MODULAR PROPERTIES OF MATRIX COEFFICIENTS OF COREPRESENTATIONS OF A LOCALLY COMPACT QUANTUM GROUP

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Abstract. Using a quantum group version of the Plancherel theorem, we derive orthogonality relations for matrix coefficients of corepresentations of a locally compact quantum group. Moreover, we prove that the modular operator and the modular conjugation that appear in the Tomita-Takesaki theorem can be expressed in terms of these matrix coefficients. As a consequence, the modular automorphism group of a unimodular quantum group can be expressed in terms of matrix coefficients. As an example, we make this expression precise for the quantum group analogue of the normaliser of $SU(1, 1)$ in $SL(2, \mathbb{C})$.

1. Introduction

The definition of locally compact quantum groups has been given by Kustermans and Vaes \cite{kustermans-vaes-2, kustermans-vaes-3} at the turn of the millennium, and we use their definition of locally compact quantum groups in this paper. We stick mainly to the von Neumann algebraic setting \cite{kustermans-vaes-3}. Since the introduction of quantum groups in the 1980ies and their theoretical development, many results known in the theory of groups have been generalised to quantum groups in some setting. In particular, the theory of compact quantum groups has been settled satisfactorily by Woronowicz establishing a analogues of the Haar measure and the Schur orthogonality relations for matrix elements of corepresentations analogous to the group case, see \cite{woronowicz} and references given there. In particular, in the Kustermans-Vaes approach to locally quantum groups there is a well-defined notion of dual locally compact quantum groups. Moreover, the double dual gives back the original locally compact quantum group. In the dual locally compact quantum group we have suitable matrix elements of corepresentations occurring the left regular corepresentation associated to the multiplicative unitary $W$.

In his thesis \cite[§3.2]{desmedt-thesis} Pieter Desmedt has generalised the Schur orthogonality relations for compact quantum groups to orthogonality relations for matrix elements of square integrable irreducible corepresentations of locally compact quantum groups. Desmedt's results in this direction have also been obtained by Buss and Meyer \cite{buss-meyer}. In particular, Desmedt shows that there exist so-called Duflo-Moore operators for each square integrable corepresentation encoding the orthogonality relations for the matrix coefficients. The orthogonality relations are with respect to the left and right Haar weight. Moreover, he proves an analogue of the Plancherel theorem for locally compact quantum groups following the classical proof as in e.g. Dixmier \cite[§13-18]{dixmier}.

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The first goal of the paper is to extend the orthogonality relations to orthogonality relations for direct integrals of matrix elements of corepresentations occurring in the left regular corepresentation involving the analogues of Desmedt’s Duflo-Moore operators for all the corepresentations occurring in the Plancherel measure. This can be obtained from Desmedt’s analogue of the Plancherel theorem for locally compact quantum groups, and for this reason we recall Desmedt’s Plancherel theorem in Theorem 2.1. For our purposes a slight generalisation is needed. The generalisation also involves a Plancherel measure for the right Haar weight and their interrelation, see Theorem 2.3. Then we prove the generalised orthogonality relations for matrix coefficients of irreducible corepresentations in Theorem 3.6 assuming some technical conditions on the locally compact quantum group comparable to the conditions in the group case [5].

In the definition of a locally compact quantum group the Haar weights and their modular properties play an important role. The orthogonality relations suggest that modular properties of direct integrals of the matrix coefficients of corepresentations of a locally compact quantum can be expressed in terms of the corresponding operators of Duflo-Moore type. We give the polar decomposition of the second operator (4.1) as in the Tomita-Takesaki theorem for a general locally compact quantum group satisfying the conditions of the Plancherel theorem, see Theorem 2.1. In the case of a unimodular locally compact quantum group, we obtain an explicit expression for the action of the modular automorphism group on matrix elements of corepresentations. This result is presented in Theorem 4.8. As a consequence we relate unimodularity and traciality of the left and right Haar weight for the locally compact quantum group and its dual.

Next we calculate the modular conjugation and the modular automorphism group for the case of the locally compact quantum group associated with the normaliser of $SU(1,1)$ in $SL(2, C)$ introduced in [9] and further studied in [6], where the explicit decomposition of the left regular corepresentation is presented. In Appendix A we prove that this examples satisfies the conditions of the Plancherel theorem. We calculate the Duflo-Moore operators for almost all corepresentations in the decomposition of the left regular corepresentation. Desmedt’s results [2, §3.5] on the Duflo-Moore operators for the discrete series corepresentations are obtained in a simpler fashion, and we extend the result to all corepresentations occurring in the left regular corepresentation. Finally, in Appendix A we collect some prerequisites for the integral decomposition of tensor products of unbounded operators and we prove that the example of Section 5 satisfies the conditions of the Plancherel theorems.

Conventions and notation. For results on weight theory on von Neumann algebras our main reference is [17]. If $\varphi$ is a weight on a von Neumann algebra $M$, we use the standard notation $N_{\varphi} = \{ x \in M \mid \varphi(x^*x) < \infty \}$ and $M_{\varphi} = N_{\varphi}^*N_{\varphi}$, $M_{\varphi}^+ = M_{\varphi} \cap M^+$. $\sigma_{\varphi}^+$ denotes the modular automorphism group of $\varphi$.

The definition of a locally compact quantum group we use is the one by Kustermans and Vaes [12], [13]. We briefly recall their notational conventions, see also [11], [18]. Let $(M, \Delta)$ be a locally compact quantum group, where $M$ denotes the von Neumann algebra and $\Delta$ the comultiplication. So $\Delta$ is normal $\ast$-homomorphism $\Delta: M \to M \otimes M$ satisfying $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$, where $\iota$ denotes the identity. Moreover, there exist two normal semi-finite faithful
weights $\varphi$, $\psi$ on $M$ so that
\[
\varphi((\omega \circ \iota)\Delta(x)) = \varphi(x)\omega(1), \quad \forall \omega \in M_+^*, \forall x \in M_+^\times \quad \text{(left invariance)},
\]
\[
\psi((\iota \circ \omega)\Delta(x)) = \psi(x)\omega(1), \quad \forall \omega \in M_+^*, \forall x \in M_+^\times \quad \text{(right invariance)}.
\]

$\varphi$ is the left Haar weight and $\psi$ the right Haar weight. $(H_\varphi, \Lambda, \pi_\varphi)$ and $(H_\psi, \Gamma, \pi_\psi)$ denote the GNS-constructions with respect to the left Haar weight $\varphi$ and the right Haar weight $\psi$ respectively. Without loss of generality we may assume that $H_\varphi = H_\psi$ and $M \subset B(H_\varphi)$. The operator $W \in B(H_\varphi \otimes H_\varphi)$ defined by $W^*(\Lambda(a) \otimes \Lambda(b)) = (\Lambda \otimes \Lambda)(\Delta(b)(a \otimes 1))$ is a unitary operator known as the multiplicative unitary. It implements the comultiplication $\Delta(x) = W^*(1 \otimes x)W$ for all $x \in M$ and satisfies the pentagonal equation $W_{12}^*W_{23} = W_{23}^*W_{12}$ in $B(H_\varphi \otimes H_\varphi \otimes H_\varphi)$. In [12], [13], see also [11], [18], it is proved that there exists a dual locally compact quantum group $(\hat{M}, \hat{\Delta})$, so that $(\hat{M}, \hat{\Delta}) = (M, \Delta)$.

A unitary corepresentation $U$ of a von Neumann algebraic quantum group on a Hilbert space $H$ is a unitary element $U \in M \otimes B(H)$ such that $(\Delta \circ \iota)(U) = U_{12}U_{23} \in M \otimes M \otimes B(H)$, where the standard leg-numbering is used in the right hand side. A closed subspace $L \subseteq H$ is an invariant subspace for the unitary corepresentation $U$ if $(\omega \otimes \iota)(U)$ preserves $L$ for all $\omega \in M_*$. A unitary corepresentation $U$ in the Hilbert space $H$ is irreducible if there are only trivial (i.e. equal to $\{0\}$ or the whole Hilbert space $H$) invariant subspaces. If $U_1$ is a corepresentation on a Hilbert space $H_1$ and $U_2$ is a corepresentation on a Hilbert space $H_2$, then $U_1$ is equivalent to $U_2$ if there is a unitary map $T : H_1 \to H_2$ such that $(\iota \otimes T)U_1 = U_2(\iota \otimes T)$. We use the notation $\text{IC}(M)$ for the equivalence classes of irreducible, unitary corepresentations of $(M, \Delta)$.

$(\hat{M}_u, \hat{\Delta}_u)$ denotes the universal dual and $(\hat{M}_c, \hat{\Delta}_c)$ denotes the reduced dual C*-algebraic quantum groups [10]. The dual weights are denoted by $\hat{\varphi}_u$ and $\hat{\psi}_u$ for $(\hat{M}_u, \hat{\Delta}_u)$ and $\hat{\varphi}_c$ and $\hat{\psi}_c$ for $(\hat{M}_c, \hat{\Delta}_c)$. Similarly, we have GNS-constructions $(H_\varphi, \Lambda_\varphi, \pi_\varphi)$ and $(H_\psi, \Lambda_\psi, \pi_\psi)$ for $(M_u, \Delta_u)$ and $(H_\varphi, \Lambda_\varphi, \pi_\varphi)$ and $(H_\psi, \Lambda_\psi, \pi_\psi)$ for $(M_c, \Delta_c)$. Recall that without loss of generality we may assume that $H_\varphi$ equals $H_\psi$.

By $\text{IR}(\hat{M}_u)$ and $\text{IR}(\hat{M}_c)$ we denote the equivalence classes of irreducible, unitary representations of $\hat{M}_u$ and $\hat{M}_c$ respectively. We recall from [10] that there is a bijective correspondence between $\text{IR}(\hat{M}_u)$ and $\text{IC}(M)$ and that $\text{IR}(\hat{M}_c)$ is contained in $\text{IR}(\hat{M}_u)$.

$W$ denotes the multiplicative unitary associated with $M$. For $\omega \in M_*$ we define $\lambda(\omega) = (\omega \circ \iota)(W) \in \hat{M}$. We set
\[
\mathcal{I} = \{\omega \in M_* \mid \Lambda(x) \mapsto \omega(x^*), x \in \mathcal{N}_\varphi \text{ is a continuous functional on } H_\varphi\}.
\]

$\mathcal{I}$ is dense in $M_*$ [12, Lemma 8.5]. By the Riesz representation theorem, for every $\omega \in \mathcal{I}$ one can associate a unique vector denoted by $\xi(\omega)$ such that $\langle \xi(\omega), \Lambda(x) \rangle = \omega(x^*)$. The set $\xi(\omega)$, $\omega \in \mathcal{I}$, is dense in $H_\varphi$ [12, Lemma 8.5]. Then the dual weight $\hat{\varphi}$ on $\hat{M}$ is the weight defined by the GNS-construction $\lambda(\omega) \mapsto \xi(\omega)$. This GNS-construction of $\hat{M}$ is denoted by $\hat{\Lambda}$. All these definitions have right analogues.

\[
\mathcal{I}_R = \{\omega \in M_* \mid \Gamma(x) \mapsto \omega(x^*), x \in \mathcal{N}_\psi \text{ is a continuous functional on } H_\psi\}.
\]
For $\omega \in \mathcal{I}_R$, there is a vector $\xi_R(\omega)$ such that $\langle \xi_R(\omega), \Gamma(x) \rangle = \omega(x^*)$. The set $\xi_R(\omega), \omega \in \mathcal{I}_R$, is dense in $H_\varphi$. Then the dual weight $\hat{\psi}$ on $M$ is the weight defined by the GNS-construction $\lambda(\omega) \mapsto \xi_R(\omega)$. This GNS-map of $M$ is denoted by $\hat{\Gamma}$.

For $\alpha \in M_+$, we denote $\overline{\alpha} \in M_+$ for the functional defined by $\overline{\alpha}(x) = \overline{\alpha(x^*)}$. Define $M^*_\alpha = \{ \alpha \in M_+ \mid \exists \theta \in M_* : (\theta \otimes \iota)(W) = (\alpha \otimes \iota)(W)^* \}$. It can be shown [10] that for every $\alpha \in M^*_\alpha$ there is a unique $\theta \in M_*$ such that $(\theta \otimes \iota)(W) = (\alpha \otimes \iota)(W)^*$ and $\theta$ is determined by $\theta(x) = \overline{\alpha}(S(x)), x \in \mathcal{D}(S)$, where $S$ is the unbounded antipode of $(M, \Delta)$. We will write $\alpha^*$ for this $\theta$.

Basic results on direct integration can be found in [4]. For direct integrals of unbounded operators we refer to [15]. If $X$ is a standard measure space with measure $\mu$, we use the notation $(H_U)_{U \in X}$ or simply $(H_U)_U$ for a field of Hilbert spaces $H_U$ over $X$. If $(H_U)_U$ is a measurable field of Hilbert spaces we denote its direct integral by $\int^\oplus_X H_U d\mu(U)$. Similarly we add subscripts to denote fields of vectors, operators and representations.

Let $H$ be a Hilbert space. We define the inner product to be linear in the first entry and anti-linear in the second entry. We denote the Hilbert-Schmidt operators on $H$ by $B_2(H)$. Recall that $B_2(H)$ is a Hilbert space itself, which is isomorphic to $H \otimes H$, the isomorphism being given by $\xi \otimes \eta : h \mapsto \langle h, \eta \rangle \xi$. Here $\overline{H}$ denotes the conjugate Hilbert space. We denote vectors in $\overline{H}$ and operators acting on $H$ with a bar. For $\xi, \eta \in H$ the normal functional $\omega_{\xi, \eta}$ on $B(H)$ is defined as $\omega_{\xi, \eta}(A) = \langle A \xi, \eta \rangle$. The domain of an (unbounded) operator $A$ on $H$ is denoted by $\mathcal{D}(A)$. The symbol $\otimes$ denotes either the tensor product of Hilbert spaces, the tensor product of operators or the von Neumann algebraic tensor product. It will always be clear from the context which tensor product is meant.

## 2. Plancherel Theorems

The classical Plancherel theorem for locally compact groups [5, Theorem 18.8.1] has a quantum group analogue, which has been proved by Desmedt in [2]. This section recalls part of Desmedt’s Plancherel theorem and elaborates a bit on minor modifications of this theorem which turn out to be useful for explicit computations in Section 5. Since [2] is unpublished, we provide the reader with some sketches of Desmedt’s proofs in order to give a better understanding of how the results can be obtained.

For two unbounded operators $A$ and $B$, we denote $A \cdot B$ for the closure of their product.

**Theorem 2.1** (Desmedt [2, Theorem 3.4.1]). Let $(M, \Delta)$ be a locally compact quantum group such that $M$ is a type I von Neumann algebra and such that $M_u$ is a separable $C^*$-algebra. There exist a standard measure $\mu$ on $IC(M)$, a measurable field $(H_U)_U$ of Hilbert spaces, a measurable field $(D_U)_U$ of self-adjoint, strictly positive operators and an isomorphism $Q_L$ of $H_\varphi$ onto $\int^\oplus B_2(H_U) d\mu(U)$ with the following properties:

1. For all $\alpha \in \mathcal{I}$ and $\mu$-almost all $U \in IC(M)$, the operator $(\alpha \otimes \iota)(U)D_U^{-1}$ is bounded and $(\alpha \otimes \iota)(U) \cdot D_U^{-1}$ is a Hilbert-Schmidt operator on $H_U$.
2. For all $\alpha, \beta \in \mathcal{I}$ one has the Parseval formula

$$\langle \xi(\alpha), \xi(\beta) \rangle = \int_{IC(M)} \text{Tr} \left( (\beta \otimes \iota)(U) \cdot D_U^{-1} \right) \cdot (\alpha \otimes \iota)(U) \cdot D_U^{-1} \right) d\mu(U),$$

where $\text{Tr}$ denotes the trace on $M_u$.
and $Q_L$ is the isometric extension of
\[ \hat{\Lambda}(\lambda(T)) = \int_{\text{IC}(M)} B_\alpha(H_U) \, d\mu(U) : \xi(\alpha) \mapsto \int_{\text{IC}(M)} (\alpha \otimes i)(U) \cdot D_U^{-1} \, d\mu(U). \]

We just give a sketch of the proof, following Desmedt [2]. The theorem is basically obtained as a corollary of the following auxiliary result. We denote $\text{IR}(A)$ for the set of equivalence classes of irreducible unitary representations of a C*-algebra $A$. For the definition of an approximately KMS-weight on a C*-algebra we refer to [12]. If $(M, \Delta)$ is a locally compact quantum group, then the Haar weights of $M_u$ are approximately KMS, see [13].

**Theorem 2.2** (Desmedt [2, Theorem 3.4.5]). Let $A$ be a separable C*-algebra with lower semi-continuous, densely defined, approximately KMS-weight $\phi$ such that $\pi_\phi(A)^\prime$ is a type I von Neumann algebra. Then there exists a positive measure $\mu$ on $\text{IR}(A)$, a measurable field of Hilbert spaces $(K_\sigma)_\sigma$, a measurable field $(\pi_\sigma)_\sigma$ of representations of $A$ on $K_\sigma$ such that $\pi_\sigma$ belongs to the class $\sigma$ for every $\sigma \in \text{IR}(A)$, a measurable field $(D_\sigma)_\sigma$ of self-adjoint, strictly positive operators and an isomorphism $\mathcal{P}$ of $H_\phi$ onto $\int_{\text{IR}(A)} K_\sigma \otimes K_\sigma d\mu(\sigma)$ with the following properties:

1. For all $x \in \mathcal{N}_\phi$ and almost all $\sigma \in \text{IR}(A)$, the operator $\pi_\sigma(x)D_\sigma^{-1}$ is bounded and $\pi_\sigma(x) \cdot D_\sigma^{-1}$ is Hilbert-Schmidt.
2. For all $a, b \in \mathcal{N}_\phi$ one has the Parseval formula
\[ \langle \Lambda_\phi(a), \Lambda_\phi(b) \rangle = \int_{\text{IR}(A)} \text{Tr} \left( (b \cdot D_\sigma^{-1})^* (a \cdot D_\sigma^{-1}) \right) \, d\mu(\sigma), \]
and $\mathcal{P}$ is the isometric extension of
\[ \Lambda_\phi(\mathcal{N}_\phi) \rightarrow \int_{\text{IR}(A)} K_\sigma \otimes K_\sigma d\mu(\sigma) : \Lambda_\phi(x) \mapsto \int_{\text{IR}(A)} \pi_\sigma(x) \cdot D_\sigma^{-1} \, d\mu(\sigma). \]

**Sketch of the proof following** [2]. Since $\pi_\phi(A)^\prime$ is type I, it is of the form $\int_X B(K_\sigma) \, d\mu(\sigma)$ where $\mu$ is a measure on the measure space $X$ and $(K_\sigma)_\sigma$ is a field of separable Hilbert spaces [16, Ch V, Theorem 1.27]. Therefore $\pi_\phi(A)^\prime$ has a canonical trace $t = \oplus_n \text{Tr}_n \otimes \int_{X_n} \, d\mu$, where $X_n$ contains all $\sigma \in X$ such that $K_\sigma$ is of dimension $n$ and $\text{Tr}_n$ is the trace on the $n$-dimensional Hilbert space ($n = \infty$ is allowed). Note that the GNS-space with respect to $t$ is given by $\int_X B_2(K_\sigma) d\mu(\sigma)$ and $\mathcal{N}_t$ equals the Hilbert-Schmidt operators in $\pi_\phi(A)^\prime$.

Now by [19] there is a self-adjoint strictly positive operator $D$ which is affiliated with $\pi_\phi(A)^\prime$ such that $D = t_{D^{-2}}$, see [19] for the definition of the W*-lift $\tilde{\phi}$ and the weight $t_{D^{-2}}$. Then by [15, Theorem 1.8], $D$ has a direct integral decomposition $D = \int_X D_\sigma d\mu(\sigma)$. Let $\mathcal{N}_\phi^0$ denote the set of $y \in \pi_\phi(A)^\prime$ such that $yD^{-1}$ is bounded and $y \cdot D^{-1} \in \mathcal{N}_t$. From the first section of [19] one finds that $\mathcal{N}_\phi^0$ is a core for $\Lambda_\phi$. Hence we have $\Lambda_\phi(y) = \Lambda_\phi(y \cdot D^{-1}) = \int_X y_\sigma \cdot D_\sigma^{-1} \, d\mu(\sigma)$. So for $y \in \mathcal{N}_\phi^0$ the map
\[ \Lambda_\phi(y) \mapsto \int_X y_\sigma \cdot D_\sigma^{-1} \, d\mu(\sigma), \]
is an isometry. Desmedt proves that actually for any \( y \in N_\phi \) and for almost all \( \sigma \in X \), the operator \( y_\sigma D_\sigma^{-1} \) is bounded and \( y_\sigma \cdot D_\sigma^{-1} \) is Hilbert-Schmidt. Furthermore, for all \( y \in \pi_\phi(A)^u \),
\[
y - \pi_\phi(y) = \pi_t(y) = \int_X \pi_{\tau_\sigma}(y_\sigma) d\mu(\sigma) = \int_X (y_\sigma \otimes 1) d\mu(\sigma).
\]
(2.1)

Since \( \pi_\phi(A) \) commutes with the diagonisable operators, we find from [16, Theorem IV.8.25] that there exists a measurable field of representations \( (\zeta_\sigma)_\sigma \) of \( A \) on \( B_2(K_\sigma) \) such that \( \pi_\phi = \int_X \zeta_\sigma d\mu(\sigma) \), and by (2.1) this implies that there exists a measurable field of representations \( \pi_\sigma \) from \( A \) in \( K_\sigma \) such that, for all \( a \in A \), \( \zeta_\sigma(a) = \pi_\sigma(a) \otimes 1 \) for almost all \( \sigma \). By the decomposition of \( \pi_\phi(A)^u \) we find that \( \pi_\sigma(A)^u = B(K_\sigma) \), hence \( \pi_\sigma(A)^u = \mathbb{C}1 \) for almost all \( \sigma \). Therefore, almost all \( \pi_\sigma \) are irreducible and [5, Lemma 8.4.1(iii)] yields non-equivalence of almost all \( \pi_\sigma \). Subtracting a negligible set from \( X \) we may assume that \( \pi_\sigma \) are non-equivalent and by [5, Proposition 8.1.8], it follows that \( \sigma \mapsto \pi_\sigma \) is a Borel map from \( X \) to \( IR(A) \) which is injective. We extend \( \mu \) under this correspondence to the whole \( IR(A) \), which proves the auxiliary theorem.

Theorem 2.1 is now proved by choosing \( A = M_u \) and using the correspondence between \( IR(M_u) \) and \( IC(M) \) [10] and the relations, see also [10],
\[
\pi_\sigma \left( (\omega \otimes \iota)(\hat{\mathcal{V}}) \right) = (\omega \otimes \iota)(U_\sigma) \quad \text{where } \sigma \in IR(M_u) \text{ corresponds to } U_\sigma \in IC(M),
\]
(2.2)
\[
\xi(\omega) = \hat{\Lambda} ((\omega \otimes \iota)(W)) = \hat{\phi}_u \left( (\omega \otimes \iota)(\hat{\mathcal{V}}) \right).
\]

Here \( \hat{\phi}_u \) is the left Haar weight on the universal dual quantum group, and \( \hat{\phi}_u \) is the GNS-embedding with respect to this weight. For the definition of the unitary corepresentation \( \hat{\mathcal{V}} \in \mathcal{M}_c \otimes M_u \), we refer to [10].

Here \( \mu \) is called the left Plancherel measure and \( Q_L \) is called the left Plancherel transform. We will be dealing with the following analogue of the above theorem as well.

**Theorem 2.3.** Let \((M, \Delta)\) be a locally compact quantum group such that \( \hat{M} \) is a type-I von Neumann algebra and such that \( \hat{M}_u \) is a separable C*-algebra. There exist a standard measure \( \mu_R \) on \( IC(M) \), a measurable field \( (K_U)_U \) of Hilbert spaces, a measurable field \( (E_U)_U \) of self-adjoint, strictly positive operators and an isomorphism \( Q_R \) of \( H_\phi \) onto \( \int \oplus B_2(K_\sigma) d\nu(U) \) with the following properties:

1. For all \( \alpha \in T_R \) and \( \mu_R \)-almost all \( U \in IC(M) \), the operator \( (\alpha \otimes \iota)(U)E_U^{-1} \) is bounded and \( (\alpha \otimes \iota)(U) \cdot E_U^{-1} \) is a Hilbert-Schmidt operator on \( K_U \).
2. For all \( \alpha, \beta \in T_R \) one has the Parseval formula
\[
\langle \xi_R(\alpha^\tau), \xi_R(\beta^\tau) \rangle = \int_{IC(M)} \text{Tr} \left( \left( (\beta \otimes \iota)(U) \cdot E_U^{-1} \right)^* \left( (\alpha \otimes \iota)(U) \cdot E_U^{-1} \right) \right) d\mu_R(U),
\]
and \( Q_R \) is the isometric extension of
\[
\hat{\Gamma}(\lambda(\mathcal{I})) \rightarrow \int_{IC(M)} B_2(H_U) d\mu_R(U) : \xi_R(\alpha^\tau) \mapsto \int_{IC(M)} (\alpha \otimes \iota)(U) \cdot E_U^{-1} d\mu_R(U).
\]
The measure $\mu_R$ can be chosen equal to the measure $\mu$ of Theorem 2.1 and the measurable field of Hilbert spaces $(K_U)_U$ can be chosen equal to $(H_U)_U$, the measurable field of Hilbert spaces of Theorem 2.1.

The theorem can be obtained from the auxiliary Theorem 2.2 and using the relations between the right Haar weights $\psi$ and the right Haar weight $\psi_u$ on the universal dual quantum group. We elaborate a bit on the third statement. Since $\hat{\phi}_u$ and $\hat{\psi}_u$ are both approximately KMS-weights on the universal dual $M_u$, their W*-lifts are n.s.f. weights so that [17, Theorem VIII.3.2] implies that the representations $\pi_{\phi}$ and $\pi_\psi$ are equivalent. Hence

\begin{equation}
\pi_{\hat{\phi}_u}(M_u)^{\text{op}} = \pi_\phi(M) \simeq \pi_\psi(M) = \pi_{\hat{\psi}_u}(M_u)^{\text{op}}.
\end{equation}

The proof of Theorem 2.2 shows that the measures $\mu$ and $\nu$ together with the measurable fields of Hilbert spaces $(H_U)_U$ and $(K_U)_U$ in Theorems 2.1 and 2.3 arise from the direct integral decompositions of $\pi_{\hat{\phi}_u}(M_u)^{\text{op}}$ and $\pi_{\hat{\psi}_u}(M_u)^{\text{op}}$, respectively. That is:

\begin{align*}
\pi_{\hat{\phi}_u}(M_u)^{\text{op}} &= \int_X B(H_\sigma)d\mu(\sigma), \\
\pi_{\hat{\psi}_u}(M_u)^{\text{op}} &= \int_Y B(K_\sigma)d\mu_R(\sigma).
\end{align*}

By (2.3) we may assume that $\mu = \mu_R$, $X = Y$ and $(H_U)_U = (K_U)_U$. Furthermore, by (2.1) we have $\pi_\phi(y) = y = \pi_\psi(y), \forall y \in M$, which shows that the correspondence between $X$ and the measurable subspace IR$(M_u)$ is the same for $\pi_{\hat{\phi}_u}$ and $\pi_{\hat{\psi}_u}$. This proves the third statement of Theorem 2.3.

Here $\mu_R$ is called the right Plancherel measure and $Q_R$ is called the right Plancherel transform. Since $\mu_R$ may be chosen equal to $\mu$, we will simply speak about the Plancherel measure $\mu$, without specifying left and right. Similarly, we identify $(K_U)_U$ with $(H_U)_U$.

Remark 2.4. Theorems 2.1 and 2.3 remain valid when the assumption that $M_c$ is separable (universal norm) is replaced by the assumption that $M_r$ is separable (reduced norm) and the measure space IC$(M)$ is replaced by the measure space IR$(M_r)$. The proof is a minor modification of the proof of [2, Theorem 3.4.1]. Here, IR$(M_u)$ can be replaced by IR$(M_c)$ and $\hat{\mathcal{V}}$ should be read as the multiplicative unitary $W$, see [10] for the definition of $\hat{\mathcal{V}}$. The proof of this modification can be obtained by using the following relations instead of (2.2)

\begin{equation}
\pi_\sigma((\omega \otimes \iota)(W)) = (\omega \otimes \iota)(U_\sigma) \text{ where } \sigma \in \text{IR}(\hat{M}_c) \text{ corresponds to } U_\sigma \in \text{IC}(M),
\end{equation}

\begin{equation}
\xi(\omega) = \Lambda((\omega \otimes \iota)(W)) = \Lambda_{\hat{\phi}_u}((\omega \otimes \iota)(W)).
\end{equation}

In [2, Theorem 3.4.8] Desmedt proves that the support of the left and right Plancherel measures equal IR$(M_c)$, which is in agreement with this observation.

Remark 2.5. The corepresentations that appear as discrete mass points in the Plancherel measure correspond to the square integrable corepresentations in the sense of [1, Definition 3.2] or the equivalent definition of left square integrable corepresentations as in [2, Definition 3.2.29]. A proof of this can be found in [2, Theorem 3.4.10].
3. Orthogonality relations

We show that there exist orthogonality relations between matrix coefficients of corepresentations of a locally compact quantum group \((M, \Delta)\). These orthogonality relations extend the known Schur orthogonality relations for compact quantum groups as proved by Woronowicz [21, Theorem 5.4]. The result is a consequence of the Plancherel theorems. A theorem that gives precise information about the domains of the operators \(D_U\) and \(E_U\) arising in the Plancherel theorems is given.

**Notation 3.1.** The following conventions are made for Sections 3 and 4. \((M, \Delta)\) is a fixed locally compact quantum group satisfying the conditions of Theorems 2.1 and 2.3. We set \(D = \int_{IC(M)}^\oplus D_U d\mu(U)\), \(E = \int_{IC(M)}^\oplus E_U d\mu(U)\) and \(H = \int_{IC(M)}^\oplus H_U d\mu(U)\), where \(\mu\) is the Plancherel measure. All (direct) integrals are taken over \(IC(M)\). In the proofs we omit this in the notation.

**Lemma 3.2.** We have the following:

1. Let \(x \in M\), such that the linear map \(f : A(A(x)) \rightarrow \mathbb{C} : \xi(x) \mapsto (\xi(x), \Lambda(x))\) is bounded. Then \(x \in D(\Lambda)\) and \(f(\xi(x)) = (\langle \xi(x), \Lambda(x) \rangle)\).

2. Let \(x \in M\), such that the linear map \(f : \Gamma(\Lambda(\mathbb{I})) \rightarrow \mathbb{C} : \xi_R(x) \mapsto (\xi_R(x), \Gamma(x))\) is bounded. Then \(x \in D(\Gamma)\) and \(f(\xi_R(x)) = (\langle \xi_R(x), \Gamma(x) \rangle)\).

**Proof.** We prove the first statement, the second being analogous. The claim is true for \(x \in \mathcal{N}_\varphi\), since \(D(\Lambda) = \mathcal{N}_\varphi\) and by definition \(\langle \xi(x), \Lambda(x) \rangle = \alpha(x^*)\), for all \(\alpha \in \mathbb{I}\). For arbitrary \(x \in M\) as in (1), let \(x_i \in \mathcal{N}_\varphi\) be a net converging \(\sigma\)-weakly to \(x\), which exists since \(\varphi\) is semi-finite. The set \(\{\xi(\alpha) | \alpha \in \mathbb{I}\}\) is dense in \(H_\varphi\) by [12, Lemma 8.5] and its subsequent remark. Hence, by the Riesz theorem, there is a \(v \in H_\varphi\) such that for every \(\alpha \in \mathbb{I}\)

\[
\langle \xi(\alpha), \Lambda(x_i) \rangle = \alpha(x_i^*) \rightarrow \alpha(x^*) = \langle \xi(\alpha), v \rangle.
\]

Hence, \(\Lambda(x_i)\) converges weakly to \(v\). Since \(\Lambda\) is closed with respect to the \(\sigma\)-weak topology on \(M\) and the weak topology on \(H_\varphi\), we have \(x \in D(\Lambda)\) and \(\Lambda(x_i)\) converges weakly to \(v = \Lambda(x)\). \(\square\)

Recall that \(\int H U d\mu(U) \approx \int H U^* d\mu(U)\). For \(\eta = \int H U d\mu(U)\), \(\xi = \int \xi_U d\mu(U) \in H\) the measurable field of vectors \((\xi_U \otimes \eta_U(U))U\) is not necessarily square integrable. If it is square integrable, \(\int \xi_U \otimes \eta_U d\mu(U) \in \int B_2(HU) d\mu(U)\).

From the above Plancherel theorems we derive the following useful lemma.

**Lemma 3.3.** Let \(\eta = \int_{IC(M)}^\oplus \xi_U d\mu(U) \in H\) and \(\xi = \int_{IC(M)}^\oplus \xi_U d\mu(U) \in H\) be such that \(\eta \in D(D^{-1})\) and \((\xi_U \otimes \eta_U(U))U\) is square integrable. Then \(IC(M) \ni U \mapsto (\xi_U \otimes \eta_U(U^*)) \in M\) is weak-* integrable with respect to \(\mu\) and \(\int_{IC(M)}^\oplus (\xi_U \otimes \eta_U(U^*)) d\mu(U) \in \mathcal{N}_\varphi\), and

\[
Q^{-1}_L \left( \int_{IC(M)}^\oplus \xi_U \otimes \eta_U d\mu(U) \right) = \Lambda \left( \int_{IC(M)}^\oplus (\xi_U \otimes \eta_U(U^*)) d\mu(U) \right).
\]
Proof. For $\alpha \in I$, Theorem 2.1 implies that
\[
\langle \xi(\alpha), \mathcal{Q}_L^{-1}(\int \xi_U \otimes \overline{\eta_U} \, d\mu(U)) \rangle = \int \langle (\alpha \otimes \iota)(U) D_U^{-1}, \xi_U \otimes \overline{\eta_U} \rangle_{\mathbb{H}^d} \, d\mu(U) \\
= \int (\alpha \otimes \omega_{D_U^{-1} \eta_U, \xi_U})(U) \, d\mu(U).
\]
The map $T : I \rightarrow L^1(\mathbb{IC}(M), \mu) : \alpha \mapsto (U \mapsto (\alpha \otimes \omega_{D_U^{-1} \eta_U, \xi_U})(U))$ is continuous, since
\[
\int |(\alpha \otimes \omega_{D_U^{-1} \eta_U, \xi_U})(U)| \, d\mu(U) \leq \|\alpha\| \int \|D_U^{-1} \eta_U\| \|\xi_U\| \, d\mu(U) \leq \|\alpha\| \|D^{-1} \eta\| \|\xi\|.
\]
Recall that $I$ is dense in $M^*$ [12, Lemma 8.5]. Therefore $T$ has a bounded extension to $M^*$. Since integrating with respect to the measure $\mu$ is a continuous functional on $L^1(\mathbb{IC}(M), \mu)$, the map $I \ni \alpha \mapsto \int (\alpha \otimes \omega_{D_U^{-1} \eta_U, \xi_U})(U) \, d\mu(U)$ determines a functional on $M^*$. Hence there is an element $x \in M = (M^*)^*$ such that $\langle \alpha(x) = \int (\alpha \otimes \omega_{D_U^{-1} \eta_U, \xi_U})(U) \, d\mu(U) \rangle$ for all $\alpha \in M^*$. By definition $x$ is the weak-* (or Gelfand) integral $\int (\iota \otimes \omega_{\eta_U, D_U^{-1} \xi_U})(U) \, d\mu(U)$, see [3, Section II] for more details. Now,
\[
\langle \xi(\alpha), \mathcal{Q}_L^{-1}(\int \xi_U \otimes \overline{\eta_U} \, d\mu(U)) \rangle = \int (\alpha \otimes \omega_{D_U^{-1} \eta_U, \xi_U})(U) \, d\mu(U) \\
= \alpha\left(\int (\iota \otimes \omega_{\eta_U, D_U^{-1} \xi_U})(U) \, d\mu(U)\right) = \alpha\left(\int (\iota \otimes \omega_{\xi_U, D_U^{-1} \eta_U})(U^*) \, d\mu(U)^*\right),
\]
so that by Lemma 3.2, $\int (\iota \otimes \omega_{\xi_U, D_U^{-1} \eta_U})(U^*) \, d\mu(U) \in \mathcal{D}(\Lambda) = \mathcal{N}_p$, and
\[
\mathcal{Q}_L^{-1}\left(\int \xi_U \otimes \overline{\eta_U} \, d\mu(U)\right) = \Lambda\left(\int (\iota \otimes \omega_{\xi_U, D_U^{-1} \eta_U})(U^*) \, d\mu(U)\right).
\]
\[\square\]

Remark 3.4. As in the proof of Lemma 3.3 we see that for $\xi = \int \xi_U \, d\mu(U) \in H$, $\eta = \int \eta_U \, d\mu(U) \in H$, the weak-* integral $\int (\iota \otimes \omega_{\xi_U, \eta_U})(U^*) \, d\mu(U) \in M$ exists, and for $\alpha \in M^*$, $|\int (\alpha \otimes \omega_{\xi_U, \eta_U})(U) \, d\mu(U)| \leq \|\alpha\| \|\xi\| \|\eta\|.$

The previous lemma shows that $\mathcal{Q}_L^{-1}$ is an analogue of what Desmedt calls the (left) Wigner map [2, Section 3.3.1]. This map is defined as
\[
B_2(H_U) \rightarrow H : \xi \otimes \overline{\eta} \mapsto \Lambda\left((\iota \otimes \omega_{\xi_U, \eta_U})(U^*)\right),
\]
where $U$ is a corepresentation on a Hilbert space $H_U$ that appears as a discrete mass point in the Plancherel measure, cf. the remarks about square integrable corepresentations at the end of Section 2. Desmedt uses this map to prove a preliminary version of his Plancherel theorems [2, Theorem 3.3.4]. This map is also considered in [1, Page 203], where it is denoted by $\Phi$.

The next Lemma is the right analogue of Lemma 3.3, the proof being similar.

Lemma 3.5. Let $\eta = \int \eta_U \, d\mu(U) \in H$ and $\xi = \int \xi_U \, d\mu(U) \in H$ be such that $\eta \in \mathcal{D}(E^{-1})$ and $(\xi_U \otimes \overline{\eta_U})_U$ is square integrable. Then $\mathbb{IC}(M) \ni U \mapsto (\iota \otimes \omega_{\xi_U, E_U^{-1} \eta_U})(U) \in M$ is
weak-* integrable with respect to $\mu$. Furthermore, $\int_{\text{IC}(M)}(t \otimes \omega_{\xi_U, E_U^{-1} \eta_U})(U) d\mu(U) \in N_\rho$ and
\[ Q_R^{-1}(\int_{\text{IC}(M)} \xi_U \otimes \overline{H_U} d\mu(U)) = \Gamma \left( \int_{\text{IC}(M)} (t \otimes \omega_{\xi_U, E_U^{-1} \eta_U})(U) d\mu(U) \right). \]

Now we state the orthogonality relations between matrix coefficients of corepresentations of $(M, \Delta)$. The result extends the known Schur orthogonality relations for compact quantum groups [21, Theorem 5.4] or [18, Proposition 5.3.8]. Furthermore, it extends the result by Desmedt [2, Theorems 3.2.16, 3.2.17, 3.2.38, 3.2.39], or Buss and Meyer [1, Theorems 3.10, 3.11], in which the relations are proved for corepresentations that appear as discrete mass points in the Plancherel measure. Of course, most of the effort of proving Theorem 3.6 is contained in Desmedt’s Theorems 2.1 and 2.3.

**Theorem 3.6** (Orthogonality relations). Let $(M, \Delta)$ be a locally compact quantum group, such that $\hat{M}_u$ is separable and $\hat{M}$ is a type I von Neumann algebra. Let $\eta = \int \eta U d\mu(U) \in H$, $\xi = \int \xi_U d\mu(U) \in H$, $\eta' = \int \eta' U d\mu(U) \in H$ and $\xi' = \int \xi' U d\mu(U) \in H$. We have the following orthogonality relations:

1. Suppose that $\eta, \eta' \in \mathcal{D}(D)$ and that $(\xi_U \overline{D_U \eta_U})_U, (\xi'_U \overline{D_U \eta'_U})_U$ are square integrable fields of vectors, then
\[ \varphi \left( \int (t \otimes \omega_{\xi_U, \eta_U})(U^*) d\mu(U) \right)^* \int (t \otimes \omega_{\xi'_U, \eta'_U})(U^*) d\mu(U) = \int (D_U \eta_U, D_U \eta'_U) \langle \xi_U, \xi'_U \rangle d\mu(U). \]

2. Suppose that $\eta, \eta' \in \mathcal{D}(E)$ and that $(\xi_U \overline{E_U \eta_U})_U, (\xi'_U \overline{E_U \eta'_U})_U$ are square integrable fields of vectors, then:
\[ \psi \left( \int (t \otimes \omega_{\xi_U, \eta_U})(U) d\mu(U) \right)^* \int (t \otimes \omega_{\xi'_U, \eta'_U})(U) d\mu(U) = \int (E_U \eta_U, E_U \eta'_U) \langle \xi_U, \xi'_U \rangle d\mu(U). \]

Here $\int (t \otimes \omega_{\xi_U, \eta_U})(U) d\mu(U), \int (t \otimes \omega_{\xi_U, \eta'_U})(U^*) d\mu(U), \int (t \otimes \omega_{\xi'_U, \eta_U})(U) d\mu(U)$, $\int (t \otimes \omega_{\xi'_U, \eta'_U})(U^*) d\mu(U)$ are defined in Lemma 3.3 and 3.5. The integrals are taken over $\text{IC}(M)$.

**Proof.** We prove the first orthogonality relation. By Lemma 3.3, $\int (t \otimes \omega_{\xi_U, \eta_U})(U^*) d\mu(U), \int (t \otimes \omega_{\xi'_U, \eta'_U})(U^*) d\mu(U) \in N_\rho$ and
\[ \varphi \left( \int (t \otimes \omega_{\xi_U, \eta_U})(U^*) d\mu(U) \right)^* \int (t \otimes \omega_{\xi'_U, \eta'_U})(U^*) d\mu(U) \]
\[ = \Lambda \left( \int (t \otimes \omega_{\xi_U, \eta_U})(U^*) d\mu(U) \right) \Lambda \left( \int (t \otimes \omega_{\xi'_U, \eta'_U})(U^*) d\mu(U) \right) \]
\[ = \int (\xi_U \overline{D_U \eta_U}, \xi'_U \overline{D_U \eta'_U}) d\mu(U) \]
\[ = \int (D_U \eta_U, D_U \eta'_U) \langle \xi_U, \xi'_U \rangle d\mu(U). \]

Part (2) follows similarly from Lemma 3.5. □
In Theorem 3.6 we have focussed on the case \( \int (t \otimes \omega_{\xi_U,D_U^{-1}}) (U^*) d\mu(U) \in \mathcal{N}_\psi \). As observed in Remark 3.4 the element \( \int (t \otimes \omega_{\xi_U,D_U^{-1}}) (U^*) d\mu(U) \in M \) exists for \( \eta = \int \xi_U d\mu(U) \in H \), \( \xi = \int \xi_d d\mu(U) \in H \) and the next theorem investigates the consequences of \( \int (t \otimes \omega_{\xi_U,D_U^{-1}}) (U^*) d\mu(U) \in \mathcal{N}_\psi \).

**Theorem 3.7.** Let \( \xi = \int \xi_d d\mu(U) \in H \) be an essentially bounded field of vectors.

1. Let \( \eta = \int \xi_d d\mu(U) \in H \) be such that \( \int \xi_d d\mu(U) \in \mathcal{N}_\psi \). Then, for almost every \( U \) in the support of \( \langle \xi_U \rangle_U \), we have \( \eta_U \in D(D_U) \).

2. Let \( \eta = \int \xi_d d\mu(U) \in H \) be such that \( \int \xi_d d\mu(U) \in \mathcal{N}_\psi \). Then, for almost every \( U \) in the support of \( \langle \xi_U \rangle_U \), we have \( \eta_U \in D(D_U) \).

**Proof.** We only give a proof of the first statement. Consider the sesquilinear form

\[
q(\eta, \eta') = \varphi \left( \int (t \otimes \omega_{\xi_U,n_U})(U^*) d\mu(U) \right) ^* \int (t \otimes \omega_{\xi_U,n_U})(U^*) d\mu(U) ,
\]

\[
D(q) = \left\{ \eta = \int \eta_U d\mu(U) \mid \int (t \otimes \omega_{\xi_U,n_U})(U^*) d\mu(U) \in \mathcal{N}_\psi \right\}.
\]

\( q \) is a closed form on \( H \). Indeed, assume that \( \eta_n \in D(q) \) converges in norm to \( \eta \in H \) and that \( q(\eta_n - \eta_m) \to 0 \). Then \( \int (t \otimes \omega_{\xi_U,n_U})(U^*) d\mu(U) \) converges to \( \int (t \otimes \omega_{\xi_U,n_U})(U^*) d\mu(U) \) \( \sigma \)-weakly. By assumption \( \Lambda(\int (t \otimes \omega_{\xi_U,n_U})(U^*) d\mu(U)) \) is a Cauchy sequence in norm. The \( \sigma \)-weak-weak closedness of \( \Lambda \) implies that \( \int (t \otimes \omega_{\xi_U,n_U})(U^*) d\mu(U) \in D(\Lambda) = \mathcal{N}_\psi \), so \( \eta \in D(q) \) and \( \Lambda(\int (t \otimes \omega_{\xi_U,n_U})(U^*) d\mu(U)) \) converges to \( \Lambda(\int (t \otimes \omega_{\xi_U,n_U})(U^*) d\mu(U)) \) weakly. Since we know that \( \Lambda(\int (t \otimes \omega_{\xi_U,n_U})(U^*) d\mu(U)) \) is a actually a Cauchy sequence in the norm topology it is norm convergent to \( \Lambda(\int (t \otimes \omega_{\xi_U,n_U})(U^*) d\mu(U)) \). This proves that \( q(\eta - \eta_n) \to 0 \).

Since \( \langle \xi_U \rangle_U \) is a square integrable, essentially bounded field of vectors, \( \int \xi_U, \eta_U d\mu(U) \in B_2(H) \). By Lemma 3.3, \( D(D) \subseteq D(q) \), so that \( q \) is densely defined. \( q \) is symmetric and positive by its definition. By [8, Theorem VI.2.23], there is a unique positive, self-adjoint, possibly unbounded operator \( A \) on \( H \) such that \( q(\eta, \eta') = \langle A\eta, A\eta' \rangle \) and \( D(A) = D(q) \). By Theorem 3.6 we see that for \( \eta, \eta' \in D(D) \) we have \( \|D_U \eta_U, D_U \eta'_U\| \xi_U \| \| d\mu(U) = \langle A\eta, A\eta' \rangle \). Since both \( A \) and \( \int \xi_U d\mu(U) \) are positive, self-adjoint operators this yields \( A = \int \xi_U \| D_U d\mu(U) \). In particular \( \eta_U \in D(D_U) \) for almost every \( U \in \text{supp} \langle \xi_U \rangle_U = \{ U \in \text{IC}(M) - \xi_U \neq 0 \} \).

4. Modular properties of matrix coefficients

In this section we work towards expressions for the modular automorphism group of the left and right Haar weight in terms of matrix elements of corepresentations, culminating in Theorem 4.8. The matrix coefficients of corepresentations are preserved under the modular automorphism group. The idea of proving this formula is to describe the polar decomposition of the conjugation operator \( \Gamma(x) \to \Lambda(x^*), x \in \mathcal{N}_\psi \cap \mathcal{N}_\psi^* \) explicitly in terms of corepresentations. Then, for a unimodular quantum group, where \( \Gamma = \Lambda \), the modular automorphism group is implemented by the absolute value of this operator. In this section we keep the conventions made in Notation 3.1.
At this point we recall the relevant results from the theory of normal, semi-finite, faithful (n.s.f.) weights and their modular automorphism groups. This is contained in [17, Chapters VI, VII, VIII]. We emphasize that the notation sometimes differs from [17].

Consider the following two operators [17, Section VIII.3]

\[ S_{\psi,0} : H_{\psi} \to H_{\psi} : \Gamma(x) \mapsto \Gamma(x^*), \quad x \in \mathcal{N}_\psi \cap \mathcal{N}_\psi^*, \]

\[ S_0 : H_{\psi} \to H_{\psi} : \Gamma(x) \mapsto \Lambda(x^*), \quad x \in \mathcal{N}_\psi \cap \mathcal{N}_\psi^*. \]

Both operators are densely defined and preclosed. We denote their closures by and respectively. and correspond to and in [17, Section VIII.3, (13)]. We denote their polar decompositions by \( S_{\psi} = J_\psi \nabla_\psi^{1/2}, S = J \nabla^{1/2} \). By construction, \( J_\psi \) and \( \nabla_\psi \) are the modular conjugation and the modular operator appearing in the Tomita-Takesaki theorem. In particular, \( \nabla_\psi \) implements the modular automorphism group \( \sigma^\psi_t \), i.e.

\[ \sigma^\psi_t(\pi_\psi(x)) = \nabla_\psi^{it} \pi_\psi(x) \nabla_\psi^{-it}, \quad x \in M, t \in \mathbb{R}. \]

Furthermore, \( \nabla_\psi^{it}, t \in \mathbb{R}, \) is a homomorphism of the left Hilbert algebra \( \Gamma(\mathcal{N}_\psi \cap \mathcal{N}_\psi^*) \), i.e.

\[ \nabla_\psi^{it} \pi_\psi(x) \nabla_\psi^{-it} \Gamma(y) = \pi_i(\nabla_\psi^{it} \Gamma(x) \Gamma(y)), \]

where \( \pi_i(a) b = a b \) for \( a,b \in \Gamma(\mathcal{N}_\psi \cap \mathcal{N}_\psi^*) \). By [17, Section VIII.3, (11) and (29)] the modular automorphism group \( \sigma^\psi_t \) is implemented by \( \nabla \), i.e.

\[ \sigma^\psi_t(x) = \nabla^{it} x \nabla^{-it}, \quad x \in M, t \in \mathbb{R}. \]

We emphasize that in general \( \nabla^{it}, t \in \mathbb{R}, \) fails to be a Hilbert algebra homomorphism of the left Hilbert algebra \( \Gamma(\mathcal{N}_\psi \cap \mathcal{N}_\psi^*) \). In case \( (M, \Delta) \) is unimodular, we find that \( \nabla = \nabla_\psi \) and \( \nabla^{it}, t \in \mathbb{R}, \) satisfies the relation (4.3). This fact will eventually lead to Theorem 4.8. However, we present the theory more general and do not suppose that \( (M, \Delta) \) is unimodular until this theorem.

It turns out that the polar decomposition of \( S \) can be expressed in terms of corepresentations by means of the Plancherel theorems. The polar decomposition of \( Q_L \circ S \circ Q_R^{-1} \) and the morphisms \( Q_L \) and \( Q_R \) give the polar decomposition of \( S \). Eventually this yields Theorems 4.6 and 4.7.

**Remark 4.1.** For \( \xi = \int \xi_U d\mu(U) \in H \) and \( \eta = \int \eta_U d\mu(U) \in H \), and \( A = \int \xi_U \otimes \eta_U d\mu(U) \), \( B = \int \xi_U \otimes \eta_U d\mu(U) \) decomposable operators on \( H \), we will use \( (\xi, \eta) \in \mathcal{D}^o(A, B) \) to mean \( \xi \in \mathcal{D}(A), \eta \in \mathcal{D}(B), \) \( (\xi_U \otimes \eta_U) d\mu(U) \) is square integrable and \( \int \xi_U \otimes \eta_U d\mu(U) \in \mathcal{D}(\int \xi_U \otimes \eta_U d\mu(U)) \). For closed operators \( A \) and \( B \) the set of \( \int \xi_U \otimes \eta_U d\mu(U) \) with \( (\xi_U, \eta_U) \in \mathcal{D}^o(A, B) \) is a core for \( \int \xi_U \otimes \eta_U d\mu(U) \) by Lemma A.1. In particular this set is dense in \( \int \xi_U \otimes \eta_U d\mu(U) \). Let \( \Sigma \) be the anti-linear flip

\[ \Sigma : \int \xi_U \otimes \eta_U d\mu(U) \to \int \xi_U \otimes \eta_U d\mu(U) : \int \xi_U \otimes \eta_U d\mu(U) \mapsto \int \eta_U \otimes \xi_U d\mu(U). \]

\( \Sigma \) is an anti-linear isometry of \( \int \xi_U \otimes \eta_U d\mu(U) \).
Lemma 4.2. For \( \eta = \int_{IC(M)}^\oplus \eta_U d\mu(U) \), \( \xi = \int_{IC(M)}^\oplus \xi_U d\mu(U) \in H \), with \( (\eta, \bar{\xi}) \in D^\oplus(E^{-1}, \overline{D}) \), we have \( Q_R^{-1} \left( \int_{IC(M)}^\oplus \xi_U \otimes \overline{\eta_U} d\mu(U) \right) \in D(S) \) and:

\[
(4.5) \quad Q_L \circ S \circ Q_R^{-1} \left( \int_{IC(M)}^\oplus \xi_U \otimes \overline{\eta_U} d\mu(U) \right) = \left( \int_{IC(M)}^\oplus E_U^{-1} \eta_U \otimes \overline{D_U\xi_U} d\mu(U) \right).
\]

Proof. By Lemma 3.5:

\[
(4.6) \quad Q_R^{-1} \left( \int_{IC(M)}^\oplus \xi_U \otimes \overline{\eta_U} d\mu(U) \right) = \Gamma \left( \int (t \otimes \omega_{\xi_U,E_U^{-1}\eta_U}) d\mu(U) \right).
\]

By Lemmas 3.3 and 3.5 we obtain

\[
\int (t \otimes \omega_{\xi_U,E_U^{-1}\eta_U}) d\mu(U) = \left( \int (t \otimes \omega_{E_U^{-1}\eta_U,E_U^{-1}\xi_U}) d\mu(U) \right) \in \mathcal{N}_\phi \cap \mathcal{N}_\phi^*.
\]

Hence, by (4.1), (4.6) and Lemma 3.3

\[
Q_L \circ S \circ Q_R^{-1} \left( \int_{IC(M)}^\oplus \xi_U \otimes \overline{\eta_U} d\mu(U) \right) = Q_L \left( \Lambda \left( \int (t \otimes \omega_{\xi_U,E_U^{-1}\eta_U}) d\mu(U) \right) \right) = \left( \int_{IC(M)}^\oplus E_U^{-1} \eta_U \otimes \overline{D_U\xi_U} d\mu(U) \right).
\]

\[\square\]

We are now able to give the polar decomposition of \( Q_L \circ S \circ Q_R^{-1} \).

Theorem 4.3. Consider \( S_Q := Q_L \circ S \circ Q_R^{-1} \) as an operator on \( \int_{IC(M)}^\oplus H_U \otimes \overline{H_U} d\mu(U) \). Then the polar decomposition of \( S_Q \) is given by the self-adjoint, strictly positive operator \( \int_{IC(M)}^\oplus D_U \otimes E_U^{-1} d\mu(U) \) and the anti-linear isometry \( \Sigma \).

Proof. Throughout this proof, let \( \eta = \int_{IC(M)}^\oplus \eta_U d\mu(U) \in H \), \( \xi = \int_{IC(M)}^\oplus \xi_U d\mu(U) \in H \), \( \eta' = \int_{IC(M)}^\oplus \eta'_U d\mu(U) \in H \) and \( \xi' = \int_{IC(M)}^\oplus \xi'_U d\mu(U) \in H \) be such that \( (\eta_U \otimes \bar{\xi}_U)_U \) and \( (\eta'_U \otimes \bar{\xi}'_U)_U \) are square integrable.

Assume \( (\eta, \bar{\xi}) \in D^\oplus(D, E^{-1}), (\xi', \bar{\eta}') \in D^\oplus(D, E^{-1}) \), so that by (4.5),

\[
\left( \int_{IC(M)}^\oplus (\xi_U \otimes \overline{\eta_U}) d\mu(U), S_Q \int_{IC(M)}^\oplus (\xi'_U \otimes \overline{\eta'_U}) d\mu(U) \right) = \left( \int_{IC(M)}^\oplus (\xi_U \otimes \overline{\eta_U}) d\mu(U), \int_{IC(M)}^\oplus (E_U^{-1} \eta'_U \otimes \overline{D_U\xi'_U}) d\mu(U) \right) = \int_{IC(M)}^\oplus \langle \xi_U, E_U^{-1} \eta'_U \rangle d\mu(U) = \int_{IC(M)}^\oplus \langle \xi'_U, \eta_U \rangle d\mu(U) = \int_{IC(M)}^\oplus \langle \xi'_U, \eta_U \rangle d\mu(U) = \int_{IC(M)}^\oplus \langle \xi'_U \otimes \overline{\eta'_U} \rangle d\mu(U), \int_{IC(M)}^\oplus (D_U\eta_U \otimes E_U^{-1} \xi_U) d\mu(U) \right).
\]

So

\[
S_Q \left( \int_{IC(M)}^\oplus (\xi_U \otimes \overline{\eta_U}) d\mu(U) \right) = \int_{IC(M)}^\oplus (D_U\eta_U \otimes E_U^{-1} \xi_U) d\mu(U).
\]

Assuming \( (\xi, \bar{\eta}) \in D^\oplus(D^2, E^{-2}) \), it follows

\[
S_Q S_Q \left( \int_{IC(M)}^\oplus (\xi_U \otimes \overline{\eta_U}) d\mu(U) \right) = \int_{IC(M)}^\oplus (D_U^2 \xi_U \otimes E_U^{-2} \overline{\eta_U}) d\mu(U).
\]
$\int D_U^2 \otimes E^{-2}_U d\mu(U)$ is a positive, self-adjoint operator for which the set
\[ C := \text{span}_C \left\{ \int (\xi_U \otimes \eta_U)d\mu(U) \mid (\xi, \eta) \in D^\oplus(D^2, E^{-2}) \right\}, \]
forms a core by Lemma A.1. Since $S_Q^* S_Q$ is self-adjoint and agrees with the self-adjoint operator $\int D_U^2 \otimes E^{-2}_U d\mu(U)$ on $C$ we find $S_Q^* S_Q = \int D_U^2 \otimes E^{-2}_U d\mu(U)$.

Assuming that $(\xi, \eta) \in D^\oplus(D, E^{-1})$, \[
\Sigma \circ \left( \int D_U \otimes E^{-1}_U d\mu(U) \right) = \Sigma \left( \int (D_U \xi_U \otimes \overline{E^{-1}_U \eta_U})d\mu(U) \right) = \int (E^{-1}_U \eta_U \otimes \overline{D_U \xi_U})d\mu(U),
\]
so that $S_Q$ and $\Sigma \circ \left( \int D_U \otimes E^{-1}_U d\mu(U) \right)$ agree on a core, cf. Remark 4.1. □

Finally we translate everything back to the level of the GNS-representations $H_\psi$ and $H_\psi$.

**Proposition 4.4.** Let \[
D_{v_0}^\frac{1}{2} = \text{span}_C \left\{ \int_{iC(M)} (\iota \otimes \omega_{\xi_U, \eta_U})(U)d\mu(U) \mid \eta \in D(E) \cap D(E^{-1}), (\xi, E\eta) \in D^\oplus(D, E^{-1}) \right\},
\]
and define $\nabla_0^\frac{1}{2} : \Gamma(D_{v_0}^\frac{1}{2}) \to H_\psi$ by \[
\Gamma(\int_{iC(M)} (\iota \otimes \omega_{\xi_U, \eta_U})(U)d\mu(U)) \mapsto \Gamma(\int_{iC(M)} (\iota \otimes \omega_{D_U \xi_U, E^{-1}_U \eta_U})(U)d\mu(U)).
\]
Then $\nabla_0^\frac{1}{2}$ is a densely defined, preclosed operator and its closure $\nabla_0^\frac{1}{2}$, is a self-adjoint, strictly positive operator satisfying $Q_R \circ \nabla_0^\frac{1}{2} \circ Q_R^{-1} = \int_{iC(M)} D_U \otimes E^{-1}_U d\mu(U)$.

**Proof.** Let $C := \text{span}_C \left\{ \int (\xi_U \otimes \overline{\eta_U})d\mu(U) \mid (\xi, \eta) \in D^\oplus(D^2, E^{-2}) \right\}$. Then $C$ is a core for $\int D_U \otimes E^{-1}_U d\mu(U)$. Indeed, $C$ is a core for $\int D_U^2 \otimes E^{-2}_U d\mu(U)$ by Lemma A.1, and hence this is a core for $\int D_U \otimes E^{-1}_U d\mu(U)$.

Now, let $\eta = \int \eta_U d\mu(U) \in H$ and $\xi = \int \xi_U d\mu(U) \in H$ be such that \[ \eta \in D(E) \cap D(E^{-1}), \quad (\xi, E\eta) \in D^\oplus(D, E^{-1}). \]
So $\eta \in D(E)$ and $(\xi_U \otimes E^{-1}_U \eta_U)U$ is square integrable, so that $\int (\iota \otimes \omega_{\xi_U, \eta_U})(U)d\mu(U) \in \mathcal{N}_\psi$ by Lemma 3.5. Similarly, since $E^{-1}_U \eta \in D(E)$ and $(D_U \xi_U \otimes \eta_U)U$ is square integrable, $\int (\iota \otimes \omega_{D_U \xi_U, E^{-1}_U \eta_U})(U)d\mu(U) \in \mathcal{N}_\psi$. Furthermore, we have the following inclusions: \[
C \subseteq Q_R(\Gamma(D_{v_0}^\frac{1}{2})) \subseteq D(\int D_U \otimes E^{-1}_U d\mu(U))
\]
and for \( x \in \Gamma(D_{\sqrt{\gamma}}) \) we have,

\[
\nabla_{\sqrt{\gamma}}(x) = Q^{-1}_R \left( \int_{\mathcal{D}(\sqrt{\gamma})} Du \otimes E^{-1}_U d\mu(U) \right) Q_R(x).
\]

Since \( Q_R \) is an isometric isomorphism, the claims follow from the fact that \( \int_{\mathcal{D}(\sqrt{\gamma})} Du \otimes E^{-1}_U d\mu(U) \) is a self-adjoint, strictly positive operator for which \( C \) is a core.

\( \square \)

**Proposition 4.5.** Let \( D_{J_0} \) be the linear space

\[
\text{span}_C \left\{ \int_{\mathcal{I}(\mathcal{M})} (\xi \otimes \omega_{\xi_U,\eta_U})(U) d\mu(U) \mid \xi \in \mathcal{D}(D^{-1}), \eta \in \mathcal{D}(E), (\xi_U \otimes \eta_U)_{\xi_U,\eta_U} \right. \text{ is sq. int.} \}
\]

and define

\[
J_0 : \Gamma(D_{J_0}) \to H_{\varphi} : \Gamma(\int_{\mathcal{I}(\mathcal{M})} (\xi \otimes \omega_{\xi_U,\eta_U})(U) d\mu(U)) \mapsto \Lambda(\int_{\mathcal{I}(\mathcal{M})} (\xi \otimes \omega_{\xi_U,\eta_U})(U) d\mu(U)^*).
\]

Then \( J_0 \) is a densely defined anti-linear isometry, and its closure, denoted by \( J \), is a surjective anti-linear isometry satisfying \( Q_L \circ J \circ Q_R^{-1} = \Sigma \).

**Proof.** Let \( C := \text{span}_C \left\{ \int_{\mathcal{I}(\mathcal{M})} (\xi \otimes \eta_U) d\mu(U) \mid (\xi,\eta) \in \mathcal{D}(D^{-1}, E) \right\} \). \( C \) is dense in \( \int_{\mathcal{H}} H_U \otimes \overline{H_U} d\mu(U) \), c.f. Remark 4.1.

For \( \eta = \int_{\mathcal{H}} \eta_U d\mu(U) \in H \) and \( \xi = \int_{\mathcal{H}} \xi_U d\mu(U) \in H \) so that \( \xi \in \mathcal{D}(D^{-1}), \eta \in \mathcal{D}(E) \) and \( (\xi_U \otimes \eta_U)_{\xi_U,\eta_U} \) is square integrable, we find \( \int_{\mathcal{I}(\mathcal{M})} (\xi \otimes \omega_{\xi_U,\eta_U})(U) d\mu(U) \in \mathcal{N}_{\varphi} \) and \( \int_{\mathcal{I}(\mathcal{M})} (\xi \otimes \omega_{\xi_U,\eta_U})(U) d\mu(U)^* \in \mathcal{N}_{\varphi}^* \) by Lemmas 3.3 and 3.5. So \( C \subseteq Q_R(\Gamma(D_{J_0})) \), and for \( x \in \Gamma(D_{J_0}) \), \( J_0(x) = Q^{-1}_L \circ \Sigma \circ Q_R(x) \). Then, since \( Q_L \) and \( Q_R \) are isomorphisms, the claim follows from \( \Sigma \) being a surjective anti-linear isometry. \( \square \)

Note that the previous proposition is an analogy of the classical situation. Suppose that \( G \) is a locally compact group for which the classical Plancherel theorem \([5, \text{Theorem } 18.8.1]\) holds. The anti-linear operator \( f \mapsto f^* \) acting on \( L^2(G) \) is transformed into the anti-linear flip acting on \( \int_{\mathcal{H}} K(\xi) \otimes \overline{K}(\xi) d\mu(\xi) \) by the Plancherel transform. Here \( f^*(x) = \overline{f(x^{-1})} \delta_\xi(x^{-1}) \) and \( \delta_\xi \) is the modular function on \( G \).

From Theorem 4.3 and Propositions 4.4 and 4.5 we obtain the following result.

**Theorem 4.6.** The polar decomposition of \( S \) is given by \( S = J \sqrt{\gamma} \).

The roles of \( \varphi \) and \( \psi \) can be interchanged. Consider the operator:

\[
S_0' : H_\varphi \to H_\psi : \Lambda(x) \mapsto \Gamma(x^*), \quad x \in \mathcal{N}_\varphi \cap \mathcal{N}_\psi^*.
\]

This operator is densely defined and preclosed. We denote its closure by \( S' \). The polar decomposition of \( S' \) can be expressed in terms of corepresentations in a similar way.

**Theorem 4.7.** Consider \( S' : H_\varphi \to H_\psi \). Let \( D'_{J_0} \) be the linear space

\[
\text{span}_C \left\{ \int_{\mathcal{I}(\mathcal{M})} (\xi \otimes \omega_{\xi_U,\eta_U})(U)^* d\mu(U) \mid \xi \in \mathcal{D}(D), \eta \in \mathcal{D}(E^{-1}), (Du \xi_U \otimes \eta_U)_{\xi_U,\eta_U} \right. \text{ is sq. int.} \}
\]

and for \( x \in \Gamma(D_{\sqrt{\gamma}}) \) we have,
and define \( J'_0 : \Lambda(D'_0) \to H_0 : \)
\[
\Lambda \left( \int_{\text{IC}(M)} (t \otimes \omega_{\xi_U, \eta_U})(U) d\mu(U) \right) \mapsto \Gamma \left( \int_{\text{IC}(M)} (t \otimes \omega_{D_U \xi_U, E_U^{-1} \eta_U})(U) d\mu(U) \right).
\]
Then \( J'_0 \) is densely defined and isometric, and its closure, denoted by \( J' \), is a surjective anti-linear isometry. Let
\[
D'_0 = \text{span}_C \left\{ \int_{\text{IC}(M)} (t \otimes \omega_{\xi_U, \eta_U})(U) d\mu(U) \mid \xi \in D(D) \cap D(D^{-1}), (D\xi, \eta) \in D^0(D^{-1}, E) \right\},
\]
and define \( \nabla'^{\frac{1}{2}}_0 : \Lambda(D'_0) \to H_0 : \)
\[
\Lambda \left( \int_{\text{IC}(M)} (t \otimes \omega_{\xi_U, \eta_U})(U) d\mu(U) \right) \mapsto \Lambda \left( \int_{\text{IC}(M)} (t \otimes \omega_{D_U^{-1} \xi_U, E_U \eta_U})(U) d\mu(U) \right).
\]
Then \( \nabla'^{\frac{1}{2}}_0 \) is a densely defined, preclosed operator and its closure, denoted by \( \nabla'^{\frac{1}{2}} \), is a self-adjoint, strictly positive operator.

Moreover, the polar decomposition of \( S' \) is given by \( S' = J' \nabla'^{\frac{1}{2}} \).

We now assume that \((M, \Delta)\) is unimodular, so that \( S = S' = S_\psi \) and Theorem 4.6 give an explicit expression for the modular operator and modular conjugation. This leads to the following expression for the modular automorphism group. In this case we write \( \sigma_t \) for \( \sigma_t^\psi \).

**Theorem 4.8.** Suppose that \((M, \Delta)\) is unimodular. Let \((\xi_U)_{U}, (\eta_U)_{U}\) be square integrable vector fields. The modular automorphism group \( \sigma_t \) of the Haar weight \( \psi \) can be expressed as:

\[
\sigma_t \left( \int_{\text{IC}(M)} (t \otimes \omega_{\xi_U, \eta_U})(U) d\mu(U) \right) = \int_{\text{IC}(M)} (t \otimes \omega_{D_U^{\mu \xi_U, E_U^{\mu \eta_U}}})(U) d\mu(U).
\]

**Proof.** For \( \eta = \int \xi_U d\mu(U), \xi = \int \xi_U d\mu(U) \in H \), \( \xi = \int \xi_U d\mu(U) \in H \), such that \((\xi_U \otimes \eta_U)_{U}\) is a square integrable field of vectors and \( \eta \in D(E) \), we find

\[
\nabla'^{\mu} \Gamma \left( \int_{\text{IC}(M)} (t \otimes \omega_{\xi_U, \eta_U})(U) d\mu(U) \right) = \Gamma \left( \int_{\text{IC}(M)} (t \otimes \omega_{D_U^{\mu \xi_U, E_U^{\mu \eta_U}}})(U) d\mu(U) \right).
\]

Indeed, \( \left( \int_{D_U \otimes E_U^{-1}} d\mu(U) \right)^{2\mu} (\xi_U \otimes \eta_U) = \int_{D_U^{2\mu} \xi_U \otimes E_U^{2\mu} \eta_U}) d\mu(U) \) by [15, Theorem 1.10], so (4.9) follows from Lemma 3.5 and Proposition 4.4. Since \( \sigma_t(\pi_\psi(x)) = \nabla'^{\mu} \pi_\psi(x) \nabla'^{-\mu}, x \in M, \) (4.3) implies

\[
\sigma_t \left( \pi_\psi \left( \int_{\text{IC}(M)} (t \otimes \omega_{\xi_U, \eta_U})(U) d\mu(U) \right) \right) = \pi_\psi \left( \int_{\text{IC}(M)} (t \otimes \omega_{D_U^{\mu \xi_U, E_U^{\mu \eta_U}}})(U) d\mu(U) \right),
\]

so the theorem follows from the identification of \( M \) with \( \pi_\psi(M) \), in this case.

Now let \( \eta = \int \xi_U d\mu(U) \in H \) and \( \xi = \int \xi_U d\mu(U) \in H \) be arbitrary. We take sequences of square integrable vector fields \( \xi_n = \int \xi_U d\mu(U), \eta_n = \int \eta_U d\mu(U) \) such that \((\xi_U \otimes \eta_U)_{U}\) is a square integrable field of vectors, \( \eta_n \in D(E) \) and such that \( \xi_n \) converges to \( \xi \) and \( \eta_n \) converges
to $\eta$. Then \( \int (t \otimes \omega_{\xi_{U,n},\eta_{U,n}})(U)d\mu(U) \) is $\sigma$-weakly convergent to \( \int (t \otimes \omega_{\xi_{U,n}})(U)d\mu(U) \) and hence

\[
\sigma_t \left( \int (t \otimes \omega_{\xi_{U,n},\eta_{U,n}})(U)d\mu(U) \right) = \lim_{n \to \infty} \sigma_t \left( \int (t \otimes \omega_{\xi_{U,n}})(U)d\mu(U) \right) = \\
\lim_{n \to \infty} \int (t \otimes \omega_D^{\eta_{U,n}})(U)d\mu(U) = \int (t \otimes \omega_D^{\eta_{U}})(U)d\mu(U).
\]

\[\square\]

We used (4.3) to obtain (4.10). The unimodularity assumption is essential for Theorem 4.8.

**Corollary 4.9.** Let \((M, \Delta)\) be unimodular. Let \(\eta = \int \eta_U d\mu(U) \in H, \xi = \int \xi_U d\mu(U) \in H,\) \(r \in \mathbb{R}\) be such that \(\eta \in \mathcal{D}(E^{2r})\) and \(\xi \in \mathcal{D}(D^{2r})\), then:

\[
\int_{\mathcal{IC}(M)} (t \otimes \omega_{\xi_{U},\eta_{U}})(U)d\mu(U) \in \mathcal{D}(\sigma_z),
\]

for all \(z\) in the strip \(S(r) := \{z \in \mathbb{C} \mid 0 \leq \text{Im}(z) \leq r, \text{or } r \leq \text{Im}(z) \leq 0\}\). In particular, if \(\eta\) is analytic for \(E\) and if \(\xi\) is analytic for \(D\), then \(\int_{\mathcal{IC}(M)} (t \otimes \omega_{\xi_{U},\eta_{U}})(U)d\mu(U) \) is analytic for the one-parameter group at.

**Proof.** For \(\alpha \in M_*,\) define

\[
F_{\alpha}(z) = \alpha \left( \int (t \otimes \omega_D^{\eta_{U}})(U)d\mu(U) \right) = \int \int \left( \int (t \otimes \omega_D^{\eta_{U}})(U)d\mu(U) \right) 2iz U d\mu(U), \left( \int \int \xi_U d\mu(U) \right) 2iz U d\mu(U)) \).
\]

Here the last equality follows from [15, Theorem 1.10]. By [17, Lemma VI.2.3], \(F_{\alpha}(z)\) is an analytic continuation of \(\alpha \left( \int (t \otimes \omega_D^{\eta_{U}})(U)d\mu(U) \right)\) to the strip \(S(r)\) such that \(F_{\alpha}(z)\) is bounded by a constant \(C\|\alpha\|\) where \(C\) is independent of \(\alpha\). Moreover, \(F_{\alpha}(z)\) is continuous on \(S(r)\) and analytic on the interior \(S(r)^0\). Therefore \(F(z) = \int (t \otimes \omega_D^{\eta_{U}})(U)d\mu(U) \) is a continuation of \(\sigma_t \left( \int (t \otimes \omega_{\xi_{U},\eta_{U}})(U)d\mu(U) \right)\) to the strip \(S(r)\) such that \(F(z)\) is bounded and \(\sigma\)-weakly continuous on \(S(r)\) and analytic on the interior \(S(r)^0\) [14, Result 1.2].

\[\square\]

We will derive some relations between the weights $\varphi, \psi, \check{\varphi}, \check{\psi}$. The following lemma occurs in [2], the proof being a direct application of the work by Kustermans and Vaes [13].

**Lemma 4.10 ([2, Lemma 1.1.11]).** Let \((M, \Delta)\) be a l.c. quantum group (without imposing the assumptions at the beginning of this section). If $\check{\varphi}$ is a trace, then \((\hat{M}, \hat{\Delta})\) is unimodular.

The lemma is symmetric, so that the dual statement follows: if $\varphi$ is a trace, then \((M, \Delta)\) is unimodular. We remark that the converse of Lemma 4.10 is false, as can be seen from [20, Proposition 5.5].

We assume again that \((M, \Delta)\) satisfies the assumptions made in the beginning of this section. The following proposition does not impose an unimodularity condition on \(M\), but rather on \(\hat{M}\).
Proposition 4.11. Suppose that \((\hat{M}, \hat{\Delta})\) is unimodular and that \(\hat{\varphi} = \hat{\psi}\) is a trace. Then \(\varphi = \psi\) is a trace.

Proof. \(M\) is unimodular by Lemma 4.10. Looking back at the proof of Theorem 2.2 and using the notation there, we see that \(t = \int \hat{\Theta} \text{Tr}_\sigma d\mu(\sigma)\), where \(\text{Tr}_\sigma\) is the canonical trace on \(K_\sigma\). Moreover, \(\hat{\varphi} = t_{D^{-2}} = \int \hat{\Theta} \text{Tr}_{\sigma D^{-2}} d\mu(\sigma)\), where \(\text{Tr}_{\sigma D^{-2}}\) is defined as in [19, Definition 1.5].

Theorem 2.1 was obtained by setting \(\phi = \hat{\varphi}_u\), so that \(\hat{\varphi} = \hat{\psi}\) is tracial. Since \(B(K_\sigma)\) has a unique n.s.f. trace up to a positive constant, we must have \(D_\sigma = d_\sigma 1_{K_\sigma}\) for \(d_\sigma \in \mathbb{R}^+\) almost everywhere. By replacing \(\mu(\sigma)\) by \(d^{-2}_\sigma(\sigma)\) if necessary, we may assume that \(D_\sigma = 1_{K_\sigma}\).

The operators \(D\) and \(E\) are uniquely determined by \(\hat{\varphi} = t_{D^{-2}}\) and \(\hat{\psi} = t_{E^{-2}}\). So the unimodularity implies \(D = E\). From Proposition 4.4, Theorem 4.6 and Theorem 4.7, we see that \(\nabla = 1\) and \(\nabla' = 1\). Then by (4.4) it follows that \(\sigma^\varphi_t\) is trivial, so that \(\varphi\) is tracial. □

Since we assume \((M, \Delta)\) to be a locally compact quantum group with \(M\) a type I von Neumann algebra and the C*-algebra \(M_u\) separable, the previous proposition is not a symmetric statement. Nevertheless, we find the following consequence of Theorem 4.8.

Proposition 4.12. \((M, \Delta)\) is unimodular and has tracial Haar weights if and only if \((\hat{M}, \hat{\Delta})\) is unimodular and has tracial Haar weights.

Proof. It remains to prove the trace property of \(\varphi\) in the implication \(\Rightarrow\), since the other implications follow from Lemma 4.10 and Proposition 4.11. So, suppose that \(\varphi = \psi\) is a trace. Then \(\sigma_t\) is trivial, so that by Theorem 4.8 for \(\eta = \int \hat{\Theta} \eta_U d\mu(U) \in H\) and \(\xi = \int \hat{\Theta} \xi_U d\mu(U) \in H\),

\[
\int (\iota \otimes \omega_{\xi_U, \eta_U})(U) d\mu(U) = \int (\iota \otimes \omega_{\xi_U, \eta_U})(U) d\mu(U), \quad \forall t \in \mathbb{R}.
\]

Assume furthermore that \(\eta \in \mathcal{D}(E)\) and \((\xi_U \otimes \overline{E_U \eta_U})_U\) is a square integrable field of vectors, we find that for \(\eta' = \int \hat{\Theta} \eta_U d\mu(U) \in H\) and \(\xi' = \int \hat{\Theta} \xi_U d\mu(U) \in H\) such that \(\eta' \in \mathcal{D}(E^2)\) and \((\xi'_U \otimes \overline{E_U \eta'_U})_U\) is a square integrable field of vectors:

\[
\int \langle \eta_U, E_U^2 \eta'_U \rangle \langle \xi'_U, \xi_U \rangle d\mu(U) = \psi \left( \left( \int (\iota \otimes \omega_{\xi_U, \eta_U})(U) d\mu(U) \right)^* \int (\iota \otimes \omega_{\xi_U, \eta_U})(U) d\mu(U) \right) = \\
\psi \left( \left( \int (\iota \otimes \omega_{D_U^{-2} \xi_U, E_U \eta_U})(U) d\mu(U) \right)^* \int (\iota \otimes \omega_{\xi_U, \eta_U})(U) d\mu(U) \right) = \\
\int \langle E_U^{2t} \eta_U, E_U^2 \eta'_U \rangle \langle \xi'_U, D_U^{2t} \xi_U \rangle d\mu(U).
\]

Since for such elements \(\xi', \eta'\), the vectors \(\int \hat{\Theta} \xi'_U \otimes \overline{E_U^2 \eta'_U} d\mu(U)\) are dense in \(\int \hat{\Theta} H_U \otimes \overline{H_U} d\mu(U)\), we have \(D_U \otimes E_U^{-1} = 1_{H_U} \otimes \overline{H_U}\) almost everywhere, so that both \(D_U\) and \(E_U\) are multiples of \(1_{H_U}\). Since in the proof of Theorem 2.2 (applied to \(M_u\)), \(D^{-2} = \int_X D^{-2}_\sigma d\mu(\sigma)\) is the Radon-Nikodym derivative of \(\hat{\varphi}\) with respect to \(t\) and it is affiliated with the center of \(\hat{M}\), \(\hat{\varphi}\) is tracial. □
5. Example

Using the theory of square integrable corepresentations, Desmedt [2] determined the operators $D_U$ and $E_U$ for the corepresentations that appear as discrete mass points of the Plancherel measure, see also Remark 2.5. In particular, his theory applies to compact quantum groups, for which every corepresentation is square integrable. As a non-compact example, Desmedt was able to determine the operators $D_U$ for the discrete series corepresentations of the quantum group analogue of the normalizer of $SU(1,1)$ in $SL(2,\mathbb{C})$, which we denote by $(M,\Delta)$ from now on, see [9] and [6]. Having the theory of Sections 3 and 4 at hand we determine the operators $D_U$ and $E_U$ for the principal series corepresentations of $(M,\Delta)$.

We refer to [9] and [6] for the relevant properties of $(M,\Delta)$ and use the same notational conventions. In [6, Theorem 5.7] a decomposition of the multiplicative unitary in terms of irreducible corepresentations is given:

\[
W = \bigoplus_{p \in q^\mathbb{Z}} \left( \int_{[-1,1]} W_{p,x} dx \otimes \bigoplus_{x \in \sigma_d(\Omega_p)} W_{p,x} \right).
\]

Here $\sigma_d(\Omega_p)$ is the discrete spectrum of the Casimir operator [6, Definition 4.5, Theorem 4.6] restricted to the subspace given in [6, Theorem 5.7]. $W_{p,x}$ is a corepresentation that is a direct sum of at most 4 irreducible corepresentations [6, Propositions 5.3 and 5.4]. An orthonormal basis for the corepresentation Hilbert space $L_{p,x}$ of $W_{p,x}$ is given by the vectors $e_m^\epsilon(p,x), \epsilon, \eta \in \{-, +\}, m \in \mathbb{Z}$. The corepresentations $W_{p,x}, p \in q^\mathbb{Z}, x \in \sigma(\Omega_p)$ are called the discrete series corepresentations and the corepresentations $W_{p,x}, p \in q^\mathbb{Z}, x \in [-1,1]$ are called the principal series corepresentations. We denote $D_{p,x}$ and $E_{p,x}$ for $D_{W_{p,x}}$ and $E_{W_{p,x}}$. The operators $D_{p,x}$ have been computed by Desmedt [2] for the discrete series. Hence we focus on the principal series. In Appendix A we verify that $(M,\Delta)$ satisfies the conditions of the Plancherel theorem, so that the theory of Sections 3 and 4 applies. Furthermore, $(M,\Delta)$ is unimodular [9]. We denote the modular automorphism group of the Haar weight by $\sigma_h$.

By [6, Lemmas 10.9] the action of the matrix elements in the GNS-space can be calculated explicitly:

\[
(t \otimes \omega_{e_m^\epsilon,\epsilon',\eta',m'}(W_{p,x}) f_{m_0,p_0,t_0} = C(\eta \epsilon x; m', \epsilon', \eta', \epsilon' | p_0 | p^{-1} q^{-m-m'}, p_0, m-m') \delta_{sgn(p_0),\eta' \epsilon' | p_0 | p^{-1} q^{-m-m'},t_0}.
\]

Fix $p \in q^\mathbb{Z}$. Let $\epsilon, \eta, m, \epsilon', \eta', m'$ be $\mu$-measurable functions of $x \in [-1,1]$, thus $\epsilon = \epsilon(x), \eta = \eta(x), \ldots$. Let $f, f'$ be $\mu$-square integrable complex functions on $[-1,1]$. Then $f(x)e_m^\epsilon(\eta, p, x)$ and $f'(x)e_m^\epsilon(p, x)$ are $\mu$-square integrable fields of vectors. Since
the modular automorphism group $\sigma_t$ is implemented by $\gamma^* \gamma$ [9, Section 4], Theorem 4.8 yields

$$
\left( \int_{[-1,1]} (t \otimes \omega_{f(x)} D_{p,x}^{\epsilon_m,\epsilon_m'} f'(x) E_{p,x}^{\epsilon_m,\epsilon_m'} (W_{p,x}) \, d\mu(x) \right) f_{m_0,p_0,t_0} = \sigma_t \left( \int_{[-1,1]} (t \otimes \omega_{f(x)} e_{\epsilon_m,\epsilon_m'} f'(x) E_{p,x}^{\epsilon_m,\epsilon_m'} (W_{p,x}) \, d\mu(x) \right) f_{m_0,p_0,t_0}
$$

$$
- |t|^{2it} \left( \int_{[-1,1]} (t \otimes \omega_{f(x)} e_{\epsilon_m,\epsilon_m'} f'(x) E_{p,x}^{\epsilon_m,\epsilon_m'} (W_{p,x}) \, d\mu(x) \right) \left| |t|^{-2it} f_{m_0,p_0,t_0} \right|
$$

$$
= \left( \frac{p_0^2}{p_0^2 p - q - 2m - 2m'} \right)^{it} \int_{[-1,1]} f(x) \left| f'(x) \right| C(\eta \varepsilon x; m', \varepsilon', \eta'; \varepsilon'(p_0)|p^{-1}q^{-m-m'}, p_0, m - m')
$$

$$
\times \delta_{sgn(p_0),qf} f_{m_0-m+m', \varepsilon' \varepsilon(p_0)|p^{-1}q^{-m-m'}, t_0} d\mu(x)
$$

$$
= \left( p^2 q^{2m+2m'} \right)^{it} \int_{[-1,1]} (t \otimes \omega_{f(x)} e_{\epsilon_m,\epsilon_m'} f'(x) E_{p,x}^{\epsilon_m,\epsilon_m'} (W_{p,x}) \, d\mu(x) \right) f_{m_0,p_0,t_0}.
$$

(5.2)

Define $A$ and $B$ as the unbounded self-adjoint operators on $L^1([-1,1], \mathcal{L}_{p,x} d\mu(x)$ determined by $A = \int_{[-1,1]} A_{p,x} d\mu(x), A_{p,x} e_{\epsilon_m,\epsilon_m'}(p,x) = p^2q^{2m} e_{\epsilon_m,\epsilon_m'}(p,x). B = \int_{[-1,1]} B_{p,x} d\mu(x), B_{p,x} e_{\epsilon_m,\epsilon_m'}(p,x) = q^{-2m} e_{\epsilon_m,\epsilon_m'}(p,x)$. So (5.2) yields

$$
\int_{[-1,1]} (t \otimes \omega_{f(x)} D_{p,x}^{\epsilon_m,\epsilon_m'} f'(x) E_{p,x}^{\epsilon_m,\epsilon_m'} (W_{p,x}) \, d\mu(x) = \int_{[-1,1]} (t \otimes \omega_{f(x)} A_{p,x}^{\epsilon_m,\epsilon_m'} f'(x) B_{p,x}^{\epsilon_m,\epsilon_m'} (W_{p,x}) \, d\mu(x),
$$

where the integrals are taken over $[-1,1]$. For bounded operators $F = \int_{[-1,1]} F_{p,x} d\mu(x), G = \int_{[-1,1]} G_{p,x} d\mu(x)$ on $\int_{[-1,1]} \mathcal{L}_{p,x} d\mu(x)$, the map $\left( \int_{[-1,1]} \mathcal{L}_{p,x} d\mu(x) \right) \otimes \left( \int_{[-1,1]} \mathcal{L}_{p,x} d\mu(x) \right) \rightarrow M$ given by

$$
v \otimes w = \int_{[-1,1]} v_x d\mu(x) \otimes \int_{[-1,1]} w_x d\mu(x) \mapsto \int_{[-1,1]} (t \otimes \omega_{F_{p,x} \mathcal{L}_{p,x} d\mu(x)} \otimes \mathcal{L}_{p,x} d\mu(x)) (W_{p,x}) \, d\mu(x)
$$

is norm-\sigma-weakly continuous since $| \int_{[-1,1]} \alpha \otimes \omega_{v_x w(x)} (W_{p,x}) d\mu(x) | \leq \| \alpha \| \| F \| \| G \| \| v \| \| w \|, \alpha \in M_\sigma$. Therefore, for $v = \int_{[-1,1]} v_x d\mu(x), w = \int_{[-1,1]} w_x d\mu(x) \in \int_{[-1,1]} \mathcal{L}_{p,x} d\mu(x)$, using [4, II.1.6, Proposition 7] and (5.3),

$$
\int_{[-1,1]} (t \otimes \omega_{D_{p,x}^{\epsilon_m,\epsilon_m'} E_{p,x}^{\epsilon_m,\epsilon_m'}} (W_{p,x}) \, d\mu(x) = \int_{[-1,1]} (t \otimes \omega_{A_{p,x}^{\epsilon_m,\epsilon_m'} B_{p,x}^{\epsilon_m,\epsilon_m'}} (W_{p,x}) \, d\mu(x).
$$

(5.4)

For $v = \int_{[-1,1]} v_x d\mu(x), w = \int_{[-1,1]} w_x d\mu(x) \in \int_{[-1,1]} \mathcal{L}_{p,x} d\mu(x)$, with $(v_x)_x$ essentially bounded, $w \in \mathcal{D} \left( \int_{[-1,1]} E_{p,x} d\mu(x) \right).$ Theorem 3.6 implies that $\int_{[-1,1]} (t \otimes \omega_{D_{p,x}^{\epsilon_m,\epsilon_m'} E_{p,x}^{\epsilon_m,\epsilon_m'}} (W_{p,x}) \, d\mu(x) \in \mathcal{N}_\psi$. By (5.4) and Theorem 3.7, $B_{p,x}^{\epsilon_m,\epsilon_m'} \in \mathcal{D}(E_{p,x})$ almost everywhere in the support of $(v_x)_x$. Theorem 3.6 implies that for $v' = \int_{[-1,1]} v_x' d\mu(x), w' = \int_{[-1,1]} w_x' d\mu(x) \in \int_{[-1,1]} \mathcal{L}_{p,x} d\mu(x)$ such
that \( w' \in D \left( \int_{[-1,1]} E_{p,x}^2 d\mu(x) \right) \) and \( \langle v'_x \otimes E_{p,x} w'_x, x \rangle \) is square integrable,

\[
\int_{[-1,1]} \langle B_{p,x}^2 w_x, E_{p,x}^2 v'_{x} \rangle \langle v'_x, A_{p,x}^2 v_x \rangle d\mu(x)
= \psi \left( \left( \int_{[-1,1]} (t \otimes \omega_{A_{p,x}^2 v_x, B_{p,x}^2 w_x}) (W_{p,x}) d\mu(x) \right)^* \int_{[-1,1]} (t \otimes \omega_{v'_x, w'_x}) (W_{p,x}) d\mu(x) \right)
= \psi \left( \left( \int_{[-1,1]} (t \otimes \omega_{D_{p,x}^2 v_x, E_{p,x}^2 v_x}) (W_{p,x}) d\mu(x) \right)^* \int_{[-1,1]} (t \otimes \omega_{v'_x, w'_x}) (W_{p,x}) d\mu(x) \right)
= \int_{[-1,1]} \langle E_{p,x}^2 w_x, E_{p,x}^2 v'_x \rangle \langle v'_x, D_{p,x}^2 v_x \rangle d\mu(x).
\]

\( E_{p,x} \) is strictly positive by the Plancherel theorem. The elements \( \int_{[-1,1]} v'_x \otimes E_{p,x}^2 w'_x d\mu(x) \) are dense in \( \int_{[-1,1]} L_{p,x} d\mu(x) \otimes \int_{[-1,1]} L_{p,x} d\mu(x) \), so \( \int_{[-1,1]} D_{p,x}^2 \otimes E_{p,x}^2 d\mu(x) = \int_{[-1,1]} A_{p,x}^2 \otimes B_{p,x}^2 d\mu(x) \).

By Stone’s theorem and [15, Theorem 1.10], \( \int_{[-1,1]} D_{p,x} \otimes E_{p,x} d\mu(x) = \int_{[-1,1]} A_{p,x}^2 \otimes B_{p,x}^2 d\mu(x) \). Hence we see that there is a positive function \( c(p, x) \), such that

\[
D_{p,x} e_m^n = pq^n c(p, x) e_m^n,
E_{p,x} e_m^n = q^{-m} c(p, x) e_m^n.
\]

The function \( c(p, x) \) depends on the choice of the Plancherel measure \( \mu \), see [2, Theorem 3.4.1, property 6]. Desmedt [2, §3.5] obtains a similar result using summations for basic hypergeometric series, a method different from the one presented here. Note that this method also applies to discrete series corepresentations.

**APPENDIX A. APPENDIX**

**Direct integrals.** For the theory of direct integrals of bounded operators we refer to [4]. We use Lance’s definition of direct integrals [15].

**Lemma A.1.** Let \((X, \mu)\) be a standard measure space. Let \((H_p)_p\) and \((K_p)_p\) be measurable fields of Hilbert spaces. Let \((A_p)_p\) and \((B_p)_p\) be measurable fields of closed operators on \((H_p)_p\) and \((K_p)_p\) respectively. Let \((e^n_p)_p, n \in \mathbb{N}\) be a fundamental sequence for \((A_p)_p\) and let \((f^n_p)_p, n \in \mathbb{N}\) be a fundamental sequence for \((B_p)_p\). Set \(A = \int_X A_p d\mu(p), B = \int_X B_p d\mu(p), H = \int_X H_p d\mu(p)\) and \(K = \int_X K_p d\mu(p)\).

(a) \((A_p \otimes B_p)_p\) is a measurable field of closed operators.

(b) The countable set

\[
R = \left\{ (e^n_p \otimes f^m_p)_p \mid n, m \in \mathbb{N} \right\},
\]

is a fundamental sequence for \((A_p \otimes B_p)_p\).

(c) The set

\[
T = \operatorname{span}_C \left\{ \int_X \xi_p \otimes \eta_p d\mu(p) \mid \xi = \int_X \xi_p d\mu(p) \in D(A), \eta = \int_X \eta_p d\mu(p) \in D(B), \int_X (\xi_p \otimes \eta_p) d\mu(p) \in D(\int_X (A_p \otimes B_p) d\mu(p)) \right\},
\]

is a core for \(\int_X (A_p \otimes B_p) d\mu(p)\).
Proof. We first prove (a) and (b). By [4, II.1.8, Proposition 10], for $(\xi_p)_p, (\eta_p)_p$ measurable fields of vectors, there is a unique measurable structure so that $(\xi_p \otimes \eta_p)_p$ is a measurable field of vectors. We check (1) - (3) of [15, Remark 1.5, (1) - (3)].

(1) $(e^n_p \otimes f^m_p)_p$ is a $\mu$-measurable field of vectors and $e^n_p \otimes f^m_p \in \mathcal{D}(A_p \otimes B_p)$ for all $p$. The function

$$ p \mapsto \langle (A_p \otimes B_p)(e^n_p \otimes f^m_p), (e^n_p \otimes f^m_p) \rangle_p = \langle A_pe^n_p, e^n_p \rangle \langle B_pf^m_p, f^m_p \rangle,$$

is $\mu$-measurable, so (2) follows. For (3) fix a $p \in X$. By definition $\{e^n_p \mid n \in \mathbb{N}\}$ is a core for $A_p$ and $\{f^m_p \mid n \in \mathbb{N}\}$ is a core for $B_p$. Then it follows from [7, Lemma 11.2.29] that $\text{span}_C \{e^n_p \otimes f^m_p \mid n, m \in \mathbb{N}\}$ is a core for $A_p \otimes B_p$, so that $R$ is total in $\mathcal{D}(A_p \otimes B_p)$ with respect to the graph norm. In all, we have proved (a) and (b).

Using [4, II.1.3, Remarque 1], we may assume that $(e^n_p)_p$ (resp. $(f^m_p)_p$) satisfies $p \mapsto \| (e^n_p)_p \|$ (resp. $p \mapsto \| (f^m_p)_p \|$) is bounded and vanishes outside a set of finite measure. Let

$$ \lambda^{n,m}_p = (\max(1, \| (A_p \otimes B_p)(e^n_p \otimes f^m_p) \|, \| A_pe^n_p \|, \| B_pf^m_p \|))^{-1},$$

so $\lambda^{n,m}_p$ is measurable and $0 < \lambda^{n,m}_p \leq 1$. Using the assumption $\lambda^{n,m}_p(e^n_p \otimes f^m_p) \in T$. Moreover, $p \mapsto \| \lambda^{n,m}_p(e^n_p \otimes f^m_p) \|^2_{\text{Graph}(A_p \otimes B_p)}$ is bounded. Let $S = \{ (\lambda^{n,m}_p(e^n_p \otimes f^m_p))_p \mid n, m \in \mathbb{N}\} \subseteq T$. Now define

$$ M = \bigcup_{f \in \mathcal{C}} m_f S,$$

where $\mathcal{C}$ is the set of bounded measurable scalar-valued functions vanishing outside a set of finite measure and $m_f$ is multiplication by $f$. Then $M \subseteq T \subseteq \mathcal{D}(\int_X^\Theta (A_p \otimes B_p) d\mu(p))$ and by [4, II.1.6, Proposition 7], $M$ is total in $\mathcal{D}(\int_X^\Theta (A_p \otimes B_p) d\mu(p))$ equipped with the graph norm. Hence $T$ is a core for $\int_X^\Theta (A_p \otimes B_p) d\mu(p)$.

The quantum group analogue of the normalizer of $SU(1,1)$ in $SL(2,\mathbb{C})$. $(M, \Delta)$ denotes the quantum group analogue of the normalizer of $SU(1,1)$ in $SL(2,\mathbb{C})$. We use the same notation as in [9] and [6]. The Casimir operator $\Theta$ is defined in [6, Definition 4.5]. $\sigma(\Theta)$ and $\sigma_d(\Theta)$ denote the spectrum and the discrete spectrum of $\Theta$ respectively.

Proposition A.2. Let $x \in [-1,1]$ and $x' \in \sigma_d(\Theta)$, so in particular $x \neq x'$. Then the irreducible summands of $W_{p,x}$ are all inequivalent from $W_{p,x'}$.

Proof. This follows from [6], since the eigenvalues of $\Theta$ when restricted to $W_{p,x}$ are contained in $\mathbb{R} \setminus [-1,1]$, whereas for $W_{p,x}$ the eigenvalues of $\Theta$ are in $[-1,1]$.

The next propositions show that $(M, \Delta)$ satisfies the conditions of the Plancherel theorem, cf. Remark 2.4.

Proposition A.3. $M$ is a type I von Neumann algebra.

Proof. We start with some preliminary remarks. The projections in $M'$ correspond to the invariant subspaces of $W$ and the minimal projections in $M'$ correspond to the irreducible subspaces of $W$. The partial isometries in $M'$ correspond to intertwiners of closed subcorepresentations of $W$. 
Let $P \in \hat{M}'$ be the projection on $\bigoplus_{p \in \mathbb{Q}^2} \int_{[-1,1]} \mathcal{L}_{p,x}$. There are no intertwiners between closed subcorepresentations of $\bigoplus_{p \in \mathbb{Q}^2} \int_{[-1,1]} W_{p,x} \, dx$ and $\bigoplus_{p \in \mathbb{Q}^2} \int_{x \in \sigma_d(\Omega)} W_{p,x}$, see Proposition A.2. Therefore, $P$ commutes with every partial isometry in $\hat{M}'$ so that $P$ is central. We have $\hat{M}' = PM'P \oplus (1 - P)\hat{M}'(1 - P)$. The von Neumann algebra $(1 - P)\hat{M}'(1 - P)$ is of type $I$ since the direct sum decomposition $\bigoplus_{p \in \mathbb{Q}^2} \int_{x \in \sigma_d(\Omega)} W_{p,x}$ together with the preliminary remarks yield that every projection majorizes a minimal projection.

Now we prove that $PM'P$ is a type $I$ von Neumann algebra. Define the Hilbert spaces

$$\mathcal{L}_x = \left( \bigoplus_{p \in \mathbb{Q}^2} \mathcal{L}_{p,x} \right) \oplus \left( \bigoplus_{p \in \mathbb{Q}^2} \mathcal{L}_{p,-x} \right), \quad x \in (0,1); \quad \mathcal{L}_0 = \bigoplus_{p \in \mathbb{Q}^2} \mathcal{L}_{p,0}.$$  

Then,

$$PK = \int_{[0,1]} \mathcal{L}_x \, dx,$$

and we let $Z$ denote the diagonizable operators with respect to this direct integral decomposition.

We claim that $Z \subseteq \hat{M}' \subseteq Z'$. For the former inclusion, note that the stepfunctions in $Z$ are linear combinations of projections onto invariant subspaces for $\hat{M}$. By the preliminary remarks we find $Z \subseteq M'$. To prove that $M' \subseteq Z'$, note that by [6, Corollary 4.11], $M'$ is the $\sigma$-strong-* closure of the linear span of elements $JQ(p_1,p_2,n)J$, $p_1,p_2 \in \mathbb{Q}^2$, $n \in \mathbb{Z}$. The operators $Q(p_1,p_2,n)$ are decomposable with respect to the direct integral decomposition (A.1) as was proved in [6]; combine [6, Proposition 10.5] together with the direct integral decomposition [6, Theorem 5.7] and the definition of $Q(p_1,p_2,n)$ [6, Equation (20)]. We prove that $J$ is a decomposable operator with respect to (A.1). It suffices to show that $J \subseteq Z'$ [4, Theorem II.2.1].

Let $B \subseteq [0,1]$ be a Borel set and let $P_B \in Z$ be the operator $P_B = \int_{[0,1]} \chi_B(x) 1_{\mathcal{L}} \, dx$, where $\chi_B$ is the indicator function on $B$. $P_B$ is a projection and we have

$$\chi_{BJ-B}(\Omega)\mathcal{K} = \chi_{BJ-B}(\Omega) \bigoplus_{p,m,\epsilon,\eta} \mathcal{K}(p,m,\epsilon,\eta) =$$

$$\bigoplus_p \left( \bigoplus_{m,\epsilon,\eta=1} \int_{x \in BJ-B} \mathcal{C} \, dx \oplus \bigoplus_{m,\epsilon,\eta=-1} \int_{-x \in BJ-B} \mathcal{C} \, dx \right) =$$

$$\bigoplus_p \int_{x \in BJ-B} \mathcal{C} \, dx \oplus \bigoplus_p \int_{x \in BJ-B} \mathcal{L}_{p,x} \, dx = \int_{x \in BJ-B} \mathcal{L} \, dx = P_B \mathcal{K},$$

where the second equation uses [15, Theorem 1.10] and the fact that there is a direct integral decomposition $\mathcal{K}(p,m,\epsilon,\eta) = \int_{[0,1]} \mathcal{C} \, dx$ such that $\chi_B(\Omega)\mathcal{K}(p,m,\epsilon,\eta) = \int_{x \in BJ-B} \mathcal{C} \, dx$, see [6, Theorem 8.13]. Other equations are a matter of changing the order and combining direct integrals.
Note that \( \Omega \) leaves the spaces \( \mathcal{K}^+ \) and \( \mathcal{K}^- \) invariant. Let \( P^+ \) and \( P^- \) be the projections onto respectively \( \mathcal{K}^+ \) and \( \mathcal{K}^- \). Write, again using the notation of [6]

\[
\Omega = \begin{pmatrix} \Omega^+ & 0 \\ 0 & \Omega^- \end{pmatrix}, \quad \Omega_0 = \begin{pmatrix} \Omega_0^+ & 0 \\ 0 & \Omega_0^- \end{pmatrix},
\]

where \( \Omega^\pm = \Omega P^\pm \) and \( \Omega_0^\pm = \Omega_0 P^\pm \). Note that \( \Omega^\pm \) is a self-adjoint extension of \( \Omega_0^\pm \). By [6, Equation (11)] we see that \( \hat{J} \) leaves the spaces \( \mathcal{K}^+ \) and \( \mathcal{K}^- \) invariant. We claim that

(A.3) \( \hat{J}|_{\mathcal{K}^+} \Omega^+ \hat{J}|_{\mathcal{K}^+} = \Omega^+ \), \quad \hat{J}|_{\mathcal{K}^-} \Omega^- \hat{J}|_{\mathcal{K}^-} = -\Omega^-.

By [6, Equations (11) and (19)] we find that \( \hat{J}\Omega_0^+ j f_{m,p,t} = \text{sgn}(pt)\Omega_0^+ f_{m,p,t} \), so that \( \hat{J}\Omega_0^+ \hat{J} = \Omega_0^+ \) and \( \hat{J}\Omega_0^- \hat{J} = -\Omega_0^- \). Hence \( \hat{J}\Omega^+ \hat{J} \supseteq \Omega^+ \), and \( \hat{J}\Omega^- \hat{J} \supseteq -\Omega^- \). Let \( x \in \hat{M}' \), and write:

\[
\hat{j} x \hat{j} = y^+ \oplus y^- , \quad y^+ = \left( \begin{array}{c} y_1^+ \\ 0 \\ y_2^+ \end{array} \right) \in M_+, \quad y^- = \left( \begin{array}{c} 0 \\ y_1^- \\ 0 \end{array} \right) \in M_- ,
\]

where the decomposition is as in [6, Proposition 4.8]. By that same proposition, we find that \( y_1^+ \Omega^+ \subseteq -\Omega^- y_1^- \), \( y_2^- \Omega^- \subseteq -\Omega^+ y_2^- \), \( y_1^+ \Omega^+ \subseteq \Omega^+ y_1^+ \) and \( y_2^- \Omega^- \subseteq \Omega^- y_2^- \). This implies the inclusion in the following computation:

\[
x \hat{j} \left( \begin{array}{cc} \Omega^+ & 0 \\ 0 & -\Omega^- \end{array} \right) \hat{j} = \hat{j} x \hat{j} \left( \begin{array}{cc} \Omega^+ & 0 \\ 0 & -\Omega^- \end{array} \right) \hat{j} = \hat{j} \left( \begin{array}{c} y_1^+ \\ 0 \\ y_2^+ \end{array} \right) \Omega^+ \left( \begin{array}{c} y_1^- \\ 0 \\ y_2^- \end{array} \right) \left( \begin{array}{cc} \Omega^+ & 0 \\ 0 & -\Omega^- \end{array} \right) \Omega^+ \left( \begin{array}{c} y_1^- \\ 0 \\ y_2^- \end{array} \right) = \hat{j} \left( \begin{array}{c} y_1^+ \\ 0 \\ -\Omega^- \end{array} \right) \left( y_1^+ \oplus y^- \right) \hat{j} = \hat{j} \left( \begin{array}{cc} \Omega^+ & 0 \\ 0 & -\Omega^- \end{array} \right) \left( y_1^+ \oplus y^- \right) \hat{j} x.
\]

So \( \hat{J}\Omega^+ \hat{J} \oplus -\hat{J}\Omega^- \hat{J} \) is a self-adjoint operator affiliated to \( \hat{M} \) extending \( \Omega_0 \). So [6, Theorem 4.6] implies that \( (\hat{J}|_{\mathcal{K}^+} \Omega^+ \hat{J}|_{\mathcal{K}^+} \oplus -\hat{J}|_{\mathcal{K}^-} \Omega^- \hat{J}|_{\mathcal{K}^-}) = \Omega \), which results in (A.3).

To prove that \( \hat{J} \subseteq Z' \), it suffices to prove that for all Borel sets \( B \subseteq [0,1] \), \( \hat{J} P_B \hat{J} = P_B \).

Indeed we have

\[
\hat{j} P_B \hat{j} = \hat{j} \chi_{B \cup \bar{B}}(\Omega) \hat{j} = \hat{j} \chi_{B \cup \bar{B}}(\Omega^+) \hat{j} |_{\mathcal{K}^+} \oplus \hat{j} \chi_{B \cup \bar{B}}(\Omega^-) \hat{j} |_{\mathcal{K}^-} = \chi_{B \cup \bar{B}}(\Omega^+) \oplus \chi_{B \cup \bar{B}}(\Omega^-) = \chi_{B \cup \bar{B}}(\Omega) = P_B.
\]

The first and last equality are due to (A.2); the third equality is due to (A.3). In all, we have proved that \( Z \subseteq \hat{M} \subseteq Z' \).

Let \( W_x = \left( \bigoplus_{p \in \mathbb{Z}^q} W_{p,x} \right) \oplus \left( \bigoplus_{p \in \mathbb{Z}^q} W_{p,-x} \right) \) for \( x \in (0,1] \) and \( W_0 = \bigoplus_{p \in \mathbb{Z}^q} W_{p,0} \). The operators \( Q(p_1,p_2,n) \) form a countable family that generates \( \hat{M} \) [6, Proposition 4.9]. We apply [4, Theorem II.3.2] and its subsequent remark, together with [4, Theorem II.3.1] to conclude that

(A.4) \( P \hat{M} P = \int_{x \in [0,1]} \hat{M}_x dx \).
where $M_x$ is generated by $\{(\omega \otimes I)(W_x) \mid \omega \in M_\lambda\}$ almost everywhere. The projections in $M'_x$ correspond to irreducible subspaces of $W_x$. Since $W_x$ decomposes as a direct sum of irreducible corepresentations [6, Proposition 5.4], every projection in $M'_x$ majorizes a minimal projection. We find that $M'_x$ is type I and by [7, Theorem 14.1.21], [16, Corollary V.2.24] and (A.4) we conclude that $PMP$ is type I.

**Proposition A.4.** $M_c$ is separable.

**Proof.** Note that if $\omega_n \in M_\lambda$ is sequence that converges in norm to $\omega \in M_\lambda$, then $\|\lambda(\omega_n) - \lambda(\omega)\| \leq \|\omega_n - \omega\|$ so that $\lambda(\omega_n)$ converges in norm to $\lambda(\omega)$. Since the norm on $M_c$ is the operator norm on the GNS-space and $M_c$ is the C*-algebra obtained as the closure of $\{\lambda(\omega) \mid \omega \in M_\lambda\}$. It suffices to check that $M_\lambda$ is separable. The Q-linear span of $\{\omega_{f_{m_0,p_0,t_0},f_{m_1,p_1,t_1}} \mid m_i \in \mathbb{Z}, p_i, t_i \in I_q, i = 0, 1\}$ is weakly dense, hence norm dense in $M_\lambda$.

**References**


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