Lambda-conductors for group rings

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1 Introduction.

This paper is part of a project which aims to provide a method for computing the Nil groups of the group rings of finite abelian groups, by refining some of the techniques used in [1] and [2] in such a way that the allowed coefficient rings include polynomial rings. For the refinement of the p-adic logarithm discussed in [3] and [4] it is assumed that the rings involved have a structure of λ-ring; we refer to these papers for generalities about λ-rings. Thus it is useful to extend as much as possible of the other techniques to the context of λ-rings. In this paper we investigate how to describe a group ring of a finite abelian group as a pull back of a diagram of rings which are more accessible to calculations in algebraic K-theory.

Let be given a commutative ring $S$ and subring $R$. For each ideal $I$ of $S$ which is contained in $R$ one has a cartesian square

$$
\begin{array}{ccc}
R & \longrightarrow & S \\
\downarrow & & \downarrow \\
R/I & \longrightarrow & S/I
\end{array}
$$

thus describing $R$ as a pull back of rings for which the $K$-theory is hopefully better understood. By taking for $I$ the sum of all such ideals one finds a diagram where the rings on the bottom row are as small as possible.

We modify this construction by assuming that $R$ has a structure of λ-ring and considering only ideals $I$ stable under the λ-operations. We call the resulting ideal the λ-conductor of $S$ into $R$. In particular we are interested in the case that $R$ is the group ring $\mathbb{Z}[G]$ of a finite abelian group, and $S$ is its normal closure in $R \otimes \mathbb{Q}$, which splits as a direct sum of rings $S_i = \mathbb{Z}[\chi_i]$ associated to equivalence classes of characters $\chi_i : G \rightarrow \mathbb{C}$.

In this situation $R$ is a λ-ring such that $\psi^n(g) = g^n$ for every $n \in \mathbb{N}$ and $g \in G$. In general $S$ is not stable under the λ-operations on $R \otimes \mathbb{Q}$, but it is stable under the associated Adams operations $\psi^n$ since they are ring homomorphisms.

We will prove that in case $G$ is a primary group its λ-conductor is precisely the intersection of the classical conductor and the augmentation ideal. We do this by exhibiting generators of the classical conductor and examining their behavior under the fundamental λ-operations.
2 The primary case

Throughout this section $G$ is a group of order $n = p^k$, where $p$ is prime. We consider representations $\rho : G \to \mathbb{C}^*$. We say that $\rho$ is of level $k$ if the image of $\rho$ has $p^k$ elements.

Two representations $\tau_1$ and $\tau_2$ are called equivalent if they have the same kernel. That means that there must be some $m \in \mathbb{Z}$ prime to $p$ such that $\tau_2(x) = \tau_1(x)^m$ for all $x \in G$. Obviously equivalent representations have the same level.

Given a representation $\tau$ of level $k > 0$ one gets a representation $\psi \tau$ of level $k - 1$ by the formula $(\psi \tau)(x) = \tau(x^p)$ for $x \in G$. If $\psi \tau_1$ and $\psi \tau_2$ are equivalent then we may replace $\tau_2$ by an equivalent representation $\tau'_2$ so that $\psi \tau'_2 = \psi \tau_1$. So we may choose a representation in each class in such a way that $\psi \tau$ and $\rho$ coincide if they are equivalent.

Let $\rho$ be a representation of level $k > 0$ and write $\omega = \exp(2\pi i / p)$. We define an element $b_\rho \in \mathbb{Z}[G]$ by the formula

$$b_\rho = \sum_{\xi=1}^{\rho(x)} x - \sum_{\rho(x) \neq \omega} \xi.$$ 

If we choose $y_\rho \in G$ such that $\rho(y) = \omega$ then we get

$$b_\rho = \left( \sum_{\rho(x)=1} x \right) (1 - y_\rho).$$

The only representation of level 0 is the trivial representation, which we denote by 1, and it gives rise to $b_1 = \sum_{x \in G} x$.

**Proposition 1.** If $\rho$ and $\tau$ are not equivalent then $b_\rho b_\tau = 0$. Furthermore $b_\rho^2 = p^{k-1} (1 - y_\rho) b_\rho$ for $\rho$ of level $k > 0$.

**Proof.** If $\ker(\rho) \neq \ker(\tau)$ we may assume that there is $g \in G$ with $\rho(g) = 1$ but $\tau(g) = \omega$. If $g$ has order $m$ then $\sum_{\rho(x)=1} x$ and thus $b_\rho$ is a multiple of $\sum_{j=0}^{m-1} g^j$, whereas $b_\tau$ is a multiple of $1 - g$. The product of these two factors is 0.

The second part follows from $(\sum_{\rho(\xi)=1} \xi)(\sum_{\rho(x)=1} x) = p^{n-k} \sum_{\rho(x)=1} x$ which is true because each $\xi$ gives the same contribution and there are $p^{n-k}$ of them. \hfill \Box

Every representation $\rho$ of level $k$ gives rise to a homomorphism $j_\rho$ from $\mathbb{Z}[G]$ to $S_\rho = \mathbb{Z}[\omega_k]$, where $\omega_k = \exp(2\pi i / p^k)$.

**Proposition 2.** If $\rho$ and $\tau$ are not equivalent then $j_\rho(b_\tau) = 0$. Furthermore $j_1(b_1) = p^n$, and $j_\rho(b_\rho) = p^{k-1} (1 - \omega)$ if $\rho$ is of level $k > 0$.

**Proof.** The second part is obvious since every $x$ in the definition of $b_\rho$ maps to 1, and $y_\rho$ maps to $\omega$. The first part follows since $j_\rho(b_\rho) j_\rho(b_\tau) = 0$ by Proposition 1 and $S_\rho$ is a domain. \hfill \Box
It is well known that the maps \( j_\rho \) (one from each equivalence class) combine to an embedding from \( R \) into its integral closure \( S = \oplus_\rho S_\rho \).

**Proposition 3.** The \( b_\rho \) generate the conductor ideal \( I \) of \( S \) into \( R \).

**Proof.** By the theorem of Jacobinski (Theorem 27.8 in [5]) the conductor is \( \oplus_\rho nD_\rho^{-1} \subset \oplus_\rho S_\rho = S \), where \( D_\rho^{-1} \) is the lattice in \( \mathbb{Q}[\omega_k] \) dual to \( \mathbb{Z}[\omega_k] \) under the trace form. It is a simple exercise that this fractional ideal is in fact generated by \( p^{-k}(1 - \omega) \), which means that \( nD_\rho^{-1} \) is just \( j(b_\rho)S = j(B_\rho)S_\rho = j(b_\rho R) \). \( \Box \)

We remind the reader that in particular \( nS \) is contained in the conductor.

The \( \lambda \)-conductor \( I_\lambda \) from \( S \) into \( R \) is defined as the largest ideal of \( S \) contained in \( R \) which is stable under the fundamental \( \lambda \)-operations \( \theta^\rho \). It is of course a subset of the largest ideal of \( S \) contained in \( R \), which is the ordinary conductor \( I \) described above. Thus we have to investigate the behaviour of the operations \( \theta^\rho \) on the generators \( b_\rho \).

**Lemma 1.** If \( \rho \) is of level \( k > 0 \) and there is no \( \tau \) with \( \psi \tau = \rho \) then \( \psi^\rho b_\rho = 0 \).

**Proof.** Write \( G \) as a direct product of cyclic groups, with generators \( g_i \). If the order of \( \rho(g_i) \) is strictly smaller than the order of \( g_i \) for all \( i \) then one can find a suitable \( \tau \) by taking for each \( \tau(g_i) \) a \( p \)-th root of \( \rho(g_i) \). If however the orders are the same for some \( i \) then there is certainly some \( h \in G \) such that \( \rho(h) = \omega \). By definition of \( b_\rho \) we have

\[
\psi^\rho b_\rho = \sum_{\rho \in \Xi} \xi^\rho - \sum_{\rho \in \omega} \eta^\rho
\]

Here the term in the second sum associated to \( \eta = h\xi \) cancels the term in the first sum associated to \( \xi \). \( \Box \)

**Proposition 4.** If \( \rho \) is of level \( k > 0 \) then

\[
\psi^\rho(b_\rho) = \sum_{\psi \tau = \rho} \psi^\rho \tau
\]

**Proof.** By definition we have

\[
\sum_{\psi \tau = \rho} b_\tau = \sum_{\psi \tau = \rho} \sum_{\tau x = 1} x - \sum_{\psi \tau = \rho} \sum_{\tau x = \omega} x
\]

We claim that all terms with \( x \notin G^\rho \) cancel. To prove this assume that the class of \( x \) in \( G/G^\rho \) is nontrivial. Then there exists a homomorphism \( \sigma : G/G^\rho \to \mathbb{C}^* \) such that \( \sigma(x) = \omega \). Now the term associated to \( \tau \) in the first sum equals the term associated to \( \tau' = \tau \cdot \sigma \) in the second sum.

So we only have to consider terms of the form \( x = \xi^\rho \) with \( \xi \in G \). The condition \( \tau(x) = 1 \) is then independent of \( \tau \) (since it is equivalent to \( \rho(\xi) = 1 \)) and the sum over all \( \tau \) with \( \psi \tau = \rho \) reduces to a multiplication with the number.
of equivalence classes of such \( \tau \). By the Lemma we may assume that this number is nonzero. Now \( \tau_1 \) and \( \tau_2 \) with \( \psi \tau_1 = \rho = \psi \tau_2 \) are equivalent iff \( \tau_2 = \tau_1^{1+m} \rho^k \) for some \( m \) with \( 0 \leq m < p \). So this number is \( 1/p \) times the number of homomorphisms \( \sigma: G/G^p \to C^* \), hence equals \( p^{r-1} \), where \( r \) denotes the rank of \( G \).

On the other hand we have

\[
\psi^p b_\rho = \sum_{\xi=1}^{\xi^p} \xi^p - \sum_{\rho \neq \omega} \rho^p
\]

Here the first sum is a certain factor times the sum over all \( x \in G \) for which there exists \( \xi \in G \) with \( x = \xi^p \) and which satisfy \( \tau(x) = 1 \) for any (and thus all) \( \tau \) with \( \psi \tau = \rho \). The factor is the number of \( \xi \) which satisfy these conditions, which equals \( p^r \).

We write \( h \) for the polynomial of degree \( p - 2 \) given by

\[
h(t) = \frac{1}{1-t} \left( p - \frac{1-t^p}{1-t} \right)
\]

**Proposition 5.**

\[
\psi^p(b_1) = b_1 + \sum_{\tau \neq 1, \psi \tau = 1} h(y_\tau) b_\tau
\]

**Proof.** We have

\[
b_\tau = \left( \sum_{x=1}^x \right) (1 - y_\tau)
\]

and thus

\[
h(y_\tau) b_\tau = \left( \sum_{x=1}^x \right) \left( p - \frac{p^1 - \sum_{j=0}^p y_j}{p-1} \right) = p \left( \sum_{x=1}^x \right) - \left( \sum_{x \in G} x \right)
\]

We must take the sum of \( \sum_{\tau=1}^{\tau x=1} x \) over all equivalence classes of \( \tau \neq 1 \) with \( \psi \tau = 1 \). Interchange the sum over \( x \) and the sum over \( \tau \). There are two cases:

- If \( x \in G^p \) then \( \tau x = 1 \) for all \( \tau \), and we must simply count the number of equivalence class of \( \tau \). There are \( p^r - 1 \) of them, with \( p - 1 \) in each class.
- If \( x \notin G^p \) then the number of \( \tau \) such that \( \tau x = 1 \) is \( p^{r-1} - 1 \), with again \( p - 1 \) in each class.

So we get

\[
\sum_{\tau \neq 1, \psi \tau = 1} h(y_\tau) b_\tau = p \left( \frac{p^r - 1}{p-1} \sum_{x \in G^p} x + \frac{p^{r-1} - 1}{p-1} \sum_{x \notin G^p} x \right) - \left( \frac{p^r - 1}{p-1} \sum_{x \in G^p} x + \frac{p^{r-1} - 1}{p-1} \sum_{x \notin G^p} x \right)
\]

\[
= p^{r} \sum_{x \in G^p} x - \sum_{x \in G} x - \sum_{\xi \in \xi^p} \xi^p - \sum_{x \in G} x = \psi^p b_1 - b_1
\]

4
Now we consider the effect of Adams operations $\psi^q$ for primes $q \neq p$. For any prime $q$ we write $f_q$ and $g_q$ for the polynomials given by

$$f_q(t) = \frac{1 - t^q}{1 - t}, \quad g_q(t) = \frac{(1 - t)^{q-1} - f_q(t)}{q}$$

**Proposition 6.** If $\rho$ is of level $k > 0$ then

$$\psi^q(b_\rho) = f_q(y_\rho) b_\rho$$

and $\psi^q(b_1) = b_1$.

**Proof.** We have $\psi^q(b_1) = \psi^q \left( \sum_{\xi \in \mathcal{G}} \xi \right) = \sum_{\xi \in \mathcal{G}} \xi = b_1$ and

$$\psi^q(b_\rho) = \psi^q \left( \sum_{\rho \neq 1} x \right) \left( 1 - y_\rho \right) = \left( \sum_{\rho \neq 1} x^q \right) \left( 1 - y_\rho^q \right) = \left( \sum_{\rho \neq 1} \xi \right) \left( 1 - y_\rho \right) f_q(y_\rho) = b_\rho f_q(y_\rho)$$

\[\square\]

**Corollary 1.** For the idempotents $e_\rho \in S_\rho$ one has

$$\psi^p(e_\rho) = \sum_{\psi_\tau = \rho} e_\rho \quad \text{if } \rho \neq 1,$$

$$\psi^p(e_1) = e_1 + \sum_{\tau \neq 1, \psi_\tau = 1} e_\tau$$

$$\psi^q(e_\rho) = e_\rho \quad \text{if } q \neq p$$

Since $R$ has no $\mathbb{Z}$-torsion the Adams operations $\psi^q$ determine the operations $\theta^q$ and we find

**Proposition 7.** If $\rho$ has level $k > 0$ then

$$\theta^p(b_\rho) = p^{(e-k)(p-1)-1}(1 - y_\rho)^{p-1} b_\rho - \sum_{\psi_\tau = \rho} b_\tau \quad \text{if } k < e$$

$$\theta^p(b_\rho) = g_p(y_\rho) b_\rho \quad \text{if } k = e$$

$$\theta^q(b_\rho) = \left( \frac{p^{(e-k)(q-1)} - 1}{q} (1 - y_\rho)^{q-1} + g_q(y_\rho) \right) b_\rho \quad \text{for } q \neq p$$

Moreover

$$\theta^p(b_1) = p^{e(p-1)-1} b_1 - p^{-1} b_1 - p^{-1} \sum_{\tau \neq 1, \psi_\tau = 1} h(y_\tau) b_\tau$$

$$\theta^q(b_1) = \frac{p^{e(q-1)} - 1}{q} b_1 \quad \text{for } q \neq p$$
Proof. This is just a matter of combining the last three Propositions with the formula $\ell \theta^\ell(a) = a^\ell - \psi^\ell a$. Note that $k = e$ can only happen if $G$ is cyclic, in which case $y_p^p = 1$, which implies that $b_p^p = (1 - y_p)^p = p(1 - y_p)g_p(y_p)$. \qed

**Theorem 1.** The $b_p$ with $p \neq 1$ generate the $\lambda$-conductor ideal $I_\lambda$. In other words $I_\lambda$ is the intersection of the augmentation ideal and the ordinary conductor ideal $I$.

**Proof.** Write $J$ for the $R$-ideal generated by the $b_p$ with $p \neq 1$. From Proposition 7 one reads of that $\theta^\ell(b_p) \in J$ for $p \neq 1$ and for every prim $\ell$. From the identity

$$
\theta^\ell(ab) = \theta^\ell(a)b^\ell + \psi^\ell(a)\theta^\ell(b)
$$

it then follows that $\theta^\ell(Rb_p) \subset J$ for $p \neq 1$ and all $\ell$. Finally from

$$
\theta^\ell(u + v) = \theta^\ell(u) + \theta^\ell(v) + \sum_{i=1}^{\ell-1} \frac{1}{\ell} \binom{\ell}{i} u^i v^{\ell-i}
$$

it follows that $\theta^\ell(J) \subset J$ for every $\ell$. Since $J \subset I$ by Proposition 3 this shows that $J \subset I_\lambda$.

Suppose that $x \in I_\lambda$ and $x \notin J$. Then $x \in I$, so by Proposition 3 there are $x_p \in R$ such that $x = \sum_{p \neq 1} x_p b_p$. Since $\sum_{p \neq 1} x_p b_p \in J \subset I_\lambda$ by the first half of the proof, it follows that $x_1 b_1 \in I_\lambda$. Since $gb_1 = b_1$ for every $g \in G$ we may assume that $x_1 \in \mathbb{Z}$. Moreover $x_1 \neq 0$ which means that its $p$-valuation $v_p(x_1)$ is a natural number. We may assume that $x$ is chosen in such a way that $v_p(x_1)$ is minimal. Now $I_\lambda$ must also contain

$$
\theta^p(x_1 b_1) = p^{-1} \left( x_1^p p^{p-1} b_1 - x_1 b_1 + \sum_{\tau \neq 1, \psi = 1} b_{\tau} \right)
$$

However the valuation of the coefficient of $b_1$ is $v_p(x_1^p p^{p-1} x_1) - 1 = v_p(x_1) - 1$, in contradiction with the way $x$ was chosen. Thus $I_\lambda \subset J$. \qed

## 3 Direct products of relatively prime order

Let $G_1$ be a group of order $n_1 = p^e$, and let $G_2$ a group of order $n_2 = q^f$, where $p$ and $q$ are different primes. We write $R_1 = \mathbb{Z}[G_1]$ and $R_2 = \mathbb{Z}[G_2]$, and denote their normal closures by $S_1$ and $S_2$ respectively. Finally we write $I_1$ for the conductor of $S_1$ into $R_1$ and $I_2$ for the conductor of $S_2$ into $R_2$. Since the $S_i$ are free abelian groups, the same is true for the other additive groups involved, and we can view $I_1 \otimes I_2$ as a subgroup of $R_1 \otimes I_2$ and of $R_1 \otimes R_2$.

**Lemma 2.**

$$
I_1 \otimes I_2 = (R_1 \otimes I_2) \cap (I_1 \otimes R_2)
$$
Proof. There are $m_1, m_2 \in \mathbb{Z}$ such that $m_1n_1 + m_2n_2 = 1$. If $x$ is an element of the left hand side then $x \in R_1 \otimes I_2$, so $n_1x \in n_1R_1 \otimes I_2 \subset n_1S_1 \otimes I_2 \subset I_1 \otimes I_2$ and therefore $m_1n_1x \in I_1 \otimes I_2$. Similarly $m_2n_2x \in I_1 \otimes I_2$ and thus $x = m_1n_1x + m_2n_2x \in I_1 \otimes I_2$. The other implication is obvious.

**Proposition 8.** The conductor $I$ of $S_1 \otimes S_2$ into $R_1 \otimes R_2$ is $I_1 \otimes I_2$.

Proof. Suppose that $x \in I$, so that $x(S_1 \otimes S_2) \subset R_1 \otimes R_2$. We write $x \in R_1 \otimes R_2$ as $\sum x_g \otimes g$, where $g$ runs through $G_2$. For any $a \in S_1$ we have $\sum (x_ga) \otimes g = (\sum x_g \otimes g)(a \otimes 1) = x(a \otimes 1) \in R_1 \otimes R_2$. Therefore $x_ga \in I_1$ for any $a \in S_1$, which means that $a_g \in I_1$ for all $g \in G_1$. Thus $x \in I_1 \otimes R_2$. Similarly $x \in R_1 \otimes I_2$. Thus $x \in I_1 \otimes I_2$ by the Lemma. The other inclusion is obvious.

We show now that for the $\lambda$-conductor a similar theorem holds:

**Theorem 2.** The $\lambda$-conductor $I_\lambda$ of $S_1 \otimes S_2$ into $R_1 \otimes R_2$ is the tensor product of the $\lambda$-conductors $I_{1\lambda}$ of $S_1$ into $R_1$ and $I_{2\lambda}$ of $S_2$ into $R_2$.

Proof. The $\lambda$-conductor $I_\lambda$ is a subset of the classical conductor $I$, which is $I_1 \otimes I_2$. However $I_1$ is the direct sum $\mathbb{Z}b_1 \oplus I_{1\lambda}$ by theorem 1. and similarly for $I_2$. Thus any $x \in I_\lambda$ can uniquely be written as

$$x = x_0(b_1 \oplus b_1) \oplus (x_1 \oplus x_1) \oplus (b_1 \otimes x_2) \oplus y$$

with $x_0 \in \mathbb{Z}$, $x_1 \in I_{1\lambda}$, $x_2 \in I_{2\lambda}$, $y \in I_{1\lambda} \otimes I_{2\lambda}$. Since $I_\lambda$ is an ideal of $S_1 \otimes S_2$, each of these four summands must be in $I_\lambda$.

Therefore we consider the intersection of $I_\lambda$ with $b_1 \otimes I_{2\lambda}$. Suppose that $a$ is an element of this intersection, say $a = b_1 \otimes x$ with $x \in I_{2\lambda}$. Then $\theta^p(a) \in I_\lambda$ too. We have

$$\theta^p(a) = p^{-1}(a^p - \psi^p a) = p^{-1}(p^{\epsilon(p-1)}b_1 \otimes x^p - (b_1 + \sum_{\tau \neq 1, \psi \tau = 1} b_{\tau}) \otimes \psi^p x)$$

and thus $p^{\epsilon(p-1)}b_1 \otimes x^p - p^{-1}b_1 \otimes \psi^p x$ should be in $I_\lambda$. Now the first term is a multiple of $a$ and thus in $I_\lambda$. So the other term $p^{-1}b_1 \otimes \psi^p x$ is in the aforementioned intersection. Since $\psi^p$ is an automorphism (of finite order) of $R_2$ this shows that the intersection is $p$-divisible. Since the intersection is a finitely generated abelian group this can only happen if it vanishes.

The same argument applies to the first and second summand of $x$. Thus $x = y \in I_{1\lambda} \otimes I_{2\lambda}$ and we have shown that $I_\lambda \subset I_{1\lambda} \otimes I_{2\lambda}$. The other inclusion is obvious.

**References**


