Dynamic Policy Programming

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Abstract

In this paper, we propose a novel policy iteration method, called dynamic policy programming (DPP), to estimate the optimal policy in the infinite-horizon Markov decision processes. We prove the finite-iteration and asymptotic \( \ell_\infty \)-norm performance-loss bounds for DPP in the presence of approximation/estimation error. The bounds are expressed in terms of the \( \ell_\infty \)-norm of the average accumulated error as opposed to the \( \ell_\infty \)-norm of the error in the case of the standard approximate value iteration (AVI) and the approximate policy iteration (API). This suggests that DPP can achieve a better performance than AVI and API since it averages out the simulation noise caused by Monte-Carlo sampling throughout the learning process. We examine this theoretical results numerically by comparing the performance of the approximate variants of DPP with existing reinforcement learning (RL) methods on different problem domains. Our results show that, in all cases, DPP-based algorithms outperform other RL methods by a wide margin.

Keywords: Approximate dynamic programming, reinforcement learning, Markov decision processes, Monte-Carlo methods, function approximation.

1. Introduction

Many problems in robotics, operations research and process control can be represented as a control problem that can be solved by finding the optimal policy using dynamic programming (DP). DP is based on the estimating some measures of the value of state-action \( Q^*(x,a) \) through the Bellman equation. For high-dimensional discrete systems or for continuous systems, computing the value function by DP is intractable. The common approach to make the computation tractable is to approximate the value function using function-approximation and Monte-Carlo sampling (Szepesvári, 2010; Bertsekas and Tsitsiklis, 1996). Examples of such approximate dynamic programming (ADP) methods are approximate policy iteration (API) and approximate value iteration (AVI) (Bertsekas, 2007; Lagoudakis and Parr, 2003; Perkins and Precup, 2002; de Farias and Roy, 2000).

ADP methods have been successfully applied to many real world problems, and theoretical results have been derived in the form of finite iteration and asymptotic performance guarantee of the induced policy (Farahmand et al., 2010; Thiery and Scherrer, 2010; Munos, 2003; Bertsekas and Tsitsiklis, 1996). The asymptotic \( \ell_\infty \)-norm performance-loss bounds of
API and AVI are expressed in terms of the supremum, with respect to (w.r.t.) the number of iterations, of the approximation errors:

$$\limsup_{k \to \infty} \|Q^* - Q_{\pi_k}\| \leq \frac{2\gamma}{(1 - \gamma)^2} \limsup_{k \to \infty} \|\epsilon_k\|,$$

where $\gamma$ denotes the discount factor, $\| \cdot \|$ is the $\ell_\infty$-norm w.r.t. the state-action pair $(x,a)$. Also, $\pi_k$ and $\epsilon_k$ are the control policy and the approximation error at round $k$ of the ADP algorithms, respectively. In many problems of interest, however, the supremum over the normed-error $\|\epsilon_k\|$ can be large and hard to control due to the large variance of estimation caused by Monte-Carlo sampling. In those cases, a bound which instead depends on the average accumulated error $\bar{\epsilon}_k = 1/(k + 1) \sum_{j=0}^{k} \epsilon_j$ is preferable. This is due to the fact that the errors associated with the variance of estimation can be considered as the instances of some zero-mean random variables. Therefore, one can show, by making use of a law of large numbers argument, that those errors are asymptotically averaged out by accumulating the approximation errors of all iterations.

In this paper, we propose a new mathematically-justified approach to estimate the optimal policy, called dynamic policy programming (DPP). We prove finite-iteration and asymptotic performance loss bounds for the policy induced by DPP in the presence of approximation. The asymptotic bound of approximate DPP is expressed in terms of the average accumulated error $\|\bar{\epsilon}_k\|$ as opposed to $\|\epsilon_k\|$ in the case of AVI and API. This result suggests that DPP may perform better than AVI and API in the presence of large variance of estimation since it can average out the estimation errors throughout the learning process. The dependency on the average error $\|\bar{\epsilon}_k\|$ follows naturally from the incremental policy update of DPP which at each round of policy update, unlike AVI and API, accumulates the approximation errors of the previous iterations, rather than just minimizing the approximation error of the current iteration.

This article is organized as follows. In Section 2, we present the notations which are used in this paper. We introduce DPP and we investigate its convergence properties in Section 3. In Section 4, we demonstrate the compatibility of our method with the approximation techniques. We generalize DPP bounds to the case of function approximation and Monte-Carlo simulation. We also introduce a new convergent RL algorithm, called DPP-RL, which relies on an approximate sample-based variant of DPP to estimate the optimal policy. Section 5 presents numerical experiments on several problem domains including the optimal replacement problem [Munos and Szepesvári, 2008] and a stochastic grid world. In Section 6, we briefly review some related work. Finally, we discuss some of the implications of our work in Section 7.

2. Preliminaries

In this section, we introduce some concepts and definitions from the theory of Markov decision processes (MDPs) and reinforcement learning (RL) as well as some standard notations. We begin by the definition of the $\ell_2$-norm (Euclidean norm) and the $\ell_\infty$-norm.

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1. The law of large numbers requires the errors to satisfy some stochastic assumptions, e.g., they need to be identically and independently distributed (i.i.d.) samples or martingale differences.
2. For further reading see Szepesvári (2010).
(supremum norm). Assume that $Y$ is a finite set. Given the probability measure $\mu$ over $Y$, for a real-valued function $g : Y \to \mathbb{R}$, we shall denote the $\ell_2,\mu$-norm of $g$ by $\|g\|_2^2 = \sum_{y \in Y} g(y)^2$ and $\|g\|_{2,\mu}^2 = \sum_{y \in Y} \mu(y)g(y)^2$, respectively. Also, the $\ell_{\infty}$-norm of $g$ is defined by $\|g\|_\infty = \max_{y \in Y} |g(y)|$.

### 2.1 Markov Decision Processes

A discounted MDP is a quintuple $(X,A,P,R,\gamma)$, where $X$ and $A$ are, respectively, the state space and the action space. $P$ shall denote the state transition distribution and $R$ denotes the reward kernel. $\gamma \in [0,1)$ denotes the discount factor. The transition $P$ is a probability kernel over the next state upon taking action $a$ from state $x$, which we shall denote by $P(\cdot|x,a)$. $R$ is a set of real-valued numbers. A reward $r(x,a) \in R$ is associated with each state $x$ and action $a$. To keep the representation succinct, we shall denote the joint state-action space $X \times A$ by $Z$.

**Assumption 1 (MDP Regularity)** We assume $X$ and $A = \{a_1,a_2,\ldots,a_L\}$ are finite sets. Also, the absolute value of the immediate reward $r(x,a)$ is bounded from above by $R_{\max} > 0$ for all $(x,a) \in Z$. We also define $V_{\max} = R_{\max}/(1 - \gamma)$.

A policy kernel $\pi(\cdot|\cdot)$ determines the distribution of the control action given the past observations. The policy is called stationary and Markovian if the distribution of the control action is independent of time and only depends on the last state $x$. Given the last state $x$, we shall denote the stationary policy by $\pi(\cdot|x)$. A stationary policy is called deterministic if for any state $x$ there exists some action $a$ such that $\pi(\cdot|x)$ concentrates on this action. Given the policy $\pi$ its corresponding value function $V^\pi : X \to \mathbb{R}$ denotes the expected value of the long-term discounted sum of rewards in each state $x$, when the action is chosen by policy $\pi$ which we denote by $V^\pi(x)$. Often it is convenient to associate value functions not with states but with state-action pairs. Therefore, we introduce $Q^\pi : Z \to \mathbb{R}$ as the expected total discounted reward upon choosing action $a$ from state $x$ and then following policy $\pi$, which we shall denote by $Q^\pi(x,a)$. We define the **Bellman operator** $\mathcal{T}^\pi$ on the action-value functions by:

$$
\mathcal{T}^\pi Q(x,a) \triangleq r(x,a) + \gamma \sum_{(x',a') \in Z} P(x'|x,a)\pi(a'|x')Q(x',a'), \quad \forall (x,a) \in Z.
$$

We also notice that $Q^\pi$ is the fixed point of $\mathcal{T}^\pi$.

The goal is to find a policy $\pi^*$ that attains the **optimal value function**, $V^*(x) \triangleq \sup_\pi V^\pi(x)$, at all states $x \in X$. The optimal value function satisfies the Bellman equation:

$$
V^*(x) = \max_{a \in A} \left[ r(x,a) + \sum_{x' \in X} P(x'|x,a)V^*(x') \right], \quad \forall x \in X. \quad (1)
$$
Likewise, the optimal action-value function $Q^*$ is defined by $Q^*(x, a) = \sup_{\pi} Q^{\pi}(x, a)$ for all $(x, a) \in \mathcal{Z}$. We shall define the Bellman optimality operator $\mathcal{T}$ on the action-value functions as:

$$\mathcal{T}Q(x, a) \triangleq r(x, a) + \gamma \sum_{x' \in \mathcal{X}} P(x'|x, a) \max_{a' \in A} Q(x', a'), \quad \forall (x, a) \in \mathcal{Z}. $$

Likewise, $Q^*$ is the fixed point of $\mathcal{T}$.

Both $\mathcal{T}$ and $\mathcal{T}^{\pi}$ are contraction mappings, w.r.t. the supremum norm, with the factor $\gamma$ (Bertsekas, 2007, chap. 1). In other words, for any two action-value functions $Q$ and $Q'$, we have:

$$\|\mathcal{T}Q - \mathcal{T}Q'\| \leq \gamma \|Q - Q'\|^1, \quad ||\mathcal{T}^{\pi}Q - \mathcal{T}^{\pi}Q'|| \leq \gamma \|Q - Q'\|. \quad (2)$$

The policy distribution $\pi$ defines the state-action transition kernel $P^{\pi} : M(\mathcal{Z}) \rightarrow M(\mathcal{Z})$, where $M$ is the space of all probability measures defined on $\mathcal{Z}$, as:

$$P^{\pi}(x', a'|x, a) \triangleq \pi(a'|x) P(x'|x, a).$$

From this kernel a right-linear operator $P^{\pi} \cdot$ is defined by:

$$(P^{\pi}Q)(x, a) \triangleq \sum_{(x', a') \in \mathcal{Z}} P^{\pi}(x', a'|x, a)Q(x', a'), \quad \forall (x, a) \in \mathcal{Z}. $$

Further, we define two other right-linear operators $\pi \cdot$ and $P \cdot$ by:

$$(\pi Q)(x) \triangleq \sum_{a \in A} \pi(a|x)Q(x, a), \quad \forall x \in \mathcal{X},$$

$$(PV)(x, a) \triangleq \sum_{x' \in \mathcal{X}} P(x'|x, a)V(x'), \quad \forall (x, a) \in \mathcal{Z}. $$

We define the max operator $M$ on the action value functions by $(MQ)(x) \triangleq \max_{a \in A} Q(x, a)$, for all $x \in \mathcal{X}$. Based on the new definitions one can rephrase the Bellman operator and the Bellman optimality operator as:

$$\mathcal{T}^{\pi}Q(x, a) = r(x, a) + \gamma (P^{\pi}Q)(x, a), \quad \mathcal{T}Q(x, a) = r(x, a) + \gamma (PMQ)(x, a). \quad (3) $$

### 3. Dynamic Policy Programming

In this section, we derive the DPP algorithm starting from the Bellman equation. We first show that by adding a relative entropy term to the reward we can control the deviations of the induced policy from a baseline policy. We then derive an iterative double-loop approach which combines value and policy updates. We reduce this double-loop iteration to just a single iteration by introducing DPP algorithm. We emphasize that the purpose of the following derivations is to motivate DPP, rather than to provide a formal characterization. Subsequently, in Subsection 3.2 and Section 4, we theoretically investigate the finite-iteration and the asymptotic behavior of DPP and prove its convergence.
3.1 From Bellman Equation to DPP Recursion

Consider the relative entropy between the policy \( \pi \) and some baseline policy \( \bar{\pi} \):

\[
g^\pi_\pi(x) \triangleq \text{KL}(\pi(\cdot|x) || \bar{\pi}(\cdot|x)) = \sum_{a \in A} \pi(a|x) \log \left( \frac{\pi(a|x)}{\bar{\pi}(a|x)} \right), \quad \forall x \in \mathcal{X}.
\]

We define a new value function \( V^\pi_\pi(x) \), for all \( x \in \mathcal{X} \), which incorporates \( g \) as a penalty term for deviating from the base policy \( \bar{\pi} \) and the reward under the policy \( \pi \):

\[
V^\pi_\pi(x) \triangleq \lim_{n \to \infty} \mathbb{E} \left[ \sum_{k=0}^{n} \gamma^k \left( r_{t+k} - \frac{1}{\eta} g^\pi_\pi(x_{t+k}) \right) \big| x_t = x \right],
\]

where \( \eta \) is a positive constant and \( r_{t+k} \) is the reward at time \( t + k \). Also, the expected value is taken w.r.t. the state transition probability distribution \( P \) and the policy \( \pi \). The optimal value function \( V^*_\pi(x) \) then satisfies the following Bellman equation for all \( x \in \mathcal{X} \):

\[
V^*_\pi(x) = \sup_{\pi} \sum_{a \in A} \pi(a|x) \left[ r(x, a) - \frac{1}{\eta} \log \frac{\pi(a|x)}{\bar{\pi}(a|x)} + \gamma(PV^*_\pi)(x, a) \right]. \tag{4}
\]

Equation (4) is a modified version of (1) where, in addition to maximizing the expected reward, the optimal policy \( \bar{\pi}^* \) also minimizes the distance with the baseline policy \( \bar{\pi} \). The maximization in (4) can be performed in closed form. Following Todorov (2006), we state Proposition [1].

**Proposition 1** Let \( \eta \) be a positive constant, then for all \( x \in \mathcal{X} \) the optimal value function \( V^*_\pi(x) \) and for all \((x, a)\) \( \in \mathcal{Z} \) the optimal policy \( \bar{\pi}^*(a|x) \), respectively, satisfy:

\[
V^*_\pi(x) = \frac{1}{\eta} \log \sum_{a \in A} \bar{\pi}(a|x) \exp \left[ \eta (r(x, a) + \gamma(PV^*_\pi)(x, a)) \right], \tag{5}
\]

\[
\bar{\pi}^*(a|x) = \frac{\bar{\pi}(a|x) \exp \left[ \eta (r(x, a) + \gamma(PV^*_\pi)(x, a)) \right]}{\exp (\eta V^*_\pi(x))}. \tag{6}
\]

**Proof** See Appendix [A].

The optimal policy \( \bar{\pi}^* \) is a function of the base policy, the optimal value function \( V^*_\pi \) and the state transition probability \( P \). One can first obtain the optimal value function \( V^*_\pi \) through the following fixed-point iteration:

\[
V^{k+1}_\pi(x) = \frac{1}{\eta} \log \sum_{a \in A} \bar{\pi}(a|x) \exp \left[ \eta (r(x, a) + \gamma(PV^k_\pi)(x, a)) \right], \tag{7}
\]

and then compute \( \bar{\pi}^* \) using (6). \( \bar{\pi}^* \) maximizes the value function \( V^*_\pi \). However, we are not, in principle, interested in quantifying \( \bar{\pi}^* \), but in solving the original MDP problem and computing \( \pi^* \). The idea to further improve the policy towards \( \pi^* \) is to replace the baseline policy with the just newly computed policy of (6). The new policy can be regarded
as a new base-line policy, and the process can be repeated again. This leads to a double-loop algorithm to find the optimal policy \( \pi^* \), where the outer-loop and the inner-loop would consist of a policy update, Equation (6), and a value function update, Equation (7), respectively.

We then follow the following steps to derive the final DPP algorithm: (i) We introduce some extra smoothness to the policy update rule by replacing the double-loop algorithm by direct optimization of both value function and policy simultaneously using the following fixed point iterations:

\[
V_{k+1}^k(x) = \frac{1}{\eta} \log \sum_{a \in A} \bar{\pi}(a|x) \exp[\eta(r(x,a) + \gamma(PV_{k}^k)(x,a))],
\]

(8)

\[
\bar{\pi}_{k+1}(a|x) = \frac{\bar{\pi}_k(a|x) \exp[\eta(r(x,a) + \gamma(PV_{k}^k)(x,a))]}{\exp(\eta V_{k+1}^k(x))}.
\]

(9)

Further, (ii) we define the action preference function \( \Psi_k \) \cite{SuttonBarto}, for all \((x,a) \in Z\) and \(k \geq 0\), as follows:

\[
\Psi_{k+1}(x,a) = \frac{1}{\eta} \log \frac{\exp(\eta \Psi_k(x,a))}{\sum_{a' \in A} \exp(\eta \Psi_k(x,a'))}.
\]

(10)

By comparing (10) with (9) and (8), we deduce:

\[
\bar{\pi}_k(a|x) = \exp(\eta \Psi_k(x,a)) / \sum_{a' \in A} \exp(\eta \Psi_k(x,a')).
\]

(11)

\[
V_k^k(x) = \frac{1}{\eta} \log \sum_{a \in A} \exp(\eta \Psi_k(x,a)).
\]

(12)

Finally, (iii) by plugging (11) and (12) into (10) we derive:

\[
\Psi_{k+1}(x,a) = \Psi_k(x,a) - \mathcal{L}_\eta \Psi_k(x) + r(x,a) + \gamma(P\mathcal{L}_\eta \Psi_k)(x,a),
\]

(13)

with \( \mathcal{L}_\eta \) operator being defined by \( \mathcal{L}_\eta \Psi(x) \triangleq 1/\eta \log \sum_{a \in A} \exp(\eta \Psi(x,a)) \). (13) is one form of the DPP equations. There is a more efficient and analytically more tractable version of the DPP equation, where we replace \( \mathcal{L}_\eta \) by the Boltzmann soft-max \( \mathcal{M}_\eta \) defined by \( \mathcal{M}_\eta \Psi(x) \triangleq \sum_{a \in A} [\exp(\eta \Psi(x,a))\Psi(x,a)/\sum_{a' \in A} \exp(\eta \Psi(x,a'))] \). In principle, we can provide formal analysis for both versions. However, the proof is somewhat simpler for the

3. Replacing \( \mathcal{L}_\eta \) with \( \mathcal{M}_\eta \) is motivated by the following relation between these two operators:

\[
|\mathcal{L}_\eta \Psi(x) - \mathcal{M}_\eta \Psi(x)| = 1/\eta H_\pi(x) \leq \frac{L}{\eta}, \quad \forall x \in X,
\]

(14)

with \( H_\pi(x) \) is the entropy of the policy distribution \( \pi \) obtained by plugging \( \Psi \) into (16). In words, \( \mathcal{M}_\eta \Psi(x) \) is close to \( \mathcal{L}_\eta \Psi(x) \) up to the constant \( L/\eta \). Also, both \( \mathcal{L}_\eta \Psi(x) \) and \( \mathcal{M}_\eta \Psi(x) \) converge to \( \mathcal{M} \Psi(x) \) when \( \eta \) goes to \( +\infty \). For the proof of (14) and further readings see \cite{MacKay}.
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\[ M_\eta \text{ case, which we will consider in the remainder of this paper. By replacing } L_\eta \text{ with } M_\eta \text{ we deduce the DPP recursion:} \]

\[
\Psi_{k+1}(x,a) = \mathcal{O} \Psi_k(x,a) \triangleq \Psi_k(x,a) + r(x,a) + \gamma PM_\eta \Psi_k(x,a) - M_\eta \Psi_k(x), \quad \forall (x,a) \in \mathcal{Z},
\]

where \( \mathcal{O} \) is an operator defined on the action preferences \( \Psi_k \) and \( \pi_k \) is the soft-max policy associated with \( \psi_k \):

\[
\pi_k(a|x) \triangleq \frac{\exp(\eta \Psi_k(x,a))}{\sum_{a' \in A} \exp(\eta \Psi_k(x,a'))}.
\]

In Subsection 3.2 we show that this iteration gradually moves the policy towards the greedy optimal policy. Algorithm 1 shows the procedure.

Finally, we would like to emphasize on an important difference between DPP and the double-loop algorithm resulted by solving (4). One may notice that DPP algorithm, regardless of the choice of \( \eta \), is always incremental in \( \psi \) and \( \pi \) even when \( \eta \) goes to \(+\infty\), whereas, in the case of double-loop update of the policy and the value function, the algorithm is reduced to standard value iteration for \( \eta = +\infty \) which is apparently not incremental in the policy \( \pi \). The reason for this difference is due to the extra smoothness introduced to DPP update rule by replacing the double-loop update with a single loop in (8) and (9).

**Algorithm 1:** (DPP) Dynamic Policy Programming

1. **Input:** Randomized action preferences \( \Psi_0(\cdot) \), \( \gamma \) and \( \eta \)
2. for \( k = 0, 1, 2, \ldots, K - 1 \) do
   3. for each \((x,a) \in \mathcal{Z}\) do
      4. for each \(a' \in A\) do
         5. \( \pi_k(a'|x) := \frac{\exp(\eta \Psi_k(x,a'))}{\sum_{a'' \in A} \exp(\eta \Psi_k(x,a''))} \);
      6. end
   7. end
   8. \( \Psi_{k+1}(x,a) := \Psi_k(x,a) + \mathcal{T}^{\pi_k} \Psi_k(x,a) - \pi_k \Psi_k(x); \)
9. end
10. for each \((x,a) \in \mathcal{Z}\) do
   11. \( \pi_K(a|x) := \frac{\exp(\eta \Psi_K(x,a))}{\sum_{a' \in A} \exp(\eta \Psi_K(x,a'))} \);
12. end
13. return \( \pi_K; \)

### 3.2 Performance Guarantee

We begin by proving a finite iteration performance guarantee for DPP:
Theorem 1 (The $\ell_\infty$-norm performance loss bound of DPP) Let Assumption 1 hold. Also, assume that $\Psi_0$ is uniformly bounded from above by $V_{\text{max}}$ for all $(x,a) \in Z$, then the following inequality holds for the policy induced by DPP at round $k \geq 0$:

$$\|Q^* - Q^{{\pi_k}}\| \leq 2\gamma \left(4V_{\text{max}} + \frac{\log(L)}{\eta}\right) \frac{(1 - \gamma)^2(k + 1)}{(1 - \gamma)^2(k + 1)}.$$  

Proof See Appendix B.

We can optimize this bound by the choice of $\eta = \infty$, for which the soft-max policy and the soft-max operator $M_\eta$ are replaced with the greedy policy and the max-operator $M$. As an immediate consequence of Theorem 1, we obtain the following result:

Corollary 2 The following relation holds in limit:

$$\lim_{k \to +\infty} Q^{\pi_k}(x,a) = Q^*(x,a), \forall (x,a) \in Z.$$  

In words, the policy induced by DPP asymptotically converges to the optimal policy $\pi^*$.

One can also show that, under some mild assumption, there exists a unique limit for the action preferences in infinity.

Assumption 2 We assume that MDP has a unique deterministic optimal policy $\pi^*$ given by:

$$\pi^*(a|x) = \begin{cases} 1 & a = a^*(x) \\ 0 & \text{otherwise} \end{cases}, \forall x \in X,$$

where $a^*(x) = \arg\max_{a \in A} Q^*(x,a)$.

Theorem 3 Let Assumption 1 and 2 hold and $k$ be a non-negative integer and let $\Psi_k(x,a)$, for all $(x,a) \in Z$, be the action preference after $k$ iteration of DPP. Then, we have:

$$\lim_{k \to +\infty} \Psi_k(x,a) = \begin{cases} V^*(x) & a = a^*(x) \\ -\infty & \text{otherwise} \end{cases}, \forall x \in X.$$  

Proof See Appendix C.

4. Dynamic Policy Programming with Approximation

Algorithm 1 (DPP) only applies to small problems with a few states and actions. One can generalize the DPP algorithm for the problems of practical scale by using function approximation techniques. Also, to compute the optimal policy by DPP an explicit knowledge of model is required. In many real world problems, this information is not available instead it may be possible to simulate the state transition by Monte-Carlo sampling and then approximately estimate the optimal policy using these samples. In this section, we provide results on the performance-loss of DPP in the presence of approximation/estimation error. We then compare $\ell_\infty$-norm performance-loss bounds of DPP with the standard results of AVI and API. Finally, We introduce new approximate algorithms for implementing DPP with Monte-Carlo sampling (DPP-RL) and linear function approximation (SADPP).
4.1 The $\ell_\infty$-norm performance-loss bounds for approximate DPP

Let us consider a sequence of action preferences $\{\Psi_0, \Psi_1, \Psi_2, \ldots\}$ such that, at round $k$, the action preferences $\Psi_{k+1}$ is the result of approximately applying the DPP operator by the means of function approximation or Monte-Carlo simulation, i.e., for all $(x,a) \in Z$: $\Psi_{k+1}(x,a) \approx O\Psi_k(x,a)$. The error term $\epsilon_k$ is defined as the difference of $O\Psi_k$ and its approximation:

$$\epsilon_k(x,a) \triangleq \Psi_{k+1}(x,a) - O\Psi_k(x,a), \quad \forall (x,a) \in Z. \quad (17)$$

The approximate DPP update rule is then given by:

$$\Psi_{k+1}(x,a) = \Psi_k(x,a) + r(x,a) + \gamma PM\eta \Psi_k(x,a) - M\eta\Psi_k(x,a) + \epsilon_k(x,a) \quad (18)$$

where $\pi_k$ is given by (16).

We begin by finite iteration analysis of the approximate DPP. The following theorem establishes an upper-bound on the performance loss of DPP in the presence of approximation error. The proof is based on generalization of the bound that we established for DPP by taking into account the error $\epsilon_k$:

**Theorem 4 ($\ell_\infty$-norm performance loss bound of approximate DPP)** Let Assumption 1 hold. Assume that $k$ is a non-negative integer and $\Psi_0$ is bounded by $V_{\text{max}}$. Further, define $\epsilon_k$ for all $k$ by (17) and the accumulated error $E_k$ as:

$$E_k(x,a) \triangleq \sum_{j=0}^{k} \epsilon_j(x,a), \quad \forall (x,a) \in Z. \quad (19)$$

Then the following inequality holds for the policy induced by approximate DPP at round $k$:

$$\|Q^* - Q^{\pi_k}\| \leq \frac{1}{(1-\gamma)(k+1)} \left[ 2\gamma \left( 4V_{\text{max}} + \frac{\log(L)}{\eta} \right) + \sum_{j=0}^{k} \gamma^{k-j} \|E_j\| \right].$$

**Proof** See Appendix D.

Taking the upper-limit yields in the following corollary of Theorem 4.

**Corollary 5 (Asymptotic $\ell_\infty$-norm performance-loss bound of approximate DPP)** Define $\bar{\epsilon} \triangleq \limsup_{k \to \infty} \|E_k\|/(k+1)$. Then, the following inequality holds:

$$\limsup_{k \to \infty} \|Q^* - Q^{\pi_k}\| \leq \frac{2\gamma}{(1-\gamma)^2} \bar{\epsilon}. \quad (20)$$

The asymptotic bound is similar to the existing results of AVI and API (Thiery and Scherrer, 2010; Bertsekas and Tsitsiklis, 1996, chap. 6):
where $\varepsilon_{\text{max}} = \limsup_{k \to \infty} \|\epsilon_k\|$. The difference is that in (20) the supremum norm of error $\varepsilon_{\text{max}}$ is replaced by the supremum norm of the average error $\bar{\varepsilon}$. In other words, unlike AVI and API, the size of error at each iteration is not a critical factor for the performance of DPP and as long as the size of average error remains close to 0, DPP can achieve a near-optimal performance even when the error itself is arbitrary large. To gain a better understanding of this result consider a case in which, for any algorithm, the sequence of errors $\{\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots\}$ are some i.i.d. zero-mean random variables bounded by $0 < U < \infty$. We then obtain the following asymptotic bound for the approximate DPP by applying the law of large numbers to Corollary 5:

$$\limsup_{k \to \infty} \|Q^* - Q^{\pi_k}\| \leq \frac{2\gamma}{(1 - \gamma)^2} \bar{\varepsilon} = 0,$$

whilst for API and AVI we have:

$$\limsup_{k \to \infty} \|Q^* - Q^{\pi_k}\| \leq \frac{2\gamma}{(1 - \gamma)^2} U.$$

In words, approximate DPP manages to cancel the i.i.d. noise and asymptotically converges to the optimal policy whereas there is no guarantee, in this case, for the convergence of API and AVI to the optimal solution. This result suggests that DPP can average out the simulation noise caused by Monte-Carlo sampling and eventually achieve a significantly better performance than AVI and API in the presence of large variance of estimation. We will show, in the the next subsection that a sampling-based variant of DPP (DPP-RL) manages to cancel the simulation noise and asymptotically converges, almost surely, to the optimal policy (see Theorem 6).

### 4.2 Reinforcement Learning with Dynamic Policy Programming

To compute the optimal policy by DPP one needs an explicit knowledge of model. In many problems we do not have access to this information but instead we can generate samples by simulating the model. The optimal policy can then be learned using these samples. In this section, we introduce a new RL algorithm, called DPP-RL, which relies on a sampling-based variant of DPP to update the policy. The update rule of DPP-RL is very similar to (15). The only difference is that, in DPP-RL, we replace the Bellman operator $T^\pi \Psi(x, a)$ with its sample estimate $T^\pi_k \Psi(x, a) \triangleq r(x, a) + \pi \Psi(y_k)$, where the next sample $y_k$ is drawn from $P(\cdot| x, a)$.

$$\Psi_{k+1}(x, a) \triangleq \Psi_k(x, a) + T^\pi_k \Psi_k(x, a) - \pi_k \Psi_k(x).$$

4. We assume, hereafter, that we have access to the generative model of MDP, i.e., given the state-action pair $(x, a)$ we can generate the next sample $y$ from $P(\cdot| x, a)$ for all $(x, a) \in \mathcal{Z}$. 

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The pseudo-code of DPP-RL algorithm is shown in Algorithm 2.

Algorithm 2: (DPP-RL) Reinforcement learning with DPP

Input: Initial the action preferences $\Psi_0(\cdot)$, discount factor $\gamma$, $\eta$ and number of iterations $K$

1. Generate a set of i.i.d. samples $\{y_1, y_2, y_3, \ldots, y_K\}$, for every $(x, a) \in \mathcal{Z}$, from $P(\cdot|x, a)$;

2. for $k := 0, 1, 2, 3, \ldots, K-1$ do
   3. for each $(x, a) \in \mathcal{Z}$ do
      4. for each $a' \in A$ do
         5. $\pi_k(a'|y_k) = \frac{\exp(\eta \Psi_k(y_k, a'))}{\sum_{a'' \in A} \exp(\eta \Psi_k(y_k, a''))}$;
      6. end
      7. $T^\pi_k \Psi_k(x, a) := r(x, a) + \gamma \pi_k \Psi_k(y_k);$;
      8. $\Psi_{k+1}(x, a) := \Psi_k(x, a) + T^\pi_k \Psi_k(x, a) - \pi_k \Psi_k(x);$;
   9. end
10. end
11. $\pi_K(a|x) = \frac{\exp(\eta \Psi_K(x, a))}{\sum_{a' \in A} \exp(\eta \Psi_K(x, a'))}$;
12. return $\pi_K$

Equation (22) is just an approximation of DPP update rule (15). Therefore, the convergence result of Corollary 2 does not hold for DPP-RL. However, the new algorithm still converges to the optimal policy since one can show that the errors associated with approximating (15) are asymptotically averaged out by DPP-RL, as postulated by Corollary 5. The following theorem establishes the asymptotic convergence of the policy induced by DPP-RL to the optimal policy.

Theorem 6 (Asymptotic convergence of DPP-RL) Let Assumption 1 hold. Assume that the initial action-value function $\Psi_0$ is uniformly bounded by $V_{\text{max}}$ and $\pi_k$ is the policy induced by Algorithm 2 at round $k$. Then, w.p. 1, the following holds:

$$\lim_{k \to \infty} Q^\pi_k(x, a) = Q^*(x, a), \quad \forall (x, a) \in \mathcal{Z}.$$ 

Proof See Appendix E.

One may notice that the update rule of DPP-RL, unlike other incremental RL methods such as Q-learning (Watkins and Dayan, 1992) and SARSA (Singh et al., 2000), does not involve any decaying learning step. This is an important difference since it is known that the convergence rate of incremental RL methods like Q-learning is very sensitive to the choice of learning step (Even-Dar and Mansour, 2003; Szepesvári, 1997) and a bad choice of the learning step may lead to significantly slow rate of convergence. DPP-RL seems to not suffer from this problem (see Figure 3) since the DPP-RL update rule is just an empirical estimate of the update rule of DPP. Therefore, one may expect that the rate of convergence of DPP-RL remains close to the fast rate of convergence of DPP established in Theorem 1.
4.3 Approximate Dynamic Policy Programming with Linear Function Approximation

In this subsection, we consider DPP with linear function approximation (LFA) and least-squares regression. Given a set of basis functions \( \mathcal{F}_\phi = \{ \phi_1, \ldots, \phi_k \} \), where each \( \phi_i : \mathbb{Z} \rightarrow \mathbb{R} \) is a bounded real-valued function, the sequence of action preferences \( \{ \Psi_0, \Psi_1, \Psi_2, \ldots \} \) are defined as a linear combination of these basis functions: 

\[
\Psi_k = \theta_k^T \Phi,
\]

where \( \Phi \) is a \( m \times 1 \) column vector with the entries \( \{ \phi_i \}_{i=1:m} \) and \( \theta_k \in \mathbb{R}^m \) is a \( m \times 1 \) vector of parameters.

The action preference function \( \Psi_{k+1} \) is an approximation of the DPP operator \( O \Psi_k \). In case of LFA the common approach to approximate DPP operator is to find a vector \( \theta_{k+1} \) that projects \( O \Psi_k \) on the column space spanned by \( \Phi \) by minimizing the loss function:

\[
J_k(\theta; \Psi) \triangleq \| \theta^T \Phi - O \Psi_k \|_{2, \mu}^2,
\]

where \( \mu \) is a probability measure on \( \mathbb{Z} \). The best solution, that minimize \( J \), is called the least-squares solution:

\[
\theta_{k+1} = \arg \min_{\theta \in \mathbb{R}^m} J_k(\theta; \Psi) = \left[ \mathbb{E}(\Phi \Phi^T) \right]^{-1} \mathbb{E}(\Phi O \Psi_k),
\]

where the expectation is taken w.r.t. \( (x, a) \sim \mu \). In principle, to compute the least squares solution equation requires to compute \( O \Psi_k \) for all states and actions. For large scale problems this becomes infeasible. Instead, we can make a sample estimate of the least-squares solution by minimizing the empirical loss \( \tilde{J}_k(\theta; \Psi) \):

\[
\tilde{J}_k(\theta; \Psi) \triangleq \frac{1}{N} \sum_{n=1}^N \left( \theta^T \Phi(X_n, A_n) - O_n \Psi_k \right)^2 + \alpha \theta^T \theta,
\]

where \( \{(X_n, A_n)\}_{n=1:N} \) is a set of \( N \) i.i.d. samples drawn from the distribution \( \mu \). Also, \( O_n \Psi_k \) denotes a single sample estimate of \( O \Psi_k(X_n, A_n) \) defined by \( O_n \Psi_k \triangleq \Psi_k(X_n, A_n) + r(X_n, A_n) + \gamma \mathcal{M}_\eta \Psi_k(X'_n) - \mathcal{M}_\eta \Psi_k(X_n) \), where \( X'_n \sim P(\cdot|X_n, A_n) \). Further, to avoid over-fitting due to the small size of data set, we add a quadratic regularization term to the loss function. The empirical least-squares solution which minimizes \( \tilde{J}_k(\theta; \Psi) \) is given by:

\[
\tilde{\theta}_{k+1} = \left[ \sum_{n=1}^N \Phi(X_n, A_n) \Phi(X_n, A_n)^T + \alpha N \mathbf{I} \right]^{-1} \sum_{n=1}^N O_n \Psi_k \Phi(X_n, A_n).
\]
Algorithm 3 presents the sampling-based approximate dynamic policy programming (SADPP) in which we rely on (25) to approximate DPP operator at each iteration.

**Algorithm 3:** (SADPP) Sampling-based approximate dynamic policy programming

Input: \( \tilde{\theta}_0, \eta, \gamma, \alpha, K \) and \( N \)

1. for \( k = 0, 1, 2, \ldots, K - 1 \) do
2. Generate a set of i.i.d. samples \( \{(X_n, A_n, X'_n)\}_{n=1:N} \) by drawing \( N \) sample from \( \mu \) and \( P(\cdot|X_n, A_n) \);
3. for \( n = 1, 2, 3, \ldots, N \) do
4. \( \Psi_k(X_n, A_n) = \tilde{\theta}_k^T \Phi(X_n, A_n) \);
5. for each \( A' \in \mathcal{A} \) do
6. \( \Psi_k(X_n, A') = \tilde{\theta}_k^T \Phi(X_n, A') \);
7. \( \Psi_k(X'_n, A') = \tilde{\theta}_k^T \Phi(X'_n, A') \);
8. end
9. \( M_\eta \Psi_k(X_n) = \sum_{A' \in \mathcal{A}} \frac{\exp[\eta \Psi_k(X_n, A')] \Psi_k(X_n, A')}{\sum_{A'' \in \mathcal{A}} \exp[\eta \Psi_k(X_n, A'')]} \); 
10. \( M_\eta \Psi_k(X'_n) = \sum_{A' \in \mathcal{A}} \frac{\exp[\eta \Psi_k(X'_n, A')] \Psi_k(X'_n, A')}{\sum_{A'' \in \mathcal{A}} \exp[\eta \Psi_k(X'_n, A'')]} \);
11. \( O_n \Psi_k = \Psi_k(X_n, A_n) - r(X_n, A_n) - \gamma (M_\eta \Psi_k)(X'_n) + (M_\eta \Psi_k)(X_n) \);
12. end
13. \( \tilde{\theta}_{k+1} = \left[ \sum_{n=1}^{N} \Phi(X_n, A_n) \Phi(X_n, A_n)^T + \alpha N I \right]^{-1} \sum_{n=1}^{N} O_n \Psi_k \Phi(X_n, A_n) \);
14. end
15. return \( \tilde{\theta}_K \)

5. Numerical Results

In this section, we analyze empirically the effectiveness of the proposed algorithms on different problem domains. We first examine the convergence properties of DPP-RL (Algorithm 2) on several discrete state-action problems and compare it with two standard algorithms: a synchronous variant of Q-learning (Even-Dar and Mansour, 2003) (QL) and the model-based Q-value iteration (VI) of Kearns and Singh (1999). Next, we investigate the finite-time performance of SADPP (Algorithm 3) in the presence of function approximation and a limited sampling budget per iteration. In this case, we consider a variant of the optimal replacement problem described in Munos and Szepesvári (2008) and compare our method with regularized least-squares fitted Q-iteration (RFQI) (Farahmand et al., 2008). The source code of all tested algorithms are freely available in [http://www.mbfys.ru.nl/~mazar/Research_Top](http://www.mbfys.ru.nl/~mazar/Research_Top).

5.1 DPP-RL

We consider the following large-scale MDPs as benchmark problems:

**Linear MDP:** this problem consists of states \( x_k \in \mathcal{X}, k = \{1, 2, \ldots, 2500\} \) arranged in a one-dimensional chain (see Figure 1). There are two possible actions \( \mathcal{A} = \{-1, +1\} \) (left/right) and every state is accessible from any other state except for the two ends of the chain, which are absorbing states. A state \( x_k \in \mathcal{X} \) is called absorbing
Figure 1: Linear MDP: Illustration of the linear MDP problem. Nodes indicate states. States $x_1$ and $x_{2500}$ are the two absorbing states and state $x_k$ is an example of interior state. Arrows indicate possible transitions of these three nodes only. From $x_k$ any other node is reachable with transition probability (arrow thickness) proportional to the inverse of the distance to $x_k$ (see the text for details).

If $P(x_k|x_k,a) = 1$ for all $a \in A$ and $P(x_l|x_k,a) = 0, \forall l \neq k$. Any transition to one of these two states has associated reward 1.

The transition probability for an interior state $x_k$ to any other state $x_l$ is inversely proportional to their distance in the direction of the selected action, and zero for all states corresponding to the opposite direction. Formally, consider the following quantity $n(x_l,a,x_k)$ assigned to all non-absorbing states $x_k$ and to every $(x_l,a) \in \mathbb{Z}$:

$$n(x_l,a,x_k) = \begin{cases} 
\frac{1}{|l-k|} & \text{for } (l-k)a > 0 \\
0 & \text{otherwise}
\end{cases}.$$

We can write the transition probabilities as:

$$P(x_l|x_k,a) = \frac{n(x_l,a,x_k)}{\sum_{x_m \in \mathcal{X}} n(x_m,a,x_k)}.$$

Any transition that ends up in one of the interior states has associated reward $-1$.

The optimal policy corresponding to this problem is to reach the closest absorbing state as soon as possible.

**Combination lock:** the combination lock problem considered here is a stochastic variant of the reset state space models introduced in [Koenig and Simmons (1993)](1993), where more than one reset state is possible (see Figure 2).

In our case we consider, as before, a set of states $x_k \in \mathcal{X}, k \in \{1, 2, \ldots, 2500\}$ arranged in a one-dimensional chain and two possible actions $A = \{-1, +1\}$. In this problem, however, there is only one absorbing state (corresponding to the state lock-opened) with associated reward of 1. This state is reached if the all-ones sequence $\{+1, +1, \ldots, +1\}$ is entered correctly. Otherwise, if at some state $x_k, k < 2500$, action $-1$ is taken, the lock automatically resets to some previous state $x_l, l < k$ randomly (in the original combination lock problem, the reset state is always the initial state $x_1$).
For every intermediate state, the rewards of actions $-1$ and $+1$ are set to $0$ and $-0.01$, respectively. The transition probability upon taking the wrong action $-1$ is, as before, inversely proportional to the distance of the states. That is

$$n(x_k, x_l) = \begin{cases} 1/k-l & \text{for } l < k \\ 0 & \text{otherwise} \end{cases}, \quad P(x_l|x_k, 0) = \frac{n(k,l)}{\sum_{x_m \in X} n(k,m)}.$$

Note that this problem is more difficult than the linear MDP since the goal state is only reachable from one state, $x_{2499}$.

**Grid world:** this MDP consists of a grid of $50 \times 50$ states. A set of four actions \{RIGHT, UP, DOWN, LEFT\} is assigned to every state $x \in X$. The location of each state $x$ of the grid is determined by the coordinates $c_x = (h_x, v_x)$, where $h_x$ and $v_x$ are some integers between 1 and 50. There are 196 absorbing firewall states surrounding the grid and another one at the center of grid, for which a reward $-1$ is assigned. The reward for the firewalls is

$$r(x,a) = -\frac{1}{\|c_x\|_2^2}, \quad \forall a \in A.$$  

Also, we assign reward $0$ to all of the remaining (non-absorbing) states.

This means that both the top-left absorbing state and the central state have the least possible reward ($-1$), and that the remaining absorbing states have reward which increases proportionally to the distance to the state in the bottom-right corner (but are always negative).

The transition probabilities are defined in the following way: taking action $a$ from any non-absorbing state $x$ results in a one-step transition in the direction of action $a$ with probability $0.6$, and a random move to a state $y \neq x$ with probability inversely proportional to their Euclidean distance $1/\|c_x - c_y\|_2$. The optimal policy then is to survive in the grid as long as possible by avoiding both the absorbing firewalls and the center of the grid. Note that because of the
difference between the cost of firewalls, the optimal control prefers the states near the bottom-right corner of the grid, thus avoiding absorbing states with higher cost.

5.1.1 Experimental Setup and Results

We describe now our experimental setting. The convergence properties of DPP-RL are compared with two other algorithms: a synchronous variant of Q-learning (Even-Dar and Mansour, 2003) (QL), which, like DPP-RL, updates the action-value function of all state-action pairs at each iteration, and the model-based Q-value iteration (VI) of Kearns and Singh (1999). VI is a batch reinforcement learning algorithm that first estimates the model using the whole data set and then performs value iteration on the learned model.

All algorithms are evaluated in terms of \( \ell_\infty \)-norm performance loss of the action-value function \( \| Q^* - Q^{\pi_k} \| \) obtained by policy \( \pi_k \) induced at iteration \( k \). We choose this performance measure in order to be consistent with the performance measure used in Section 4. The discount factor \( \gamma \) is fixed to 0.995 and the optimal action-value function \( Q^* \) is computed with high accuracy through value iteration.

We consider QL with polynomial learning step \( \alpha_k = 1/(k + 1)^\omega \) where \( \omega \in \{0.51, 0.75\} \) and the linear learning step \( \alpha_k = 1/(k + 1) \). Note that \( \omega \) needs to be larger than 0.5, otherwise QL can asymptotically diverge (see Even-Dar and Mansour, 2003, for the proof).

To achieve the best rate of convergence for DPP-RL, we fix \( \eta \) to +\( \infty \) (see Section 3.2). This replaces the soft-max operator \( M_\eta \) in the DPP-RL update rule with the max operator \( M \), resulting in a greedy policy \( \pi_k \).

To have a fair comparison of the three algorithms, since each algorithm requires different number of computations per iteration, we fix the total computational budget of the algorithms to the same value for each benchmark. The computation time is constrained to 30 seconds in the case of linear MDP and the combination lock problems. For the grid world, which has twice as many actions as the other benchmarks, the maximum run time is fixed to 60 seconds. We also fix the total number of samples, per state-action, to \( 1 \times 10^5 \) samples for all problems and algorithms. Significantly less number of samples leads to a dramatic decrease of the quality of the obtained solutions using all the approaches.

Algorithms were implemented as MEX files (in C++) and ran on a Intel core i5 processor with 8 GB of memory. cpu time was acquired using the system function \texttt{times()} which provides process-specific cpu time. Randomization was implemented using \texttt{gsl\_rng\_uniform()} function of the GSL library, which is superior to the standard \texttt{rand()} function. Sampling time, which is the same for all algorithms, were not included in cpu time.

Figure 3 shows the performance-loss in terms of elapsed cpu time for the three problems and algorithms. The results are averages over 50 runs, where at the beginning of each run (i) the action-value function and the action preferences are randomly initialized in the interval \([-V_{\text{max}}, V_{\text{max}}]\), and (ii) a new set of samples is generated from \( P(\cdot|x,a) \) for all \((x,a) \in \mathbb{Z}\). Results correspond to the average error computed after a small fixed amount of iterations.

First, we see that DPP-RL converges very fast achieving near optimal performance after a few seconds. DPP-RL outperforms both QL and VI in all the three benchmarks. The minimum and maximum errors are attained for the linear MDP problem and the Grid

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Figure 3: A comparison between DPP-RL, Q-Learning and model-based VI. Each plot compares the performance loss of the policy induced by the algorithms for a different MDP averaged over 50 different runs (see the text for details).

world, respectively. We also observe that the difference between DPP-RL and QL is very significant, about two orders of magnitude, in both the linear MDP and the Combination lock problems. In the grid world DPP-RL’s performance is more than 4 times better than that of QL.

QL shows the best performance for \( \omega = 0.51 \). The quality of the QL solution degrades as a function of \( \omega \). Concerning VI, its error shows a sudden decrease on the first error caused by the model estimation.

The standard deviations of the performance-loss give an indication of how robust are the solutions obtained by the algorithms. Table 1 shows the final numerical outcomes of DPP-RL, QL and VI (standard deviations between parenthesis). We can see that the variance of estimation of DPP-RL is substantially smaller than those of QL and VI.

Table 1: A Comparison between DPP-RL, Q-learning (QL) and the model-based value iteration (VI) given a fixed computational and sampling budget. Table 1 shows error means and standard deviations (between parenthesis) at the end of the simulations for three different algorithms (columns) and three different benchmarks (rows)
These results show that, as suggested in Theorem 6 and 4, DPP-RL manages to average out the simulation noise caused by sampling and converges, rapidly, to a near optimal solution, which is very robust. In addition, we can conclude that DPP-RL performs significantly better than QL and VI in the three presented benchmarks for our choice of experimental setup.

5.2 SADPP

In this subsection, we illustrate the performance of the SADPP algorithm in the presence of function approximation and limited sampling budget per iteration. We compare SADPP with a modification of regularized fitted Q-iteration (RFQI) (Farahmand et al., 2008) which make use of a fixed number of basis functions. RFQI can be regarded as a Monte-Carlo sampling implementation of approximate value iteration with action-state representation. We compare SADPP with RFQI since both methods make use of \( \ell_2 \)-regularization. The purpose of this subsection is to analyze numerically the sample complexity, i.e., the number of samples required to achieve a near optimal performance with a low variance, of SADPP. The benchmark we consider is a variant of the optimal replacement problem presented in Munos and Szepesvári (2008). In the following subsection we describe the problem and subsequently we present the results.

5.2.1 Optimal replacement problem

This problem is an infinite-horizon, discounted MDP. The state measures the accumulated use of a certain product and is represented as a continuous, one-dimensional variable. At each time-step \( t \), either the product is kept \( a(t) = 0 \) or replaced \( a(t) = 1 \). Whenever the product is replaced by a new one, the state variable is reset to zero \( x(t) = 0 \), at an additional cost \( C \). The new state is chosen according to an exponential distribution, with possible values starting from zero or from the current state value, depending on the latest action:

\[
p(y|x, a = 0) = \begin{cases} \beta e^{\beta(y-x)} & \text{if } y \geq x \\ 0 & \text{if } y < 0 \end{cases} \quad \quad p(y|x, a = 1) = \begin{cases} \beta e^{\beta y} & \text{if } y \geq 0 \\ 0 & \text{if } y < 0 \end{cases}.
\]

The reward function is a monotonically increasing function of the state \( x \) if the product is kept \( r(x, 0) = -c(x) \) and constant if the product is replaced \( r(x, 1) = -C - c(0) \).

The optimal action is to keep as long as the accumulated use is below a threshold or to replace otherwise:

\[
a^*(x) = \begin{cases} 0 & \text{if } x \in [0, \bar{x}] \\ 1 & \text{if } x > \bar{x} \end{cases}.
\]

Following Munos and Szepesvári (2008), \( \bar{x} \) can be obtained exactly via the Bellman equation and is the unique solution to

\[
C = \int_0^{\bar{x}} \frac{c'(y)}{1 - \gamma} \left( 1 - \gamma e^{-\beta(1-\gamma)y} \right) dy.
\]
5.2.2 Experimental setup and results

For both SADPP and RFQI we map the state-action space using 20 radial basis functions (10 for the continuous one-dimensional state variable $x$, spanning the state space $\mathcal{X}$, and 2 for the two possible actions). Other parameter values where chosen to be the same as in [Munos and Szepesvári (2008)], that is, $\gamma = 0.6$, $\beta = 0.5$, $C = 30$ and $c(x) = 4x$, which results in $\bar{x} \simeq 4.8665$. We also fix an upper bound for the states, $x_{\text{max}} = 10$ and modify the problem definition such that if the next state $y$ happens to be outside of the domain $[0, x_{\text{max}}]$ then the product is replaced immediately, and a new state is drawn as if action $a = 1$ were chosen in the previous time step.

To compare both Algorithms we discretize the state space in $K = 100$ bins and use the following error measure:

$$\text{error} = \frac{\sum_{k=1}^{K} |a^*(x_k) - \hat{a}(x_k)|}{K}, \quad (26)$$

where $\hat{a}$ is the action selected by the Algorithm. Note that, unlike RFQI which selects the action by choosing the action with the highest action-value function, SADPP induces a stochastic policy, that is, a distribution over actions. We select $\hat{a}$ for SADPP by choosing the most probable action from the induced soft-max policy, and then use this to compute (26). Both algorithms were implemented in MatLab and executed under the same hardware specifications of the previous section.

We analyze the effect of using different number of samples $N$ per iteration, $N \in \{50, 150, 500\}$. The results are averages over 200 runs, where at the beginning of each run the vector $\theta$ is initialized in the interval $[-1, 1]$ for both algorithms. The rest of the parameters, including the regularization factor $\alpha$ and $\eta$, were optimized for the best asymptotic performance for each $N$ independently.

Figure 4 shows averages and standard deviations of the errors. First, we observe that for large $N$, after an initial transient, both SADPP and RFQI reach a near optimal solution. We observe that SADPP asymptotically outperforms RFQI on average in all cases. The average error and the variance of estimation of the resulting solutions decreases with $N$ in both approaches. A comparison of the variances after the transient suggests that the sample complexity of SADPP is significantly smaller than RFQI. Remarkably, the variance of SADPP using $N = 50$ samples is comparable to the one provided by RFQI using $N = 500$ samples. Further, the variance of SADPP is reduced faster with increasing $N$. These results allow to conclude that SADPP can have positive effects in reducing the effect of simulation noise, as postulated in Section 4.

6. Related Work

There are other methods which rely on a incremental update of the policy. One well-known algorithm of this kind is the actor-critic method (AC), in which the actor uses the value function computed by the critic to guide the policy search (Sutton and Barto, 1998, chap. 6.6). An important extension of AC, the policy-gradient actor critic (PGAC), extends the idea of AC to problems of practical scale (Sutton et al., 1999; Peters and Schaal, 1999).
Figure 4: Numerical results for the optimal replacement problem. Each plot shows a comparison of the error between SDAPP and RFQI Algorithms using certain number of samples $N$. Error is defined as in equation (26). Areas indicate averages plus/minus standard deviations of the error starting from 200 uniformly distributed initial conditions (see the text for details).
SAEM (Vlassis and Toussaint, 2009), which rely on EM algorithm to maximize a lower bound for the expected return in an iterative fashion. This lower-bound also can be written as a KL-divergence between two distributions. Another relevant study is relative entropy policy search (REPS) (Peters et al., 2010) which relies on the idea of minimizing the relative entropy to control the size of policy update. The main differences are: (i) the REPS algorithm is an actor-critic type of algorithm, while DPP is more a policy iteration type of method. (ii) In REPS the inverse temperature $\eta$ needs to be optimized while DPP converges to the optimal solution for any inverse temperature $\eta$, and (iii) here we provide a convergence analysis of DPP, while there is no convergence analysis in REPS.

7. Discussion and Future Works

We have presented a new approach, dynamic policy programming (DPP), to compute the optimal policy in infinite-horizon discounted-reward MDPs. We have theoretically proven the convergence of DPP to the optimal policy for the tabular case. We have also provided performance-loss bounds for DPP in the presence of approximation. The bounds have been expressed in terms of supremum norm of average accumulated error as opposed to standard results for AVI and API which expressed in terms of supremum norm of the errors. We have then introduced a new incremental model-free RL algorithm, called DPP-RL, which relies on a sample estimate instance of DPP update rule to estimate the optimal policy. We have proven the asymptotic convergence of DPP-RL to the optimal policy and then have compared its, numerically, with the standard RL methods. Experimental results on various MDPs have been provided showing that, in all cases, DPP-RL is superior to other RL methods in terms of convergence rate. This may be due to the fact that DPP-RL, unlike other incremental RL methods, does not rely on stochastic approximation for estimating the optimal policy and therefore it does not suffer from the slow convergence caused by the presence of the decaying learning step in stochastic approximation.

In this work, we are only interested in the estimation of the optimal policy and not the problem of exploration. Therefore, we have not compared our algorithms to the PAC-MDP methods (Strehl et al., 2009), in which the choice of the exploration policy impacts the behavior of the learning algorithm. Also, in this paper, we have not compared our results with those of (PG)AC since they rely on a different kind of sampling strategy: Both DPP-RL and SADPP rely on a generative model for sampling, whereas AC makes use of some trajectories of the state-action pairs, generated by Monte-Carlo simulation, to estimate the optimal policy.

In this study, we provide $\ell_\infty$-norm performance-loss bounds for approximate DPP. However, most supervised learning and regression algorithms rely on minimizing some form of $\ell_p$-norm error. Therefore, it is natural to search for a kind of performance bound that relies on the $\ell_p$-norm of approximation error. Following Munos (2005), $\ell_p$-norm bounds for approximate DPP can be established by providing a bound on the performance loss of each component of value function under the policy induced by DPP. This would be a topic for future research.

Another direction for future work is to provide finite-sample probably approximately correct (PAC) bounds for SADPP and DPP-RL in the spirit of previous theoretical results available for fitted value iteration and fitted $Q$-iteration (Munos and Szepesvári, 2008).
In the case of SADPP, this would require extending the error propagation result of Theorem 4 to an $\ell_2$-norm analysis and combining it with the standard regression bounds.

Finally, an important extension of our results would be to apply DPP for large-scale action problems. In that case, we need an efficient way to approximate $M_\eta \Psi_k(x)$ in update rule (15) since computing the exact summations become expensive. One idea is to sample estimate $M_\eta \Psi_k(x)$ using Monte-Carlo simulation [38], since $M_\eta \Psi_k(x)$ is the expected value of $\Psi_k(x,a)$ under the soft-max policy $\pi_k$.

Appendix A. Proof of Proposition 1

We first introduce the Lagrangian function $L(x; \lambda_x) : X \rightarrow \mathbb{R}$:

\[
L(x; \lambda_x) = \sum_{a \in A} \pi(a|x) [r(x,a) + \gamma (PV^*_\eta)(x,a)] - \frac{1}{\eta} KL (\pi(.|x)||\bar{\pi}(.|x)) - \lambda_x \left[ \sum_{a \in A} \pi(a|x) - 1 \right].
\]

The maximization in (4) can be expressed as maximizing the Lagrangian function $L(x, \lambda_x)$. The necessary condition for the extremum with respect to $\pi(a|x)$ is:

\[
0 = \frac{\partial L(x, \lambda_x)}{\partial \pi(a|x)} = r(x,a) + \gamma (PV^*_\eta)(x,a) - \frac{1}{\eta} - \frac{1}{\eta} \log \left( \frac{\pi(a|x)}{\bar{\pi}(a|x)} \right) - \lambda_x,
\]

which leads to:

\[
\bar{\pi}^*(a|x) = \bar{\pi}(a|x) \exp (\eta \lambda_x - 1) \exp [\eta (r(x,a) + \gamma (PV^*_\eta)(x,a))], \quad \forall x \in X. \tag{27}
\]

By plugging (28) into (27) we deduce:

\[
\bar{\pi}^*(a|x) = \frac{\bar{\pi}(a|x) \exp (\eta (r(x,a) + \gamma (PV^*_\eta)(x,a)))}{\sum_{a \in A} \bar{\pi}(a|x) \exp (\eta (r(x,a) + \gamma (PV^*_\eta)(x,a)))}, \quad \forall (x, a) \in Z. \tag{29}
\]

The Lagrange multipliers can then be solved from the constraints:

\[
1 = \sum_{a \in A} \bar{\pi}^*(a|x) = \exp (\eta \lambda_x - 1) \sum_{a \in A} \bar{\pi}(a|x) \exp [\eta (r(x,a) + \gamma (PV^*_\eta)(x,a))],
\]

\[
\lambda_x = \frac{1}{\eta} \log \sum_{a \in A} \bar{\pi}(a|x) \exp [\eta (r(x,a) + \gamma (PV^*_\eta)(x,a))] - \frac{1}{\eta}. \tag{28}
\]

The results then follows by substituting (29) in (4).

Appendix B. Proof of Theorem 1

In this section, we provide a formal analysis of the convergence behavior of DPP. Our objective is to establish a rate of convergence for the value function of the policy induced by DPP.
Our main result is in the form of following finite-iteration performance-loss bound, for all \( k \geq 0 \):

\[
\| Q^* - Q_{\pi_k} \| \leq \frac{2\gamma \left( 4V_{\text{max}} + \frac{\log(L)}{\eta} \right)}{(1 - \gamma)^2(k + 1)}.
\]

(30)

Here, \( Q_{\pi_k} \) is the action-values under the policy \( \pi_k \) and \( \pi_k \) is the policy induced by DPP at step \( k \).

To derive (30) one needs to relate \( Q_{\pi_k} \) to the optimal \( Q^* \). Unfortunately, finding a direct relation between \( Q_{\pi_k} \) and \( Q^* \) is not an easy task. Instead, we relate \( Q_{\pi_k} \) to \( Q^* \) via an auxiliary action-value function \( Q_k \), which we define below. In the remainder of this Section we take the following steps: (i) we express \( \Psi_k \) in terms of \( Q_k \) in Lemma 7, (ii) we obtain an upper bound on the normed error \( \| Q^* - Q_k \| \) in Lemma 8. Finally, (iii) we use these two results to derive a bound on the normed error \( \| Q^* - Q_{\pi_k} \| \). For the sake of readability, we skip the formal proofs of the Lemmas in this section since we prove a more general case in Section D. Further, in the sequel, we repress the state(-action) dependencies in our notation wherever these dependencies are clear, e.g., \( \Psi(x,a) \) becomes \( \Psi \), \( Q(x,a) \) becomes \( Q \).

Now let us define the auxiliary action-value function \( Q_k \). The sequence of auxiliary action-value functions \( \{Q_0, Q_1, Q_2, \ldots \} \) is obtained by iterating the initial \( Q_0 = \Psi_0 \) from the following recursion:

\[
Q_k = k - 1 + \frac{1}{k} \pi_{k-1} - 1 \frac{1}{k} Q_{k-1} + 1 \frac{1}{k} Q_{k-1} + 1 Q_0,
\]

(31)

where \( \pi_k \) is the policy induced by the \( k \)th iterate of DPP.

Lemma 7 relates \( \Psi_k \) with \( Q_k \):

**Lemma 7** Let \( k \) be a positive integer. Then, we have:

\[
\Psi_k = kQ_k + Q_0 - \pi_{k-1}((k - 1)Q_{k-1} + Q_0).
\]

(32)

Now we focus on relating \( Q_k \) and \( Q^* \):

**Lemma 8** Let Assumption 7 hold and \( L \) denotes the cardinality of \( \mathcal{A} \) and \( k \) be a positive integer, also assume that \( \| \Psi_0 \| \leq V_{\text{max}} \) then the following inequality holds:

\[
\| Q^* - Q_k \| \leq \frac{\gamma \left( 4V_{\text{max}} + \frac{\log(L)}{\eta} \right)}{(1 - \gamma)^2}.
\]

(33)

Lemma 8 provides an upper bound on the normed-error \( \| Q_k - Q^* \| \). We make use of Lemma 8 to prove the main result of this Section:

\[
\| Q^* - Q_{\pi_k} \| = \| Q^* - Q_{k+1} + Q_{k+1} - \pi_{k}Q^* + \pi_{k}Q^* - Q_{\pi_k} \|
\leq \| Q^* - Q_{k+1} \| + \| Q_{k+1} - \pi_{k}Q^* \| + \| \pi_{k}Q^* - \pi_{k}Q_{\pi_k} \|
\leq \| Q^* - Q_{k+1} \| + \| Q_{k+1} - \pi_{k}Q^* \| + \gamma \| Q^* - Q_{\pi_k} \|.
\]
By collecting terms we obtain:

\[\|Q^* - Q^{\pi_k}\| \leq \frac{1}{1 - \gamma} \left( \|Q^* - Q_{k+1}\| + \|Q_{k+1} - \mathcal{T}^{\pi_k} Q^*\| \right)\]

\[= \frac{1}{1 - \gamma} \left( \|Q^* - Q_{k+1}\| + \left\| \frac{k}{k+1} \mathcal{T}^{\pi_k} Q_k + \frac{1}{k+1} \mathcal{T}^{\pi_k} Q_0 - \mathcal{T}^{\pi_k} Q^* \right\| \right)\]

\[\leq \frac{1}{1 - \gamma} \left( \|Q^* - Q_{k+1}\| + \frac{k}{k+1} \|\mathcal{T}^{\pi_k} Q^* - \mathcal{T}^{\pi_k} Q_k\| + \frac{1}{k+1} \|\mathcal{T}^{\pi_k} Q^* - \mathcal{T}^{\pi_k} Q_0\| \right)\]

\[\leq \frac{1}{1 - \gamma} \left( \|Q^* - Q_{k+1}\| + \frac{\gamma k}{k+1} \|Q^* - Q_k\| + \frac{\gamma}{k+1} \|Q^* - Q_0\| \right).\]

This combined with Lemma 8 completes the Proof.

**Appendix C. Proof of Theorem 3**

First, we note that \(Q_k\) converges to \(Q^*\) (Lemma 8) and subsequently \(\pi_k\) converges to \(\pi^*\) by (37). Therefore, there exists a limit for \(\Psi_k\) since \(\Psi_k\) writes in terms of \(Q_k\), \(Q_0\) and \(\pi_{k-1}\) (Lemma 7).

Now, we compute the limit of \(\Psi_k\). \(Q_k\) converges to \(Q^*\) with a linear rate from Lemma 8. Also, we have \(V^* = \pi^* Q^*\) by definition of \(V^*\) and \(Q^*\). Then, by taking the limit of (32) we deduce:

\[\lim_{k \to \infty} \Psi_k(x, a) = \lim_{k \to \infty} \left[ k Q^*(x, a) + Q_0(x, a) - (k - 1) V^*(x) - (\pi^* Q_0)(x) \right] \]

\[= \lim_{k \to \infty} k (Q^*(x, a) - V^*(x)) \]

\[+ Q_0(x, a) - (\pi^* Q_0)(x) + V^*(x).\]

This combined with Assumption 2 completes the Proof.

**Appendix D. Proof of Theorem 4**

This Section provides a formal theoretical analysis of the performance of dynamic policy programming in the presence of approximation.

Consider a sequence of the action preferences \(\{\Psi_0, \Psi_1, \Psi_2, \ldots\}\) as the iterates of (18). Our goal is to establish an \(\ell_\infty\)-norm performance loss bound of the policy induced by approximate DPP. The main result is that at iteration \(k \geq 0\) of approximate DPP, we have:

\[\|Q^* - Q^{\pi_k}\| \leq \frac{2}{(1 - \gamma)(k + 1)} \left[ \gamma \left( \frac{4V_{\text{max}} + \log(L)}{\eta} \right) + \sum_{j=1}^{k+1} \gamma^{k-j+1} \|E_{j-1}\| \right], \quad (34)\]

where \(E_k = \sum_{j=0}^{k} \epsilon_k\) is the cumulative approximation error up to step \(k\). Here, \(Q^{\pi_k}\) denotes the action-value function of the policy \(\pi_k\) and \(\pi_k\) is the soft-max policy associated with \(\Psi_k\).

As in the proof of Theorem 1, we relate \(Q^*\) with \(Q^{\pi_k}\) via an auxiliary action-value function \(Q_k\). In the sequel, we first express \(\Psi_k\) in terms of \(Q_k\) in Lemma 9. Then, we obtain an upper bound on the normed error \(\|Q^* - Q_k\|\) in Lemma 10. Finally, we use these two results to derive (34).
Now, let us define the auxiliary action-value function $Q_k$. The sequence of auxiliary action-value function $\{Q_0, Q_1, Q_2, \ldots\}$ is resulted by iterating the initial action-value function $Q_0 = \Psi_0$ from the following recursion:

$$ Q_k = \frac{k - 1}{k} \Psi_{k-1} Q_{k-1} + \frac{1}{k} (\Psi_{k-1} Q_0 + E_{k-1}), $$

(35)

where (35) may be considered as an approximate version of (31). Lemma 9 relates $\Psi_k$ with $Q_k$:

**Lemma 9** Let $k$ be a positive integer and $\pi_k$ denotes the policy induced by the approximate DPP at iteration $k$. Then we have:

$$ \Psi_k = kQ_k + Q_0 - \pi_{k-1} ((k - 1)Q_{k-1} + Q_0). $$

(36)

**Proof** We rely on induction for the Proof of this Theorem. The result holds for $k = 1$ since one can easily show that (36) reduces to (18). We then show that if (36) holds for $k$ then it also holds for $k + 1$. From (18) we have:

$$ \Psi_{k+1} = \Psi_k + \Psi\pi_k \Psi_k - \pi_k \Psi_k + \epsilon_k $$

$$ = kQ_k + Q_0 - \pi_{k-1} ((k - 1)Q_{k-1} + Q_0) + T_{\pi_k}(kQ_k + Q_0 - \pi_{k-1} ((k - 1)Q_{k-1} + Q_0)) $$

$$ - \pi_k(kQ_k + Q_0 - \pi_{k-1} ((k - 1)Q_{k-1} + Q_0)) + \epsilon_k $$

$$ = kQ_k + Q_0 + T_{\pi_k}(kQ_k + Q_0 - \pi_{k-1} ((k - 1)Q_{k-1} + Q_0)) - \pi_k(kQ_k + Q_0) $$

$$ + E_k - E_{k-1}, $$

where in the last step we make use of the following:

$$ \pi_{k\pi_k - 1}(\cdot) = \pi_{k-1}(\cdot), \quad T_{\pi_k \pi_k - 1}(\cdot) = T_{\pi_k - 1}(\cdot). $$

By collecting terms we deduce:

$$ \Psi_{k+1} = kQ_k - (k - 1)T_{\pi_k - 1} Q_{k-1} - T_{\pi_k - 1} Q_0 - E_{k-1} + kT_{\pi_k} Q_k + T_{\pi_k} Q_0 + E_k $$

$$ + Q_0 - \pi_k(kQ_k + Q_0) $$

$$ = (k + 1)Q_{k+1} + Q_0 - \pi_k(kQ_k + Q_0), $$

(36)

Thus (36) holds for $k + 1$, and is thus true for all $k \geq 1$.

Based on Lemma 9, one can express the policy induced by DPP, $\pi_k$, in terms of $Q$:

$$ \pi_k(a|x) = \frac{\exp(\eta(kQ_k(x,a) + Q_0(x,a) - \pi_{k-1} ((k - 1)Q_{k-1} + Q_0)(x)))}{Z(x)} $$

$$ = \frac{\exp(\eta(kQ_k(x,a) + Q_0(x,a)))}{Z'(x)}, $$

(37)

where $Z'(x) = Z(x) \exp(\eta\pi_{k-1}((k - 1)Q_{k-1} + Q_0)(x))$ is the normalization factor. Equation (37) expresses $\pi_k$ in terms of $Q_k$ and $Q_0$. In an analogy to Lemma 8 we state the following lemma that establishes a bound on $\|Q^* - Q_k\|$
Lemma 10 (\(\ell_\infty\)-norm bound on \(Q^* - Q_k\)) Let Assumption 4 hold. Define \(Q_k\) by (35). Let \(L\) denote the cardinality of \(A\) and \(k\) be a non-negative integer, also, assume that \(\|\Psi_0\| \leq V_{\text{max}}\), then the following inequality holds:

\[
\|Q^* - Q_k\| \leq \frac{\gamma \left( 4V_{\text{max}} + \frac{\log(L)}{\eta} \right)}{(1 - \gamma)^k} + \frac{1}{k} \sum_{j=1}^{k} \gamma^{k-j} \|E_{j-1}\|.
\]

We make use of the following results to prove Lemma 10.

Lemma 11 Let \(\eta > 0\) and \(Y\) be a finite set with cardinality \(L\). Also assume that \(\mathcal{F}\) denotes the space of measurable functions on \(Y\) with \(\mathcal{Y}_{\text{max}} \subset Y\) set of all entries of \(Y\) which maximize \(f\). Then the following inequality holds for all \(f \in \mathcal{F}\):

\[
\max_{y \in Y} f(y) - \sum_{y \in Y} \frac{\exp(\eta f(y)) f(y)}{\sum_{y' \in Y} \exp(\eta f(y'))} \leq \frac{\log(L)}{\eta}.
\]

Lemma 12 Let \(\eta > 0\) and \(k\) be a positive integer. Assume \(\|Q_0\| \leq V_{\text{max}}\), then the following holds:

\[
\|k\mathcal{T}Q_k + \mathcal{T}Q_0 - k\mathcal{T}^\pi_k Q_k - \mathcal{T}^\pi_k Q_0\| \leq \gamma \left( 2V_{\text{max}} + \frac{\log(L)}{\eta} \right).
\]

Proof (Proof of Lemma 10) We rely on induction for the proof of this Lemma. Obviously the result holds for \(k = 0\). Then we need to show that if (33) holds for \(k\) it also holds for \(k + 1\):

\[
\|Q^* - Q_{k+1}\| = \|TQ^* - \left( \frac{k}{k+1} T^\pi_k Q_k + \frac{1}{k+1} (\mathcal{T}^\pi_k Q_0 + E_k) \right)\|
\]

\[
= \left\| \frac{1}{k+1} (TQ^* - T^\pi_k Q_0) + \frac{k}{k+1} (TQ^* - T^\pi_k Q_k) - \frac{1}{k+1} E_k \right\|
\]

\[
= \frac{1}{k+1} \|TQ^* - TQ_0 + TQ_0 - T^\pi_k Q_0 + k(TQ^* - TQ_k + TQ_k - T^\pi_k Q_k)\|
\]

\[
+ \frac{1}{k+1} \|E_k\|
\]

\[
\leq \frac{1}{k+1} \|TQ^* - TQ_0\| + \|kTQ_k + TQ_0 - kT^\pi_k Q_k - T^\pi_k Q_0\|
\]

\[
+ \frac{k}{k+1} \|TQ^* - TQ_k\| + \frac{1}{k+1} \|E_k\|
\]

\[
\leq \frac{1}{k+1} \gamma \|Q^* - Q_0\| + \frac{k}{k+1} \|TQ_k + TQ_0 - kT^\pi_k Q_k - T^\pi_k Q_0\|
\]

\[
+ \frac{k}{k+1} \|Q^* - Q_k\| + \frac{1}{k+1} \|E_k\|,
\]

(38)
Now based on Lemma 12 and by plugging (33) into (38) we have:

\[
\|Q^* - Q_{k+1}\| \leq \frac{\gamma}{k+1} \left[ 4V_{\text{max}} + \frac{\log L}{\eta} \right] + \frac{\gamma k}{k+1} \left[ \frac{\gamma \left( 4V_{\text{max}} + \frac{\log(L)}{\eta} \right)}{k(1 - \gamma)} + \frac{1}{k} \sum_{j=1}^{k} \gamma^{k-j} \|E_{j-1}\| \right] \\
+ \frac{1}{k+1} \|E_k\| \\
= \frac{\gamma \left( 4V_{\text{max}} + \frac{\log(L)}{\eta} \right)}{(1 - \gamma)(k+1)} + \frac{1}{k+1} \sum_{j=1}^{k+1} \gamma^{k-j+1} \|E_{j-1}\|.
\]

The result then follows, for all \( k \geq 0 \), by induction.

\[\text{Proof} \ (\text{Proof of Lemma 11}) \text{ For any } f \in \mathcal{F} \text{ we have:}
\]

\[
\max_{y \in Y} f(y) - \sum_{y \in Y} \frac{\exp(\eta f(y)) f(y)}{\sum_{y' \in Y} \exp(\eta f(y'))} = \sum_{y \in Y} \frac{\exp(-\eta g(y)) g(y)}{\sum_{y' \in Y} \exp(-\eta g(y'))},
\]

with \( g(y) = \max_{y \in Y} f(y) - f(y) \). According to MacKay (2003, chap. 31):

\[
\sum_{y \in Y} \frac{\exp(-\eta g(y)) g(y)}{\sum_{y' \in Y} \exp(-\eta g(y'))} = -\frac{1}{\eta} \log \sum_{y \in Y} \exp(-\eta g(y)) + \frac{1}{\eta} H_p,
\]

where \( H_p \) is the entropy of probability distribution \( p \) defined by:

\[
p(y) = \frac{\exp(-\eta g(y))}{\sum_{y' \in Y} \exp(-\eta g(y'))}.
\]

The following steps complete the proof.

\[
\sum_{y \in Y} \frac{\exp(-\eta g(y)) g(y)}{\sum_{y' \in Y} \exp(-\eta g(y'))} \leq -\frac{1}{\eta} \log \left[ 1 + \sum_{y \in Y_{\text{max}}} \exp(-\eta g(y)) \right] + \frac{1}{\eta} H_p \\
\leq \frac{1}{\eta} H_p \leq \frac{\log(L)}{\eta}.
\]

\[\text{Proof} \ (\text{Proof of Lemma 12}) \text{ We have, by definition of operator } \mathcal{J}:
\]

\[
\|k\mathcal{J}Q_k + \mathcal{J}Q_0 - k\mathcal{J}^{\pi_k}Q_k - \mathcal{J}^{\pi_k}Q_0\| \leq \gamma \|kPMQ_k + PMQ_0 - kP^{\pi_k}Q_k - P^{\pi_k}Q_0\| \\
= \gamma \|P(MkQ_k + MQ_0 - \pi_k(kQ_k + Q_0))\| \\
\leq \gamma \|MkQ_k + MQ_0 - \pi_k(kQ_k + Q_0)\| \\
\leq \gamma \|2MQ_0 + M(kQ_k + Q_0) - \pi_k(kQ_k + Q_0)\| \\
\leq \gamma (2\|Q_0\| + \|M(kQ_k + Q_0) - M_0(kQ_k + Q_0)\|),
\]
where in the last line we make use of Equation (37). The result then follows by comparing (39) with Lemma 11.

Lemma 11 provides an upper-bound on the normed-error \( \|Q^* - Q_k\| \). We make use of this result to derive a bound on the performance loss \( \|Q^* - Q_k^\pi\| : \)

\[
\|Q^* - Q_k^\pi\| = \|Q^* - Q_{k+1} - T^\pi_k Q^* + T^\pi_k Q^* - Q_k^\pi\|
\leq \|Q^* - Q_{k+1}\| + \|Q_{k+1} - T^\pi_k Q^*\| + \|T^\pi_k Q^* - T^\pi_k Q_k^\pi\|
\leq \|Q^* - Q_{k+1}\| + \|Q_{k+1} - T^\pi_k Q^*\| + \gamma \|Q^* - Q_k^\pi\|.
\]

By collecting terms we obtain:

\[
\|Q^* - Q_k^\pi\| \leq \frac{1}{1 - \gamma} \left( \|Q^* - Q_{k+1}\| + \|Q_{k+1} - T^\pi_k Q^*\| \right)
= \frac{1}{1 - \gamma} \left( \|Q^* - Q_{k+1}\| + \frac{k}{k+1} T^\pi_k Q_k + \frac{1}{k+1} (T^\pi_k Q_0 + E_k) - T^\pi_k Q^* \right)
\leq \frac{1}{1 - \gamma} \left( \|Q^* - Q_{k+1}\| + \frac{k}{k+1} \|T^\pi_k Q^* - T^\pi_k Q_k\| + \frac{1}{k+1} \|T^\pi_k Q^* - T^\pi_k Q_0\| \right)
+ \frac{1}{(1 - \gamma)(k+1)} \|E_k\|
\leq \frac{1}{1 - \gamma} \left( \|Q^* - Q_{k+1}\| + \frac{\gamma k}{k+1} \|Q^* - Q_k\| + \frac{1}{k+1} \|E_k\| + \frac{\gamma}{k+1} \|Q^* - Q_0\| \right).
\]

This combined with Lemma 11 completes the proof.

Appendix E. Proof of Theorem 6

We begin the analysis by introducing some new notations. We define the estimation error associated with the \(k\)th iterate of DPP-RL as the difference between the Bellman operator \(T^\pi_k \Psi_k(x,a)\) and its sample estimate:

\[
\epsilon_k(x,a) = T^\pi_k \Psi_k(x,a) - T^\pi_k \Psi_k(x,a), \quad \forall (x,a) \in \mathcal{Z}.
\]

The DPP-RL update rule can then be re-expressed in form of the more general approximate DPP update rule:

\[
\Psi_{k+1}(x,a) = \Psi_k(x,a) + T^\pi_k \Psi_k(x,a) - \pi_k \Psi_k(x,a) + \epsilon_k(x,a).
\]

Now let us define \( \mathcal{F}_k \) as the filtration generated by the sequence of all random variables \(\{y_1, y_2, y_3, \ldots, y_k\}\) drawn from the distribution \(P(\cdot | x,a)\) for all \((x,a) \in \mathcal{Z}\). We have the property that \(E(\epsilon_k(x,a) | \mathcal{F}_{k-1}) = 0\) which means that for all \((x,a) \in \mathcal{Z}\) the sequence of estimation errors \(\{\epsilon_1, \epsilon_2, \ldots, \epsilon_k\}\) is a martingale difference sequence w.r.t. the filtration \(\mathcal{F}_k\). The asymptotic converge of DPP-RL to the optimal policy follows by extending the result of (21) to the case of bounded martingale differences. For that we need to show that the sequence of estimation errors \(\{\epsilon_j\}_{j=0:k}\) is uniformly bounded:
Lemma 13 (Stability of DPP-RL) Let Assumption 4 hold and assume that the initial action-preference function $\Psi_0$ is uniformly bounded by $V_{\text{max}}$, then we have, for all $k \geq 0$,

$$
\|\mathcal{T}_k^{\pi_k} \Psi_k\| \leq \frac{2\gamma \log L}{\eta(1-\gamma)} + V_{\text{max}}, \quad \|\epsilon_k\| \leq \frac{4\gamma \log L}{\eta(1-\gamma)} + 2V_{\text{max}}.
$$

Proof We first prove that $\|\mathcal{T}_k^{\pi_k} \Psi_k\| \leq V_{\text{max}}$ by induction. Let us assume that the bound $\|\mathcal{T}_k \Psi_k\| \leq V_{\text{max}}$ holds. Thus

$$
\|\mathcal{T}_{k+1}^{\pi_k} \Psi_{k+1}\| \leq \|r\| + \gamma \|P^{\pi_k} \Psi_{k+1}\| \leq \|r\| + \gamma \|\mathcal{M}_\eta \Psi_{k+1}\|
$$

$$
= \|r\| + \gamma \|\mathcal{M}_\eta (\Psi_k + \mathcal{T}_k^{\pi_k} \Psi_k - \mathcal{M}_\eta \Psi_k)\|
$$

$$
\leq \|r\| + \gamma \|\mathcal{M}_\eta (\Psi_k + \mathcal{T}_k^{\pi_k} \Psi_k - \mathcal{M}_\eta \Psi_k) - \mathcal{M} (\Psi_k + \mathcal{T}_k^{\pi_k} \Psi_k - \mathcal{M}_\eta \Psi_k)\|
$$

$$
+ \gamma \|\mathcal{M} (\Psi_k + \mathcal{T}_k^{\pi_k} \Psi_k - \mathcal{M}_\eta \Psi_k)\|
$$

$$
\leq \|r\| + \frac{\gamma \log L}{\eta} + \gamma \|\mathcal{M} (\Psi_k + \mathcal{T}_k^{\pi_k} \Psi_k - \mathcal{M}_\eta \Psi_k)\|
$$

$$
= \|r\| + \frac{\gamma \log L}{\eta} + \gamma \|\mathcal{M} (\Psi_k + \mathcal{T}_k^{\pi_k} \Psi_k - \mathcal{M}_\eta \Psi_k + \mathcal{M} \Psi_k - \mathcal{M} \Psi_k)\|
$$

$$
\leq \|r\| + \frac{2\gamma \log L}{\eta} + \gamma \|\mathcal{T}_k^{\pi_k} \Psi_k\| \leq \|r\| + \frac{2\gamma \log L}{\eta} + \frac{2\gamma^2 \log (L)}{\eta(1-\gamma)} + \gamma V_{\text{max}}
$$

$$
\leq \frac{2\gamma \log L}{\eta(1-\gamma)} + R_{\text{max}} + \gamma V_{\text{max}} = \frac{2\gamma \log L}{\eta(1-\gamma)} + V_{\text{max}},
$$

where we make use of Lemma 11 to bound the difference between the max operator $\mathcal{M}(\cdot)$ and the soft-max operator $\mathcal{M}_\eta(\cdot)$. Now, by induction, we deduce that for all $k \geq 0$, $\|\mathcal{T}_k \Psi_k\| \leq 2\gamma \log L/(\eta(1-\gamma)) + V_{\text{max}}$. The bound on $\epsilon_k$ is an immediate consequence of this result. 

Now based on Lemma 13 and Corollary 5 we prove the main result. We begin by recalling the result of Corollary 5

$$
\limsup_{k \to \infty} \|Q^* - Q^{\pi_k}\| \leq \frac{2\gamma}{(1-\gamma)^2} \lim_{k \to \infty} \frac{1}{k+1} \|E_k\|,
$$

Thus to prove the convergence of DPP-RL we only need to show that $1/(k+1) \|E_k\|$ asymptotically converges to $0$ w.p. 1. For this we rely on the strong law of large numbers for martingale differences [Hoffmann-Jørgensen and Pisier, 1976], which states that the average of a sequence of martingale differences asymptotically converges, almost surely, to 0 if the second moments of all entries of the sequence are bounded by some $0 \leq U \leq \infty$. This is the case for the sequence of martingales $\{\epsilon_1, \epsilon_2, \ldots\}$ since we already have proven the boundedness of $\|\epsilon_k\|$ in Lemma 13. Thus, we deduce:

$$
\lim_{k \to \infty} \frac{1}{k+1} \|E_k(x,a)\| = 0, \quad \text{w.p. 1}.
$$
Thus:
\[
\lim_{k \to \infty} \frac{1}{k+1} \| E_k \| = 0, \quad \text{w.p. 1.} \tag{40}
\]

The result then follows by combining (40) with Corollary 5.

References


