Ranking Functions for Loops with Disjunctive Exit-Conditions

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Introduction

Basic Procedure

Piecewise Ranking Functions

Condition Jumping

Conclusions
• Decreases in every basic block
• Here: in every loop iteration
• Bounded by zero

1 while (i < 15) {
2   i++;
3 }

• Ranking function for the loop above is $15 - i$
Motivation and Aim

- Prove termination
- Bounding runtime
- Compiler optimisations
- Resource Analysis

1 while (i < 15) {
2   consumeResource();
3   i++;
4 }
Motivation and Aim

- Prove termination
- Bounding runtime
- Compiler optimisations
- Resource Analysis

```plaintext
1 while (i < 15) {
2     consumeResource();
3     i++;
4 }
```
Motivation and Aim

• Prove termination
• Bounding runtime
• Compiler optimisations
• **Resource Analysis**

```java
1 while (i < 15) {
2    consumeResource();
3    i++;
4 }
```
Presentation Outline

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Conclusions
O. Shkaravska, R. Kersten, M. van Eekelen.

Test-Based Inference of Polynomial Loop-Bound Functions.

PPPJ’10: Proceedings of the 8th International Conference on the Principles and Practice of Programming in Java.
Applicable Loops

• The basic method considers loops with conditions in the following form:

\[ C := sC \mid C_1 \land C_2 \]

\[ sC := e_1 [\lt, \gt, \leq, \geq, =, \neq] e_2 \]

• where \( e_i \) are arithmetical expressions
• i.e. conjunctions over arithmetical (in)equalities
Test-Based Approach

1. Instrument loop with a counter
2. Do test runs for a set of $N_d^k = \binom{d+k}{k}$ input values satisfying NCA and the exit condition
3. Interpolate a polynomial from the results
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Test-Based Approach

1. Instrument loop with a counter
2. Do test runs for a set of $N_d^k = {d+k\choose k}$ input values satisfying NCA and the exit condition
3. Interpolate a polynomial from the results
Quadratic Example

```java
class Example {
    public int m(int a, int b, int c) {
        int count = 0;
        while (a > 0 && c <= b && c > 0) {
            if (c == b) { a--; c = 0; }
            c++;
            count++;
        }
        return count;
    }
}
```

Test runs

1\textsuperscript{st} group: degree 2 NCA on plane
- \(a=1, b=1, c=1\) => count=1
- \(a=1, b=1, c=2\) => count=2
- \(a=1, b=1, c=3\) => count=3
- \(a=1, b=2, c=2\) => count=1
- \(a=1, b=2, c=3\) => count=2
- \(a=1, b=3, c=3\) => count=1

2\textsuperscript{nd} group: degree 1 NCA on plane
- \(a=2, b=1, c=1\) => count=2
- \(a=2, b=1, c=2\) => count=4
- \(a=2, b=2, c=2\) => count=3

3\textsuperscript{rd} group: degree 0 NCA on plane
- \(a=3, b=1, c=1\) => count=3

Find the interpolating polynomial and generate the method annotated with the corresponding ranking function:

\[ RF(a, b, c) = a \cdot b - c + 1 \]
Soundness

- The procedure itself is unsound
- Use external prover to verify the inferred ranking functions
- KeY: http://www.key-project.org/
- Ranking function can be expressed in JML as a decreases clause

```java
//@ decreases i < 15 ? 15 - i : 0;
while (i < 15) {
    i++;
}
```
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Any loop ranking function is piecewise...

```
1 while (i < 15) {
2    i++;  
3 }
```

Its ranking function is actually:

\[
\begin{cases} 
15 - i & \text{if } (i < 15) \\
0 & \text{else}
\end{cases}
\]
Non-Trivial Example

1 \textbf{while} ((i>0 \ \&\& \ i<20) \ || \ i>50) \ { \\
2 \quad \textbf{if} \ (i>50) \ i--; \\
3 \quad \textbf{else} \ i++; \\
4 \ }

It’s ranking function is non-trivially piecewise:

\[
\begin{aligned}
20 - i & \quad \text{if} \ (i > 0) \land (i < 20) \\
i - 50 & \quad \text{if} \ i > 50 \\
0 & \quad \text{else}
\end{aligned}
\]
Expressing Piecewise Ranking Functions in JML

```java
1   //@ decreases (i>0 && i<20) ? 20-i : (i>50 ? i-50 : 0);
2   while ((i>0 && i<20) || i>50) {
3       if (i>50) i--;
4       else i++;
5     }
```
• The extended method considers loops with conditions in the following form:

\[ C := sC \mid C_1 \land C_2 \mid C_1 \lor C_2 \]
\[ sC := e_1 [<, >, \leq, \geq, =, \neq] e_2 \]

• where \( e_i \) are arithmetical expressions
• i.e. first-order propositional logic expressions over arithmetical (in)equalities
Extending the Basic Procedure: Example

while (\((i > 0 \land i < 20) \lor i > 50\) ) {
    if (\(i > 50\)) \(i--;\)
    else \(i++;\)
}

1. Split up the condition into disjunctive parts:
   - \(i > 0 \land i < 20 \land \neg (i > 50)\)
   - \(i > 50 \land \neg (i > 0 \land i < 20)\)
   - \(i > 0 \land i < 20 \land i > 50\)

2. Execute the basic procedure separately for each of the pieces.
Extending the Basic Procedure: Example

```plaintext
1 while ((i>0 && i<20) || i>50) {
2   if (i>50) i--;
3   else i++;
4 }
```

1 Split up the condition into disjunctive parts:
   - \( i > 0 \land i < 20 \land \neg(i > 50) \)
   - \( i > 50 \land \neg(i > 0 \land i < 20) \)
   - \( i > 0 \land i < 20 \land i > 50 \)

2 Execute the basic procedure separately for each of the pieces.
Extending the Basic Procedure: Example

1 while ((i>0 && i<20) || i>50) {
2   if (i>50) i--;  
3   else i++;  
4 }

1 Split up the condition into disjunctive parts:
   • \( i > 0 \land i < 20 \lor (i > 50) \)
   • \( i > 50 \lor (i > 0 \land i < 20) \)
   • \( i > 0 \land i < 20 \land i > 50 \)

2 Execute the basic procedure separately for each of the pieces.
Extending the Basic Procedure: Example

```
while ((i>0 && i<20) || i>50) {
    if (i>50) i--;
    else i++;
}
```

1. Split up the condition into disjunctive parts:
   - $i > 0 \land i < 20$
   - $i > 50$

2. Execute the basic procedure separately for each of the pieces.
Extending the Basic Procedure: Example

1 `while ((i>0 && i<20) || i>50) {`
2 `if (i>50) i--;`
3 `else i++;`
4 `}

1 Split up the condition into disjunctive parts:
   - $i > 0 \land i < 20$
   - $i > 50$

2 Execute the basic procedure separately for each of the pieces
Extending the Basic Procedure: Example

```c
1 while ((i > 0 && i < 20) || i > 50) {
2    if (i > 50) i--; 
3    else i++; 
4 }
```

\[
\begin{align*}
20 - i & \quad \text{if } (i > 0) \land (i < 20) \\
i - 50 & \quad \text{if } i > 50 \\
0 & \quad \text{else}
\end{align*}
\]
Extending the Basic Procedure: Generic

1. Put the condition in Disjunctive Normal Form
2. Split up the condition into its disjunctive pieces
3. Execute the basic procedure separately for each of the pieces
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Example

```
1 while ((i>0 && i<20) || i>22) {
2     if (i>22) i--;  
3     else i+=4;    
4 }                     

\[
\begin{cases} 
    \lceil \frac{(20 - i)}{4} \rceil & \text{if } (i > 0) \land (i < 20) \\
    i - 22 & \text{if } i > 22 \\
    0 & \text{else} \\
\end{cases}
\]
```
Example

```c
while ((i > 0 && i < 20) || i > 22) {
    if (i > 22) i--;  
    else i += 4;
}
```

\[
\begin{cases}
\lfloor (20 - i)/4 \rfloor + 1 & \text{if } (i > 0) \land (i < 20) \land i \mod 4 = 3 \\
\lfloor (20 - i)/4 \rfloor & \text{if } (i > 0) \land (i < 20) \land i \mod 4 \neq 3 \\
i - 22 & \text{if } i > 22 \\
0 & \text{else}
\end{cases}
\]
What happens...
Introduction Basic Procedure Piecewise Ranking Functions Condition Jumping Conclusions

Detection of Condition Jumping: Example

1 while ((i>0 && i<20) || i>22) {
2   if (i>22) i--;
3   else i+=4;
4 }

\[\text{next}_i(i) = \begin{cases} 
  i - 1 & \text{if } i > 22 \\
  i + 4 & \text{if } \neg(i > 22)
\end{cases}\]

1 (declare-fun i () Int)
2 (define-fun nexti ((x Int)) Int
3   (ite (> x 22) (¬ x 1) (+ x 4)))
4 (assert (and (and (> i 0) (< i 20))
5     (> (nexti i) 22)))
6 (check-sat)
Detection of Condition Jumping: Example

```plaintext
1 while ((i>0 && i<20) || i>22) {
2   if (i>22) i--; 
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\[ next_i(i) = \begin{cases} 
  i - 1 & \text{if } i > 22 \\
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\end{cases} \]

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Detection of Condition Jumping: Example

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5   (> (nexti i) 22)))
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Detection of Condition Jumping: Generic

- Symbolically execute the loop body to find a next function for each program variable
- Use SMT-solver to search for a model that satisfies one piece first and another after execution of the loop body
Finding Models for Condition Jumping: Example

Find all nodes that jump from the piece with condition $i > 0 \land i < 20$ into the piece with condition $i > 22$. Using an SMT-solver:

1. Find all nodes that jump directly into the other piece: $\{19\}$
2. Find all nodes that can jump to $\{19\}$, $\{3, 7, 11, 15\}$ and add them to the list of jumping nodes:
   $\{3, 7, 11, 15, 19\} = \{i \mid i \mod 4 = 3 \land i > 0 \land i < 20\}$
Finding Models for Condition Jumping: Example

Find all nodes that jump from the piece with condition \( i > 0 \land i < 20 \) into the piece with condition \( i > 22 \). Using an SMT-solver:

1. Find all nodes that jump directly into the other piece: \{19\}
2. Find all nodes that can jump to \{19\}, \{3, 7, 11, 15\} and add them to the list of jumping nodes: 
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\{3, 7, 11, 15, 19\} = \{i \mid i \mod 4 = 3 \land i > 0 \land i < 20\}
\]
Find all nodes that jump from the piece with condition $i > 0 \land i < 20$ into the piece with condition $i > 22$. Using an SMT-solver:

1. Find all nodes that jump directly into the other piece: $\{19\}$
2. Find all nodes that can jump to $\{19\}$, $\{3, 7, 11, 15\}$ and add them to the list of jumping nodes:
   
   $\{3, 7, 11, 15, 19\} = \{i \mid i \mod 4 = 3 \land i > 0 \land i < 20\}$
Finding Models for Condition Jumping: Generic

$J$ is the set of models of which it is known that condition jumping occurs, $Q$ is a queue of models, find all models that jump from $b_1$ to $b_2$:

1. **Is there a model $\bar{v}$ for which $b_1(\bar{v}) \land b_2(\text{next}(\bar{v})) \land \bar{v} \not\in J$?**
   - SAT $\rightarrow$ Add $\bar{v}$ to $J$ and $Q$, goto 1.
   - UNSAT $\rightarrow$ Goto 2.

2. **Q empty?**
   - Yes $\rightarrow$ Done.
   - No $\rightarrow$ Goto 3.

3. **Pop a model $\bar{q}$ off the queue $Q$. Is there a model $\bar{v}$ for which $b_1(\bar{v}) \land \text{next}(\bar{v}) = \bar{q} \land \bar{v} \not\in J$?**
   - SAT $\rightarrow$ Add $\bar{v}$ to $J$ and $Q$, goto 3.
   - UNSAT $\rightarrow$ Goto 2.
Generating Ranking Functions: Example

- We now know the set \( \{ i \mid i \mod 4 = 3 \land i > 0 \land i < 20 \} \) for which jumping occurs.
- So, we can split the condition \( i > 0 \land i < 20 \) into two:
  \( i > 0 \land i < 20 \land i \mod = 3 \) and \( i > 0 \land i < 20 \land i \mod \neq 3 \)
- We can then apply the basic method separately to each of these disjunctive pieces.
• We now know the set $D_{1,2}$ for which jumping occurs
• So, we can split the condition $b_1$ into two: $b_1(\bar{v}) \land \bar{v} \in D_{1,2}$ and $b_1(\bar{v}) \land \bar{v} \not\in D_{1,2}$
• We can then apply the basic method to each of these disjunctive pieces
Multi-Jumping

1. DNF-split into $n$ conditions
2. For each $i$ and $j$, $1 \leq i < j \leq n$, detect jumping from $D_i$ to $D_j$. Build a list $J$ of jumping pairs $(D_x, D_y)$ for which condition jumping from $D_x$ to $D_y$ can occur.
3. If there are no more jumping pairs $(D_x, D_y)$ for which $D_x$ is unflagged, done! Else, goto 4.
4. Pop a jumping pair $(D_x, D_y)$ off $J$, for which $D_x$ is unflagged.
5. Find the set $D_x, 2$ of all nodes in $D_x$ from which jumping to $D_y$ occurs and, dually, the set $D_x, 1$ for which no jumping to $D_y$ occurs. Replace any condition pair $(D_x, D_z)$ in $J$ by $(D_x, 1, D_z)$. Add $(D_x, 2, D_y)$ to $J$.
   - If $D_x, 1 = \emptyset$, flag $D_x, 2$ as complete, goto 3.
   - Else, for any jumping pair $(D_z, D_x)$ in $J$ (i.e. for which jumping from $D_z$ to $D_x$ can occur), unflag $D_z$, detect jumping into $D_x, 1$ and $D_x, 2$ and update $J$ accordingly. Goto 3.
Presentation Outline

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- Condition Jumping
- Conclusions
Conclusions

- Extension to the method presented at PPPJ’10, which can infer \textit{polynomial} ranking functions:
  - Definition of Condition Jumping
  - Detection of Condition Jumping
  - Infer ranking functions for loops in which condition jumping occurs

- Ranking functions for loops can be used in the creation of a \textit{global} ranking function in order to prove termination

- If the body of a loop with ranking function $RF(\bar{v})$ consumes $n$ resources, then we know that the whole loop consumes $RF(\bar{v}) \cdot n$ resources
Implementation: ResAna

http://resourceanalysis.cs.ru.nl/resana

• The basic procedure and DNF-splitting (minus removal of unsatisfiable pieces) have been implemented
• Future work: implement condition jumping solution