Ranking Functions for Loops with Disjunctive Exit-Conditions

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Presentation Outline

Introduction

Basic Procedure

Piecewise Ranking Functions

Condition Jumping

Conclusions
• Decreases in every basic block
• Here: in every loop iteration
• Bounded by zero

```c
1 while (i < 15) {
2    i++;   
3 }
```

• Ranking function for the loop above is $15 - i$
Motivation and Aim

- Prove termination
- Bounding runtime
- Compiler optimisations
- Resource Analysis

```java
1 while (i < 15) {
2     consumeResource();
3     i++;
4 }
```
Motivation and Aim

- Prove termination
- Bounding runtime
- Compiler optimisations
- **Resource Analysis**

```plaintext
1 while (i < 15) {
2   consumeResource();
3   i++;
4 }
```
Motivation and Aim

- Prove termination
- Bounding runtime
- Compiler optimisations
- **Resource Analysis**

1 while (i < 15) {
2   consumeResource();
3   i++;  
4 }

```
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Inference of Polynomial Loop Ranking Functions

Applicable Loops

• The basic method considers loops with conditions in the following form:

\[ C := sC \mid C_1 \land C_2 \]
\[ sC := e_1 [<, >, \leq, \geq, =, \neq] e_2 \]

• where \( e_i \) are arithmetical expressions

• i.e. conjunctions over arithmetical (in)equalities
Test-Based Approach

1. Instrument loop with a counter
2. Do test runs for a set of $N_d^k = \binom{d+k}{k}$ input values satisfying NCA and the exit condition
3. Interpolate a polynomial from the results
Test-Based Approach

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Test-Based Approach

1. Instrument loop with a counter
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3. Interpolate a polynomial from the results
Quadratic Example

```
public int m(int a, int b, int c) {
    int count = 0;
    while (a > 0 && c <= b && c > 0) {
        if (c == b) { a--; c = 0; }
        count++;
        c++;
    }
    return count;
}
```

Test runs

1\textsuperscript{st} group: degree 2 NCA on plane
- a=1, b=1, c=1 => count=1
- a=1, b=1, c=2 => count=2
- a=1, b=1, c=3 => count=3
- a=1, b=2, c=2 => count=1
- a=1, b=2, c=3 => count=2
- a=1, b=3, c=3 => count=1

2\textsuperscript{nd} group: degree 1 NCA on plane
- a=2, b=1, c=1 => count=2
- a=2, b=1, c=2 => count=4
- a=2, b=2, c=2 => count=3

3\textsuperscript{rd} group: degree 0 NCA on plane
- a=3, b=1, c=1 => count=3

Expected degree of polynomial (here: d=2)

Find the interpolating polynomial and generate the method annotated with the corresponding ranking function:

\[ RF(a, b, c) = a \times b - c + 1 \]
Soundness

- The procedure itself is unsound
- Use external prover to verify the inferred ranking functions
- KeY: http://www.key-project.org/
- Ranking function can be expressed in JML as a decreases clause

```jml
//@ decreases i < 15 ? 15 − i : 0;
while (i < 15) {
  i++;
}
```
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Any loop ranking function is piecewise...

1 \textbf{while} (i < 15) {
2  \quad i++; \\
3 \}

Its ranking function is actually:

\[
\begin{cases}
15 - i & \text{if (} i < 15 \text{)} \\
0 & \text{else}
\end{cases}
\]
Non-Trivial Example

```c
while ((i>0 && i<20) || i>50) {
    if (i>50) i--;
    else i++;
}
```

It’s ranking function is non-trivially piecewise:

\[
\begin{cases} 
20 - i & \text{if } (i > 0) \land (i < 20) \\
i - 50 & \text{if } i > 50 \\
0 & \text{else}
\end{cases}
\]
Expressing Piecewise Ranking Functions in JML

```java
  1  //@ decreases (i>0&&i<20) ? 20−i : (i>50 ? i−50 : 0);
  2  while ((i>0 && i<20) || i>50) {
  3      if (i>50) i--;
  4      else i++;
  5  }
```
Applicable Loops

• The extended method considers loops with conditions in the following form:

\[
C := sC \mid C_1 \land C_2 \mid C_1 \lor C_2 \\
\text{sC} := e_1 [\lt, \gt, \leq, \geq, =, \neq] e_2
\]

• where \( e_i \) are arithmetical expressions
• i.e. first-order propositional logic expressions over arithmetical (in)equalities
Extending the Basic Procedure: Example

```java
while ((i > 0 && i < 20) || i > 50) {
    if (i > 50) i--;
    else i++;
}
```

1. Split up the condition into disjunctive parts:
   - \( i > 0 \land i < 20 \land \neg (i > 50) \)
   - \( i > 50 \land \neg (i > 0 \land i < 20) \)
   - \( i > 0 \land i < 20 \land i > 50 \)

2. Execute the basic procedure separately for each of the pieces.
Extending the Basic Procedure: Example

1 while ((i>0 && i<20) || i>50) {
2   if (i>50) i--;
3   else i++;
4 }

1. Split up the condition into disjunctive parts:
   - \( i > 0 \land i < 20 \land \neg (i > 50) \)
   - \( i > 50 \land \neg (i > 0 \land i < 20) \)
   - \( i > 0 \land i < 20 \land i > 50 \)

2. Execute the basic procedure separately for each of the pieces.
Extending the Basic Procedure: Example

```c
while ((i>0 && i<20) || i>50) {
    if (i>50) i--;
    else i++;
}
```

1. Split up the condition into disjunctive parts:
   - \( i > 0 \land i < 20 \lor (i > 50) \)
   - \( i > 50 \lor (i > 0 \land i < 20) \)
   - \( i > 0 \land i < 20 \land i > 50 \)

2. Execute the basic procedure separately for each of the pieces.
Extending the Basic Procedure: Example

1 \textbf{while} ((i>0 \land i<20) \lor i>50) \{ \\
2 \quad \textbf{if} (i>50) i--; \\
3 \quad \textbf{else} i++; \\
4 \}

1 Split up the condition into disjunctive parts:
   - \( i > 0 \land i < 20 \)
   - \( i > 50 \)

2 Execute the basic procedure separately for each of the pieces
Extending the Basic Procedure: Example

1 \textbf{while} ((i>0 \&\& i<20) || i>50) {
2 \quad \textbf{if} (i>50) i--; \\
3 \quad \textbf{else} i++; \\
4 \}

1 Split up the condition into disjunctive parts:
   \begin{itemize}
   \item $i > 0 \land i < 20$
   \item $i > 50$
   \end{itemize}

2 Execute the basic procedure separately for each of the pieces
Extending the Basic Procedure: Example

```plaintext
1 while ((i>0 && i<20) || i>50) {
2    if (i>50) i--;
3    else i++;
4 }
```

\[
\begin{cases}
20 - i & \text{if } (i > 0) \land (i < 20) \\
i - 50 & \text{if } i > 50 \\
0 & \text{else}
\end{cases}
\]
1. Put the condition in Disjunctive Normal Form
2. Split up the condition into its disjunctive pieces
3. Execute the basic procedure separately for each of the pieces
Extending the Basic Procedure: Generic

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Conclusions
Example

1 \textbf{while } ((i>0 \text{ } \&\& \text{ } i<20) \text{ } || \text{ } i>22) \text{ } \{ \\
2 \text{ if } (i>22) \text{ } i--; \\
3 \text{ else } i+=4; \\
4 \} \\

\[\begin{cases}
\lceil(20 - i)/4\rceil & \text{if } (i > 0) \land (i < 20) \\
i - 22 & \text{if } i > 22 \\
0 & \text{else}
\end{cases}\]
Example

```plaintext
while ((i > 0 && i < 20) || i > 22) {
  if (i > 22) i--;  
  else i += 4; 
}
```

\[
\begin{cases}
  \left(\frac{20 - i}{4}\right) + 1 & \text{if } (i > 0) \land (i < 20) \land i \mod 4 = 3 \\
  \left(\frac{20 - i}{4}\right) & \text{if } (i > 0) \land (i < 20) \land i \mod 4 \neq 3 \\
  i - 22 & \text{if } i > 22 \\
  0 & \text{else}
\end{cases}
\]
Detection of Condition Jumping: Example

\[
\text{next}_i(i) = \begin{cases} 
  i - 1 & \text{if } i > 22 \\
  i + 4 & \text{if } \neg(i > 22)
\end{cases}
\]

```lisp
1 (declare-fun i () Int)
2 (define-fun nexti ((x Int)) Int
3  (ite (> x 22) (- x 1) (+ x 4)))
4 (assert (and (and (> i 0) (< i 20))
5  (> (nexti i) 22)))
6 (check-sat)
```
Detection of Condition Jumping: Example

```plaintext
1 while ((i > 0 && i < 20) || i > 22) {
2   if (i > 22) i--;
3   else i += 4;
4 }

next_i(i) = \begin{cases} 
    i - 1 & \text{if } i > 22 \\
    i + 4 & \text{if } \neg(i > 22)
\end{cases}
```

```plaintext
1 (declare-fun i () Int)
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```
Detection of Condition Jumping: Example

```
while ((i > 0 && i < 20) || i > 22) {
  if (i > 22) i --;
  else i += 4;
}
```

\[\text{next}_i(i) = \begin{cases} 
  i - 1 & \text{if } i > 22 \\
  i + 4 & \text{if } \neg(i > 22) 
\end{cases} \]

```
declare-fun i () Int
define-fun nexti ((x Int)) Int
  (ite (> x 22) (– x 1) (+ x 4)))
(assert (and (and (> i 0) (< i 20))
  (> (nexti i) 22)))
(check-sat)
```
Detection of Condition Jumping: Generic

- Symbolically execute the loop body to find a next function for each program variable
- Use SMT-solver to search for a model that satisfies one piece first and another after execution of the loop body
Finding Models for Condition Jumping: Example

Find all nodes that jump from the piece with condition \( i > 0 \land i < 20 \) into the piece with condition \( i > 22 \). Using an SMT-solver:

1. Find all nodes that jump directly into the other piece: \( \{19\} \)

2. Find all nodes that can jump to \( \{19\}, \{3, 7, 11, 15\} \) and add them to the list of jumping nodes:

\[
\{3, 7, 11, 15, 19\} = \{i \mid i \mod 4 = 3 \land i > 0 \land i < 20\}
\]
Finding Models for Condition Jumping: Example

Find all nodes that jump from the piece with condition \( i > 0 \land i < 20 \) into the piece with condition \( i > 22 \). Using an SMT-solver:

1. Find all nodes that jump directly into the other piece: \{19\}

2. Find all nodes that can jump to \{19\}, \{3, 7, 11, 15\} and add them to the list of jumping nodes:
\[
\{3, 7, 11, 15, 19\} = \{i \mid i \text{ mod } 4 = 3 \land i > 0 \land i < 20\} \]
Finding Models for Condition Jumping: Example

Find all nodes that jump from the piece with condition $i > 0 \land i < 20$ into the piece with condition $i > 22$. Using an SMT-solver:

1. Find all nodes that jump directly into the other piece: \{19\}
2. Find all nodes that can jump to \{19\}, \{3, 7, 11, 15\} and add them to the list of jumping nodes:

\{3, 7, 11, 15, 19\} = \{i \mid i \text{ mod } 4 = 3 \land i > 0 \land i < 20\}
Finding Models for Condition Jumping: Generic

$J$ is the set of models of which it is known that condition jumping occurs, $Q$ is a queue of models, find all models that jump from $b_1$ to $b_2$:

1. Is there a model $\bar{v}$ for which $b_1(\bar{v}) \land b_2(\text{next}(\bar{v})) \land \bar{v} \notin J$?
   - SAT $\rightarrow$ Add $\bar{v}$ to $J$ and $Q$, goto 1.
   - UNSAT $\rightarrow$ Goto 2.

2. $Q$ empty?
   - Yes $\rightarrow$ Done.
   - No $\rightarrow$ Goto 3.

3. Pop a model $\bar{q}$ off the queue $Q$. Is there a model $\bar{v}$ for which $b_1(\bar{v}) \land \text{next}(\bar{v}) = \bar{q} \land \bar{v} \notin J$?
   - SAT $\rightarrow$ Add $\bar{v}$ to $J$ and $Q$, goto 3.
   - UNSAT $\rightarrow$ Goto 2.
Generating Ranking Functions: Example

- We now know the set \( \{ i \mid i \mod 4 = 3 \land i > 0 \land i < 20 \} \) for which jumping occurs.
- So, we can split the condition \( i > 0 \land i < 20 \) into two:
  \( i > 0 \land i < 20 \land i \mod = 3 \) and \( i > 0 \land i < 20 \land i \mod \neq 3 \)
- We can then apply the basic method separately to each of these disjunctive pieces.
• We now know the set $D_{1,2}$ for which jumping occurs.
• So, we can split the condition $b_1$ into two: $b_1(\bar{v}) \land \bar{v} \in D_{1,2}$ and $b_1(\bar{v}) \land \bar{v} \not\in D_{1,2}$.
• We can then apply the basic method to each of these disjunctive pieces.
Multi-Jumping

1. DNF-split into $n$ conditions
2. For each $i$ and $j$, $1 \leq i < j \leq n$, detect jumping from $D_i$ to $D_j$. Build a list $J$ of jumping pairs $(D_x, D_y)$ for which condition jumping from $D_x$ to $D_y$ can occur.
3. If there are no more jumping pairs $(D_x, D_y)$ for which $D_x$ is unflagged, done! Else, goto 4.
4. Pop a jumping pair $(D_x, D_y)$ off $J$, for which $D_x$ is unflagged.
5. Find the set $D_{x,2}$ of all nodes in $D_x$ from which jumping to $D_y$ occurs and, dually, the set $D_{x,1}$ for which no jumping to $D_y$ occurs. Replace any condition pair $(D_x, D_z)$ in $J$ by ($D_{x,1}$, $D_z$). Add ($D_{x,2}$, $D_y$) to $J$.
   - If $D_{x,1} = \emptyset$, flag $D_{x,2}$ as complete, goto 3.
   - Else, for any jumping pair $(D_z, D_x)$ in $J$ (i.e. for which jumping from $D_z$ to $D_x$ can occur), unflag $D_z$, detect jumping into $D_{x,1}$ and $D_{x,2}$ and update $J$ accordingly. Goto 3.
Conclusions

- Extension to the method presented at PPPJ’10, which can infer *polynomial* ranking functions:
  - Definition of Condition Jumping
  - Detection of Condition Jumping
  - Infer ranking functions for loops in which condition jumping occurs
- Ranking functions for loops can be used in the creation of a *global* ranking function in order to prove termination
- If the body of a loop with ranking function $RF(\bar{v})$ consumes $n$ resources, then we know that the whole loop consumes $RF(\bar{v}) \cdot n$ resources
Implementation: ResAna

http://resourceanalysis.cs.ru.nl/resana

- The basic procedure and DNF-splitting (minus removal of unsatisfiable pieces) have been implemented
- Future work: implement condition jumping solution