Ranking Functions for Loops with Disjunctive Exit-Conditions

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Introduction

Basic Procedure

Piecewise Ranking Functions

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Ranking Function

- Decreases in every basic block
- Here: in every loop iteration
- Bounded by zero

```plaintext
while (i < 15) {
    i++;
}
```

- Ranking function for the loop above is $15 - i$
Motivation and Aim

- Prove termination
- Bounding runtime
- Compiler optimisations
- Resource Analysis

```java
while (i < 15) {
    consumeResource();
    i++;
}
```
Motivation and Aim

- Prove termination
- Bounding runtime
- Compiler optimisations
- **Resource Analysis**

```c
1 while (i < 15) {
2     consumeResource();
3     i++;
4 }
```
Motivation and Aim

• Prove termination
• Bounding runtime
• Compiler optimisations
• **Resource Analysis**

```java
1 while (i < 15) {
2     consumeResource();
3     i++;
4 }
```
Presentation Outline

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Inference of Polynomial Loop Ranking Functions

O. Shkaravska, R. Kersten, M. van Eekelen.
Test-Based Inference of Polynomial Loop-Bound Functions.
PPPJ’10: Proceedings of the 8th International Conference on the Principles and Practice of Programming in Java
Applicable Loops

• The basic method considers loops with conditions in the following form:

\[ C := sC \mid C_1 \land C_2 \]
\[ sC := e_1 \ [<, >, \leq, \geq, =, \neq] e_2 \]

• where \( e_i \) are arithmetical expressions

• i.e. conjunctions over arithmetical (in)equalities
Test-Based Approach

1. Instrument loop with a counter
2. Do test runs for a set of $N_d^k = \binom{d+k}{k}$ input values satisfying NCA and the exit condition
3. Interpolate a polynomial from the results
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Test-Based Approach

1. Instrument loop with a counter
2. Do test runs for a set of $N_d^k = \binom{d+k}{k}$ input values satisfying \textbf{NCA} and the exit condition
3. Interpolate a polynomial from the results
Quadratic Example

```java
public int m(int a, int b, int c) {
    int count = 0;
    while (a > 0 && c <= b && c > 0) {
        if (c == b) { a--; c = 0; }
        c++;
        count++;
    }
    return count;
}
```

Test runs

1st group: degree 2 NCA on plane
- a=1, b=1, c=1 => count=1
- a=1, b=1, c=2 => count=2
- a=1, b=1, c=3 => count=3
- a=1, b=2, c=2 => count=1
- a=1, b=2, c=3 => count=2
- a=1, b=3, c=3 => count=1

2nd group: degree 1 NCA on plane
- a=2, b=1, c=1 => count=2
- a=2, b=1, c=2 => count=4
- a=2, b=2, c=2 => count=3

3rd group: degree 0 NCA on plane
- a=3, b=1, c=1 => count=3

Expected degree of polynomial (here: d=2)

Find the interpolating polynomial and generate the method annotated with the corresponding ranking function:

RF(a, b, c) = a*b – c + 1
Soundness

- The procedure itself is unsound
- Use external prover to verify the inferred ranking functions
- KeY: http://www.key-project.org/
- Ranking function can be expressed in JML as a decreases clause

```jml
//@ decreases i < 15 ? 15 - i : 0;
while (i < 15) {
  i++;  
}
```
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Any loop ranking function is piecewise...

```plaintext
1 while (i < 15) {
2      i++;
3 }
```

Its ranking function is actually:

\[
\begin{cases} 
15 - i & \text{if } (i < 15) \\
0 & \text{else}
\end{cases}
\]
Non-Trivial Example

```c
while ((i > 0 && i < 20) || i > 50) {
    if (i > 50) i--;  
    else i++;  
}
```

It's ranking function is non-trivially piecewise:

\[
\begin{cases} 
20 - i & \text{if } (i > 0) \land (i < 20) \\
 i - 50 & \text{if } i > 50  \\
 0 & \text{else}
\end{cases}
\]
Expressing Piecewise Ranking Functions in JML

```java
//@ decreases (i>0&&i<20) ? 20−i : (i>50 ? i−50 : 0);
while ((i>0 && i<20) || i>50) {
    if (i>50) i--;  
    else i++; 
}
```
Applicable Loops

- The extended method considers loops with conditions in the following form:

\[
C := sC \mid C_1 \land C_2 \mid C_1 \lor C_2 \\
sC := e_1 [\lt, \gt, \leq, \geq, =, \neq] e_2
\]

- where \( e_i \) are arithmetical expressions
- i.e. first-order propositional logic expressions over arithmetical (in)equalities
Extending the Basic Procedure: Example

```c
while ((i>0 && i<20) || i>50) {
    if (i>50) i--;
    else i++;
}
```

1. Split up the condition into disjunctive parts:
   - \( i > 0 \land i < 20 \land \neg (i > 50) \)
   - \( i > 50 \land \neg (i > 0 \land i < 20) \)
   - \( i > 0 \land i < 20 \land i > 50 \)

2. Execute the basic procedure separately for each of the pieces.
Extending the Basic Procedure: Example

1 \textbf{while} \ ((i>0 \ \&\& \ i<20) \ || \ i>50) \ \{ \\
2 \ \ \ \textbf{if} \ (i>50) \ i--; \\
3 \ \ \ \textbf{else} \ i++; \\
4 \ \} \\

1 Split up the condition into disjunctive parts:
   - \( i > 0 \ \&\& \ i < 20 \ \&\& \neg(i > 50) \)
   - \( i > 50 \ \&\& \neg(i > 0 \ \&\& i < 20) \)
   - \( i > 0 \ \&\& i < 20 \ \&\& i > 50 \)

2 Execute the basic procedure separately for each of the pieces
Extending the Basic Procedure: Example

```java
while ((i > 0 && i < 20) || i > 50) {
    if (i > 50) i--;
    else i++;
}
```

1. Split up the condition into disjunctive parts:
   - $i > 0 \land i < 20 \lor (i > 50)$
   - $i > 50 \lor (i > 0 \land i < 20)$
   - $i > 0 \land i < 20 \land i > 50$

2. Execute the basic procedure separately for each of the pieces.
Extending the Basic Procedure: Example

1 \textbf{while} ((\textit{i}>0 \land \textit{i}<20) \lor \textit{i}>50) \{ \\
2 \quad \textbf{if} \ (\textit{i}>50) \textit{i}--; \\
3 \quad \textbf{else} \ \textit{i}++; \\
4 \} \\

1 Split up the condition into disjunctive parts:
   - \textit{i} > 0 \land \textit{i} < 20
   - \textit{i} > 50

2 Execute the basic procedure separately for each of the pieces
Extending the Basic Procedure: Example

1 while ((i>0 && i<20) || i>50) {
2    if (i>50) i--;
3    else i++;
4 }

1 Split up the condition into disjunctive parts:
   - $i > 0 \land i < 20$
   - $i > 50$

2 Execute the basic procedure separately for each of the pieces
Extending the Basic Procedure: Example

```
1 while ((i>0 && i<20) || i>50) {
2     if (i>50) i--;
3     else i++;
4 }
```

\[
\begin{align*}
20 - i & \quad \text{if } (i > 0) \land (i < 20) \\
    i - 50 & \quad \text{if } i > 50 \\
        0 & \quad \text{else}
\end{align*}
\]
Extending the Basic Procedure: Generic

1. Put the condition in Disjunctive Normal Form
2. Split up the condition into its disjunctive pieces
3. Execute the basic procedure separately for each of the pieces
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Example

```plaintext
1 while ((i>0 && i<20) || i>22) {
2     if (i>22) i--;
3     else i+=4;
4 }
```

\[
\begin{cases}
\left\lceil \frac{(20 - i)}{4} \right\rceil & \text{if } (i > 0) \land (i < 20) \\
i - 22 & \text{if } i > 22 \\
0 & \text{else}
\end{cases}
\]
Example

```python
1 while ((i>0 && i<20) || i>22) {
2     if (i>22) i--;  
3     else i+=4;
4 }
```

\[
\begin{cases}
\left\lceil \frac{(20 - i)}{4} \right\rceil + 1 & \text{if } (i > 0) \land (i < 20) \land i \mod 4 = 3 \\
\left\lceil \frac{(20 - i)}{4} \right\rceil & \text{if } (i > 0) \land (i < 20) \land i \mod 4 \neq 3 \\
i - 22 & \text{if } i > 22 \\
0 & \text{else}
\end{cases}
\]
What happens...
Detection of Condition Jumping: Example

1 \textbf{while} ((i>0 && i<20) || i>22) {
2 \quad \textbf{if} (i>22) i--; 
3 \quad \textbf{else} i+=4;
4 \}

\[
next_i(i) = \begin{cases} 
  i - 1 & \text{if } i > 22 \\
  i + 4 & \text{if } \neg(i > 22) 
\end{cases}
\]

1 (declare-fun i () Int)
2 (define-fun nexti ((x Int)) Int
3 \quad (ite (> x 22) (− x 1) (+ x 4)))
4 (assert (and (and (> i 0) (< i 20))
5 \quad (> (nexti i) 22)))
6 (check-sat)
Detection of Condition Jumping: Example

1 while ((i>0 && i<20) || i>22) {
2   if (i>22) i--;
3   else i+=4;
4 }

\[
next_i(i) = \begin{cases} 
  i - 1 & \text{if } i > 22 \\
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Detection of Condition Jumping: Example

1 while ((i>0 && i<20) || i>22) {
2  if (i>22) i--;
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\text{next}_i(i) = \begin{cases} 
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1 (declare-fun i () Int)
2 (define-fun nexti ((x Int)) Int
3   (ite (> x 22) (− x 1) (+ x 4)))
4 (assert (and (and (> i 0) (< i 20))
5   (> (nexti i) 22)))
6 (check-sat)
Detection of Condition Jumping: Generic

- Symbolically execute the loop body to find a `next` function for each program variable
- Use SMT-solver to search for a model that satisfies one piece first and another after execution of the loop body
Finding Models for Condition Jumping: Example

Find all nodes that jump from the piece with condition \( i > 0 \land i < 20 \) into the piece with condition \( i > 22 \). Using an SMT-solver:

1. Find all nodes that jump directly into the other piece: \( \{19\} \)
2. Find all nodes that can jump to \( \{19\}, \{3, 7, 11, 15\} \) and add them to the list of jumping nodes:
   \[
   \{3, 7, 11, 15, 19\} = \{i \mid i \mod 4 = 3 \land i > 0 \land i < 20\}
   \]
Finding Models for Condition Jumping: Example

Find all nodes that jump from the piece with condition $i > 0 \land i < 20$ into the piece with condition $i > 22$. Using an SMT-solver:

1. Find all nodes that jump directly into the other piece: $\{19\}$
2. Find all nodes that can jump to $\{19\}$, $\{3, 7, 11, 15\}$ and add them to the list of jumping nodes:

$$\{3, 7, 11, 15, 19\} = \{i \mid i \mod 4 = 3 \land i > 0 \land i < 20\}$$
Finding all nodes that jump from the piece with condition \( i > 0 \land i < 20 \) into the piece with condition \( i > 22 \). Using an SMT-solver:

1. Find all nodes that jump directly into the other piece: \( \{19\} \)
2. Find all nodes that can jump to \( \{19\} \), \( \{3, 7, 11, 15\} \) and add them to the list of jumping nodes:
\[
\{3, 7, 11, 15, 19\} = \{i \mid i \mod 4 = 3 \land i > 0 \land i < 20\}
\]
Finding Models for Condition Jumping: Generic

\( J \) is the set of models of which it is known that condition jumping occurs, \( Q \) is a queue of models, find all models that jump from \( b_1 \) to \( b_2 \): 

1. Is there a model \( \bar{v} \) for which \( b_1(\bar{v}) \land b_2(\text{next}(\bar{v})) \land \bar{v} \notin J \)?
   - SAT \( \rightarrow \) Add \( \bar{v} \) to \( J \) and \( Q \), goto 1.
   - UNSAT \( \rightarrow \) Goto 2.

2. Q empty?
   - Yes \( \rightarrow \) Done.
   - No \( \rightarrow \) Goto 3.

3. Pop a model \( \bar{q} \) off the queue \( Q \). Is there a model \( \bar{v} \) for which \( b_1(\bar{v}) \land \text{next}(\bar{v}) = \bar{q} \land \bar{v} \notin J \)?
   - SAT \( \rightarrow \) Add \( \bar{v} \) to \( J \) and \( Q \), goto 3.
   - UNSAT \( \rightarrow \) Goto 2.
Generating Ranking Functions: Example

• We now know the set \( \{ i \mid i \mod 4 = 3 \land i > 0 \land i < 20 \} \) for which jumping occurs.

• So, we can split the condition \( i > 0 \land i < 20 \) into two:
  \[ i > 0 \land i < 20 \land i \mod = 3 \quad \text{and} \quad i > 0 \land i < 20 \land i \mod \neq 3 \]

• We can then apply the basic method separately to each of these disjunctive pieces.
• We now know the set $D_{1,2}$ for which jumping occurs
• So, we can split the condition $b_1$ into two: $b_1(\bar{v}) \land \bar{v} \in D_{1,2}$
  and $b_1(\bar{v}) \land \bar{v} \not\in D_{1,2}$
• We can then apply the basic method to each of these disjunctive pieces
Multi-Jumping

1. DNF-split into \( n \) conditions
2. For each \( i \) and \( j \), \( 1 \leq i < j \leq n \), detect jumping from \( D_i \) to \( D_j \). Build a list \( J \) of jumping pairs \((D_x, D_y)\) for which condition jumping from \( D_x \) to \( D_y \) can occur.
3. If there are no more jumping pairs \((D_x, D_y)\) for which \( D_x \) is unflagged, done! Else, goto 4.
4. Pop a jumping pair \((D_x, D_y)\) off \( J \), for which \( D_x \) is unflagged.
5. Find the set \( D_{x,2} \) of all nodes in \( D_x \) from which jumping to \( D_y \) occurs and, dually, the set \( D_{x,1} \) for which no jumping to \( D_y \) occurs. Replace any condition pair \((D_x, D_z)\) in \( J \) by \((D_{x,1}, D_z)\). Add \((D_{x,2}, D_y)\) to \( J \).
   - If \( D_{x,1} = \emptyset \), flag \( D_{x,2} \) as complete, goto 3.
   - Else, for any jumping pair \((D_z, D_x)\) in \( J \) (i.e. for which jumping from \( D_z \) to \( D_x \) can occur), unflag \( D_z \), detect jumping into \( D_{x,1} \) and \( D_{x,2} \) and update \( J \) accordingly. Goto 3.
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Conclusions

• Extension to the method presented at PPPJ’10, which can infer *polynomial* ranking functions:
  - Definition of Condition Jumping
  - Detection of Condition Jumping
  - Infer ranking functions for loops in which condition jumping occurs

• Ranking functions for loops can be used in the creation of a *global* ranking function in order to prove termination

• If the body of a loop with ranking function $RF(\overline{v})$ consumes $n$ resources, then we know that the whole loop consumes $RF(\overline{v}) \cdot n$ resources
http://resourceanalysis.cs.ru.nl/resana

- The basic procedure and DNF-splitting (minus removal of unsatisfiable pieces) have been implemented
- Future work: implement condition jumping solution