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The adjoint group of an Alexander quandle.

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To an abelian group $M$ equipped with an automorphism $T$ one can associate a quandle $A(M,T)$ called its Alexander quandle. It is given by the set $M$ together with the quandle operation $*$ defined by $y * x = Ty + x - Tx$. To any quandle $Q$ one can associate a group $\text{Adj}(Q)$ called the adjoint group of $Q$. It is defined as the abstract group with one generator $e_x$ for each $x \in Q$ and one relation $e_{yx} = e_x^{-1}e_y e_x$ for each $x, y \in Q$.

It is the purpose of this note to show that the adjoint group of an Alexander quandle $Q(M,T)$ has an elegant description in terms of $M$ and $T$, at least if the quandle is connected, which is the case if $1 - T$ is invertible. From this description one gets a similar description of the fundamental group of $Q(M,T)$ based at $0 \in M$. This note can be viewed as an exercise inspired by [2], to which we refer for motivation and definitions.

The adjoint group $A = \text{Adj}(A(M,T))$ acts from the right on $M$ by the formula $p \cdot e_x = p * x$. This defines a homomorphism $\rho$ from $A$ to the group $G$ of quandle automorphisms of $A(M,T))$. Thus $p \cdot e_0^{-1} = T^{-1}p$ and $p \cdot e_0^{-1} e_x = p + x - Tx$. From one sees that

$$p \cdot e_0^{-1} e_x e_0^{-1} e_y = p + x - Ty + y - Ty = p \cdot e_0^{-1} e_{x+y}$$

Therefore

$$e_0^{-1} e_{x+y} = \gamma(x,y) e_0^{-1} e_x e_0^{-1} e_y$$

for some $\gamma(x,y) \in \text{Adj}(Q)$ which acts trivially on $M$ and thus is an element of $K = \text{ker}(\rho)$. The group $K$ is a central subgroup of $A$ as explained in [2].

From the definition of $\gamma(x,y)$ we see that $\gamma(0,y) = 1$ and $\gamma(x,0) = 1$ for all $x$ and $y$. Furthermore the formulas

$$e_0^{-1} e_{x+y+z} = \gamma(x,y+z) e_0^{-1} e_x e_0^{-1} e_y e_0^{-1} e_z$$

$$e_0^{-1} e_{x+y+z} = \gamma(x+y,z) e_0^{-1} e_x e_0^{-1} e_y e_0^{-1} e_z$$

show that

$$\gamma(x,y+z) \gamma(y,z) = \gamma(x+y,z) \gamma(x,y)$$

for all $x,y,z \in M$
This shows that $\gamma$ is a group 2-cocycle for the group $M$ with values in $K$. We will not use this: our purpose is not to show that $\gamma$ is a coboundary, but to show that it vanishes to a certain degree, by exploiting its relation with $T$. However if $\gamma$ were a coboundary then in particular $\gamma(x, y)$ would be symmetric in $x$ and $y$. This is one of the motivations to consider the map $\lambda: M \times M \to K$ defined by

$$\lambda(x, y) = \gamma(y, x)^{-1} \gamma(x, y) = [e_0^{-1} e_y, e_0^{-1} e_x]$$

(3)

The defining relation for $A$ shows that $e_0 e_x e_0^{-1} = e_T e_x$ or equivalently $e_x e_0^{-1} = e_0^{-1} e_T e_x$ for $x \in M$. So we can rewrite $e_{x+y} = \gamma(x, y)e_x e_0^{-1} e_y$ as $e_{x+y} = \gamma(x, y)e_0^{-1} e_T e_x e_y$. In other words

$$e_u e_v = \gamma(Tu, v)^{-1} e_0 e_{Tu+v}$$

for all $u, v \in M$ (4)

If we substitute this twice in the defining relation we find that

$$\gamma(Tu, v)^{-1} e_0 e_{Tu+v} = e_u e_v = e_v e_{Tu+v-Tv}$$

$$= \gamma(Tv, Tu+v-Tv)^{-1} e_0 e_{Tv+Tu+v-Tv}$$

This implies that $\gamma(Tu, v) = \gamma(Tv, Tu+v-Tv)$ for $u, v \in M$, in other words

$$\gamma(x, y) = \gamma(Ty, x+y-Ty)$$

for $x, y \in M$ (5)

and in particular

$$\gamma(Ty, y-Ty) = 1$$

for $y \in M$ (6)

We now switch to additive notation for $K$. From (5) and the cocycle relation we find

$$\gamma(u, v) + \gamma(v-Tv, u) = \gamma(Tv, v-Tv+u) + \gamma(v-Tv, u)$$

and in particular

$$\lambda(u, v) = \gamma(u, v) - \gamma(v, u) = -\gamma(v-Tv, u)$$

(7)

Thus if $\gamma$ were symmetric then $\lambda$ would vanish, and so would $\gamma$ since $1-T$ is assumed to be invertible.

Now we look at the consequences for $\lambda$ of the cocycle condition for $\gamma$. If we substitute (7) in the cocycle condition for $\gamma$ we find

$$\lambda((1-T)^{-1}(x+y), z) + \lambda((1-T)^{-1}x, y) = \lambda((1-T)^{-1}x, y+z) + \lambda((1-T)^{-1}y, z)$$

and putting $x = u - Tu$, $y = v - Tv$ this yields

$$\lambda(u+v, z) + \lambda(u, v-Tv) = \lambda(u, v-Tv+z) + \lambda(v, z)$$

(8)
On the other hand subtracting two instances of the cocycle condition for $\gamma$

$$\gamma(u, v + z) + \gamma(v, z) = \gamma(u + v, z) + \gamma(u, v)$$

$$\gamma(z, v + u) + \gamma(v, u) = \gamma(z + v, u) + \gamma(z, v)$$

we find

$$\lambda(u + v, z) + \lambda(u, v) = \lambda(u, v + z) + \lambda(v, z) \quad (9)$$

Subtracting (9) from (8) we find

$$\lambda(u, v - Tv) - \lambda(u, v) = \lambda(u, v - Tv + z) - \lambda(u, v + z) \quad (10)$$

This means that the right hand side of (10) does not depend on $z$; in particular it has the same value for $z = -v$. Thus using the fact that $\lambda(u, 0) = 0$ we can rewrite (10) as

$$\lambda(u, -Tv) = \lambda(u, v - Tv + z) - \lambda(u, v + z) \quad (11)$$

Substituting $a = v + z$ and $b = -Tv$ this yields

$$\lambda(u, b) = \lambda(u, a + b) - \lambda(u, a) \quad (12)$$

We have just proved that $\lambda$ is additive in its second coordinate. Since $\lambda$ is skew-symmetric it is in fact bi-additive. Thus (7) and the invertibility of $1 - T$ imply that $\gamma$ is bi-additive. Moreover using (6) we can simplify (5) to

$$\gamma(x, y) = \gamma(Ty, x) \text{ for all } x, y \quad (13)$$

This motivates the following definition and theorem.

**Definition 1.** Define $\tau: M \otimes M \to M \otimes M$ by the formula $\tau(x \otimes y) = Ty \otimes x$. Define $S(M, T)$ as $\text{coker}(1 - \tau)$. Thus $\gamma$ can be viewed as a map from $S(T, M)$ to $K$. Finally define $F(M, T)$ as the set $Z \times M \times S(M, T)$ with the multiplication given by

$$(k, x, a)(m, y, b) = (k + m, Tmx + y, a + \tilde{b} + [Tmx \otimes y])$$

**Theorem 1.** The groups $\text{Adj}(A(M, T))$ and $F(M, T)$ are isomorphic.

**Proof.** We define $\phi: \text{Adj}(A(M, T)) \to F(M, T)$ by setting $\phi(e_x) = (1, x, 0)$. To see that this is well defined we have to check the following:

$$\phi(e_x)\phi(e_y) = (1, x, 0)(1, Ty + x - Tx, 0)$$

$$= (2, Tx + (Ty + x - Tx), [Tx \otimes (Ty + x - Tx)])$$

$$= (2, Ty + x, [Ty \otimes x]) = (1, y, 0)(1, x, 0) = \phi(e_y)\phi(e_x)$$

which is the case since $[Tx \otimes Ty] = [Ty \otimes x]$ and $[Tx \otimes Tx] = [Tx \otimes x]$. We define $\psi: F(M, T) \to \text{Adj}(A(M, T))$ by setting $\psi(k, x, \alpha) = e_0^{-1} e_x \gamma(\alpha)^{-1}$. 

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To see that $\psi$ is a homomorphism we have to check the following:

$$
\psi(k, x, \alpha)\psi(m, y, \beta) = e_k^{-1}e_x\gamma(\alpha)^{-1}e_0^{-1}e_y\gamma(\beta)^{-1} \\
e_k^{-1}e_x e_0^{-1}e_y \gamma(\alpha)^{-1}\gamma(\beta)^{-1} = e_k^{-1}e_m e_T e_0^{-1}e_y \gamma(\alpha + \beta)^{-1} \\
e_k^{-1} e_T e_x + y \gamma(T^m x \otimes y)^{-1} \gamma(\alpha + \beta)^{-1} \\
= \psi(k + m, T^m x + y, \alpha + \beta + [T^m x \otimes y])
$$

which is the case $e_z e_0^{-1} e_y = e_z e_0^{-1} e_y [z \otimes y]^{-1}$ for $z = T^m x$ by (1).

From $\psi(\phi(e_x)) = \psi(1, x, 0) = e_x$ we see that $\psi \phi = 1$. The other composition requires more work; first we compute

$$
\phi(\gamma[u \otimes v]^{-1}) = \phi(e_0^{-1}e_u e_0^{-1}e_v) = (1, u + v, 0)^{-1}(1, u, 0)(1, 0, 0)^{-1}(1, v, 0) \\
= (-1, -T^{-1}(u + v), [(u + v) \otimes (u + v)](1, u, 0)(-1, 0, 0)(1, v, 0) \\
= (-1, -T^{-1}(u + v), [(u + v) \otimes (u + v)](1, u + v, [u \otimes v]) = (0, 0, [u \otimes v])
$$

which shows that $\phi(\gamma(\alpha)^{-1}) = (0, 0, \alpha)$ for all $\alpha$. From this we get

$$
\phi(\psi(k, x, \alpha)) = \phi(e_k^{-1})\phi(e_x)\phi(\gamma(\alpha)^{-1}) = (k - 1, 0, 0)(1, x, 0)(0, 0, \alpha) = (k, x, \alpha)
$$

so we find that $\psi \phi = 1$.

For any quandle $Q$ there is a unique homomorphism $\epsilon: \text{Adj}(Q) \to \mathbb{Z}$ such that $\epsilon(e_x) = 1$ for all $x \in Q$; the kernel is denoted by $\text{Adj}(Q)^\circ$. It is clear that $\epsilon(\alpha) = 0$ for all $\alpha$, so $\epsilon(\psi(k, x, \alpha)) = k$. Therefore under $\psi$ the subgroup $\text{Adj}(A(M, T))^\circ$ of $\text{Adj}(A(M, T))$ corresponds to the subgroup $F(M, T)^\circ$ of $F(M, T)$ consisting of the triples $(0, x, \alpha)$. Note that on $F(M, T)^\circ$ the multiplication simplifies to

$$(0, x, \alpha)(0, y, \beta) = (0, x + y, \alpha + \beta + [x \otimes y])$$

For any quandle the fundamental group based at $q \in Q$ is defined as $\pi_1(Q, q) = \{g \in \text{Adj}(Q)^\circ \mid q \cdot g = q\}$. For these definitions we refer to [2]. In order to describe this in terms of $(M, T)$ for the case $Q = A(M, T)$ we need to translate the action of $\text{Adj}(A(M, T))$ on $M$ into an action of $F(M, T)$ on $M$.

One can easily check that $0 \cdot \psi(k, x, \alpha) = x - Tx$ for all $k$, $x$, and $\alpha$. This implies that $0 \cdot (0, x, \alpha) = 0$ if and only if $x = 0$, which means that $\pi_1(A(M, T), 0)$ is isomorphic to $S(M, T)$.

**Example 1.** Let $F$ be a field, let $M = F[t]/(t^2 + at + b)$ and let $T$ be multiplication by the class of $t$. Then $T$ is an automorphism if $b \neq 0$ and $A(M, T)$ is connected if $1 + a + b \neq 0$. In this case $S(M, T)$ isomorphic to $K/(b^2 + ab - a - 1)$. Thus $A(M, T)$ is simply connected if $b^2 + ab - a - 1 \neq 0$. The entry for $F = \mathbb{Z}/(3)$ and $f(t) = t^2 - t + 1$ in the table on page 49 of [1] is not compatible with this, but it is a misprint.
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