Abstract. In this paper we show that the image of any locally finite $k$-derivation of the polynomial algebra $k[x, y]$ in two variables over a field $k$ of characteristic zero is a Mathieu subspace. We also show that the two-dimensional Jacobian conjecture is equivalent to the statement that the image $\text{Im} D$ of every $k$-derivation $D$ of $k[x, y]$ such that $1 \in \text{Im} D$ and $\text{div} D = 0$ is a Mathieu subspace of $k[x, y]$.

1. Introduction

Kernels of derivations have been studied in many papers. On the other hand, only a few results are known concerning images of derivations.

In this paper we consider the question if the image of a derivation of a polynomial algebra in two variables over a field $k$ is a Mathieu subspace of the polynomial algebra.

The notion of the Mathieu subspaces was introduced recently by the third-named author in [Z2] in order to study the Mathieu conjecture [M], the image conjecture [Z1] and the Jacobian conjecture (see [BCW] and [E1]). We will recall its definition in Section 2 below.

Throughout this paper we fix the following notation: $k$ is a field of characteristic zero and $x, y$ are two free commutative variables. We denote by $A$ the polynomial algebra $k[x, y]$ over the field $k$.

The contents of the paper are arranged as follows.

In Section 2 we recall some facts concerning Mathieu subspaces and show that the image of a $k$-derivation of $A$ needs not be a Mathieu subspace (see Example 2.4).
In Section 3 we prove in Theorem 3.1 that for every locally finite $k$-derivation $D$ of $A$, the image $\text{Im } D$ is a Mathieu subspace. Finally in Section 4 we show in Theorem 4.3 that the two-dimensional Jacobian conjecture is equivalent to the following: if $D$ is a $k$-derivation of $A$ with $\text{div } D = 0$ such that $1 \in \text{Im } D$, then $\text{Im } D$ is a Mathieu subspace of $A$.

2. Preliminaries

We start with the following notion introduced in [Z2].

**Definition 2.1.** Let $R$ be any commutative $k$-algebra and $M$ a $k$-subspace of $R$. Then $M$ is a Mathieu subspace of $R$ if the following condition holds: if $a \in R$ is such that $a^m \in M$ for all $m \geq 1$, then for any $b \in R$, there exists an $N \in \mathbb{N}$ such that $ba^m \in M$ for all $m \geq N$.

Obviously every ideal of $R$ is a Mathieu subspace of $R$. However not every Mathieu subspace of $R$ is an ideal of $R$. Before we give some examples, we first recall the following simple lemma proved in Lemma 4.5, [Z2], which will be very useful for our later arguments. For the sake of completeness, we here also include a proof.

**Lemma 2.2.** If $M$ is a Mathieu subspace of $R$ and $1 \in M$, then $M = R$.

**Proof:** Since $1 \in M$, it follows that $1^m = 1 \in M$ for all $m \geq 1$. Then for every $a \in R$, $a = a1^m \in M$ for all large $m$. Hence $R \subseteq M$ and $R = M$. \qed

**Example 2.3.** Let $R := k[t, t^{-1}]$ be the algebra of Laurent polynomials in the variable $t$. For each $c \in k$, let $D_c$ be the differential operator $\frac{d}{dt} + ct^{-1}$ of $R$. Then $\text{Im } D_c := D_c R$ is a Mathieu subspace of $R$ if and only if $c \notin \mathbb{Z}$ or $c = -1$.

Note that the conclusion above follows directly by applying Lefschetz’s principle to Proposition 2.6 [Z2]. Since Proposition 2.6 in [Z2] is for multi-variable case and its proof is quite involved, we here include a self-contained proof for the one variable case.

**Proof:** Note first that for any $m \in \mathbb{Z}$, $D_c t^m = (m + c)t^{m-1}$. So, if $c \notin \mathbb{Z}$, then $\text{Im } D_c = R$. Hence a Mathieu subspace of $R$.

If $c \in \mathbb{Z}$ but $c \neq -1$, then $D_c t = (1 + c) \neq 0$. So $1 \in \text{Im } D_c$. Since $D_c t^{-c} = (-c + c)t^{-c-1} = 0$, it is easy to see that $t^{-c-1} \notin \text{Im } D_c$. Hence $\text{Im } D_c \neq R$. Then by Lemma 2.2, $\text{Im } D_c$ is not a Mathieu subspace of $R$. 
Finally, assume \( c = -1 \). Since \( D_{-1} t^m = (m-1)t^{m-1} \) for all \( m \in \mathbb{Z} \), it is easy to see that \( \text{Im} \, D_{-1} \) is the subspace of the Laurent polynomials in \( R \) without constant term. Then by the Duistermaat-van der Kallen theorem \([DK]\), \( M \) is a Mathieu subspace of \( R \). \( \square \)

Note that when \( c = -1 \), \( \text{Im} \, D_{-1} \) is a Mathieu subspace of \( R \). But it clearly is not an ideal of \( R \). For more examples of Mathieu subspaces which are not ideals, see Section 4 in \([Z2]\).

When \( c = 0 \), we see that \( \text{Im} \, d/dt \) is not a Mathieu subspace of \( R \).

Now observe that 

\[
 k[t, t^{-1}] \cong k[[x, y]]/(xy - 1),
\]

where \( t \) corresponds to the class of \( x \) and \( t^{-1} \) to the class of \( y \). Then the derivation \( d/dt \) of \( R \) can be lifted to a \( k \)-derivation \( D \) of \( k[[x, y]] \), which maps \( x \) to \( \frac{d}{dt} t = 1 \) and \( y \) to \( \frac{d}{dt} t^{-1} = -t^{-2} \), i.e., \( -y^2 \). This leads to the following example.

**Example 2.4.** Let \( D = \partial_x - y^2 \partial_y \). Then \( \text{Im} \, D \) is not a Mathieu subspace of \( k[[x, y]] \).

**Proof:** Note that \( 1 = Dx \in \text{Im} \, D \). However \( y \notin \text{Im} \, D \) since for any \( g \in k[[x, y]] \) the \( y \)-degree of \( Dg \) can not be 1. So by Lemma 2.2 \( \text{Im} \, D \) is not a Mathieu subspace of \( k[[x, y]] \). \( \square \)

The following lemma will also be needed in Section 3.

**Lemma 2.5.** Let \( R \) be any \( k \)-algebra, \( L \) a field extension of \( k \) and \( M \) a \( k \)-subspace of \( R \). Assume that \( L \otimes_k M \) is a Mathieu subspace of the \( L \)-algebra \( L \otimes_k R \). Then \( M \) is a Mathieu subspace of the \( k \)-algebra \( R \).

**Proof:** We view \( L \otimes_k R \) as a \( k \)-algebra in the obvious way. Since \( L \otimes_k M \) is a Mathieu subspace of the \( L \)-algebra \( L \otimes_k R \), from Definition 2.1 it is easy to see that \( L \otimes_k M \) (as a \( k \)-subspace) is also a Mathieu subspace of the \( k \)-algebra \( L \otimes_k R \).

Now we identify \( R \) with the \( k \)-subalgebra \( 1 \otimes_k R \) of the \( k \)-algebra \( L \otimes_k R \). Then from Definition 2.1 again, it is easy to check that the intersection \( (L \otimes_k M) \cap R = M \) is a Mathieu subspace of \( R \). \( \square \)

Note that by the lemma above, when we prove that a \( k \)-subspace of a polynomial algebra over \( k \) is a Mathieu subspace of the polynomial algebra, we may freely replace \( k \) by any field extension of \( k \). For instance, we may assume that \( k \) is algebraically closed.

To conclude this section we recall a result from \([EWZ]\) which will be used in Section 3 below.

Let \( z = (z_1, z_2, \ldots, z_n) \) be \( n \) commutative free variables and \( k[z, z^{-1}] \) the algebra of Laurent polynomials in \( z_i \) \((1 \leq i \leq n)\). For any non-zero \( f(z) = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha z^\alpha \in k[z, z^{-1}] \), we denote by \( \text{Supp} \,(f) \) the support of
$f(z)$, i.e., the set of all $\alpha \in \mathbb{Z}^n$ such that $c_\alpha \neq 0$, and $\text{Poly}(f)$ the (Newton) polytope of $f(z)$, i.e., the convex hull of $\text{Supp}(f)$ in $\mathbb{R}^n$.

**Theorem 2.6.** ([EWZ]) Let $0 \neq f \in k[z, z^{-1}]$ and $u$ any rational point, i.e., a point with all coordinates being rational, of $\text{Poly}(f)$. Then there exists $m \geq 1$ such that $(\mathbb{R}_+u) \cap \text{Supp}(f^m) \neq \emptyset$.

### 3. Images of Locally Finite Derivations of $k[x, y]$

Let $D$ be any $k$-derivation of $A(= k[x, y])$. Then $D$ is said to be locally finite if for every $a \in A$ the $k$-vector space spanned by the elements $D^i a$ ($i \geq 1$) is finite dimensional.

The main result of this section is the following theorem.

**Theorem 3.1.** Let $D$ be any locally finite $k$-derivation of $A$. Then $\text{Im} D$ is a Mathieu subspace of $A$.

To prove this theorem, we need the following result, which is Corollary 4.7 in [E2].

**Proposition 3.2.** Let $D$ be any locally finite $k$-derivation of $A$. Then up to the conjugation by a $k$-automorphism of $A$, $D$ has one of the following forms:

i) $D = (ax + by)\partial_x + (cx + dy)\partial_y$ for some $a, b, c, d \in k$;

ii) $D = \partial_x + by\partial_y$ for some $b \in k$;

iii) $D = ax\partial_x + (x^m + amy)\partial_y$ for some $a \in k$ and $m \geq 1$;

iv) $D = f(x)\partial_y$ for some $f(x) \in k[x]$.

**Lemma 3.3.** With the same notations as in Proposition 3.2, the following statements hold.

(a) If $D$ is of type ii), then $D$ is surjective.

(b) If $D$ is of type iii), then

\[
\text{Im } D = \begin{cases}
(x^m) & \text{if } a = 0, \\
(x, y) & \text{if } a \neq 0.
\end{cases}
\]

(c) If $D$ is of type iv), then $\text{Im } D = (f(x))$.

**Proof:** (a) is well-known, see [C] or [F] (p. 96). (c) is obvious, so it remains to prove (b).

If $a = 0$, then $D = x^m\partial_y$, and hence $\text{Im } D = (x^m)$. So assume $a \neq 0$. Replacing $D$ by $a^{-1}D$ (without changing the image $\text{Im } D$), we may assume that $D = (x\partial_x + my\partial_y) + bx^m\partial_y$ for some nonzero $b \in k$.

Observe that for any $i, j \in \mathbb{N}$, we have

\[
D(x^i y^j) = (i + m j)x^i y^j + j bx^{m+i} y^{j-1}.
\]
Next we use induction on $j \geq 0$ to show that $x^iy^j \in \text{Im } D$ whenever $i + j > 0$.

First, assume $j = 0$. Then by Eq. (3.2), we have $Dx^i = ix^i$, and hence $x^i \in \text{Im } D$ for all $i \geq 1$.

Now assume $j \geq 1$. Since $m \geq 1$, we have $m + i \geq 1$ for all $i \geq 0$. Combining this fact with Eq. (3.2), we get $x^iy^j \in \text{Im } D$ since $i + mj \neq 0$ for all $i \geq 0$. Hence we have proved that $x^iy^j \in \text{Im } D$ if $i + j > 0$. Note that 1 does not lie in $\text{Im } D$ since this space is contained in the ideal generated by $x$ and $y$. Therefore we have $\text{Im } D = (x, y)$.

**Lemma 3.4.** Let $z = (z_1, z_2, ..., z_n)$ be $n$ free commutative variables and $D := \sum_{i=1}^{n} a_i z_i \partial_{z_i}$ for some $a_i \in k$ ($1 \leq i \leq n$). Then $\text{Im } D$ is a Mathieu subspace of $k[z]$.

Note that $D$ in the lemma is a locally finite derivation of the polynomial algebra $k[z]$. To show the lemma, let’s first recall the following well-known results.

**Lemma 3.5.** For any polynomials $f, g \in k[z]$ and a positive integer $m \geq 1$, we have

\[
(3.3) \quad \text{Poly } (fg) = \text{Poly } (f) + \text{Poly } (g),
\]

\[
(3.4) \quad \text{Poly } (f^m) = m \text{Poly } (f),
\]

where the sum in the first equation above denotes the Minkowski sum of polytopes.

**Proof:** Eq. (3.3) is well-known, which was first proved by A. M. Ostrowski [O1] in 1921 (see also Theorem VI, p. 226 in [O2] or Lemma 2, p. 11 in [Stu]). To show Eq. (3.4), one can first check easily that the polytope $m \text{Poly } (f)$ and the polytope obtained by taking the Minkowski sum of $m$ copies of $\text{Poly } (f)$ actually share the same set of extremal vertices, namely, the set of the vertices $mv_i$, where $v_i$ runs through all extremal vertices of $\text{Poly } (f)$. Consequently, these two polytopes coincide. Then from this fact and Eq. (3.3), we see that Eq. (3.4) follows.

**Proof of Lemma [3.4]**: If all $a_i$’s are zero, then $D = 0$ and $\text{Im } D = 0$. Hence the lemma holds in this case. So, we assume that not all $a_i$’s are zero.

Let $S$ be the set of integral solutions $\beta \in \mathbb{Z}^n$ of the linear equation $\sum_{i=1}^{n} a_i \beta_i = 0$. Note that $S \neq \emptyset$ (since $0 \in S$) and is a finitely generated $\mathbb{Z}$-module. Let $V$ be the subspace of $\mathbb{R}^n$ spanned by elements of $S$ over
Then \( V \) is a \( \mathbb{R} \)-subspace of \( \mathbb{R}^n \) with \( r := \dim_{\mathbb{R}} V < n \). Furthermore, \( V \) can be described as the set of common solutions of some linear equations with rational coefficients, since clearly the \( \mathbb{Q} \)-vector space generated by the \( \mathbb{Z} \)-generators of \( S \) can.

Note also that for any \( \beta = (\beta_1, \beta_2, ..., \beta_n) \in \mathbb{N}^n \), we have \( Dz^\beta = (\sum_{i=1}^n a_i \beta_i)z^\beta \). Hence, for any \( \beta \in \mathbb{N}^n \), the monomial \( z^\beta \in \text{Im} \, D \) iff \( \beta \not\in S \), or equivalently, \( \beta \not\in V \). Consequently, for any \( 0 \neq h(z) \in \mathbb{C}[z] \), we have

\[
(3.5) \quad h(z) \in \text{Im} \, D \iff \text{Supp} (h) \cap V = \emptyset.
\]

Now, let \( 0 \neq f(z) \in \mathbb{C}[z] \) such that \( f^m \in \text{Im} \, D \) for all \( m \geq 1 \). We claim \( \text{Poly} (f) \cap V = \emptyset \).

Assume otherwise. Since all vertices of the polytope \( \text{Poly} (f) \) are rational (actually integral), every face of \( \text{Poly} (f) \) can be described as the set of common solutions of some linear equations with rational coefficients. Since this is also the case for \( V \) (as pointed above) and \( \text{Poly} (f) \cap V \neq \emptyset \) (by our assumption), it is easy to see that there exists at least one rational point \( u \in \text{Poly} (f) \cap V \). Then by Theorem 2.6 there exists \( m \geq 1 \) such that \( (\mathbb{R}, u) \cap \text{Supp} (f^m) \neq \emptyset \), and by Eq. (3.5), \( f^m \not\in \text{Im} \, D \). Hence, we get a contradiction. Therefore, the claim holds.

Finally, we show that \( \text{Im} \, D \) is a Mathieu subspace as follows.

Let \( f(z) \) be as above and \( d \) the distance between \( V \) and \( \text{Poly} (f) \). Then by the claim above and the fact that \( \text{Poly} (f) \) is a compact subset of \( \mathbb{R}^n \), we have \( d > 0 \). Furthermore, for any \( m \geq 1 \), by Eq. (3.4) we have \( \text{Poly} (f^m) = m \text{Poly} (f) \). Hence, the distance between \( V \) and \( \text{Poly} (f^m) \) is given by \( dm \).

Now let \( h(z) \) be an arbitrary element of \( k[z] \). Note that by Eqs. (3.3) and (3.4) we have \( \text{Poly} (f^m h) = m \text{Poly} (f) + \text{Poly} (h) \) for all \( m \geq 1 \). Hence, for large enough \( m \), the distance between \( V \) and \( \text{Poly} (f^m h) \) is positive, whence \( \text{Poly} (f^m h) \cap V = \emptyset \). In particular, \( \text{Supp} (f^m h) \cap V = \emptyset \), and by Eq. (3.5), \( f^m h \in \text{Im} \, D \) when \( m \gg 0 \). Then by Definition 2.1 we see that \( \text{Im} \, D \) is indeed a Mathieu subspace of \( k[z] \).

Now we can prove the main theorem of this section as follows.

**Proof of Theorem 3.1.** First, by Proposition 3.2, we only need to show that \( \text{Im} \, D \) is a Mathieu subspace of \( A \) in each of the four cases in Proposition 3.2. Furthermore, by Lemma 3.3 it only remains to prove case i). So assume \( D = (ax + by) \partial_x + (cx + dy) \partial_y \) for some \( a, b, c, d \in k \).

Second, by Lemma 2.5, we may assume that \( k \) is algebraically closed.

Third, note that \( D \) preserves the subspace \( H := kx + ky \subset A \), so its restriction \( D|_H \) on \( H \) is a linear endomorphism of \( H \). Since \( k \) is
algebraically closed, there exists a linear automorphism $\sigma$ of $H$ such that the conjugation $\sigma(D|_H)\sigma^{-1}$ gives the Jordan form of $D|_H$. Let $\tilde{\sigma}$ be the unique extension of $\sigma$ to an automorphism of $A$. Then it is easy to see that $\tilde{\sigma}D\tilde{\sigma}^{-1}$ is also a $k$-derivation of $A$.

Note that $\text{Im} \, \tilde{\sigma}D\tilde{\sigma}^{-1} = \tilde{\sigma}(\text{Im} \, D)$ and in general Mathieu subspaces are preserved by $k$-algebra automorphisms. Therefore, we may replace $D$ by $\tilde{\sigma}D\tilde{\sigma}^{-1}$, if necessary, and assume that $D = a(x\partial_x + y\partial_y) + x\partial_y$ (in case that the Jordan form of $D|_H$ is an $2 \times 2$ Jordan block) or $D = ax\partial_x + by\partial_y$ (in case that the Jordan form of $D|_H$ is diagonal).

For the former case, by Lemma 3.3 (b) with $m = 1$, we see that $\text{Im} \, D$ is an ideal, and hence a Mathieu subspace of $A$. For the latter case, it follows from Lemma 3.4 that $\text{Im} \, D$ also a Mathieu subspace of $A$. Therefore, the theorem holds. □

4. Connection with the Two-Dimensional Jacobian Conjecture

In the previous section we showed that the image of every locally finite $k$-derivation of $A$ is a Mathieu subspace of $A$. However, as we have shown in Example 2.4, $\text{Im} \, D$ needs not to be a Mathieu subspace of $A$ for every $k$-derivation $D$ of $A$. This leads to the question of which $k$-derivations $D$ of $A$ have the property that $\text{Im} \, D$ is a Mathieu subspace of $A$. More precisely, we can ask

**Question 4.1.** Let $D$ be any $k$-derivation of $A$ such that $\text{div} \, D = 0$, where for any $D = p\partial_x + q\partial_y$ ($p, q \in A$), $\text{div} \, D := \partial_x p + \partial_y q$. Is $\text{Im} \, D$ a Mathieu subspace of $A$?

Adding one more condition, we get

**Question 4.2.** Let $D$ be any $k$-derivation of $A$ such that $\text{div} \, D = 0$. If $1 \in \text{Im} \, D$, is $\text{Im} \, D$ a Mathieu subspace of $A$?

Note that by Lemma 2.2, this question is equivalent to asking if $D$ is surjective under the further condition $1 \in \text{Im} \, D$.

The motivation of the two questions above come from the following theorem.

**Theorem 4.3.** Question 4.2 has an affirmative answer iff the two-dimensional Jacobian conjecture is true.

**Proof:** ($\Rightarrow$) Assume that Question 4.2 has an affirmative answer. Let $F = (f, g) \in k[x, y]^2$ with $\det JF = 1$. Consider the $k$-derivation $D := g_y \partial_x - g_x \partial_y$. Then $\text{div} \, D = 0$ and $1 = \det JF = Df \in \text{Im} \, D$. Since by our hypothesis $\text{Im} \, D$ is a Mathieu subspace of $A$, it follows
from Lemma 2.2 that $\text{Im} D = A$, i.e., $D$ is surjective. Then it follows from a theorem of Stein \cite{Ste} (see also \cite{C}) that $D$ is locally nilpotent.

Since $D = \partial/\partial f$, $\ker D = \ker \partial/\partial f = k[g]$ by Proposition 2.2.15 in \cite{E1}. Since $D$ has a slice $f$, it follows that $A = k[g][f]$, i.e., $F$ is invertible over $k$. So the two-dimensional Jacobian conjecture is true.

$(\Leftarrow)$ Assume that the two-dimensional Jacobian conjecture is true. Let $D = p \partial_x + q \partial_y$ ($p, q \in A$) be a $k$-derivation of $A$ such that $\text{div} D = 0$ and $1 \in \text{Im} D$.

Since $\text{div} D = 0$, we have $\partial_x p = \partial_y (-q)$. Then by Poincaré’s lemma, there exists $g \in A$ such that $p = \partial_y g$ and $q = -\partial_x g$. So $D = g \partial_x - g_x \partial_y$.

Since $1 \in \text{Im} D$, we get $1 = Df$ for some $f \in A$. Let $F := (f, g) \in k[x, y]^2$. Then we have $\det JF = Df = 1$. Since by our hypothesis $F$ is invertible, it follows that $k[x, y] = k[f, g]$. Hence, we have

$$\text{Im} D = \text{Im} \frac{\partial}{\partial f} = \frac{\partial}{\partial f}(k[f, g]) = k[f, g] = A.$$ 

In particular, $\text{Im} D$ is a Mathieu subspace of $A$. \hfill $\Box$

**References**


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