Ranking Functions for Loops with Disjunctive Exit-Conditions

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Presentation Outline

Introduction

Basic Procedure

Piecewise Ranking Functions

Condition Jumping

Conclusions
Ranking Function

- Decreases in every basic block
- Here: in every loop iteration
- Bounded by zero

```c
1 while (i < 15) {
2   i++;
3 }
```

- Ranking function for the loop above is $15 - i$
Motivation and Aim

- Prove termination
- Bounding runtime
- Compiler optimisations
- Resource Analysis

```java
1 while (i < 15) {
2     consumeResource();
3     i++;
4 }
```
Motivation and Aim

- Prove termination
- Bounding runtime
- Compiler optimisations
- **Resource Analysis**

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4 }
```
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- Bounding runtime
- Compiler optimisations
- **Resource Analysis**

```plaintext
while (i < 15) {
    consumeResource();
    i++;
}
```
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Inference of Polynomial Loop Ranking Functions

O. Shkaravska, R. Kersten, M. van Eekelen.
Test-Based Inference of Polynomial Loop-Bound Functions.
PPPJ’10: Proceedings of the 8th International Conference on the Principles and Practice of Programming in Java
Applicable Loops

- The basic method considers loops with conditions in the following form:

\[
C := \text{sC} \mid C_1 \land C_2
\]

\[
\text{sC} := e_1 [\lt, \gt, \le, \ge, =, \neq] e_2
\]

- where \( e_i \) are arithmetical expressions
- i.e. conjunctions over arithmetical (in)equalities
Test-Based Approach

1 Instrument loop with a counter
2 Do test runs for a set of $N_d^k = \binom{d+k}{k}$ input values satisfying NCA and the exit condition
3 Interpolate a polynomial from the results
Test-Based Approach

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3. Interpolate a polynomial from the results
Quadratic Example

```java
public int m(int a, int b, int c) {
    int count=0;
    while (a > 0 && c <= b && c > 0) {
        if ( c == b ) { a--; c = 0; }
        count++;
        c++;
    }
    return count;
}
```

Test runs

1st group: degree 2 NCA on plane
- a=1, b=1, c=1 => count=1
- a=1, b=1, c=2 => count=2
- a=1, b=1, c=3 => count=3
- a=1, b=2, c=2 => count=1
- a=1, b=2, c=3 => count=2
- a=1, b=3, c=3 => count=1

2nd group: degree 1 NCA on plane
- a=2, b=1, c=1 => count=2
- a=2, b=1, c=2 => count=4
- a=2, b=2, c=2 => count=3

3rd group: degree 0 NCA on plane
- a=3, b=1, c=1 => count=3

Find the interpolating polynomial and generate the method annotated with the corresponding ranking function:

\[ RF(a, b, c) = a \cdot b - c + 1 \]
Soundness

• The procedure itself is unsound
• Use external prover to verify the inferred ranking functions
• KeY: http://www.key-project.org/
• Ranking function can be expressed in JML as a decreases clause

```
1  //@ decreases i < 15 ? 15 - i : 0;
2  while (i < 15) {
3    i++;
4  }
```
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Any loop ranking function is piecewise...

1 \textbf{while} (i < 15) {
  2 \quad i++; \\
  3 \}

Its ranking function is actually:

\[
\begin{cases} 
15 - i & \text{if } (i < 15) \\
0 & \text{else}
\end{cases}
\]
Non-Trivial Example

1 \textbf{while } ((i>0 \, \&\& \, i<20) \, || \, i>50) \{ \\
2 \quad \textbf{if } (i>50) \, i--; \\
3 \quad \textbf{else } i++; \\
4 \}

It’s ranking function is non-trivially piecewise:

\[
\begin{cases}
20 - i & \text{if } (i > 0) \, \&\& \, (i < 20) \\
i - 50 & \text{if } i > 50 \\
0 & \text{else}
\end{cases}
\]
Expressing Piecewise Ranking Functions in JML

```plaintext
1 // @ decreases (i > 0 && i < 20) ? 20 - i : (i > 50 ? i - 50 : 0);
2 while ((i > 0 && i < 20) || i > 50) {
3     if (i > 50) i--;
4     else i++;
5 }
```
Applicable Loops

- The extended method considers loops with conditions in the following form:

$$ C := sC \mid C_1 \land C_2 \mid C_1 \lor C_2 $$
$$ sC := e_1 [<, >, \leq, \geq, =, \neq] e_2 $$

- where $e_i$ are arithmetical expressions
- i.e. first-order propositional logic expressions over arithmetical (in)equalities
Extending the Basic Procedure: Example

while ((i>0 && i<20) || i>50) {
  if (i>50) i--;
  else i++;
}

1. Split up the condition into disjunctive parts:
   - $i > 0 \land i < 20 \land \neg(i > 50)$
   - $i > 50 \land \neg(i > 0 \land i < 20)$
   - $i > 0 \land i < 20 \land i > 50$

2. Execute the basic procedure separately for each of the pieces
Extending the Basic Procedure: Example

```java
1 while ((i>0 && i<20) || i>50) {
2   if (i>50) i--;
3   else i++;
4 }
```

1 Split up the condition into disjunctive parts:
   - \( i > 0 \land i < 20 \land \neg(i > 50) \)
   - \( i > 50 \land \neg(i > 0 \land i < 20) \)
   - \( i > 0 \land i < 20 \land i > 50 \)

2 Execute the basic procedure separately for each of the pieces.
Extending the Basic Procedure: Example

1 while ((i>0 && i<20) || i>50) {
2     if (i>50) i--;
3     else i++;  
4 }

1 Split up the condition into disjunctive parts:

   - \( i > 0 \land i < 20 \land \neg (i > 50) \)
   - \( i > 50 \land \neg (i > 0 \land i < 20) \)
   - \( i > 0 \land i < 20 \land i > 50 \)

2 Execute the basic procedure separately for each of the pieces.
Extending the Basic Procedure: Example

1 \textbf{while} ((i>0 \&\& i<20) || i>50) \{  
2 \hspace{1em} \textbf{if} (i>50) \ i--;  
3 \hspace{1em} \textbf{else} \ i++;  
4 \}\n
1 \hspace{1em} \textbf{Split up the condition into disjunctive parts:}\n1 \hspace{1em} \begin{itemize}  
2 \hspace{1em} \bullet \ i > 0 \land i < 20  
2 \hspace{1em} \bullet \ i > 50  
4 \end{itemize}  

2 \hspace{1em} \textbf{Execute the basic procedure separately for each of the pieces}
Extending the Basic Procedure: Example

1 while ((i>0 && i<20) || i>50) {
2   if (i>50) i--;
3   else i++;
4 }

1 Split up the condition into disjunctive parts:
   - $i > 0 \land i < 20$
   - $i > 50$

2 Execute the basic procedure separately for each of the pieces
Extending the Basic Procedure: Example

1 while ((i>0 && i<20) || i>50) {
2     if (i>50) i--;
3     else i++;
4 }

\[
\begin{cases}
20 - i & \text{if } (i > 0) \land (i < 20) \\
i - 50 & \text{if } i > 50 \\
0 & \text{else}
\end{cases}
\]
Extending the Basic Procedure: Generic

1. Put the condition in Disjunctive Normal Form
2. Split up the condition into its disjunctive pieces
3. Execute the basic procedure separately for each of the pieces
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Example

```c
1 while ((i > 0) && (i < 20) || i > 22) {
2     if (i > 22) i--; 
3     else i += 4;
4 }
```

\[
\begin{cases} 
\lceil (20 - i) / 4 \rceil & \text{if } (i > 0) \land (i < 20) \\
i - 22 & \text{if } i > 22 \\
0 & \text{else}
\end{cases}
\]
**Example**

```c
1 while ((i>0 && i<20) || i>22) {
2   if (i>22) i--;
3   else i+=4;
4 }
```

\[
\begin{cases}
\lceil(20 - i)/4\rceil + 1 & \text{if } (i > 0) \land (i < 20) \land i \mod 4 = 3 \\
\lceil(20 - i)/4\rceil & \text{if } (i > 0) \land (i < 20) \land i \mod 4 \neq 3 \\
i - 22 & \text{if } i > 22 \\
0 & \text{else}
\end{cases}
\]
What happens...

D_{1,2} \quad D_{1,1} \quad D_1 \quad D_2
Detection of Condition Jumping: Example

1 while ((i>0 && i<20) || i>22) {
2   if (i>22) i--;
3   else i+=4;
4 }

\[
next_i(i) = \begin{cases} 
  i - 1 & \text{if } i > 22 \\
  i + 4 & \text{if } \neg(i > 22)
\end{cases}
\]

1 (declare-fun i () Int)
2 (define-fun nexti ((x Int)) Int
3   (ite (> x 22) (− x 1) (+ x 4)))
4 (assert (and (and (> i 0) (< i 20))
5   (>(nexti i) 22)))
6 (check-sat)
Detection of Condition Jumping: Example

1 \textbf{while} ((i>0 && i<20) \textbf{||} i>22) \{ \\
2 \hspace{1em} \textbf{if} (i>22) i--; \\
3 \hspace{1em} \textbf{else} i+=4; \\
4 \} \\

next_i(i) = \begin{cases} 
    i - 1 & \text{if } i > 22 \\
    i + 4 & \text{if } \neg (i > 22) 
\end{cases}

1 \textbf{(declare-fun i () Int)} \\
2 \textbf{(define-fun nexti ((x Int)) Int} \\
3 \hspace{1em} \textbf{(ite (> x 22) (- x 1) (+ x 4)))} \\
4 \textbf{(assert (and (and (> i 0) (< i 20))} \\
5 \hspace{1em} (> (nexti i) 22)))} \\
6 \textbf{(check-sat)}
Detection of Condition Jumping: Example

```plaintext
1 while ((i>0 && i<20) || i>22) {
2   if (i>22) i--;
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next_i(i) = \begin{cases} 
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3   (ite (> x 22) (− x 1) (+ x 4))
4 (assert (and (and (> i 0) (< i 20))
5   (> (nexti i) 22))
6 (check-sat)
```
Detection of Condition Jumping: Generic

- Symbolically execute the loop body to find a \texttt{next} function for each program variable
- Use SMT-solver to search for a model that satisfies one piece first and another after execution of the loop body
Finding Models for Condition Jumping: Example

Find all nodes that jump from the piece with condition $i > 0 \land i < 20$ into the piece with condition $i > 22$. Using an SMT-solver:

1. Find all nodes that jump directly into the other piece: $\{19\}$
2. Find all nodes that can jump to $\{19\}$, $\{3, 7, 11, 15\}$ and add them to the list of jumping nodes:

$\{3, 7, 11, 15, 19\} = \{i \mid i \mod 4 = 3 \land i > 0 \land i < 20\}$
Finding Models for Condition Jumping: Example

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   $\{3, 7, 11, 15, 19\} = \{i \mid i \mod 4 = 3 \land i > 0 \land i < 20\}$
Finding Models for Condition Jumping: Generic

\( J \) is the set of models of which it is known that condition jumping occurs, \( Q \) is a queue of models, find all models that jump from \( b_1 \) to \( b_2 \):

1. Is there a model \( \bar{v} \) for which \( b_1(\bar{v}) \land b_2(\text{next}(\bar{v})) \land \bar{v} \notin J \)?
   - SAT \( \rightarrow \) Add \( \bar{v} \) to \( J \) and \( Q \), goto 1.
   - UNSAT \( \rightarrow \) Goto 2.

2. \( Q \) empty?
   - Yes \( \rightarrow \) Done.
   - No \( \rightarrow \) Goto 3.

3. Pop a model \( \bar{q} \) off the queue \( Q \). Is there a model \( \bar{v} \) for which \( b_1(\bar{v}) \land \text{next}(\bar{v}) = \bar{q} \land \bar{v} \notin J \)?
   - SAT \( \rightarrow \) Add \( \bar{v} \) to \( J \) and \( Q \), goto 3.
   - UNSAT \( \rightarrow \) Goto 2.
We now know the set \( \{ i \mid i \mod 4 = 3 \land i > 0 \land i < 20 \} \) for which jumping occurs.

So, we can split the condition \( i > 0 \land i < 20 \) into two:

\[
i > 0 \land i < 20 \land i \mod = 3 \land i > 0 \land i < 20 \land i \mod \neq 3
\]

We can then apply the basic method separately to each of these disjunctive pieces.
Generating Ranking Functions: Generic

- We now know the set $D_{1,2}$ for which jumping occurs
- So, we can split the condition $b_1$ into two: $b_1(\bar{v}) \land \bar{v} \in D_{1,2}$ and $b_1(\bar{v}) \land \bar{v} \not\in D_{1,2}$
- We can then apply the basic method to each of these disjunctive pieces
Multi-Jumping

1. DNF-split into $n$ conditions
2. For each $i$ and $j$, $1 \leq i < j \leq n$, detect jumping from $D_i$ to $D_j$. Build a list $J$ of jumping pairs $(D_x, D_y)$ for which condition jumping from $D_x$ to $D_y$ can occur.
3. If there are no more jumping pairs $(D_x, D_y)$ for which $D_x$ is unflagged, done! Else, goto 4.
4. Pop a jumping pair $(D_x, D_y)$ off $J$, for which $D_x$ is unflagged.
5. Find the set $D_{x,2}$ of all nodes in $D_x$ from which jumping to $D_y$ occurs and, dually, the set $D_{x,1}$ for which no jumping to $D_y$ occurs. Replace any condition pair $(D_x, D_z)$ in $J$ by $(D_{x,1}, D_z)$. Add $(D_{x,2}, D_y)$ to $J$.
   - If $D_{x,1} = \emptyset$, flag $D_{x,2}$ as complete, goto 3.
   - Else, for any jumping pair $(D_z, D_x)$ in $J$ (i.e. for which jumping from $D_z$ to $D_x$ can occur), unflag $D_z$, detect jumping into $D_{x,1}$ and $D_{x,2}$ and update $J$ accordingly. Goto 3.
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- Extension to the method presented at PPPJ’10, which can infer polynomial ranking functions:
  - Definition of Condition Jumping
  - Detection of Condition Jumping
  - Infer ranking functions for loops in which condition jumping occurs
- Ranking functions for loops can be used in the creation of a global ranking function in order to prove termination
- If the body of a loop with ranking function $RF(\bar{v})$ consumes $n$ resources, then we know that the whole loop consumes $RF(\bar{v}) \cdot n$ resources
Implementation: ResAna

http://resourceanalysis.cs.ru.nl/resana

- The basic procedure and DNF-splitting (minus removal of unsatisfiable pieces) have been implemented
- Future work: implement condition jumping solution