega) := \sum_{\sigma \subset \omega \cap A} f(\sigma) = \langle f_A, \mu_\omega \rangle = \langle f_A, \lambda \rangle + \langle f_A, \mu_\omega - \lambda \rangle = U(\langle f_A, \lambda \rangle \oplus f_A \oplus 0 \oplus \cdots)(\omega)$$

have finite mean and variance satisfying

$$\mathbb{E}F_A = \|f_A\|_1 \leq \|f\|_1, \text{Var } F_A = \mathbb{E}F_A^2 - (\mathbb{E}F_A)^2 = (\|f_A\|_1^2 + \|f_A\|_2^2) - \|f_A\|_1^2 = \|f_A\|_2^2 \leq \|f\|_2^2.$$

Next, let $\psi \in \mathcal{F}$ be a finite particle vector, i.e., $\psi(\sigma) = 0$ for $\#(\sigma) > n$ for some $n \in \mathbb{N}$. Let $\zeta = \langle f, \lambda \rangle \oplus f \oplus 0 \oplus 0 \oplus \cdots$. Then the condition of Theorem 2.2 is satisfied:

$$\begin{aligned} G(\sigma) &:= \sum_{\substack{\alpha \cup \beta \cup \gamma = \sigma \\ \alpha, \beta, \gamma \text{ disjoint}}} \int_{\Gamma(S)} e^{\lambda(d\eta)} |\zeta(\alpha \cup \gamma \cup \eta) \psi(\alpha \cup \beta \cup \eta)| = |\langle f, \lambda \rangle \psi(\sigma)| \\ &\quad + \sum_{x \in \sigma} |f(x) \psi(\sigma \setminus \{x\})| + \sum_{x \in \sigma} |f(x) \psi(\sigma)| + \int_S \lambda(dx) |f(x) \psi(\sigma \cup \{x\})| \\ &= \{|\langle f, \lambda \rangle| + V_{|f|}(\sigma)\} |\psi(\sigma)| + \{b(|f|) + b(|f|)^*\} |\psi(\sigma)| \end{aligned}$$

is square integrable, ψ being contained in the intersection of the domains of $V(|f|)$, $b(|f|)$, and $b(|f|)^*$. By Theorem 2.2 the result follows. $\square$

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