Homogeneous Keller maps

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Homogeneous Keller maps

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door

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Voor Kees-Harm

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Preface

This theses has come from a research project during the last years. It is very technical in nature and not written as a novel. Nonetheless I have done my best to make it interesting for non-specialists as well, for chapters 1 and 2 and appendix B are meant for mathematicians in general. Section 1.3 does even have a narrative style. In between chapter 2 and appendix B, it is really technical for mathematicians outside the area of specialization, but maybe there is still a piece that appeals to the reader. At last, there is a Curriculum Vitae that can be read by everyone.

When I started as a Ph.D. student under the supervision of Arno van den Essen, it was not because the Jacobian conjecture was very appealing to me. Furthermore, I had specialized myself more in discrete mathematics, computer algebra and algorithmics in my previous education. No, the reason that I started working with Arno was that he was known as a very good and stimulating advisor. This was supported by my experiences with him during my study. As a matter of that, he had good contact with his students and during the breaks of his lectures, he was able to narrate very enthousiastically about the newest developments concerning the Jacobian conjecture, like the enormous calculation by his former Master student Engelbert Hubbers.

Arno has more than fulfilled his reputation as an animating supervisor, and for that I wish to thank him very much indeed. Besides of that, I wish to thank three other persons especially much. At first my coach Kees-Harm Korrelboom, who has supported me and stood by me all these years. Unfortunately, this is not possible any more, for he is no longer with us. Next I wish to express the utmost gratitude to my parents, Koos de Bondt and Irene Boon, because they both have supported me very much, in their own ways, with the creation of this thesis. Furthermore, I wish to thank my brother Pepijn de Bondt and his wife Bernadette Janssen for their moral support.

In addition, my gratitude goes to the following persons, in alphabetical order: Harm Derksen, Engelbert Hubbers, Drew Lewis, Stefan Maubach, Roel Willems, David Wright, Gaetano Zampieri, and last but not least, Wenhua Zhao, who took the efforts of reading this thesis, for their exertions and the valuable comments they provided. Besides of that, I wish to thank my roommates, former roommates and other colleages of the department of mathematics for the pleasant time during my research.

At last, I wish to thank He Tong for sharing some of his preprints, which have led to some of the results in this thesis.

Voorwoord

Dit proefschrift is voortgekomen uit een onderzoeksproject gedurende de afgelopen jaren. Het is erg technisch van aard en niet geschreven als een roman. Toch heb ik mijn best gedaan het ook interessant te maken voor niet-specialisten, want hoofdstuk 1 en 2 en appendix B zijn bedoeld voor wiskundigen in het algemeen. Paragraaf 1.3 heeft zelfs wel een verhalende stijl. Tussen hoofdstuk 2 en appendix B in is het erg technisch voor wiskundigen buiten het vakgebied, maar misschien zit er nog wel een stukje bij dat de lezer aanspreekt. Tot slot is er een Curriculum Vitae dat voor iedereen te lezen is.

Toen ik begon als junior onderzoeker bij Arno van den Essen, was dat niet omdat het Jacobivermoeden (eigenlijk het Jacobiaanvermoeden) me zo aansprak. Verder had ik me in mijn vooropleiding meer gespecialiseerd in discrete wiskunde, computeralgebra en algoritmiek dan in affiene algebraische meetkunde. Nee, de reden dat ik bij Arno begon was dat hij bekend stond als een zeer goede, stimulerende begeleider. Dit werd onderschreven door mijn ervaringen met hem tijdens mijn studie. Zo had hij goed contact met zijn studenten en kon hij gedurende de pauzes van de colleges heel enthousiast vertellen over de nieuwste ontwikkelingen aangaande het Jacobivermoeden, zoals de enorme berekening door zijn toenmalige doctoraalstudent Engelbert Hubbers.

Arno heeft zijn reputatie als animerende begeleider meer dan waargemaakt en daarvoor wil ik hem heel erg bedanken. Daarnaast wil ik nog drie andere mensen extra bedanken. Ten eerste mijn coach Kees-Harm Korrelboom, die mij in al die jaren gesteund en bijgestaan heeft. Helaas is dit niet meer mogelijk, want hij is niet meer bij ons. Vervolgens wil ik mijn ouders, Koos de Bondt en Irene Boon, enorm bedanken, omdat ze mij allebei op hun eigen manier heel erg gesteund hebben bij de totstandkoming van dit proefschrift.

Verder wil ik mijn broer Pepijn de Bondt en zijn vrouw Bernadette Janssen bedanken voor hun morele steun.

Ook gaat mijn dank uit naar volgende personen, in alfabetische volgorde: Harm Derksen, Engelbert Hubbers, Drew Lewis, Stefan Maubach, Roel Willems, David Wright, Gaetano Zampieri en 'last but not least' Wenhua Zhao, die zich de moeite hebben getroost om dit proefschrift door te lezen, voor hun inspanningen en de waardevolle aanmerkingen die ze hebben geleverd. Daarnaast wil ik mijn kamergenoten, oud-kamergenoten en andere collega's van de afdeling wiskunde bedanken voor de prettige tijd tijdens mijn onderzoek.

Tot slot wil ik He Tong bedanken voor het mij doen toekomen van enkele van zijn preprints, welke hebben geleid tot enkele van de resultaten in dit proefschrift.

Summary

The Jacobian conjecture

The Jacobian conjecture has been formulated by O.H. Keller in 1939. To describe it, let A be a commutative ring and assume $F = (F_1, F_2, \ldots, F_n)$ such that $F_i \in A[x] = A[x_1, x_2, \ldots, x_n]$ for all i. Define the Jacobian $\mathcal{J}F$ of F by

$$\mathcal{J}F := \begin{pmatrix} \frac{\partial}{\partial x_1}F_1 & \frac{\partial}{\partial x_2}F_1 & \cdots & \frac{\partial}{\partial x_n}F_1 \\ \frac{\partial}{\partial x_1}F_2 & \frac{\partial}{\partial x_2}F_2 & \cdots & \frac{\partial}{\partial x_n}F_2 \\ \vdots & \vdots & & \vdots \\ \frac{\partial}{\partial x_1}F_n & \frac{\partial}{\partial x_2}F_n & \cdots & \frac{\partial}{\partial x_n}F_n \end{pmatrix}$$

Since $\mathcal{J}F$ is a square matrix, one can compute its determinant. We say that F is a Keller map or that F satisfies the Keller condition if $\det \mathcal{J}F$ is a unit in $A[x]$. If A is a reduced ring, then the units in $A[x]$ are exactly those in A. Now the Jacobian conjecture asserts that F is an automorphism in case F is a Keller map, i.e. there exists a polynomial map $G = (G_1, G_2, \ldots, G_n)$ such that $G_i \in A[x] = A[x_1, x_2, \ldots, x_n]$ for all i and the composition map $G(F)$ is the identity map.

Keller's orginal formulation was for the ring $A = \mathbb{Z}$ only. Later on, the question was generalized to rings A that are contained in a commutative ring with $\mathbb{Q}$, and nowadays, a field of characteristic zero is taken for A in the above formulation in most cases, especially $A = \mathbb{C}$. This is only a matter of taste, because it has already been proved that the Jacobian conjecture is equivalent for all rings A that are contained in a commutative ring with $\mathbb{Q}$.

The Jacobian conjecture does not hold for commutative rings A in general. For instance $F = x_1 - x_1^q$ is a Keller map in dimension 1 over $\mathbb{F}_q$, the finite field of q elements, but $F(\mathbb{F}_q) = 0$. For more information about the Jacobian

Conjecture, I refer to section 1.2 in the introduction.

Results of this thesis

Most of the results are over the base ring $\mathbb{C}$, although many of them can easily be generalized to arbitrary fields of characteristic zero. Some results are proved for other base rings as well, but all results listed below are over $\mathbb{C}$ unless specified otherwise.

In chapter 2, the Jacobian conjecture is reduced to the case that the Jacobian of the map at hand has some symmetry. It is shown for instance that it suffices to prove the Jacobian conjecture for maps $F = x + H$ such that the symmetry group of $\mathcal{J}H$ is the full dihedral group of the square. Over the field $\mathbb{R}$, the symmetry group may be any subgroup of the full dihedral group of the square that contains a reflection, however not one that contains the reflection in the main diagonal. For some other types of symmetries of $\mathcal{J}H$, it is shown that the Jacobian conjecture is trivially satisfied for polynomial maps $F = x + H$. If for example $F = x + H$ such that $\mathcal{J}H$ is symmetric in horizontal direction and antisymmetric in vertical direction or vice versa, then F is invertible. In fact, F is a so-called quasi-translation in that case.

In chapter 3, quasi-translations are studied. Quasi-translations are polynomial automorphisms $x + H$ with inverse $x - H$. Furthermore, quasi-translations are unipotent Keller maps, i.e. $\mathcal{J}H$ is nilpotent, and there is a connection between quasi-translations and singular Hessians. Especially linear dependence between the components of H is investigated. All quasi-translations $x + H$ in dimensions ≤ 3 and all quasi-translations $x + H$ with $\mathrm{rk}\mathcal{J}H = 1$ are classified. It is shown that for these quasi-translations $x + H$, the components of H are linearly dependent over $\mathbb{C}$. But in dimension 4 and up, examples of quasi-translations $x + H$ are given for which the components of H are not linearly dependent over $\mathbb{C}$.

For homogeneous H, all quasi-translations $x + H$ in dimensions 3 and 4 and all quasi-translations $x + H$ with $\mathrm{rk}\mathcal{J}H = 2$ are classified. It is shown that for these quasi-translations $x + H$, there exists even two independent linear relations over $\mathbb{C}$ between the components of H. But for dimension 6 and up, homogeneous H with linearly independent components over $\mathbb{C}$ are given such that $x + H$ is a quasi-translation. Furthermore, quasi-translations $x + H$ in dimension 5 with H homogeneous are studied. Those with $\mathrm{rk}\mathcal{J}H \ge 3$ are classified into three existing types and for two of these types, the components of H are linearly dependent over $\mathbb{C}$.

In chapter 4, the existence of linear dependences between the components of homogeneous H with $\mathcal{J}H$ nilpotent is investigated. It is shown that the components of such H are linearly dependent over $\mathbb{C}$ in case the dimension is at most 4 and $\operatorname{rk}\mathcal{J}H \leq 2$. Furthermore, it is shown that homogeneous H in dimension 3, such that $\mathcal{J}H$ is nilpotent, are linearly triangularizable over $\mathbb{C}$. In order to obtain these results, a formula is given for homogeneous H such that $\operatorname{rk}\mathcal{J}H \leq 2$.

But for dimension 5, a homogeneous H of degree 6 with linearly independent components over $\mathbb{C}$, such that $\mathcal{J}H$ is nilpotent, is constructed. For dimension 9 and up, cubic homogeneous H with linearly independent components over $\mathbb{C}$, for which $\mathcal{J}H$ is nilpotent, are given. The first such H in dimension 9 was made by Gaetano Zampieri, out of a similar H in dimension 10 by the author. The latter H was the first counterexample of degree 3 to the dependence problem.

Furthermore, all quadratic homogeneous maps H in dimension 5 with $\mathcal{J}H$ nilpotent are being classified. In particular, it is shown that $x + H$ is tame and that the components of H are linearly dependent over $\mathbb{C}$. For a large part, this is done with computations in Maple, a computer algebra program. How the computations are performed is described in appendix A. Another computation that is described is that of maps H of degree 4 in dimension 3 such that $\mathcal{J}H$ is nilpotent.

In chapter 5, Hessians of small rank and nilpotent Hessians are investigated. It is shown that homogeneous h for which $\mathcal{H}h$ has rank r can always be expressed as a polynomial in r linear forms, if and only if $r \leq 3$. Furthermore, all h for which $\mathcal{H}h$ has rank $r \leq 2$ are being classified, as well as all homogeneous h in dimension 5 for which $\det \mathcal{H}h = 0$. In addition, singular Hessians in dimension 4 are studied. The results about quasi-translations are used to obtain these results, by way of the connection between quasi-translations and singular Hessians mentioned previously.

For nilpotent Hessians in dimensions ≤ 4 and homogeneous nilpotent Hessians in dimensions ≤ 5, it is proved that the corresponding gradient map is linearly triangularizable. Similar results are obtained for nilpotent Hessians of rank ≤ 2, homogeneous nilpotent Hessians of rank ≤ 3 and homogeneous nilpotent Hessians in dimension 6 from which the rows are linearly dependent over $\mathbb{C}$. For all $n \geq 5$ and all r with $3 \leq r \leq n-1$, h with nilpotent Hessians of rank r in dimension n, such that ∇h is not linearly triangularizable, are given. For all $n \geq 7$ and all r with $4 \leq r \leq n-1$, homogeneous h with nilpotent Hessians of rank r in dimension n, for which ∇h is not linearly triangularizable, are constructed.

In chapter 6, maps of the form

$$(Ax)^{*d} = \begin{pmatrix} (A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n)^d \\ (A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n)^d \\ \cdots \\ (A_{n1}x_1 + A_{n2}x_2 + \cdots + A_{nn}x_n)^d \end{pmatrix}$$

are studied, where A is a matrix over the base ring. We call these maps power linear (of degree d). A cubic linear map with nilpotent Jacobian and linearly independent components over $\mathbb{C}$ is constructed. Furthermore, Gorni-Zampieri pairing is studied, especially the connection with linear (in)dependence of the components of the maps involved. It is shown that if the components of H are linearly dependent over $\mathbb{C}$ and H and G are GZ-paired, then the components of G are linearly dependent over $\mathbb{C}$ as well. Furthermore, it is proved that H is linearly triangularizable, if and only if G is, in case H and G are GZ-paired.

For large d compared to the corank of A, it is shown that power linear maps $(Ax)^{*d}$ with nilpotent Jacobians are symmetrically triangularizable, i.e. that there exists a permutation P such that $P^{-1}(APx)^{*d}$ has a lower triangular Jacobian. Under similar conditions, it is shown that Zhao graphs, graphs from which the vertices are the rows of A, are totally disconnected. Zhao graphs were introduced by W. Zhao to describe homogeneous nilpotent Hessians. In order to obtain these results, a generalization of Fermat's last theorem for polynomials is used. This result is obtained from a version of Mason's theorem in appendix B. For all n, all r and all d, it is determined whether linearly triangularizable power linear maps $(Ax)^{*d}$ in dimension n, with nilpotent Jacobians and $\mathrm{rk} A = r$, are always symmetrically triangularizable or not. Furthermore, similar results with 'ditto linearly triangularizable' instead of 'symmetrically triangularizable' are derived, where 'ditto linearly triangularizable' means that some of the linear conjugations with a triangular Jacobian are power linear as well.

In chapter 7, it is proved that power linear Keller maps $(Ax)^{*d}$ with $d \geq 3$ and $\mathrm{cork} A = 3$ are ditto linearly triangularizable. Furthermore, it is proved that for any $d \geq 1$, power linear maps $(Ax)^{*d}$ over $\mathbb{C}$ with nilpotent Jacobians are linearly triangularizable in case $n \leq 7$. For $d = 3$, this result is extended to $n \leq 8$. Furthermore, all Zhao graphs (from which the vertices are the rows of A) are characterized for the case that $d \geq 3$ and $\mathrm{rk}A \geq N-3$, where N is the number of rows of A. Similar results are obtained for the case $d = 2$ and $\mathrm{rk}A \geq N-1$.

Samenvatting

Het Jacobivermoeden

Het Jacobivermoeden, of eigenlijk het Jacobiaanvermoeden, is geformuleerd door O.H. Keller in 1939. Voor de beschrijving ervan, zij A een commutatieve ring en neem aan dat $F = (F_1, F_2, \ldots, F_n)$ zodat $F_i \in A[x] = A[x_1, x_2, \ldots, x_n]$ voor alle i. Definieer de Jacobiaan $\mathcal{J}F$ van F als

$$\mathcal{J}F := \begin{pmatrix} \frac{\partial}{\partial x_1}F_1 & \frac{\partial}{\partial x_2}F_1 & \cdots & \frac{\partial}{\partial x_n}F_1 \\ \frac{\partial}{\partial x_1}F_2 & \frac{\partial}{\partial x_2}F_2 & \cdots & \frac{\partial}{\partial x_n}F_2 \\ \vdots & \vdots & & \vdots \\ \frac{\partial}{\partial x_1}F_n & \frac{\partial}{\partial x_2}F_n & \cdots & \frac{\partial}{\partial x_n}F_n \end{pmatrix}$$

Aangezien $\mathcal{J}F$ een vierkante matrix is kan men de determinant ervan berekenen. We zeggen dat F een Kellerafbeelding is of dat F voldoet aan de Kellervoorwaarde, indien $\det \mathcal{J}F$ een eenheid in $A[x]$ is. Als A een gereduceerde ring is, dan zijn de eenheden in $A[x]$ precies die van A. Nu houdt het Jacobivermoeden in dat F een automorfisme is in het geval dat F een Kellerafbeelding is, d.w.z. er bestaat een veeltermafbeelding $G = (G_1, G_2, \ldots, G_n)$ zodat $G_i \in A[x] = A[x_1, x_2, \ldots, x_n]$ voor alle i en de samengestelde afbeelding $G(F)$ de identieke afbeelding is.

Kellers oorspronkelijke formulering was alleen voor de ring $A = \mathbb{Z}$. Later werd de vraag veralgemeniseerd tot ringen A die bevat zijn in een commutatieve ring met $\mathbb{Q}$, en vandaag de dag wordt meestal een lichaam van karakteristiek nul genomen in bovenstaande formulering, in het bijzonder $A = \mathbb{C}$. Dit is slechts een kwestie van smaak, omdat al bewezen is dat het Jacobivermoeden equivalent is voor alle ringen A die bevat zijn in een commutatieve ring met $\mathbb{Q}$.

Het Jacobivermoeden geldt niet voor commutatieve ringen A in het algemeen. Bijvoorbeeld $F = x_1 - x_1^q$ is een Kellerafbeelding in dimensie 1 over $\mathbb{F}_q$, het eindige lichaam van q elementen, maar $F(\mathbb{F}_q) = 0$. Voor meer informatie over het Jacobivermoeden verwijs ik naar paragraaf 1.2 in de introductie.

Resultaten van dit proefschrift

De meeste resultaten zijn over de grondring $\mathbb{C}$, alhoewel veel ervan gemakkelijk kunnen worden veralgemeniseerd tot willekeurige lichamen van karakteristiek nul. Sommige resultaten worden ook bewezen voor andere grondringen, maar alle resultaten die hieronder opgesomd staan zijn over $\mathbb{C}$, tenzij anders wordt vermeld.

In hoofdstuk 2 wordt het Jacobivermoeden gereduceerd tot het geval dat de Jacobiaan van de beschouwde afbeelding een of andere symmetrie heeft. Er wordt bijvoorbeeld bewezen dat het volstaat om het Jacobivermoeden te bewijzen voor Kellerafbeeldingen $F = x + H$ zodat de symmetriegroep van $\mathcal{J}H$ de volledige diëdergroep van het vierkant is. Over het lichaam $\mathbb{R}$ mag de symmetriegroep elke ondergroep van de volledige diëdergroep van het vierkant zijn die een spiegeling bevat, maar juist niet de spiegeling in de hoofddiagonaal bevat. Voor enkele andere typen symmetrieën van $\mathcal{J}H$ word aangetoond dat het Jacobivermoeden op een triviale wijze geldt voor veeltermafbeeldingen $F = x + H$. Als bijvoorbeeld $F = x + H$ zodat $\mathcal{J}H$ symmetrisch is in horizontale richting en antisymmetrisch in verticale richting of andersom, dan is F inverteerbaar. In feite is F in dat geval een zogenaamde quasi-translatie.

In hoofdstuk 3 worden quasi-translaties bestudeerd. Quasi-translaties zijn veeltermautomorfismen $x + H$ met inverse $x - H$. Verder zijn quasi-translaties unipotente Kellerafbeeldingen, d.w.z. $\mathcal{J}H$ is nilpotent, en is er een verband tussen quasi-translaties en singuliere Hessianen. In het bijzonder wordt lineaire afhankelijkheid tussen de componenten van H onderzocht. Alle quasi-translaties $x + H$ in dimensie ≤ 3 en alle quasi-translaties $x + H$ met $\mathrm{rk}\mathcal{J}H = 1$ worden geclassificeerd. Er wordt aangetoond dat voor deze quasi-translaties $x + H$ de componenten van H lineair afhankelijk zijn over $\mathbb{C}$. Maar in dimensie 4 en hoger worden voorbeelden van quasi-translaties $x + H$ gegeven waarvan de componenten van H niet lineair afhankelijk zijn over $\mathbb{C}$.

Voor homogene H worden alle quasi-translaties $x + H$ in dimensie 3 en 4 en alle quasi-translaties $x + H$ met $\mathrm{rk}\mathcal{J}H = 2$ geclassificeerd. Er wordt aangetoond dat er voor deze quasi-translaties $x + H$ zelfs twee onafhankelijke lineaire relaties over $\mathbb{C}$ bestaan tussen de componenten van H. Maar voor dimension 6 en hoger worden homogene H met lineair onafhankelijke componenten over $\mathbb{C}$ gegeven zodat $x+H$ een quasi-translatie is. Verder worden quasi-translaties $x + H$ in dimension 5 met H homogeneen bestudeerd. Die met $\mathrm{rk}\mathcal{J}H \geq 3$ worden geclassificeerd in drie bestaande typen en voor twee van deze typen zijn de componenten van H lineair afhankelijk over $\mathbb{C}$.

In hoofdstuk 4 wordt het bestaan van lineaire afhankelijkheden tussen de componenten van homogene H met $\mathcal{J}H$ nilpotent onderzocht. Er wordt aangetoond dat de componenten van zulke H lineair afhankelijk zijn over $\mathbb{C}$ in het geval de dimensie ten hoogste 4 is en $\mathrm{rk}\mathcal{J}H \leq 2$. Verder wordt aangetoond dat homogene H in dimensie 3 zodat $\mathcal{J}H$ nilpotent is, lineair op driehoeksvorm te brengen zijn over $\mathbb{C}$. Om deze resultaten te verkrijgen wordt een formule gegeven voor homogene H zodat $\mathrm{rk}\mathcal{J}H \leq 2$.

Maar voor dimensie 5 wordt een homogene H van graad 6 met lineair onafhankelijke componenten over $\mathbb{C}$, zodat $\mathcal{J}H$ nilpotent is, geconstrueerd. Voor dimensie 9 en hoger worden homogene H van graad 3 met lineair onafhankelijke componenten over $\mathbb{C}$, waarvoor $\mathcal{J}H$ nilpotent is, gegeven. De eerste dergelijke H in dimensie 9 werd gemaakt door Gaetano Zampieri, uit een soortgelijke H in dimensie 10 door de auteur. De laatste H was het eerste tegenvoorbeeld van graad 3 tegen het afhankelijkheidsvermoeden.

Voorts worden alle kwadratisch homogene afbeeldingen H in dimensie 5 met $\mathcal{J}H$ nilpotent geclassificeerd. In het bijzonder wordt aangetoond dat $x + H$ tam is en dat de componenten van H lineair afhankelijk zijn over $\mathbb{C}$. Voor een groot deel wordt dit gedaan met berekeningen in Maple, een computeralgebra-programma. Hoe de berekeningen worden uitgevoerd staat beschreven in appendix A. Een andere berekening die wordt beschreven is die van afbeeldingen H van graad 4 in dimensie 3 zodat $\mathcal{J}H$ nilpotent is.

In hoofdstuk 5 worden Hessianen van kleine rang en nilpotente Hessianen onderzocht. Er wordt bewezen dat homogene h waarvoor $\mathcal{H}h$ rang r heeft altijd kunnen worden uitgedrukt als een veelterm in r lineaire vormen, dan en slechts dan als $r \leq 3$. Verder worden alle h waarvoor $\mathcal{H}h$ rang $r \leq 2$ heeft geclassificeerd, en evenzo alle homogenene h in dimensie 5 waarvoor $\det \mathcal{H}h = 0$. Bovendien worden singuliere Hessianen in dimension 4 bestudeerd. De resultaten over quasi-translaties worden gebruikt om deze resultaten te verkrijgen, door middel van het eerder vermelde verband tussen quasi-translaties and singuliere Hessianen.

Voor nilpotente Hessianen in dimensie ≤ 4 en homogene nilpotente Hessianen in dimensie ≤ 5 wordt bewezen dat de overeenkomende gradiëntafbeelding lineair op driehoeksvorm te brengen is. Soortgelijke resulten worden ook verkregen voor nilpotente Hessianen van rang ≤ 2, homogene nilpotente Hessianen van rang ≤ 3 en homogene nilpotente Hessianen in dimension 6 waarvan de rijen lineair afhankelijk zijn over $\mathbb{C}$. Voor alle $n \geq 5$ en alle r met $3 \leq r \leq n-1$ worden h met nilpotente Hessianen van rang r in dimensie n, zodat ∇h niet lineair op driehoeksvorm te brengen is, gegeven. Voor alle $n \geq 7$ en alle r met $4 \leq r \leq n-1$ worden homogene h met nilpotente Hessianen van rang r in dimensie n, waarvoor ∇h niet lineair op driehoeksvorm te brengen is, geconstrueerd.

In hoofdstuk 6 worden afbeeldingen van de vorm

$$(Ax)^{*d} = \begin{pmatrix} (A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n)^d \\ (A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n)^d \\ \cdots \\ (A_{n1}x_1 + A_{n2}x_2 + \cdots + A_{nn}x_n)^d \end{pmatrix}$$

bestudeerd, waar A een matrix over de grondring is. We noemen deze afbeeldingen machts-lineair (van graad d). Er wordt een machts-lineaire afbeelding van graad 3 met nilpotente Jacobiaan en lineair onafhankelijke componenten over $\mathbb{C}$ geconstrueerd. Verder wordt Gorni-Zampieri-paring bestudeerd, vooral het verband met lineaire (on)afhankelijkheid van de componenten van de betrokken afbeeldingen. Er wordt aangetoond dat indien de componenten van H lineair onafhankelijk zijn over $\mathbb{C}$ en H en G GZ-gepaard zijn, de componenten van G ook lineair onafhankelijk zijn over $\mathbb{C}$. Verder wordt bewezen dat H lineair op driehoeksvorm te brengen is, dan en slechts dan als G dat is, in het geval dat H and G GZ-gepaard zijn.

Voor grote d vergeleken met de corang van A wordt aangetoond dat machts-lineaire afbeeldingen $(Ax)^{*d}$ met nilpotente Jacobianen symmetrisch op driehoeksvorm te brengen zijn, d.w.z. dat er een permutatie P bestaat zodat de Jacobiaan van $P^{-1}(APx)^{*d}$ een benedendriehoeksmatrix is. Onder soortgelijke voorwaarden wordt aangetoond dat Zhao-grafen, grafen waarvan de punten de rijen van A zijn, totaal onsamenhangend zijn. Zhao-grafen werden geïntroduceerd door W. Zhao om homogene nilpotente Hessianen te beschrijven. Om deze resultaten te bereiken, wordt een veralgemenisering van de laatste stelling van Fermat voor veeltermen gebruikt. Dit resultaat wordt verkregen uit een versie van de stelling van Mason in appendix B. Voor alle n, alle r en alle d wordt bepaald of machts-lineaire afbeeldingen $(Ax)^{*d}$ in dimensie n met een nilpotente Jacobiaan en $\mathrm{rk} A = r$, die lineair op driehoeksvorm te brengen zijn, altijd symmetrisch op driehoeksvorm te brengen zijn of niet. Verder worden soortgelijke resultaten met 'lineair op ditto-driehoeksvorm' in plaats van 'symmetrisch op driehoeksvorm' afgeleid, waar 'lineair op ditto-driehoeksvorm' betekent dat sommige van de lineaire conjugaties met een Jacobiaan op driehoeksvorm tevens machts-lineair zijn.

In hoofdstuk 7 wordt bewezen dat machts-lineaire Kellerafbeeldingen $(Ax)^{*d}$ met $d \geq 3$ en $\mathrm{cork} A = 3$ lineair op ditto-driehoeksvorm te brengen zijn. Verder wordt bewezen dat voor welke $d \geq 1$ dan ook, machts-lineaire afbeeldingen $(Ax)^{*d}$ over $\mathbb{C}$ met een nilpotente Jacobiaan lineair op driehoeksvorm te brengen zijn in het geval dat $n \leq 7$. Voor $d = 3$ wordt dit resultaat uitgebreid tot $n \leq 8$. Verder worden alle Zhao-grafen (waarvan de punten de rijen van A zijn) gekarakteriseerd voor het geval dat $d \geq 3$ en $\mathrm{rk} A \geq N - 3$, waar N het aantal rijen van A is. Soortgelijke resultaten worden ook bereikt voor het geval $d = 2$ en $\mathrm{rk} A \geq N - 1$.

Chapter 1

Introduction

1.1 About this thesis

When you look at this thesis, you will probably think it is quite large, at least for a Ph.D. thesis in mathematics. This is because we met many results on our journey for the Jacobian conjecture. See section 1.3 for a travel story. The second chapter, appendix B and this introduction are meant for a wide audience of mathematicians, like a plenary talk, but the rest of the thesis is quite technical. However, most of it is elementary.

On one hand, the subject of my thesis makes that things are elementary. On the other hand, I have chosen to have my thesis as elementary as possible without unnecessary fancy things. Therefore, I did not introduce derivations in my thesis, because they are not needed for the presentation.

But against my principle of not using fancy things, I do use some geometry in chapter 3, because I need it to get certain results, especially about homogeneous quasi-translations in dimension 5.

There are several notations that are likely not to be understood. One of them is to denote $x_1, x_2, \ldots, x_n$ by x, so x depends on n implicitly. The same holds for y and z.

Another thing is the *reverse* of a matrix or vector. The reverse of a vector $v = (v_1, v_2, \ldots, v_m)$ is $v^{\mathrm{r}} = (v_m, v_{m-1}, \ldots, v_1)$ and with the reverse of a matrix M, we see M as a vector of its rows, so M^{r} has rows $M_m, M_{m-1}, \ldots, M_1$ if m is the height of M.

Other notations are more likely to be understood. See section 1.4 for more notations.

This chapter has its own list of references, namely to articles that are part of the content of this manuscript and hence not included in the general list of references. All of the references are articles by Arno van den Essen and the author, with two exceptions: Van den Essen cheated with S. Washburn and the author with H. Tong. In this thesis, some of the results are proved with different methods than in those articles, mostly because some results of this thesis are stronger and therefore require another proof.

Appendix B has its own reference list as well, because it is not about the Jacobian conjecture. Some of the results in it are used in chapter 6, though. As contrasted to the other reference lists, the reference list of this chapter is chronological instead of alphabetical.

1.2 Some history of the Jacobian conjecture

1.2.1 Keller's Jacobian conjecture

Let A be a commutative ring and assume $F = (F_1, F_2, \ldots, F_m)$ such that $F_i \in A[x] = A[x_1, x_2, \ldots, x_n]$ for all i. Define the Jacobian $\mathcal{J}F$ of F by

$$\mathcal{J}F := \begin{pmatrix} \frac{\partial}{\partial x_1}F_1 & \frac{\partial}{\partial x_2}F_1 & \cdots & \frac{\partial}{\partial x_n}F_1 \\ \frac{\partial}{\partial x_1}F_2 & \frac{\partial}{\partial x_2}F_2 & \cdots & \frac{\partial}{\partial x_n}F_2 \\ \vdots & \vdots & & \vdots \\ \frac{\partial}{\partial x_1}F_m & \frac{\partial}{\partial x_2}F_m & \cdots & \frac{\partial}{\partial x_n}F_m \end{pmatrix}$$

If $m = n$, then $\mathcal{J}F$ is a square matrix, whence one can compute its determinant. We say that F is a Keller map or F satisfies the Keller condition if $\det \mathcal{J}F$ is a unit in $A[x]$.

If A is a reduced ring, i.e. A has no nilpotent elements other than zero, then the units of $A[x]$ are exactly those in A. This is because $f \in A[x]$ is a unit in $A[x]$, if and only if the constant term of f is a unit in A and all other coefficients are nilpotent. See [2, Ch. 1, Exc. 3 i)]. The term 'Keller condition' or 'Jacobian condition' is due to Gary Meisters, but since he only considered fields, he did not need to distinguish between $\det \mathcal{J}F \in A^*$ and $\det \mathcal{J}F \in (A[x])^*$. The reason for allowing units in $A[x]$ will be given later.

In 1939, Keller asked the following question for $A = \mathbb{Z}$. Assume $m = n$ and $\det \mathcal{J}F$ is a unit in A? Is F an automorphism? In other words, does there exist a polynomial map $G = (G_1, G_2, \ldots, G_n)$ such that $G_i \in A[y] = A[y_1, y_2, \ldots, y_n]$ for all i and $G(F) = x$ is the identity map? Later on, the question was generalized to rings A contained in commutative rings with $\mathbb{Q}$.

This question is still unsolved for $A = \mathbb{Z}$, except the case $n = 1$. But for commutative rings $A \neq \{0\}$ that are not contained in a commutative ring B containing $\mathbb{Q}$, the answer is no in general. More precisely, we will show below that the answer is no if A/η is not contained in a commutative ring with $\mathbb{Q}$, where $\eta = \mathfrak{r}(0)$ is the ideal of nilpotent elements of A, the so-called nilradical of A.

The characteristic of an arbitrary ring is the smallest positive integer that is zero in the ring if such an integer exists, and 0 otherwise. For rings of characteristic $c > 1$, the polynomial map $F = x_1 - x_1^c$ in dimension 1 satisfies the Keller condition, but $F(0) = F(1)$. For that reason, Pascal Adjamagbo has formulated extra conditions for F in case A is a finite field. See [24] above Proposition 10.3.17 or [1]. For a fixed dimension n, the adapted Jacobian conjecture over $\mathbb{F}_p$ for all primes p implies the Jacobian conjecture over $\mathbb{Z}$. In [24, Prop. 10.3.17], all dimensions n are taken together, but its proof gives the stronger result that one can take the dimension n fixed.

In order to prove the Jacobian conjecture for a ring A of characteristic zero, we may divide out the nilradical of A on account of [24, Lm.1.1.9] and a variant of [24, Lm.1.1.9] for the Keller condition instead of invertibility. After dividing out the nilradical of A, we obtain a reduced ring. The variant of [24, Lm. 1.1.9] for the Keller condition instead of invertibility is only satisfied in case units in $A[x]$ are allowed in the definition of Keller condition. So the Jacobian conjecture is not affected by allowing units in $A[x]$ in the definition of Keller condition, and that is the reason why we choose this definition as such.

We show now that the Jacobian conjecture is not satisfied over commutative rings $A \supseteq \mathbb{Z}$ if A/η is not contained in a commutative ring with $\mathbb{Q}$, where $\eta = \mathfrak{r}(0)$ is the ideal of nilpotent elements of A, the so-called nilradical of A. As mentioned above, we may replace A by the reduced ring A/η. So assume $A \supseteq \mathbb{Z}$ is not contained in a ring $B \supseteq \mathbb{Q}$. Now make B from A by way of localization with respect to $\mathbb{Z}^* = \mathbb{Z}\backslash\{0\}$. If $A \ni a \mapsto c^{-1}a \in B$ is injective for all $c \in \mathbb{Z}^*$, then $B \supseteq \mathbb{Q}$. So $A \ni a \mapsto c^{-1}a \in B$ is not injective for some $c \in \mathbb{Z}^*$.

Now one can show that there exists a $c \in \mathbb{Z}^*$ that is a zero divisor in A (the latter c is divisible by the one above), say that $ca = 0$ for some nonzero

$a \in A$. By removing factors of c and migrating factors of c to a, we can obtain that c is prime. Take $F = x_1 - ax_1^c$ to obtain that $\frac{\partial}{\partial x_1} F = 1$, just as in [24, (1.1.17)]. Using that $a\binom{c}{i} = 0$ in case $i \nmid c$, one can show that the inverse power series of F is equal to

$$\sum_{k=0}^{\infty} a^{(c^k-1)/(c-1)} x_1^{c^k}$$

which is a proper power series, because A is reduced.

If A has characteristic 1, then $A = \{0\} = \{1\}$ is the trivial ring and the Jacobian conjecture is satisfied over A for every n because every polynomial over A is the same. So we can restrict ourselves to reduced rings of characteristic zero that are contained in a ring $B \supseteq \mathbb{Q}$. Now we have the following remarkable result for such rings.

Theorem 1.2.1. *Assume both A and $\tilde{A}$ are commutative rings contained in rings $B \supseteq \mathbb{Q}$ and $\tilde{B} \supseteq \mathbb{Q}$ respectively. Then the Jacobian conjecture over A for all dimensions n is equivalent to the Jacobian conjecture over $\tilde{A}$ for all dimensions n! Furthermore, the Jacobian conjecture over $\mathbb{C}$ for a fixed dimension n implies the Jacobian conjecture over A for dimension n.*

Proof. The first assertion follows from [24, Th. 1.1.18], which is based on [24, Prop. 1.1.19], a result from Edwin Connell and Lou van den Dries in [11]. That this assertion is even valid if the exclamation point after n is read as taking faculty is left as an exercise to the reader. The second assertion follows from [24, Prop. 1.1.12], which comes originally from [4] by Edwin Connell, Hyman Bass and David Wright. □

The Jacobian conjecture appears as problem 16 of a list of 18 mathematical problems for the 21st century by Steve Smale. This list is an analog of David Hilbert's list of 23 problems for the 20-th century. See [49].

1.2.2 Partial results of the Jacobian conjecture

One of the partial results of the Jacobian conjecture is that in dimension 2, it is satisfied for maps F of degree 101 at most. This is due to Tzuong Tsieng Moh in [42]. For more results on the Jacobian conjecture in dimension 2, I refer to [24, §10.2].

Another result is that the Jacobian conjecture is satisfied for maps F of degree 2 at most. This was proved by Stuart Sui-Sheng Wang at first in [51]. Theorem 1.2.3 below generalizes this result somewhat. The proof of the theorem 1.2.3 is based on the following lemma.

Lemma 1.2.2. *Assume A is an integral domain such that the integer $d \geq 2$ is nonzero in A. Assume t is a single indeterminate and let $F \in A[t]^n$ such that $\deg F \leq d$ and assume that $F(a_1) = F(a_2) = \cdots = F(a_d)$ for distinct $a_i \in A$.*

Assume in addition that polynomials in $A[t]$ of degree $d-1$ with leading coefficient d have a root in A. Then there exists a $b \in A$ such that $(\mathcal{J}_t F)|_{t=b} = 0$.

Proof. Say that $F(a_i) = c$ for each i. Notice that $\deg(F_i - c_i) \leq d$ for each i. Since $F_i - c_i$ has d distinct zeros in A in addition, it follows that $F_i - c_i$ is linearly dependent over A of the monic polynomial

$$f := (t - a_1)(t - a_2) \cdots (t - a_d)$$

for each i. Consequently, $\frac{\partial}{\partial t} F_i$ is dependent of $\frac{\partial}{\partial t} f$ for each i. Since $\deg \frac{\partial}{\partial t} f = d - 1$ and the leading coefficient of $\frac{\partial}{\partial t} f$ is equal to d, we have $(\frac{\partial}{\partial t} f)|_{t=b} = 0$ for some $b \in A$ by assumption. This gives the desired result. $\square$

Theorem 1.2.3. *Assume A is an integral domain and $d \geq 2$ an integer. Let $F \in A[x]^n$ such that $\deg F \leq d$ and assume that $F(p_1) = F(p_2) = \cdots = F(p_d)$ for distinct collinear $p_i \in A^n$. Then there exist a $q \in A^n$ on the same line as the p_i's, such that $(\mathcal{J}F)|_{x=q}$ has determinant zero, in each of the following cases:*

i) $d = 2$ is a unit in A,

ii) $A = \mathbb{R}$ and d is even,

iii) A is an algebraically closed field and d is a unit in A.

Proof. Notice that in each of the cases i), ii) and iii), the condition that polynomials in $A[t]$ of degree $d-1$ with leading coefficient d have a root in A, is fulfilled. Now set

$$G := F\big(p_1 + t(p_2 - p_1)\big)$$

then by the above lemma, there exists a $b \in A$ such that $(\mathcal{J}_t G)|_{t=b} = 0$. By the chain rule, we obtain

$$\mathcal{J}_t G = (\mathcal{J} F)_{x=p_1+t(p_2-p_1)} \cdot (p_2 - p_1)$$

So if we take $q := p_1 + b(p_2 - p_1)$, then the columns of $(\mathcal{J}F)|_{x=q}$ are dependent. This gives the desired result. □

Corollary 1.2.4. *Assume A is an integral domain and the integer $d \geq 2$ is nonzero in A. Let $F \in A[x]^n$ such that $\deg F \leq d$ and assume that $F(p_1) = F(p_2) = \cdots = F(p_d)$ for distinct collinear $p_i \in A^n$. Then F is not a Keller map.*

In particular, quadratic Keller maps over an integral domain with $\frac{1}{2}$ are injective.

Proof. Let K be the algebraic closure of $\mathbb{Q}(A)$. From iii) of theorem 1.2.3 and the definition of algebraically closed, it follows that F is not a Keller map over K, as desired.

The last assertion follows from the fact that two points p_1 and p_2 are always collinear. □

In [12], the authors Kamil Rusek and Sławomir Cynk prove that in order to show that a Keller map over $\mathbb{C}$ is invertible, it suffices to show that it is injective. The authors prove a more general result, but the less general result is easier to prove: see [46] and [24, §4.3]. But over fields of characteristic zero in general, injective polynomial maps are not always invertible, even if you assume that the Jacobian determinant does not vanish anywhere. Take for instance

$$F = x_1^3 - 3ax_1$$

then $F' = 3(x_1^2 - a)$. Since a is arbitrary, the field at hand must be closed under taking square root, in order to have a Jacobian determinant that vanishes somewhere.

For the ring $\mathbb{Z}$ of Keller, it is sufficient to show that a Keller map is surjective in order to prove that it is an automorphism. This was proved by Lou van den Dries and Ken McKenna in [17], see also [24, Cor. 10.3.9].

Another question is whether maps for which the Jacobian determinant does not vanish anywhere are automatically injective. But in 1994, Sergey Pinchuk gave a counterexample over $\mathbb{R}$ to this question in dimension $n = 2$.

See [44] or [24, §10.1]. But for quadratic maps over integral domains with $\frac{1}{2}$, the answer to this question is affirmative, as we have seen in i) of theorem 1.2.3.

An interesting question is whether those quadratic maps, that are injective on account of corollary 1.2.4, are automatically automorphisms. I did not have much time thinking about this question, since it just crossed my mind writing this introduction. The map $3x_1$ shows that the answer is negative over the ring $\mathbb{Z}[\frac{1}{2}]$, so let us assume that the base ring is a field with $\frac{1}{2}$. Wang's theorem shows that the answer is affirmative for algebraically closed fields.

1.2.3 Reductions of the Jacobian conjecture

Several reductions have been made to the Jacobian conjecture. Assume A is a commutative ring with nilradical η, such that $A/\eta \subseteq B \supseteq \mathbb{Q}$. Notice that the Jacobian conjecture for all dimensions n is equivalent for all rings like A.

The first reduction is that one may assume that in order to prove the Jacobian conjecture over A for all n, it suffices to consider maps F such that $\deg F = 3$. The next reduction is that in order to prove the Jacobian conjecture over A for all n, it suffices to consider maps F of the form $F = x + H$ such that H is homogeneous of degree d, where d is an arbitrary fixed integer greater than 2. Both reductions can be found in [4] by Bass, Connell and Wright and [58] by A. V. Yagžev. See also [24, §6.3].

A subsequent reduction is due to Ludwik Drużkowski in [18]. He shows that in addition, one may assume that each component of H is a d-th power of a linear form. Actually, he only shows the case $d = 3$, but the general case is similar. See also chapter 6.

The homogeneity reduction is done after the degree 3 reduction, but the homogeneity reduction can be separated from it. More precisely, for any $d' \geq d$, the Jacobian conjecture for maps of degree d in dimension n follows from the Jacobian conjecture for maps $x + H$ in dimension $(d-1)n + 1$ with H homogeneous of degree d'. In order to describe the homogeneity reduction, we make some definitions.

Definition 1.2.5. Let $\mathrm{JC}(A, n, d)$ denote the question whether the Jacobian conjecture is satisfied for Keller maps F over A in dimension n of degree $\leq d$. Let $\mathrm{UJC}(A, n, d)$ denote the question whether the Jacobian conjecture is satisfied for Keller maps F as above such that F is the of the form $F = x+H$, where $\mathcal{J}H$ is nilpotent. The letter U stands for unipotent. Let $\mathrm{HJC}(A, n, d)$ denote the question whether the Jacobian conjecture is satisfied for Keller maps F as above such that F is the of the form $F = x + H$, where H is homogeneous of degree d. The letter H stands for homogeneous.

Proposition 1.2.6. *Let A be a commutative ring and take $n \geq 1$ and $d \geq 2$ fixed. Then*

i) $\mathrm{UJC}(A, n(d-1), d)$ *implies* $\mathrm{JC}(A, n, d)$,

ii) $\mathrm{HJC}(A, n+1, d)$ *implies* $\mathrm{UJC}(A, n, d)$,

iii) $\mathrm{UJC}(A, n, d)$ *implies* $\mathrm{HJC}(A, n, d)$.

Proof.

i) By composition with translations and linear maps, we see that it suffices to consider instances F of the form $x+H$ of $\mathrm{JC}(A, n, d)$, such that H does not have terms of degree less than 2. In [24, Prop. 6.2.13], an instance of $\mathrm{UJC}(A, n(d-1), d)$ is made out of $(F, x_{n+1}, x_{n+2}, \ldots, x_{(d-1)n}$ by composition with tame maps. This gives the desired result.

ii) The case $d = 3$ follows from [24, Th. 6.3.1], and the general case is similar.

iii) Assume $F = x - H$ is an instance of $\mathrm{HJC}(A, n, d)$. Since F is a Keller map, it follows from Cramer's rule that $\mathcal{J}F \in \mathrm{GL}_n(A[x])$. So the inverse of $\mathcal{J}F$ as matrix over $A[[x]]$ must be a matrix over $A[x]$, i.e.

$$(\mathcal{J}\tilde{F})^{-1} = (I_n - \mathcal{J}H)^{-1} = I_n + \mathcal{J}H + \mathcal{J}H^2 + \mathcal{J}H^3 + \cdots$$

must have bounded degree. Since H is homogeneous of degree $d \geq 2$, this is only possible if $\mathcal{J}H$ is nilpotent. It follows that F is an instance of $\mathrm{UJC}(A, n, d)$ as well. This gives the desired result. □

In [4] by Bass, Connell and Wright, it is shown that $\mathrm{UJC}(\mathbb{C}, 2, d)$ has an affirmative answer for all d. $\mathrm{HJC}(\mathbb{C}, 3, 3)$ is proved by David Wright in [55] and $\mathrm{HJC}(\mathbb{C}, 4, 3)$ by Engelbert Hubbers in [36]. They use computations to obtain their results. See also chapter 4 and appendix A.

1.2.4 The dependence problem

One problem that arose with the study of the Jacobian conjecture is the linear dependence problem for homogeneous Jacobians, or shortly the dependence problem. The problem was first formulated for quadratic maps by Rusek in [46]. Later, Gary Meisters and Czesław Olech formulated the cubic variant of the problem, in [40] and [43] respectively. The latter offered a bottle of Polish vodka for either a proof or a counterexample of this problem.

Definition 1.2.7. Let K be a field of characteristic zero. Then UDP(K, n, d) is the question whether for maps H of degree $\le d$ in dimension n such that $\mathcal{J}H$ is nilpotent, the rows of $\mathcal{J}H$ are dependent over K. HDP(K, n, d) is the same question except that H is homogeneous of degree d instead.

Along with the affirmative answers for UJC$(\mathbb{C}, 2, d)$ for all d and HJC$(\mathbb{C}, n, 3)$ for $n \le 4$, affirmative answers were given to UDP$(\mathbb{C}, 2, d)$ and HDP$(\mathbb{C}, n, 3)$ for $n \le 4$ as well. In fact, showing UDP$(\mathbb{C}, 2, d)$ was the essential part of showing the corresponding part UJC$(\mathbb{C}, 2, d)$ of the Jacobian conjecture. HJC$(\mathbb{C}, 3, 3)$ and HDP$(\mathbb{C}, 3, 3)$ were proved by David Wright in [55] by way of a full classification of all cubic homogeneous H in dimension 3 respectively such that $\mathcal{J}H$ is nilpotent. Hubbers did a similar thing to prove HJC$(\mathbb{C}, 4, 3)$ and HDP$(\mathbb{C}, 4, 3)$.

Meisters and Olech showed in [39] that HDP$(\mathbb{C}, n, 2)$ has an affirmative answer for $n \le 4$ as well. In fact, Meisters showed something stronger, namely that for quadratic homogeneous H in dimensions $n \le 4$, $\mathcal{J}H$ is so-called strongly nilpotent. See chapter 5 for the definition of strong nilpotency.

Affirmative answers to both UDP$(\mathbb{C}, 3, 2)$ and UDP$(\mathbb{C}, 3, 3)$ can be obtained easily from the classification of cubic homogeneous maps in dimension 4 by Hubbers that gave HJC$(\mathbb{C}, 4, 3)$ and HDP$(\mathbb{C}, 4, 3)$. The affirmative answer to UDP$(\mathbb{C}, 3, 2)$ can be derived easily from the above strong nilpotency result in dimension 4 as well.

The dependence problem inspired Van den Essen and Hubbers to find a class of polynomial automorphisms. Some polynomial maps of this class are counterexamples to the Markus Yamabe conjecture. See [27] for more information.

1.2.5 Polynomial maps with Jacobian determinant zero

As contrasted to Keller maps, polynomial maps with Jacobian determinant zero are better understood, at least for integral domains $A \supseteq \mathbb{Q}$. For $A = \mathbb{C}$, we have proposition 1.2.9 below, and by way of Lefschetz' principle, we can obtain the same result for any integral domain $A \supseteq \mathbb{Q}$. See for instance the proof of [24, Lm. 1.1.14] for a demonstration of Lefschetz' principle.

Lemma 1.2.8. *Assume $H \in \mathbb{C}[x]^m$. Then*

$$\begin{aligned} 0 &\le \operatorname{rk}\mathcal{J}(H_1,\ldots,H_i,H_{i+1}) - \operatorname{rk}\mathcal{J}(H_1,\ldots,H_i) \\ &\le \operatorname{trdeg}_{\mathbb{C}}\mathbb{C}(H_1,\ldots,H_i,H_{i+1}) - \operatorname{trdeg}_{\mathbb{C}}\mathbb{C}(H_1,\ldots,H_i) \\ &\le 1 \end{aligned}$$

for all $i < m$.

Proof. Notice that only the inequality in the middle needs explanation. This inequality can only be violated if H_{i+1} is algebraically dependent over $\mathbb{C}$ of $H_1,\ldots,H_i$, say that

$$R(H_1,\ldots,H_i,H_{i+1}) = 0$$

for some $R \in \mathbb{C}[y_1,\ldots,y_i,y_{i+1}]$ such that $R(H_1,\ldots,H_i,y_{i+1}) \ne 0$.
Choose R of minimal degree. Since $R(H_1,\ldots,H_i,y_{i+1}) \ne 0$ and $R(H_1,\ldots,H_i,y_{i+1})|_{y_{i+1}=H_{i+1}} = 0$, it follows that

$$\left(\frac{\partial R}{\partial y_{i+1}}\right)(H_1,\ldots,H_i,y_{i+1}) \ne 0$$

and by the minimality of the degree of R,

$$\left(\frac{\partial R}{\partial y_{i+1}}\right)(H_1,\ldots,H_i,H_{i+1}) \ne 0$$

From the chain rule, it follows that

$$\begin{aligned} 0 &= \mathcal{J}(R(H_1,\ldots,H_i,H_{i+1})) \\ &= (\mathcal{J}_{y_1,\ldots,y_i,y_{i+1}}R)|_{y_1=H_1,\ldots,y_i=H_i,y_{i+1}=H_{i+1}} \cdot \mathcal{J}(H_1,\ldots,H_i,H_{i+1}) \end{aligned}$$

so the last row of $\mathcal{J}(H_1,\ldots,H_i,H_{i+1})$ is dependent of the other rows. This gives the desired result. $\square$

Proposition 1.2.9. *Assume $H \in \mathbb{C}[x]^m$. Then*

$$\mathrm{rk}\mathcal{J}(H_1, H_2, \ldots, H_m) = \mathrm{trdeg}_{\mathbb{C}}\mathbb{C}(H_1, H_2 \ldots, H_m)$$

Proof. Put $G := (0, H, x)$. Then G has $1 + m + n$ component and can be made by adding $m + n$ components to the map (0). Notice that $\mathrm{rk}\mathcal{J}(0) = 0 = \mathrm{trdeg}_{\mathbb{C}}\mathbb{C}(0)$, so by induction, it follows from the above lemma that

$$\mathrm{rk}\mathcal{J}G \leq \mathrm{trdeg}_{\mathbb{C}}\mathbb{C}(G)$$

But both the left hand side and the right hand side are equal to n. It follows from the above lemma that we have equality in the intermediate stages as well, in particular after adding m components. This gives the desired result. □

The above proposition generalizes [24, Prop. 1.2.9]. It seems that Paul Gordan and Max Nöther already knew this proposition in 1876. They use it for Hessians in [34]. Another thing Gordan and Nöther already knew is the following geometrical interpretation of $\mathrm{trdeg}_{\mathbb{C}}\mathbb{C}(H)$.

Proposition 1.2.10. *Assume $H \in \mathbb{C}[x]^m$ and let W be the Zariski closure of $H(\mathbb{C}^n)$. Then*

$$\dim W = \mathrm{trdeg}_{\mathbb{C}}\mathbb{C}(H_1, H_2 \ldots, H_m)$$

Proof. From the second part of [35, Th. 1.8A], it follows that $\dim \mathbb{C}[H] = \mathrm{trdeg}_{\mathbb{C}}\mathbb{C}(H_1, H_2 \ldots, H_m)$. From [35, Prop. 1.7], it follows that it suffices to show that $\mathbb{C}[H]$ is the coordinate ring of W.

Let $\mathfrak{r}$ be the ideal $\mathfrak{r} = (R \in \mathbb{C}[y_1, y_2, \ldots, y_m] \mid R(H) = 0)$. Then one can show that $C[y_1, y_2, \ldots, y_m]/\mathfrak{r}$ is the coordinate ring of W and that $y_i + \mathfrak{r} \mapsto H_i$ defines an isomorphism between $C[y_1, y_2, \ldots, y_m]/\mathfrak{r}$ and $\mathbb{C}[H]$. The details are left as an exercise to the reader. This gives the desired result. □

1.3 The journey for the Jacobian conjecture

1.3.1 Quasi-translations from singular Hessians

Our journey started in the beginning of 2003, when Arno came up with an article from 1876, namely [34], which he had gotten from Sherwood Washburn.

Although this article is older than the formulation of the Jacobian conjecture by O.H. Keller, it plays a central role in our research. [34] is about homogeneous polynomials for which the Hessian has determinant zero.

The Hessian $\mathcal{H}h$ of a polynomial h is the matrix of second derivatives of h, i.e.

$$\mathcal{H}h := \begin{pmatrix} \frac{\partial^2}{\partial x_1^2}h & \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_1}h & \cdots & \frac{\partial}{\partial x_n}\frac{\partial}{\partial x_1}h \\ \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}h & \frac{\partial^2}{\partial x_2^2}h & \cdots & \frac{\partial}{\partial x_n}\frac{\partial}{\partial x_2}h \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_n}h & \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_n}h & \cdots & \frac{\partial^2}{\partial x_n^2}h \end{pmatrix}$$

Notice that Hessians are symmetric Jacobians. Over a commutative ring with $\mathbb{Q}$, the converse is true as well, i.e. every symmetric Jacobian is a Hessian. See for instance [24, Lm. 1.3.53].

If $g \in \mathbb{C}[x_1, x_2, \ldots, x_{n-1}]$, then the last row and column of $\mathcal{H}g$ are zero and hence $\det \mathcal{H}g = 0$. Furthermore, one can prove that

$$\mathcal{H}g(Tx) = T^{\mathrm{t}} \cdot (\mathcal{H}g)|_{x=Tx} \cdot T$$

for all $T \in \mathrm{Mat}_n(\mathbb{C})$, where $(\mathcal{H}g)|_{x=Tx}$ means substitution of Tx for x in $\mathcal{H}g$. So if $h = g(Tx)$ for some $g \in \mathbb{C}[x_1, x_2, \ldots, x_{n-1}]$ and a $T \in \mathrm{Mat}_n(\mathbb{C})$, then $\det \mathcal{H}h = 0$. In [34], the authors Paul Gordan and Max Nöther prove that for dimensions $n \leq 4$, the converse holds as well, provided $h \in \mathbb{C}[x]$ is homogeneous.

So if $h \in \mathbb{C}[x]$ is a homogeneous polynomial in $n \leq 4$ variables such that $\det \mathcal{H}h = 0$, then $h = g(Tx)$ for some $g \in \mathbb{C}[x_1, x_2, \ldots, x_{n-1}]$ and a $T \in \mathrm{Mat}_n(\mathbb{C})$. But for dimensions $n \geq 5$, this statement is no longer true. In [34], the authors classify all homogeneous polynomials in dimension 5 for which the Hessian has determinant zero. One case is that $g \in \mathbb{C}[x_1, x_2, x_3, x_4]$ as above, but the other case is that g is of the form

$$g = f(x_1, x_2, a(x_1, x_2)x_3 + b(x_1, x_2)x_4 + c(x_1, x_2)x_5)$$

where f, a, b, and c are polynomials. Again, $h = g(Tx)$ for some $T \in \mathrm{Mat}_5(\mathbb{C})$. One can show that the example

$$h = x_1^2 x_3 + x_1 x_2 x_4 + x_2^2 x_5$$

is not of the form $g(Tx)$ with $g \in \mathbb{C}[x_1, x_2, x_3, x_4]$ and $T \in \mathrm{Mat}_5(\mathbb{C})$.

Now you might wonder what [34] has to do with the Jacobian conjecture. The first step that shows a connection was made by Arno van den Essen and Sherwood Washburn, when they showed that polynomial maps of the form $x + H$ with H homogeneous and $\mathcal{J}H$ symmetric satisfy the Jacobian conjecture in case $n \le 4$. Since symmetric Jacobians are Hessians, we have that H is of the form

$$H = \left(\frac{\partial}{\partial x_1}h, \frac{\partial}{\partial x_2}h, \ldots, \frac{\partial}{\partial x_n}h\right) =: \nabla h$$

in case $\mathcal{J}H$ is symmetric. By the Keller condition on $x + H$, one can show that $\mathcal{J}H$ is nilpotent. In particular, $\mathcal{J}H = \mathcal{H}h$ has determinant zero. Now Arno van den Essen and Sherwood Washburn used that for $n \le 4$, $h = g(Tx)$ for some $g \in \mathbb{C}[x_1, x_2, \ldots, x_{n-1}]$ and a $T \in \mathrm{Mat}_n(\mathbb{C})$ to obtain that polynomial maps of the form $x+H$ with H homogeneous and $\mathcal{J}H$ symmetric satisfy the Jacobian conjecture in case $n \le 4$.

After that, we advanced with $n = 5$. We started by trying to show that $h = g(Tx)$ for some $g \in \mathbb{C}[x_1, x_2, x_3, x_4]$ and a $T \in \mathrm{Mat}_n(\mathbb{C})$. This is not automatically true if $\det \mathcal{H}h = 0$, but we may assume that $\mathcal{H}h$ is nilpotent, which is more restricted than just $\det \mathcal{H}h = 0$. An additional condition is for instance

$$\mathrm{tr}\mathcal{H}h = \frac{\partial^2}{\partial x_1^2}h + \frac{\partial^2}{\partial x_2^2}h + \frac{\partial^2}{\partial x_3^2}h + \frac{\partial^2}{\partial x_4^2}h + \frac{\partial^2}{\partial x_5^2}h = 0$$

The trace condition is an easy condition to work with, whence we threw it against the classification formula

$$g = f\big(x_1, x_2, a(x_1, x_2)x_3 + b(x_1, x_2)x_4 + c(x_1, x_2)x_5\big)$$

of Gordan and Nöther, but not directly, since linear transformations might affect the nilpotency of the Hessian. However, we were able to understand and to deal with this transformation problem.

But in some cases, the trace condition is not enough. If for instance

$$h = (x_2 + \mathrm{i}x_4)^2 x_3 + (x_1 + \mathrm{i}x_3)^2 x_4 + (x_1 + \mathrm{i}x_3)(x_2 + \mathrm{i}x_4)x_5$$

then $\det \mathcal{H}h = 0$ and $\mathrm{tr}\mathcal{H}h = 0$, but there does not exist $T \in \mathrm{Mat}_n(\mathbb{C})$ such that $h = g(Tx)$ and $g \in \mathbb{C}[x_1, x_2, x_3, x_4]$. But we were able to reduce to the case

$$\begin{aligned} g = f\big(&x_1 + \mathrm{i}x_3, x_2 + \mathrm{i}x_4, a(x_1 + \mathrm{i}x_3, x_2 + \mathrm{i}x_4)x_3 + \\ &b(x_1 + \mathrm{i}x_3, x_2 + \mathrm{i}x_4)x_4 + c(x_1 + \mathrm{i}x_3, x_2 + \mathrm{i}x_4)x_5\big) \end{aligned}$$

with $\deg_{x_3} f \le 1$, using the trace condition and additional arguments.

Next, we solved the latter case with other arguments. These arguments have lead to the discovery that the Jacobian conjecture for all dimensions is equivalent to the Jacobian conjecture for symmetric Jacobians for all dimensions. More precisely, the Jacobian conjecture in dimension n follows from the symmetric Jacobian conjecture in dimension $2n$. This equivalence is proved in chapter 2. You may additionally assume that the map F is of the form $x + H$ with H homogeneous of an arbitrary fixed degree $d \ge 3$, as we do here. With the above result that the Jacobian conjecture is satisfied for homogeneous Keller maps in dimension 4 with symmetric Jacobians, we obtain that the Jacobian conjecture is satisfied for homogeneous Keller maps in dimension 2, which was already shown in [4].

The proof that $h = g(Tx)$ for some $g \in \mathbb{C}[x_1, x_2, x_3, x_4]$ and a $T \in \mathrm{Mat}_n(\mathbb{C})$ if h is homogeneous and $n = 5$ can be found in chapter 5. Unfortunately, the proof of the latter case above has been replaced by other arguments. But the same argument still applies for the case

$$g = a(x_1 + \mathrm{i}x_3, x_2 + \mathrm{i}x_4)x_3 + b(x_1 + \mathrm{i}x_3, x_2 + \mathrm{i}x_4)x_4 + c(x_1 + \mathrm{i}x_3, x_2 + \mathrm{i}x_4)$$

in dimension $n = 4$, where we do *not* assume that h and hence g neither are homogeneous. See the end of chapter 2 and subcase (5.15) in case 3 (of the proof of theorem 5.7.1) in section 5.7.

So we proved that $h = g(Tx)$ for some $g \in \mathbb{C}[x_1, x_2, x_3, x_4]$ and a $T \in \mathrm{Mat}_n(\mathbb{C})$ if h is homogeneous, $\mathcal{H}h$ is nilpotent and $n = 5$. This was the first step to obtain that polynomial maps of the form $x+H$ with H homogeneous and $\mathcal{J}H$ symmetric satisfy the Jacobian conjecture in case $n = 5$. We completed the proof of this result by using a theorem of Zhiqing Wang, a former Ph.D. student of Daniel Daigle. This theorem can be formulated in several ways.

One way to put it is that for a quasi-translation $x + H$ in dimension $n = 3$ over $\mathbb{C}$, the components H_1, H_2, H_3 of H are linearly dependent over $\mathbb{C}$. A quasi-translation $x + H$ is a polynomial map such that $x - H$ is the inverse polynomial map, so H acts as the constant part of a translation.

But this is not the way Wang formulates his theorem. He formulates it as follows. Assume

$$D = H_1 \frac{\partial}{\partial x_1} + H_2 \frac{\partial}{\partial x_2} + H_3 \frac{\partial}{\partial x_3}$$

is a derivation over $\mathbb{C}$ in dimension 3 such that

$$D^2 x_1 = D^2 x_2 = D^2 x_3 = 0$$

Then the components H_1, H_2, H_3 of H are linearly dependent over $\mathbb{C}$. Variants of this theorem in dimensions 1 and 2 are satisfied as well. In the beginning of chapter 3, it is shown that the assumptions on H in both formulations are equivalent.

Now you might wonder how Wang's theorem helped us. In other words, what is the connection between nilpotent Hessians and quasi-translations? The connection is that nilpotent Hessians have determinant zero and that polynomials h over $\mathbb{C}$ for which the Hessian has determinant zero, have a quasi-translation associated to them, and I shall tell you how.

For polynomials $h \in \mathbb{C}[x]$ for which the Hessian $\mathcal{H}h$ has determinant zero, we know by proposition 1.2.9 that there exists a nonzero $R \in \mathbb{C}[y]$ such that

$$R\left(\frac{\partial}{\partial x_1}h, \frac{\partial}{\partial x_2}h, \ldots, \frac{\partial}{\partial x_n}h\right) = 0$$

Now one can differentiate the left hand side with respect to x_i. By using the chain rule and interchanging partial derivatives, one can show that for

$$Q := (\nabla_y R)\left(\frac{\partial}{\partial x_1}h, \frac{\partial}{\partial x_2}h, \ldots, \frac{\partial}{\partial x_n}h\right)$$

and $D := \sum_{i=1}^{n} Q_i \frac{\partial}{\partial x_i}$, we have $D^2 x_i = 0$. See chapter 3 for the details.

Now the quasi-translation $x + Q$ tells a lot about h, so knowing things about quasi-translations is very useful. In [34], Gordan and Nöther already discovered the connection between Hessians with determinant zero and (the D^2 formulation of) quasi-translations. A large part of their article [34] is about such quasi-translations.

One thing they show is that for homogeneous quasi-translations $x + H$ (i.e. H homogeneous) in dimensions 3 and 4, there are two independent linear relations between the components of H. Having proved that, it is not so hard any more to show that if h is a homogeneous polynomial in $n \le 4$ variables, then $h = g(Tx)$ for some $g \in \mathbb{C}[x_1, x_2, \ldots, x_{n-1}]$ and a $T \in \mathrm{Mat}_n(\mathbb{C})$.

The article [34] of Gordan and Nöther is written in an old-fashioned style that makes it hard to read. But Arno van den Essen more or less reconstructed the proof that for homogeneous quasi-translations $x + H$ (i.e. H

homogeneous) with $\text{rk}\mathcal{J}H = 2$, there are two independent linear relations between the components of H. From this, one can derive that there are also two independent linear relations between the components of H if the dimensions is 3 or 4. The results for dimension $n = 5$ were harder to understand.

But let me first tell how Gordan and Nöther classify homogeneous polynomials in dimension 5 for which the Hessian has determinant zero. Their classification is based on showing that for the associated quasi-translations $x + H$, there are two independent linear relations between the components of H. This is not true for homogeneous quasi-translations $x + H$ (i.e. H homogeneous) in dimension 5 in general, but we did not know this yet at that time.

In order to show that for homogeneous quasi-translations $x+H$ in dimension 5 that come from singular Hessians, there are two independent linear relations between the components of H, Gordan and Nöther use geometrical arguments to obtain crucial information about homogeneous quasi-translations in dimension 5. But these arguments were very hard to follow. It is with this thesis that I hope to have a reconstruction in modern language of their arguments, that is less hard to follow.

Since quasi-translations are used to understand singular Hessians, in particular nilpotent Hessians, and not vice versa, the chapter about quasi-translations (chapter 3) precedes that about nilpotent Hessians (chapter 5). But let us continue with the story about homogeneous polynomials in dimension 5 for which the Hessian is nilpotent.

Wang's theorem is about quasi-translations in dimension 3, so the question is still how this theorem helped us to obtain that polynomial maps of the form $x + H$ with H homogeneous and $\mathcal{J}H$ symmetric satisfy the Jacobian conjecture in dimension $n = 5$. The reason for that is that we reduced the Jacobian conjecture for these maps in dimension $n = 5$ to maps $x + H$ in dimension $n = 3$ with $\mathcal{J}H$ symmetric and nilpotent, but H not necessarily homogeneous.

Now Wang's theorem seems more natural, since it is about quasi-translations in dimension 3 that do not need to be homogeneous either. Using Wang's theorem, one can classify all polynomials h in dimension 3 for which the Hessian has determinant zero, and again by threshing with the trace condition, one can prove that $\deg(h - g(Tx)) \le 1$ for some $g \in \mathbb{C}[x_1, x_2]$ and a $T \in \text{Mat}_3(\mathbb{C})$. We can only get $\deg(h - g(Tx)) \le 1$ instead of $h = g(Tx)$, because h is not homogeneous: linear terms of h are differentiated away when we take the Hessian.

Next we showed that the Jacobian conjecture is satisfied for maps $x + H$ in dimension $n = 3$ with $\mathcal{J}H$ symmetric and nilpotent. What about $n = 4$. Notice that Van den Essen and Washburn solved the case that H is homogeneous. A first question that comes into mind is whether Wang's theorem extends to dimension 4. The answer to that question is no, but just as above, we did not know this yet at that time.

However, I was able to prove that Wang's theorem holds for quasi-translations in dimension 4 that come from nilpotent (singular) Hessians. So if h is a polynomial in dimension 4 such that $\mathcal{H}h$ is nilpotent, then for the quasi-translation $x + Q$ that can be made from h by virtue of $\det \mathcal{H}h = 0$, the components of Q are linearly dependent over $\mathbb{C}$. With additional arguments that rely on the nilpotency of $\mathcal{H}h$, we obtained that $\deg(h - g(Tx)) \leq 1$ for some $g \in \mathbb{C}[x_1, x_2, x_3]$ and a $T \in \mathrm{Mat}_4(\mathbb{C})$. Next we showed that the Jacobian conjecture is satisfied for maps $x+H$ in dimension $n = 4$ with $\mathcal{J}H$ symmetric and nilpotent.

Chapter 5 is called 'Nilpotent Hessians', but the first half of it is about Hessians that are only singular. The results of chapter 3 about quasi-translations are used to study these Hessians.

1.3.2 Quasi-translations and the dependence problem

Notice that in case $g \in \mathbb{C}[x_1, x_2, \ldots, x_{n-1}]$, the last column and last row of $\mathcal{H}g$ are zero. If $T \in \mathrm{Mat}_n(\mathbb{C})$, then there exists a nonzero vector λ such that $T\lambda$ is dependent of e_n. Since

$$\mathcal{H}g(Tx) = T^{\mathrm{t}}(\mathcal{H}g)|_{x=Tx}T$$

it follows that $\mathcal{H}g(Tx)\lambda = 0$ and $\lambda^{\mathrm{t}}\mathcal{H}g(Tx) = 0$. If $h = g(Tx)$ or $\deg(h - g(Tx)) \leq 1$, then $\mathcal{H}h\lambda = 0 = \lambda^{\mathrm{t}}\mathcal{H}h$ as well. So both the rows and the columns of $\mathcal{H}h$ are dependent over $\mathbb{C}$ in case $\deg(h - g(Tx)) \leq 1$ and $g \in \mathbb{C}[x_1, x_2, \ldots, x_{n-1}]$.

On the other hand, if the rows of $\mathcal{H}h$ are dependent over $\mathbb{C}$, then the columns are dependent over $\mathbb{C}$ as well due to the symmetry, and $\deg(h-g(Tx)) \leq 1$ for some $g \in \mathbb{C}[x_1, x_2, \ldots, x_{n-1}]$ and a $T \in \mathrm{Mat}_n(\mathbb{C})$. In other words, showing that $\deg(h-g(Tx)) \leq 1$ for some $g \in \mathbb{C}[x_1, x_2, \ldots, x_{n-1}]$ and a $T \in \mathrm{Mat}_n(\mathbb{C})$ is nothing else than showing that ∇h satisfies the dependence problem.

So in chapter 5, which is described above, we showed for several gradient maps that they satisfy the dependence problem. We proved the invertibility of maps $x + H$ in dimension 5 with H homogeneous and $\mathcal{J}H$ symmetric and nilpotent, by showing the corresponding dependence problem and showing the invertibility of maps $x + H$ in dimension 3 with $\mathcal{J}H$ symmetric and nilpotent, but H not necessarily homogeneous.

In a similar manner, the invertibility of maps $x + H$ in dimension 6 with H homogeneous and $\mathcal{J}H$ symmetric and nilpotent follows from an affirmative answer to the corresponding dependence problem and the invertibility of maps $x + H$ in dimension 4 with $\mathcal{J}H$ symmetric and nilpotent, but H not necessarily homogeneous. We already showed the latter assertion, but the dependence problem for homogeneous H in dimension 6 with $\mathcal{J}H$ symmetric and nilpotent is still open. Since this dependence problem implies the dependence problem for homogeneous H in dimension 3 by way of corollary 2.2.6, we started studying this latter problem.

Notice that the dependence problem for homogeneous H in dimension 3 was already proved by David Wright for the case $d = 3$ by computations. By further computations we came as far as $d = 7$. We noticed that homogeneous H in dimension 3 of degree 7 at most with $\mathcal{J}H$ nilpotent are so-called linearly triangularizable. Next, we proved that an affirmative answer to the dependence problem for homogeneous maps in dimension 3 implies that such maps are linearly triangularizable.

Definition 1.3.1. We call a map H over the base ring A *linearly triangularizable*, if there exists a $T \in \mathrm{GL}_n(A)$ such that the Jacobian of $T^{-1}H(Tx)$ is triangular, i.e. zero on one side of the main diagonal.

Assume the Jacobian of $T^{-1}H(Tx) = T^{-1}(\mathcal{J}H)_{x=Tx}T$ is triangular, then the eigenvalues of $(\mathcal{J}H)_{x=Tx}$ appear on the diagonal of $\mathcal{J}(T^{-1}H(Tx))$. So the diagonal of $\mathcal{J}(T^{-1}H(Tx))$ is zero in case $\mathcal{J}H$ is nilpotent and the base ring A is reduced.

Notice that for homogeneous quasi-translations $x + H$ in dimension 3 over $\mathbb{C}$, Gordan and Nöther already proved the the rows of $\mathcal{J}H$ are dependent over $\mathbb{C}$. So in order to show the dependence problem for homogeneous Jacobians in dimension three, the case that $x + H$ is not a quasi-translation remained.

A first step in showing this case was a classification theorem for homogeneous maps H over $\mathbb{C}$ such that $\mathrm{rk}\mathcal{J}H \le 2$. We obtained this theorem from Bertini's irreducibility theorem, but the current proof uses Lüroth's theorem. See section 4.3. Next, we proved the dependence problem for homogeneous Jacobians in dimension three. The whole process of proving this dependence problem took about a month, including the classification theorem for homogeneous maps H over $\mathbb{C}$ such that $\mathrm{rk}\mathcal{J}H \leq 2$.

Months later, we got rid of the trace condition, that is, instead of assuming that all three eigenvalues of $\mathcal{J}H$ are zero, we only needed that two eigenvalues are zero. With only two eigenvalues being zero, the quasi-translation case $\mathcal{J}H \cdot H = 0$ becomes $\mathcal{J}H \cdot H = \mathrm{tr}\mathcal{J}H \cdot H$. For the other case, the trace condition was already unused in the original proof of the dependence problem for homogeneous Jacobians in dimension three.

So it was only the quasi-translation case that required another proof. At the end, we got rid of the trace condition for all homogeneous quasi-translations $x + H$ over $\mathbb{C}$ with $\mathrm{rk}\mathcal{J}H = 2$. For such quasi-translations, Gordan and Nöther already proved that there are two independent relations over $\mathbb{C}$ between the rows of $\mathcal{J}H$, but the condition that $x + H$ is a quasi-translation can be replaced by $\mathcal{J}H \cdot H = \mathrm{tr}\mathcal{J}H \cdot H$. We use the above classification theorem for homogeneous Jacobians of rank 2 to obtain this result.

Next, we started trying to find counterexamples to the dependence problem. We first found a non-homogeneous quasi-translation $x + H$ in dimension 4 over $\mathbb{C}$ for which the rows of $\mathcal{J}H$ are not dependent over $\mathbb{C}$. See section 3.6. Now it seemed interesting to find a way to homogenize quasi-translations in order to get a homogeneous counterexample. One might be able to homogenize a quasi-translation if it has homogeneous invariants (or kernel elements if we see them as derivations).

A quasi-translation $x + H$ of degree d with homogeneous invariants p and q of degree 2 and 3 respectively can be homogenized because

$$x + \tilde{H} = x + q^d H\left(\frac{p}{q}x\right)$$

is another quasi-translation and $\tilde{H}$ is homogeneous of degree $2d$. Homogeneous invariants of degree 1 are not so desirable, since they correspond to linear relations between the rows of $\mathcal{J}H$.

So we started looking for quasi-translations with homogeneous invariants, homogeneous invariants of degree 2 to be precise. Now having x_1x_2 as an invariant is not such a good idea, because its factors x_1 and x_2 are automatically invariants as well. So it is a better idea to take $x_1x_2 + x_3^2$ as a candidate invariant. We took the general linear map L in dimension 3 and computed which solutions are possible for its coefficients if $x + L$ must be a quasi-translation with $x_1x_2 + x_3^2$ as an invariant. This did not give any solutions except the identity.

Next, we did the same thing in dimension 4 with $x_1x_2 - x_3x_4$ as an invariant. This did give solutions besides the identity. Notice first that the above invariant can be permuted to $x_1x_4 - x_2x_3$. One of the solutions for $x_1x_4 - x_2x_3$ as invariant was

$$H := \big(b(ax_1 - bx_2), a(ax_1 - bx_2), b(ax_3 - bx_4), a(ax_3 - bx_4)\big)$$

Then I saw (up to permutation) that the map

$$(x, y) + \big(H(x), H(y)\big)\big|_{a=x_1x_4-x_2x_3, b=y_1y_4-y_2y_3}$$

in dimension $2n = 8$ was a counterexample to the dependence problem for homogeneous Jacobians. This was on July 20, 2004. Next, I glued x_3 and y_3 on one hand and x_4 and y_4 on the other together to obtain

$$(x, y_1, y_2) + \big(H(x), H_1(y), H_2(y)\big)\big|_{a=x_1x_4-x_2x_3, b=y_1x_4-y_2x_3}$$

which is the quasi-translation in dimension 6 of example 3.7.3. But that was on July 21, 2004, because midnight had passed. This led to the results in section 3.7.

Now the question was whether our results were sufficient for a bottle of Polish vodka, because is was not clear if the counterexample had to be cubic. But new discoveries followed 40 days later.

With H as above and $n = 5$, $x + (H|_{a=x_1x_4-x_2x_3, b=x_5^2}, 0)$ is a homogeneous quasi-translation. Now $x + (H|_{a=x_1x_4-x_2x_3, b=x_5^2}, x_5^5)$ is not a quasi-translation, but $(H|_{a=x_1x_4-x_2x_3, b=x_5^2}, x_5^5)$ is homogeneous and $n-1$ of the n eigenvalues of its Jacobian are zero.

So the trace of the Jacobian of $(H|_{a=x_1x_4-x_2x_3, b=x_5^2}, x_5^5)$ is nonzero. In order to avoid that, I tried to replace the last component x_5^5 by a power of $x_1x_4 - x_2x_3$: the other homogeneous invariant of $x + (H|_{a=x_1x_4-x_2x_3, b=x_5^2}, 0)$. This is impossible without preserving homogeneity because the degree of x_5^5 is odd. But a small variation on the idea lead to

$$\tilde{H} := \big(x_5 H|_{a=x_1x_4-x_2x_3, b=x_5^2}, (x_1x_4 - x_2x_3)^3\big)$$

So the trace of $\mathcal{J}\tilde{H}$ was zero. A computation revealed that all other symmetric functions in the eigenvalues of $\mathcal{J}\tilde{H}$ were zero as well, i.e. that $\mathcal{J}\tilde{H}$

was nilpotent. After some thinking, I was able to prove with arguments that $\mathcal{J}\tilde{H}$ is nilpotent. This was on August 30, 2004.

With the same arguments, I was able to prove corollary 4.2.2 in section 4.2. Using this corollary, I found homogeneous counterexamples to the dependence problem, of degree 4 in dimension 6 and up, and counterexamples of degree 3 in dimension 10 and up. The cubic counterexamples were found on August 31, 2004, because midnight had passed again. See section 4.2.

So the Vodka question was not relevant any more. The bottle of Vodka has been passed to me by Arno van den Essen during his lecture on the A.M.S. conference in Mainz on June 2005. It was passed to him during a sanitary break of mine, so I was completely surprised.

So homogeneous quasi-translations play an important role in the counterexamples to the dependence problem for homogeneous Jacobians. Notice that for homogeneous quasi-translations, the inverse map is homogeneous as well. The converse is true as well: homogeneous Keller maps over a commutative ring that have an inverse map of homogeneous Keller type are quasi-translations.

This is because for power series $x + H$ such that H does not have terms of degree less than 2, the inverse power series $x - G$, which exists on account of [24, Th. 1.1.2], has the property that the lowest degree parts of G and H are the same. The reader may show this.

1.3.3 Power linear Keller maps

Power linear maps are maps of the form $(Ax)^{*d}$ where A is a matrix over the base ring. Power linear Keller maps are of the form $x + (Ax)^{*d}$ and satisfy the Keller condition, with A as above.

After the results of the previous subsection, there was a period that I did not find interesting new results. Then I met a preprint by He Tong about power linear Keller maps. We wrote an article together where we showed that in some situations, such maps are not only linearly triangularizable, but the triangularization $x + T^{-1}(ATx)^{*d}$ can be chosen power linear as well.

We called such power linear maps ditto linearly triangularizable. Power linear maps $x + (Ax)^{*d}$ such that $d \geq 3$ and the rank or corank of A is less than 3 are ditto linearly triangularizable. But there exists a quadratic linear Keller map in dimension 6 with $\mathrm{rk} A = 3$ that is linearly triangularizable, but not ditto linearly triangularizable.

Due to Tong, I had become acquainted with power linear Keller maps. The next thing I did was making a cubic linear counterexample to the dependence problem for homogeneous Jacobians, one in dimension 53. I called this example the Herbie example after the Volkswagen Beetle Herbie with the number 53 on it.

After that I investigated Gorni-Zampieri pairing. Gorni and Zampieri formulated a set of rules to define that a homogeneous map and a power linear map are paired, thus making the reduction from homogeneous maps to power linear maps by Drużkowski more explicit. I proved that if H and G are GZ-paired, then G satisfies the linear dependence problem in case H does, but not the other way around. Furthermore, I showed that H is linearly triangularizable, if and only if G is, in case H and G are GZ-paired.

On a conference in Hanoi on October 2006, Tong conjectured that for large d compared to the corank of A, power linear Keller maps $x + (Ax)^{*d}$ are ditto linearly triangularizable. In the plane back home, we saw that it was a good idea trying to do something with Mason's theorem, a theorem that Stefan Maubach had just used in his research.

After improving a generalized version of this theorem, by replacing the pairwise relatively prime condition by a condition for vanishing subsums, we were able to use this theorem to prove Tong's conjecture and a similar result for so-called Zhao graphs. Zhao graphs were introduced by Wenhua Zhao to describe homogeneous nilpotent Hessians. Furthermore, Maubach's research was benefited by this adaptation as well. See appendix B for Mason's theorem. One can find a reference to a preprint of appendix B in [28] by David Finston and Stefan Maubach.

Another preprint of Tong led me to the question in which cases linear triangularizability of power linear Keller maps $x + (Ax)^{*d}$ implies ditto linear triangularizability. This problem is now completely solved in the sense that for each n, d, r, I can either construct a power linear Keller map $x + (Ax)^{*d}$ with $r = \mathrm{rk} A$ that is linear triangularizable but not ditto, or prove the above implication for such maps.

All of the above results about power linear Keller maps are in chapter 6. Later on, I proved that power linear Keller maps $(Ax)^{*d}$ with $d \geq 3$ and $\mathrm{cork} A = 3$ are ditto linearly triangularizable. This is the main result of chapter 7. Furthermore, I proved that for any $d \geq 1$, power linear maps $(Ax)^{*d}$ over $\mathbb{C}$ with nilpotent Jacobians are linearly triangularizable in case $n \leq 7$. If $d = 3$, then this result can be improved to $n \leq 8$. This and some more results can be found in chapter 7. Some of those results are about Zhao graphs instead of power linear maps, because just as in chapter 6, some results for power linear Keller maps and Zhao graphs can be proved with the same techniques.

1.3.4 Writing this thesis

I first wrote chapters 3 to 6 in that order and next chapter 2 and this introduction. At last, I added chapter 7. The new thing in the results of chapter 3 was that I have evaded derivations. Section 4 contains a new result, namely that the dependence problem for homogeneous H has an affirmative answer in dimension 4 if $\mathrm{rk}\mathcal{J}H \le 2$ and at least three of the four eigenvalues of $\mathcal{J}H$ are zero.

In addition, Gaetano Zampieri improved the cubic homogeneous counterexample to the dependence problem from dimension 10 to dimension 9. After understanding the new counterexample, I constructed another cubic homogeneous one in dimension 9. For the latter map H, $x+H$ does not have cubic homogeneous invariants. And of course no linear invariants, because it is a counterexample. See section 4.2. With Ricardo dos Santos Freire Jr. and Gianluca Gorni, Zampieri constructed a cubic homogeneous Keller map in dimension 11 without linear or quadratic homogeneous invariants: see [47].

Chapter 5 contains the most new results. First, I showed that homogeneous polynomials h over $\mathbb{C}$ with $\mathrm{rk}\mathcal{H}h \le 3$ can be expressed as a polynomial in $\mathrm{rk}\mathcal{H}h$ linear forms (and vice versa, but that is trivial). Another new result is the classification of all polynomials h over $\mathbb{C}$ such that $\mathrm{rk}\mathcal{H}h \le 2$. Next, I proved the dependence problem and the Jacobian conjecture for $x + H$, where $H = \nabla h$ with h as above. That is, either $\mathrm{rk}\mathcal{H}h \le 2$ or h homogeneous and $\mathrm{rk}\mathcal{H}h = 3$.

Another thing I did was looking whether unipotent Keller maps with symmetric Jacobians are linearly triangularizable in case the dimension or the Jacobian rank is small. Notice that triangular symmetric Jacobians are always diagonal matrices. But for nilpotent symmetric Jacobians that are linearly triangularizable, one can prove that the triangularization can be chosen in such a way that its Jacobian is symmetric with respect to the anti-diagonal. This is shown in section 5.8. The results about linear triangularizability of unipotent Keller maps over $\mathbb{C}$ with nilpotent symmetric Jacobians are summarized in the last theorem of chapter 5. All the $H = \nabla h$

in the above paragraph are linearly triangularizable.

Another result that I obtained when I wrote this thesis is the classification of quadratic homogeneous maps with nilpotent Jacobians in dimension 5. Fifteen years ago, Engelbert Hubbers started these computations, but he had to give up due to lack of computer memory. But as time progresses, so did computer hardware. The calculations are described in appendix A. Another result that is obtained by calculations is the classification of unipotent maps of degree 4 in dimension 3. Those calculations were done a few years ago.

In chapter 2, I added several symmetry reductions of the Jacobian conjecture to the above-mentioned reduction of the Jacobian conjecture to Hessians. For instance, the Jacobian conjecture for maps $F = X + H$ in dimension $4n$ over $\mathbb{C}$, such that the symmetry group of $\mathcal{J}H$ is the full dihedral group of the square, implies the Jacobian conjecture for maps $F = x + H$ in dimension n over $\mathbb{C}$, where $X = (x_1, x_2, \ldots, x_{4n})$. Over base ring $\mathbb{R}$, the symmetry group may be any subgroup of the full dihedral group of the square that contains a reflection, but without the reflection in the main diagonal. On the other hand, there are symmetry conditions for $\mathcal{J}H$, such that the Jacobian conjecture is trivially satisfied for maps of the form $F = x + H$. If for example $F = x + H$, such that $\mathcal{J}H$ is symmetric in horizontal direction and antisymmetric in vertical direction or vice versa, then F is a quasi-translation and hence invertible.

At last, I wrote chapter 7. Most of the results in it were obtained before writing this thesis, but some new results were added, especially results about Zhao graphs.

1.4 Notations

x is the vector $(x_1, x_2, \ldots, x_n)$ or just the sequence $x_1, x_2, \ldots, x_n$, depending on the context. For y and z we have similar conventions. So x, y and z depend implicitly on n, a natural number greater than zero.

For instance, in $\mathbb{C}[x]$ we have that x is a sequence and in $\mathcal{J}H \cdot x = dH$, we have that x is a vector. We shall mix up sequences and vectors in more occasions.

$\mathcal{J}_x H$ and $\mathcal{J}_y H$ are the Jacobian of H with respect to x and y respectively, et cetera. $\mathcal{J}H$ is an abbreviation of $\mathcal{J}_x H$ and $\nabla h = \nabla_x h$ is the transpose of

$\mathcal{J}h$, where h has only one component. So the Hessian $\mathcal{H}h$ of h satisfies

$$\mathcal{H}h = \mathcal{J}(\nabla h)$$

By $M|_{a=b}$ we mean the substitution of b into a in M, where M is a matrix in most occasions.

The reverse of a vector $v = (v_1, v_2, \ldots, v_m)$ is $v^{\mathrm{r}} = (v_m, v_{m-1}, \ldots, v_1)$. We see a matrix A as a vector of its rows $A_1, A_2, \ldots, A_m$, so $A = (A_1, A_2, \ldots, A_m)$ and $A^{\mathrm{r}} = (A_m, A_{m-1}, \ldots, A_1)$. The transpose of a matrix M is denoted as M^{t}. Vectors are seen as column matrices and hence, v^{t} is a row matrix if v is a vector.

Matrix rows are not seen as vectors, because matrix rows A_i of A are seen as row matrices and vectors are seen as column matrices. e_i denotes the i-th standard basis vector, but not always the one of dimension n. I_m is the identity matrix of size m. So we have that

$$A_i = e_i^{\mathrm{t}} A$$

is the i-th row of A and Ae_i is the i-th column of A.

i denotes the imaginary unit and e is Euler's number, for instance

$$\mathrm{e}^{\pi \mathrm{i}} + 1 = 0$$

Notice that i and e and also the superscript t and r for the transpose and reverse are in roman font, i.e. not italic. An italic i is nearly always an index and superscript italic is nearly always exponentiation.

$\deg f$ is the total degree of f or the total degree of f with respect to x: if I am not mistaken, then there are no situations that those degrees of f are different. f might have more components: in that case, the largest degree is considered. $\deg_x H$ and $\deg_y H$ are the degree of H with respect to x and y respectively, et cetera.

If t is an indeterminate, then we have the the following definitions.

$$\begin{aligned}
f(t) = \mathrm{O}(t^i) &\iff \deg_t f(t) \le i \\
f(t) = \Theta(t^i) &\iff \deg_t f(t) = i \\
f(t) = \Omega(t^i) &\iff \deg_t f(t) \ge i
\end{aligned}$$

But if we have a number ϵ that is close to zero instead of t, then we have the following.

$$f(\epsilon) = \mathrm{O}(\epsilon^i) \iff \deg_t f\left(\frac{1}{t}\right) \le -i$$
$$f(\epsilon) = \Theta(\epsilon^i) \iff \deg_t f\left(\frac{1}{t}\right) = -i$$
$$f(\epsilon) = \Omega(\epsilon^i) \iff \deg_t f\left(\frac{1}{t}\right) \ge -i$$

The connection with the definition of Landau symbols is that t is large and ϵ is small. i might be negative for so-called Laurant polynomials.

Endomorphisms and polynomial maps are essentially the same, but we view things as polynomial maps. So if we write for instance $\phi_1 \circ \phi_2$ for automorphisms ϕ_1 and ϕ_2, we mean $(x|_{x=\phi_1(x)})|_{x=\phi_2(x)}$, since the order of invertible polynomial maps is opposite to that of automorphisms.

For homogeneous maps (of degree d), the components must be homogeneous of a fixed degree (d) or zero. But for homogeneous Keller maps F, we define that $F - x$ is homogeneous. This is because only linear maps are Keller maps that are homogeneous in the usual sense. For nilpotent Keller maps F, we define that $\mathcal{J}(F - x)$ is nilpotent, since there are no Keller maps with nilpotent Jacobians. We call such maps F unipotent.

We use $*$ for the coordinate-wise product of vectors. Furthermore, we define v^{*m} as the coordinate-wise product of m copies of v. We do a similar thing for other operators instead of $*$, for instance $F^{\circ 100}$ is F iterated 100 times. The Jacobian conjecture is concerned with the existence of $F^{\circ(-1)}$. If R is a ring and we write R^m, then it would be more consistent to write $R^{\times m}$, since the Cartesian product of m copies of R is what we mean. This is because without reading this section, R^m is more likely to be understood than $R^{\times m}$.

With $\mathrm{GL}_n(A)$ ($\mathrm{SL}_n(A)$), we mean the general (special) linear group of dimension n over A. Similarly, we write $\mathrm{GO}_n(A)$ ($\mathrm{SO}_n(A)$) for the general (special) orthogonal group of dimension n over A. We write $\mathbb{Q}(A)$ for the quotient field of an integral domain A.

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Chapter 2

Symmetric Jacobians

2.1 Symmetric variants of the Jacobian conjecture

This section is about polynomial maps with a certain symmetry in their Jacobians, and whether the Jacobian conjecture is satisfied for such maps, or whether it is sufficient to prove the Jacobian conjecture for such maps. We will look at polynomial maps of the form

$$F = x + H \tag{2.1}$$

without constant part, where x is the linear part of F.

Other conditions on F depend on the interests of the reader. There are several options. One possibility is assuming that H is homogeneous of degree ≥ 2. For unipotent maps $F = x + H$ in general (i.e. $\mathcal{J}H$ nilpotent), x does not need to be the linear part of F, but in that case, one can prove that $x + tH$ is a Keller map. If we consider t as a variable when determining the linear part, then x is the linear part of $x + tH$, (provided H has no constant part). On the other hand, if $x + tH$ is a Keller map, then $\mathcal{J}H$ is nilpotent. So the reader who is interested in unipotent maps can be served with assuming that F has the form

$$F = x + tH \tag{2.2}$$

Another option is to take your favorite integer $d \geq 2$ and consider F as in (2.1) or (2.2) of degree d only, possibly combined with some of the conditions above.

In this chapter, we write $x_{n+i} = y_i$ for all $i \in \{1, 2 \ldots, n\}$, so $y = (x_{n+1}, x_{n+2}, \ldots, x_{2n})$.

Definition 2.1.1. □(K, n) means that the Jacobian conjecture is satisfied for n-dimensional maps F as above over the field K.

⧅(K, n) and ⧄(K, n) mean that the Jacobian conjecture is satisfied for n-dimensional maps F as above that have a symmetric Jacobian with respect to the diagonal and the anti-diagonal respectively.

⬔(K, n) and ◩(K, n) mean that the Jacobian conjecture is satisfied for n-dimensional maps F as above for which $\mathcal{J}H$ is anti-symmetric (i.e. applying the symmetry negates the matrix) with respect to the diagonal and the anti-diagonal respectively.

In the definition of ⧅(K, n), the symmetry is partially an antisymmetry, namely where colors on opposite sides of the diagonal do not match.

Theorem 2.1.2 (Meng)**.** *Assume $K \in \{\mathbb{C}, \mathbb{R}\}$. Then* ⧄$(K, 2n)$ *implies* □(K, n).

Proof. Assume $F = x + H$ is an instance of □(K, n). Put $f := \sum_{i=1}^{n} y_i F_i$. Then

$$\mathcal{H}_{x,y} f = \begin{pmatrix} * & (\mathcal{J}F)^{\mathrm{t}} \\ \mathcal{J}F & 0 \end{pmatrix} = \begin{pmatrix} * & \rotatebox{90}{\reflectbox{$\mathcal{J}F$}} \\ \mathcal{J}F & 0 \end{pmatrix}$$

and

$$\mathcal{H}_{x,y} f(x, y^{\mathrm{r}}) = \begin{pmatrix} * & ((\mathcal{J}F)^{\mathrm{r}})^{\mathrm{t}} \\ (\mathcal{J}F)^{\mathrm{r}} & 0 \end{pmatrix} = \begin{pmatrix} * & \rotatebox{-90}{$\mathcal{J}F$} \\ \reflectbox{$\mathcal{J}F$} & 0 \end{pmatrix}$$

Since the constant part of the latter matrix is $I_{2n}^{\mathrm{r}} = \reflectbox{I_{2n}}$, $\reflectbox{$\mathcal{H}_{x,y} f(x, y^{\mathrm{r}})$} = (\mathcal{H}_{x,y} f(x, y^{\mathrm{r}}))^{\mathrm{r}}$ has constant part I_{2n}. So

$$\reflectbox{$\mathcal{H}_{x,y} f(x, y^{\mathrm{r}})$} = \big(\mathcal{H}_{x,y} f(x, y)\big)^{\mathrm{r}} = \begin{pmatrix} \mathcal{J}F & 0 \\ * & (((\mathcal{J}F)^{\mathrm{r}})^{\mathrm{t}})^{\mathrm{r}} \end{pmatrix} = \begin{pmatrix} \mathcal{J}F & 0 \\ * & \rotatebox{90}{$\mathcal{J}F$} \end{pmatrix}$$

is equal to

$$I_{2n} + \begin{pmatrix} \mathcal{J}H & 0 \\ * & \rotatebox{90}{$\mathcal{J}H$} \end{pmatrix} = I_{2n} + \begin{pmatrix} \mathcal{J}H & 0 \\ * & (((\mathcal{J}H)^{\mathrm{r}})^{\mathrm{t}})^{\mathrm{r}} \end{pmatrix}$$

It follows that $\nabla_{y^{\mathrm{r}},x^{\mathrm{r}}} f(x, y^{\mathrm{r}}) = (\nabla_{x,y} f(x, y^{\mathrm{r}}))^{\mathrm{r}}$ is an instance of $\boxslash(K, 2n)$. Notice that

$$\det \mathcal{J}(\nabla_{y^{\mathrm{r}},x^{\mathrm{r}}} f(x, y^{\mathrm{r}})) = (\det \mathcal{J} F)^2$$

Since the first n components of $\nabla_{y^{\mathrm{r}},x^{\mathrm{r}}} f(x, y^{\mathrm{r}})$ are exactly those of F, if follows that F is invertible in case $\nabla_{y^{\mathrm{r}},x^{\mathrm{r}}} f(x, y^{\mathrm{r}})$ is. This gives the desired result. □

In [41], the author G. Meng constructs the map $f := \sum_{i=1}^{n} y_i F_i$ in the above proof. The corresponding gradient map $\nabla_{x,y} f$ has symmetry $\boxbslash$, but its linear part is (y, x) in case F has linear part x. In order to restore the linear part to (x, y), we composed $\nabla_{x,y} f$ with linear maps in the above proof, resulting $\nabla_{y^{\mathrm{r}},x^{\mathrm{r}}} f(x, y^{\mathrm{r}})$ with linear part (x, y) and symmetry $\boxslash$. That is why the above theorem is considered to be due to Meng.

Let G be the group generated by

$$\begin{pmatrix} \emptyset & I_n \\ I_n & \emptyset \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} I_n^{\mathrm{r}} & \emptyset \\ \emptyset & I_n \end{pmatrix}$$

Notice that $(y^{\mathrm{r}}, x^{\mathrm{r}}) = (x^{\mathrm{r}}, y) \circ (y, x) \circ (x^{\mathrm{r}}, y) \in G$.

Theorem 2.1.3. *$\boxbslash(\mathbb{C}, N)$ and $\boxslash(\mathbb{C}, N)$ are equivalent.*

Proof. Assume first that N is even, say $N = 2n$. Let $f(x, y) = \sum_{i=1}^{n} y_i x_i + h(x, y)$. Then

$$\mathcal{H}_{x,y} f(x, y^{\mathrm{r}}) = \begin{pmatrix} 0 & I_n^{\mathrm{r}} \\ I_n^{\mathrm{r}} & 0 \end{pmatrix} + \mathcal{H}_{x,y} h = I_{2n}^{\mathrm{r}} + \mathcal{H}_{x,y} h(x, y^{\mathrm{r}})$$

and by reversing the components, we see that $\nabla_{y^{\mathrm{r}},x^{\mathrm{r}}} f(x, y^{\mathrm{r}}) = (\nabla_{x,y} f(x, y^{\mathrm{r}}))^{\mathrm{r}}$ is an instance of $\boxslash(\mathbb{C}, 2n)$, if we put the right conditions on f.

Now substitute $(x, y) = (x + \mathrm{i}y, x - \mathrm{i}y)$ in $\sum_{i=1}^{n} y_i x_i$ and take the Hessian:

$$\mathcal{H}_{x,y} \left(\sum_{i=1}^{n} (x_i - \mathrm{i}y_i)(x_i + \mathrm{i}y_i) \right) = \mathcal{H}_{x,y} \left(\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2 \right) = 2I_{2n}$$

So $\nabla(\frac{1}{2} f(x + \mathrm{i}y, x - \mathrm{i}y))$ is an instance of $\boxbslash(\mathbb{C}, 2n)$, if we put the right conditions on f.

Since $\nabla_{y^{\mathrm{r}},x^{\mathrm{r}}} f(x, y^{\mathrm{r}})$ and $\nabla(\frac{1}{2} f(x + \mathrm{i}y, x - \mathrm{i}y))$ can be obtained from each other by composition with linear maps in the group G above, the desired result follows. The case that N is odd is similar, because if $N = 2n+1$, then

$x_{n+2}, x_{n+3}, \ldots, x_{2n+1}$ play the role of $x_{n+1}, x_{n+2}, \ldots, x_{2n}$ in the above case $N = 2n$, and x_{n+1} is an extra variable, but since x_{n+1} is in the center, it is reflected onto itself and hence does not affect the symmetry. $\square$

Corollary 2.1.4. *$\boxbslash(\mathbb{C}, 2n)$ implies $\square(\mathbb{C}, n)$.*

Proof. This follows from the above theorem and theorem 2.1.2. $\square$

Theorem 2.1.5. *Assume $F = x + H$ is a polynomial map over $\mathbb{R}$. If $\mathcal{J}H$ is symmetric and nilpotent, then F is invertible.*

Proof. Assume $(\mathcal{J}H)^r = 0$ and $(\mathcal{J}H)^{r-1} \neq 0$. If $r \geq 2$, then

$$0 = (\mathcal{J}H)^{2r-2} = (\mathcal{J}H)^{r-1} \cdot (\mathcal{J}H)^{r-1} = (\mathcal{J}H)^{r-1} \cdot \big((\mathcal{J}H)^{r-1}\big)^{\mathrm{t}} = (\mathcal{J}H)^{r-1} \cdot \rotatebox{-90}{$(\mathcal{J}H)^{r-1}$}$$

Substituting reals in the variables x_i, y_j in the rows of $(\mathcal{J}H)^{r-1}$, we obtain rows of real numbers that are isotropic (self-orthogonal), and hence zero. Contradiction, so $r = 1$ and $H = 0$. So $F = x$ is invertible. $\square$

Theorem 2.1.6. *Assume $K \in \{\mathbb{C}, \mathbb{R}\}$. Then $\boxbslash(K, n)$ and $\boxslash(K, n)$ have affirmative answers.*

Proof. Assume $F = x + H$ is an instance of $\boxbslash(K, n)$. Since the diagonal of $\mathcal{J}H$ is zero, H_i does not contain x_i. Assume $\deg_x H \geq 2$. Then H_i has a term divisible by $x_j x_k$ (j may be k) for some i. Consequently H_j has a term divisible by $x_i x_k$, so $j \neq k$. In addition, H_k has a term divisible by $x_i x_j$.

But the coefficients of these three terms are pairwise mutually opposite. Contradiction, so $\deg_x H \leq 1$ and F is invertible. The proof for $\boxslash(K, n)$ is similar. $\square$

Theorem 2.1.7. *Assume $K \in \{\mathbb{C}, \mathbb{R}\}$. Then $\boxslash(K, 2n)$ and $\boxbslash(K, 2n)$ are equivalent.*

Proof. Let F be an instance of $\boxslash(K, 2n)$ and set

$$S := \begin{pmatrix} I_n & I_n^{\mathrm{r}} \\ -I_n^{\mathrm{r}} & I_n \end{pmatrix} = \begin{pmatrix} I_n & \rotatebox{180}{I_n^{r}} \\ -\rotatebox{180}{I_n^{r}} & I_n \end{pmatrix}$$

Now the Jacobian of $I_{2n}^{\mathrm{r}} F = \rotatebox{180}{\reflectbox{F}}$ has the symmetry $\boxbslash$, whence

$$\mathcal{J}\big(S^{\mathrm{t}} I_{2n}^{\mathrm{r}} F(Sx)\big) = S^{\mathrm{t}} \Big(\mathcal{J}\big(I_{2n}^{\mathrm{r}} F\big)\Big)\Big|_{x=Sx} S$$

has the symmetry $\boxbslash$ as well. But the linear part of $S^{\mathrm{t}} I^{\mathrm{r}}_{2n} F(S(x,y))$ is not equal to (x,y). However

$$\begin{aligned}
\begin{pmatrix} -\frac{1}{2}I_n & \emptyset \\ \emptyset & \frac{1}{2}I_n \end{pmatrix} S^{\mathrm{t}} I^{\mathrm{r}}_{2n} F\big(S(x,y)\big) &= \begin{pmatrix} -\frac{1}{2}I_n & \frac{1}{2}I^{\mathrm{r}}_n \\ \frac{1}{2}I^{\mathrm{r}}_n & \frac{1}{2}I_n \end{pmatrix} I^{\mathrm{r}}_{2n} F\big(S(x,y)\big) \\
&= \begin{pmatrix} \frac{1}{2}I_n & -\frac{1}{2}I^{\mathrm{r}}_n \\ \frac{1}{2}I^{\mathrm{r}}_n & \frac{1}{2}I_n \end{pmatrix} F\big(S(x,y)\big) \\
&= S^{-1} F\big(S(x,y)\big)
\end{aligned}$$

has (x,y) as linear part, and its symmetry is $\boxbslash$. The converse is similar. $\square$

Corollary 2.1.8 (Drużkowski)**.** *Assume $K \in \{\mathbb{C}, \mathbb{R}\}$. Then $\boxbslash(K,2n)$ implies $\square(K,n)$.*

Proof. This follows immediately from the above theorem and theorem 2.1.2. $\square$

In fact, Drużkowski considers maps with symmetry $\boxbslash$, but linear part $(-x,y)$ in [21]. Negating the first half of the map restores the linear part, and the symmetry becomes $\boxbslash$.

Definition 2.1.9. $\boxtimes(K,n)$ means that the Jacobian conjecture is satisfied for maps F as above that have a symmetric Jacobian with respect to both the diagonal and the anti-diagonal.

In the definitions of $\boxtimes(K,n)$, $\boxtimes(K,n)$ and $\boxtimes(K,n)$, some symmetries are anti-symmetries, namely when colors on opposite sides of the symmetry axis do not match.

In the definitions of $\boxtimes(K,n)$, $\boxtimes(K,n)$, $\boxtimes(K,n)$ and $\boxtimes(K,n)$, the symmetries are partially anti-symmetries.

Notice that $\boxtimes(K,n)$, $\boxtimes(K,n)$ and $\boxtimes(K,n)$ have affirmative answers as well as $\boxbslash(K,n)$ and $\boxslash(K,n)$, because the corresponding symmetries are stronger than at least one of those of $\boxbslash$ and $\boxslash$ in theorem 2.1.6.

Theorem 2.1.10. *Let $K \in \{\mathbb{C}, \mathbb{R}\}$. Then $\boxtimes(K,n)$, $\boxtimes(K,n)$, $\boxtimes(K,n)$ and $\boxtimes(K,n)$ have affirmative answers.*

Proof. Let $F = (x,y) + H(x,y)$ be an instance of $\boxtimes(K,n)$ or $\boxtimes(K,n)$ (the result for $\boxtimes(K,n)$ and $\boxtimes(K,n)$ is similar). We show that $\deg_{x,y} H \leq 1$. For that purpose, notice that above the anti-diagonal, $\mathcal{J}_{x,y}H$ is anti-symmetric with respect to the diagonal. Now the assumption that x_1 appears above the anti-diagonal in $\mathcal{J}_{x,y}H$ leads to a contradiction in a similar manner as in the proof of theorem 2.1.6, because the argument in this proof remains above the anti-diagonal of $\mathcal{J}_{x,y}H$, where $\mathcal{J}_{x,y}H$ is anti-symmetric.

So the first column of $\mathcal{J}_{x,y}H$ does not contain x or y. By the symmetry conditions, no border entries of $\mathcal{J}_{x,y}H$ contain x or y. The entries of $\mathcal{J}_{x,y}H$ that are not on the border do not contain y_n, and form a matrix with the same symmetry as $\mathcal{J}_{x,y}H$ itself. So by induction on n, it follows that $\deg_{x,y} H \leq 1$. So F is invertible, as desired. $\square$

Definition 2.1.11. $\boxplus(K,n)$, $\boxplus(K,n)$, $\boxplus(K,n)$, $\boxplus(K,n)$ have horizontal and vertical (anti-)symmetries in their definitions.

$\boxtimes(K,n)$, $\boxtimes(K,n)$, $\boxtimes(K,n)$, $\boxtimes(K,n)$ have horizontal, vertical, and diagonal (anti-)symmetries in their definitions.

$\boxdot(K,n)$ means that the Jacobian conjecture is satisfied for n-dimensional maps F as above that have Jacobians that are symmetric with respect to the center.

In the definition of $\boxdot(K,n)$, $\boxdot(K,n)$ and $\boxdot(K,n)$, the central point symmetry is (partially) an anti-symmetry.

Theorem 2.1.12. *Assume $K \in \{\mathbb{C}, \mathbb{R}\}$. Then $\square(K,n)$, $\boxplus(K,2n)$, and $\boxplus(K,2n)$ are equivalent.*

Proof. A conjugation with the map $(x,-y)$ shows that $\boxplus(K,2n)$ and $\boxplus(K,2n)$ are equivalent.

Let $F = x + 2H$ be an instance of $\square(K,n)$. Then F in invertible, if and only if (F,y) is invertible. Furthermore F satisfies the Keller condition, if and only if (F,y) does. Now

$$\begin{aligned}
&\frac{1}{2}\begin{pmatrix} I_n & -I_n \\ I_n & I_n \end{pmatrix}\begin{pmatrix} F(x) \\ y \end{pmatrix}\Bigg|_{x=x+y,y=y-x} \\
&\quad = \frac{1}{2}\begin{pmatrix} I_n & -I_n \\ I_n & I_n \end{pmatrix}\begin{pmatrix} x+y+2H(x+y) \\ y-x \end{pmatrix} \\
&\quad = \begin{pmatrix} x+H(x+y) \\ y+H(x+y) \end{pmatrix}
\end{aligned}$$

and conjugation with (x, y^{r}) gives the desired result. $\square$

Corollary 2.1.13. *Assume $K \in \{\mathbb{C}, \mathbb{R}\}$. Then $\boxdot(K, 2n)$ implies $\square(K, n)$.*

Proof. $\boxdot(K, N)$ implies $\boxplus(K, N)$ and $\boxplus(K, N)$, because the symmetry of $\boxdot$ is a subsymmetry of both $\boxplus$ and $\boxplus$. Now apply the above theorem. $\square$

Now assume that $F = x + H$ is power linear of *even* degree. Then the construction of an instance of $\boxplus(K, 2n)$ out of the instance F of $\square(K, n)$ gives a map that is power linear of even degree again, say $(x, y) + (B(x, y))^{*d}$. Since d is even, we can assume that B has symmetry $\boxplus$ instead of $\boxplus$. But that means that $B^2 = 0$.

In the general case, we can make $(x, y) + (B(x, y))^{*d}$ out of $F = x + (Ax)^{*d}$, where

$$B := \begin{pmatrix} ab & -b^2 \\ a^2 & -ab \end{pmatrix} \otimes A = \begin{pmatrix} abA & -b^2A \\ a^2A & -abA \end{pmatrix}$$

and $B^2 = 0$ because the left factor of the Kronecker tensor product squares to zero as well. More precisely, if

$$T := \begin{pmatrix} a \sqrt[d-1]{ab^d - a^d b} & -b \sqrt[d-1]{ab^d - a^d b} \\ a^d & -b^d \end{pmatrix}$$

then

$$T^{-1} = \left(\frac{1}{\sqrt[d-1]{ab^d - a^d b}} \right)^d \cdot \begin{pmatrix} b^d & -b \sqrt[d-1]{ab^d - a^d b} \\ a^d & -a \sqrt[d-1]{ab^d - a^d b} \end{pmatrix}$$

and one can compute now that $T^{-1}(F, y)|_{(x,y)=T(x,y)}$ is of the form $(x, y) + (B(x, y))^{*d}$. See also [20].

Theorem 2.1.14. *$\boxbslash(K, n)$, $\boxtimes(K, 2n)$, and $\boxtimes(K, 2n)$ are equivalent.*

Proof. The proof is similar to that of the equivalence of $\square(K, n)$, $\boxplus(K, 2n)$, and $\boxplus(K, 2n)$. $\square$

Corollary 2.1.15. *$\boxtimes(\mathbb{C}, 4n)$ implies $\square(\mathbb{C}, n)$.*

Proof. $\boxtimes(K, N)$ implies $\boxtimes(K, N)$ and $\boxtimes(K, N)$, because the symmetry of $\boxtimes$ is a subsymmetry of both $\boxtimes$ and $\boxtimes$. Now apply the above theorem and corollary 2.1.4. $\square$

Theorem 2.1.16. *$\boxplus(K, N)$, $\boxplus(K, N)$, $\boxtimes(K, N)$, and $\boxtimes(K, N)$, have affirmative answers.*

Proof. Let $F = x + H$ be an instance of any of them. Then one can compute that

$$\mathcal{J}H \cdot H = 0 \tag{2.3}$$

In the next chapter, we will see that (2.3) is equivalent to the statement that $x - H$ is the inverse of F. So F is invertible, as desired. $\square$

Theorem 2.1.17. *$\boxdot(\mathbb{C}, 2n)$ and $\boxdot(\mathbb{C}, 2n)$ are equivalent. A similar result holds for $\boxdot(\mathbb{C}, 2n)$ and $\boxdot(\mathbb{C}, 2n)$.*

Proof. Conjugation with $(x, \mathrm{i}y)$ adapts the symmetry in the desired manner. $\square$

The following theorem shows that complex polynomial maps can be seen as real polynomial maps with a certain Jacobian symmetry.

Theorem 2.1.18. *$\square(\mathbb{C}, n)$ and $\boxdot(\mathbb{R}, 2n)$ are equivalent.*

Proof. Let F be an instance of $\square(\mathbb{C}, n)$. If F is invertible or of Keller type, then $(F(x), F(y))$ is invertible or of Keller type respectively as well. Furthermore,

$$\begin{aligned}
\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2}\mathrm{i} & \frac{1}{2}\mathrm{i} \end{pmatrix} \begin{pmatrix} F(x+\mathrm{i}y) \\ F(x-\mathrm{i}y) \end{pmatrix} &= \begin{pmatrix} \frac{1}{2}F(x+\mathrm{i}y) + \frac{1}{2}F(x-\mathrm{i}y) \\ \frac{1}{2}\mathrm{i}F(x+\mathrm{i}y) - \frac{1}{2}\mathrm{i}F(x-\mathrm{i}y) \end{pmatrix} \\
&= \begin{pmatrix} \operatorname{Re} F(x+\mathrm{i}y) \\ \operatorname{Im} F(x+\mathrm{i}y) \end{pmatrix} \qquad (2.4)
\end{aligned}$$

if x and y are considered to be real variables. Since the Jacobian of the map in the middle is

$$\begin{pmatrix} \frac{1}{2}(\mathcal{J}F)|_{x=x+\mathrm{i}y} + \frac{1}{2}(\mathcal{J}F)|_{x=x-\mathrm{i}y} & \frac{1}{2}\mathrm{i}(\mathcal{J}F)|_{x=x+\mathrm{i}y} - \frac{1}{2}\mathrm{i}(\mathcal{J}F)|_{x=x-\mathrm{i}y} \\ -\frac{1}{2}\mathrm{i}(\mathcal{J}F)|_{x=x+\mathrm{i}y} + \frac{1}{2}\mathrm{i}(\mathcal{J}F)|_{x=x-\mathrm{i}y} & \frac{1}{2}(\mathcal{J}F)|_{x=x+\mathrm{i}y} + \frac{1}{2}(\mathcal{J}F)|_{x=x-\mathrm{i}y} \end{pmatrix}$$

a conjugation with (x, y^{r}) gives an instance of $\boxdot(\mathbb{R}, 2n)$.

So we can make an instance of $\boxdot(\mathbb{R}, 2n)$ that is invertible or of Keller type from one of $\square(\mathbb{C}, n)$ that is invertible or of Keller type respectively. Notice that this construction is injective and that we are done as soon as we have shown that this construction is bijective. For that purpose, we show that the dimension of the vector space over $\mathbb{R}$ of instances of degree $\leq d$ of $\boxdot(\mathbb{R}, 2n)$ does not exceed that of $\square(\mathbb{C}, n)$.

Notice that the dimension of the vector space over $\mathbb{R}$ of polynomials of degree $\le d$ in $\mathbb{C}[x]$ is twice that of polynomials of degree $\le d$ in $\mathbb{R}[x]$, because we can split complex polynomials in real and imaginary parts. So the vector space over $\mathbb{R}$ of $2n$-tuples of polynomials of degree $\le d$ in $\mathbb{R}[x]$ is equal to that of n-tuples of polynomials of degree $\le d$ in $\mathbb{C}[x]$.

Consequently, it suffices to show that instances F of $\boxdot(\mathbb{R}, 2n)$ are completely determined by $F(x, 0)$. This follows by induction on the degree with respect to y: for every term t, the coefficient of $y_i t$ in F_j is determined by the coefficient of $x_i t$ in F_{2n+1-j} due to the symmetry of $\mathcal{J}F$. This gives the desired result. $\square$

2.2 Symmetric variants of the dependence problem

This section is about polynomial maps H with a certain symmetry in their Jacobians, and whether the (linear) dependence problem (for Jacobians) is satisfied for such maps. We say that $H \in K[x]^n$ satisfies the dependence problem if

$$\lambda^{\mathrm{t}} \mathcal{J}H = 0$$

for some nonzero $\lambda \in K^n$, or equivalently

$$\lambda_1 H_1 + \lambda_2 H_2 + \cdots + \lambda_n H_n = 0$$

in case H has no constant part. Notice that composition of H with invertible linear maps does not change whether H satisfies the dependence problem. See subsection 1.2.4 for more information about the dependence problem.

We will look at polynomial maps H of degree ≥ 2 without constant part. Other conditions on H depend on the interests of the reader. There are several options. One possibility is assuming that $\mathcal{J}H$ is nilpotent. Or just that $\det \mathcal{J}H = 0$. Or one of both combined with that H is homogeneous. Another options is to take your favorite integer $d \ge 2$ and consider H of degree d only, possibly combined with some of the conditions above.

One can assume that H is any map without linear terms as well, but in that case, H does not need to satisfy the dependence problem even for $n = 1$. But the condition $\det \mathcal{J}H = 0$ is more interesting. With $\det \mathcal{J}H = 0$, the dependence problem is satisfied for symmetric Jacobians up to $n = 2$ and even up to $n = 4$ if H is homogeneous in addition. We will prove this in section 5.

As a consequence of corollary 2.2.6 below (an analog of corollary 2.1.4), the dependence problem with $\det \mathcal{J}H = 0$ is satisfied for arbitrary Jacobians in dimension 1 and for homogeneous Jacobians in dimension 2. This is easier to prove directly, but the interesting thing is that corollary 2.2.6 gives all dimensions for which arbitrary and homogeneous Jacobians $\mathcal{J}H$ with $\det \mathcal{J}H = 0$ satisfy the dependence problem.

Definition 2.2.1. $\square[K, n]$ means that the dependence problem is satisfied for n-dimensional maps H as above over the field K.

$\boxbslash[K, n]$ and $\boxslash[K, n]$ mean that the Jacobian conjecture is satisfied for n-dimensional maps H as above that have a symmetric Jacobian with respect to the diagonal and anti-diagonal respectively.

Et cetera. We replace the parenthesis of the symmetric variants of the Jacobian conjecture by square brackets all the time.

Theorem 2.2.2. *$\square[\mathbb{C}, n]$ and $\boxdot[\mathbb{R}, 2n]$ are equivalent.*

Proof. Assume H is an instance of $\square[\mathbb{C}, n]$. Then by the proof of theorem 2.1.18, $(\operatorname{Re} H(x + \mathrm{i}y), \operatorname{Im} H(x + \mathrm{i}y))$ is an instance of $\boxdot[\mathbb{R}, 2n]$. Assume the components of the last map are linearly dependent over $\mathbb{R}$. Then by the proof of theorem 2.1.18, the components of $(H(x + \mathrm{i}y), H(x - \mathrm{i}y))$ are dependent over $\mathbb{C}$. By substituting $y = \pm\mathrm{i}x$, we obtain that the components of $H(2x)$ and hence those of $H(x)$ are dependent over $\mathbb{C}$. The converse is similar. $\square$

Corollary 2.2.3. *$\boxdot[\mathbb{C}, 2n]$ implies $\square[\mathbb{C}, n]$.*

Proof. From the proof of theorem 2.1.17, the equivalence of $\boxdot[\mathbb{C}, 2n]$ and $\boxdot[\mathbb{C}, 2n]$ follows (because composition with invertible linear maps does not change the state of the dependence problem for a polynomial map). Now apply the above theorem, using that $\mathbb{C} \supseteq \mathbb{R}$. $\square$

Theorem 2.2.4. *$\boxdot[\mathbb{R}, 4n]$ implies $\square[\mathbb{C}, n]$.*

Proof. Notice first that the proof of corollary 2.1.13 does not work to obtain that $\boxdot[K, 2n]$ implies $\square[K, n]$, because instances of $\boxplus[K, n]$ always have components that are linearly dependent over K.

By the above corollary, it suffices to show that $\boxdot[\mathbb{R}, 4n]$ implies $\boxdot[\mathbb{C}, 2n]$. Assume H is an instance of $\boxdot[\mathbb{C}, 2n]$. Then by theorem 2.2.2 above, we can make an instance of $\boxdot[\mathbb{C}, 4n]$, but that is the wrong symmetry. But the instance of $\boxdot[\mathbb{C}, 4n]$ has symmetry $\boxdot$ on every quadrant of its Jacobian. Now let P be the permutation

$$\left(\begin{array}{cccccc|cccccc}
1 & & & & & & 0 & & & & & \\
& 0 & & & \emptyset & & & 1 & & & \emptyset & \\
& & 1 & & & & & & 0 & & & \\
& & & 0 & & & & & & 1 & & \\
& & & & \ddots & & & & & & \ddots & \\
& \emptyset & & & 1 & & & \emptyset & & & 0 & \\
& & & & & 0 & & & & & & 1 \\ \hline
0 & & & & & & 1 & & & & & \\
& 1 & & & \emptyset & & & 0 & & & \emptyset & \\
& & 0 & & & & & & 1 & & & \\
& & & 1 & & & & & & 0 & & \\
& & & & \ddots & & & & & & \ddots & \\
& \emptyset & & & 0 & & & \emptyset & & & 1 & \\
& & & & & 1 & & & & & & 0
\end{array}\right)$$

By conjugation with $P = P^{-1}$, we can interchange the global symmetry and the quadrant symmetry. This gives the desired result. $\square$

Theorem 2.2.5. *Let $K \in \{\mathbb{C}, \mathbb{R}\}$. Then $\boxslash[K, 2n]$ implies $\square[K, n]$.*

Proof. Assume H is an instance of $\square[K, n]$. With the proof of theorem 2.1.2, we obtain a map $(H(x), G(x, y))$ that is an instance of $\boxslash[K, 2n]$, and

$$\mathcal{J}\begin{pmatrix} H(x) \\ G(x,y) \end{pmatrix} = \begin{pmatrix} \mathcal{J}H & 0 \\ * & (((\mathcal{J}H)^{\mathrm{r}})^{\mathrm{t}})^{\mathrm{r}} \end{pmatrix} = \begin{pmatrix} \mathcal{J}H & 0 \\ * & \rotatebox{-90}{$\mathcal{J}H$} \end{pmatrix}$$

Notice that the nilpotency (or the vanishing of the determinant) of the Jacobian of $(H(x), G(x, y))$ is completely determined by the nilpotency (or the vanishing of the determinant) of $\mathcal{J}H$. So we get another instance of $\boxslash[K, 2n]$ if we replace the part of G that has terms without y only by other terms without y. But we must not forget to preserve the symmetry. We do this by replacing the part of G that has terms without y only by $(x^{\mathrm{r}})^{*d} = (x_n^d, x_{n-1}^d, \ldots, x_2^d, x_1^d)$, where $d \geq 2$.

Now assume that $\boxslash[K, 2n]$ is satisfied. Then the components of (H, G) are linearly dependent over K, say that

$$\lambda^{\mathrm{t}} H + \mu^{\mathrm{t}} G = 0$$

where $\lambda, \mu \in K^n$ are not both zero. If $\mu = 0$, then the components of H are linearly dependent over K, as desired, so it suffices to show that $\mu = 0$. Since H has no terms with y, $\lambda^{\mathrm{t}} \mathcal{J}_y H = 0$ and we obtain that $\mu^{\mathrm{t}} \mathcal{J}_y G = 0$. So $\lambda^{\mathrm{t}} \mathcal{J}_y H \lambda^{\mathrm{r}} = \mu^{\mathrm{t}} \mathcal{J}_y G \lambda^{\mathrm{r}} = 0$ and by symmetry of $\mathcal{J}_{(x,y)}(H, G)$, $\lambda^{\mathrm{t}} \mathcal{J}_x H \mu^{\mathrm{r}} = 0$ as well. So

$$0 = (\lambda, \mu)^{\mathrm{t}} \mathcal{J} H (\lambda, \mu)^{\mathrm{r}} = \lambda^{\mathrm{t}} \mathcal{J}_y H \lambda^{\mathrm{r}} + \lambda^{\mathrm{t}} \mathcal{J}_x H \mu^{\mathrm{r}} + \mu^{\mathrm{t}} \mathcal{J}_y G \lambda^{\mathrm{r}} + \mu^{\mathrm{t}} \mathcal{J}_x G \mu^{\mathrm{r}}$$

Since the first three terms on the right hand side vanish, $\mu^{\mathrm{t}} \mathcal{J}_x G \mu^{\mathrm{r}} = 0$ follows. Looking at the part of $\mathcal{J}_x G$ with terms without y only, we obtain $\mu^{\mathrm{t}} \mathcal{J}_x (x^{\mathrm{r}})^{*d} \mu = 0$. This is only possible if $\mu = 0$, as desired. $\square$

Corollary 2.2.6. *$\boxbslash[\mathbb{C}, 2n]$ implies $\blacksquare[\mathbb{C}, n]$.*

Proof. This follows from the proof of theorem 2.1.3 and the result of the above theorem. $\square$

Notice that there is no converse of theorems 2.1.2 and 2.2.5. But if we define $\boxslash(K, n)$ and $\boxslash[K, n]$ as $\boxslash(K, n)$ and $\boxslash[K, n]$ respectively with the extra condition that the upper right quadrant of the Jacobian is zero, then we do have a converse. One can formulate extra conditions for $\boxbslash[\mathbb{C}, 2n]$ as well to get a converse of the above corollary. See subcase (5.15) in case 3 (of the proof of theorem 5.7.1) in section 5.7 to get some inspiration for this.

If we assume that $\det \mathcal{J} H = 0$ instead of that $\mathcal{J} H$ is nilpotent, then we can transform symmetries more freely, because we do not need to conjugate. We can just compose with maps in $\mathrm{GL}_n(K)$, where $K \in \{\mathbb{R}, \mathbb{C}\}$. Or with maps in $\mathrm{GL}_n(A)$, where A is an integral domain. Now if $\mathcal{J} H$ has symmetry $\boxslash$, then its trace is zero. So if $\mathcal{J} H$ has dimension 2, then $\mathcal{J} H$ is nilpotent, if and only if $\det \mathcal{J} H = 0$ and $\mathcal{J} H$ has symmetry $\boxslash$. We can use this observation to give an alternate proof of [24, Th. 7.2.25] by Van den Essen and Hubbers. See also [26].

Theorem 2.2.7. *Assume $H \in A[x]^n$ is a map in dimension $n = 2$ over a unique factorization domain $A \supseteq \mathbb{Q}$, such that $\mathcal{J} H$ is nilpotent. Then H has the form*

$$H = \big(bg(ax_1 - bx_2) + d, ag(ax_1 - bx_2) + c\big)$$

where $a, b, c, d \in A$ and $g \in A[y_1]$. In particular, the rows of $\mathcal{J}H$ are linearly dependent over A: $a\mathcal{J}H_1 - b\mathcal{J}H_2 = 0$.

Proof. Since $\mathcal{J}H$ is nilpotent, we have $\det \mathcal{J}H = 0$ and $\mathrm{tr}\mathcal{J}H = 0$. Notice that the relation $\mathrm{tr}\mathcal{J}H = 0$ is an anti-symmetry of the diagonal. If we can change this into a symmetry of the anti-diagonal, then we have a symmetric Jacobian, i.e. a Hessian.

In order to change the diagonal into an anti-diagonal, we must turn the matrix 90 degrees. So let us try applying a rotation of 90 degrees: you never know your luck. For that purpose, set

$$T := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and define $\tilde{H} := TH$. Indeed, we have

$$(\mathcal{J}\tilde{H})_{12} = T_{12} \cdot (\mathcal{J}H)_{22} = -T_{21} \cdot -(\mathcal{J}H)_{11} = T_{21} \cdot (\mathcal{J}H)_{11} = (\mathcal{J}\tilde{H})_{21}$$

so $\tilde{H}$ is symmetric. Now apply corollary 5.1.2 to obtain that $\tilde{H}$ is of the form $\nabla(g(ax_1 - bx_2) + (cx_1 - dx_2))$, i.e.

$$\tilde{H} = \big(ag'(ax_1 - bx_2) + c, -bg'(ax_1 - bx_2) - d\big)$$

and hence

$$H = T^{-1}\tilde{H} = \big(bg'(ax_1 - bx_2) + d, ag'(ax_1 - bx_2) + c\big)$$

So H is of the desired form. $\square$

Chapter 3

Quasi-translations

3.1 Introduction

Definition 3.1.1. A polynomial map F over a commutative ring A is called a *quasi-translation* if $2x - F$ is the inverse polynomial map of F.

So a quasi-translation is a polynomial map $x + H$ such that $x - H$ is its inverse. Our study of quasi-translations began with an article from 1876, by P. Gordan and M. Nöther. In that article, the authors' primary goal is to study Hessians of homogeneous polynomials that have determinant zero. Before we can explain the connection, we first need to prove the following fact.

Proposition 3.1.2. *Let A be a commutative ring with $\mathbb{Q}$ and $H : A^n \to A^n$ be a polynomial map. Then the following properties are equivalent:*

i) *$x + H$ is a quasi-translation, i.e. $x - H$ is the inverse polynomial map of $x + H$,*

ii) *$H(x + tH) = H$, where t is a new indeterminate,*

iii) *$\mathcal{J}H \cdot H = 0$.*

Proof.

i) ⇒ ii) Assume $x + H$ is a quasi-translation. Then $(x - H) \circ (x + H) = x$, whence

$$H(x + H) = \big(x - (x - H)\big) \circ (x + H) = x + H - x = H$$

We prove by induction on m that $H(x+mH) = H(x)$ for all $m \in \mathbb{N}$. Assume that $H(x+mH) = H(x)$ for some m. Substituting $x = x+H$ in it gives

$$H\big(x+(m+1)H\big) = H\big(x+H+mH(x+H)\big) = H(x+H) = H$$

and $H(x+mH) = H$ follows for all $m \in \mathbb{N}$ by induction.

Now let $d := \deg H$ and write $H_i(x+tH) - H_i = c_d t^d + c_{d-1}t^{d-1} + \cdots + c_1 t + c_0$. Since $H_i(x+mH) - H_i = 0$ for all $m \in \{0, 1, 2, \ldots, d\}$,

$$\begin{pmatrix} 1 & 0 & 0^2 & \cdots & 0^d \\ 1 & 1 & 1^2 & \cdots & 1^d \\ 1 & 2 & 2^2 & \cdots & 2^d \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & d & d^2 & \cdots & d^d \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Since $\mathbb{Q} \subseteq A$, the matrix on the left hand side is invertible, and $c_0 = c_1 = c_2 = \cdots = c_d = 0$ follows. So $H_i(x+tH) = H_i$ for all i, as desired.

ii) $\Rightarrow$ iii) By differentiating $H(x+tH) = H$ with respect to t, we obtain $\mathcal{J}H|_{x=x+tH} \cdot H = 0$. Now substitute $t = 0$ to get the desired result.

iii) $\Rightarrow$ i) Put $F := x+tH$ and let (G, t) be the power series inverse of (F, t) (this exists on account of [24, Th. 1.1.2]). Substituting $x = G$ in F, we obtain $x = F(G) = G + tH(G)$, so

$$G = x - tH(G) \tag{3.1}$$

Differentiating this with respect to t gives

$$\frac{\partial}{\partial t}G = -H(G) - t\frac{\partial}{\partial t}H(G)$$

So

$$\begin{aligned} \frac{\partial}{\partial t}H(G) &= \mathcal{J}H|_{x=G} \cdot \frac{\partial}{\partial t}G \\ &= \mathcal{J}H|_{x=G} \cdot \big(-H(G) - t\frac{\partial}{\partial t}H(G)\big) \\ &= -(\mathcal{J}H \cdot H)|_{x=G} - t\mathcal{J}H|_{x=G}\frac{\partial}{\partial t}H(G) \\ &= -t\mathcal{J}H|_{x=G}\frac{\partial}{\partial t}H(G) \end{aligned}$$

Looking at terms with minimal degree with respect to t, we obtain that $\frac{\partial}{\partial t}H(G) = 0$. It follows from (3.1) that

$$H(G) = H(G)|_{t=0} = H(x - tH(G))|_{t=0} = H$$

So $G = x - tH(G) = x - tH$. Now substitute $t = 1$ to get the desired result. $\square$

The idea of using a Vandermonde determinant has been shamelessly stolen from a student's homework, for she has chosen not to go for a Ph.D. and thus not to joy the world with her research qualities herself.

Corollary 3.1.3. *If $x + H$ is a quasi-translation over a commutative ring $A \supseteq \mathbb{Q}$, then $x + tH$ is a quasi-translation over $A[t]$ and $\mathcal{J}H$ is nilpotent.*

Proof. Assume $x+H$ is a quasi-translation. Then by ii) of proposition 3.1.2,

$$(x - tH) \circ (x + tH) = x + tH - tH(x + tH) = x + tH - tH = x$$

so $x + tH$ is a quasi-translation. Taking determinants, it follows from the chain rule that

$$\det\big(I_n - t(\mathcal{J}H)_{x=x+tH}\big) \cdot \det(I_n + t\mathcal{J}H) = 1$$

So $\det(I_n + t\mathcal{J}H)$ is a unit in $A[t]$ and $(I_n + t\mathcal{J}H) \in \mathrm{GL}_n(A[t])$. Since $(I_n + t\mathcal{J}H)^{-1} = I_n - t\mathcal{J}H + t^2(\mathcal{J}H)^2 - \cdots$ over $A[[t]] \supset A[t]$, it follows that $\mathcal{J}H$ is nilpotent. $\square$

Definition 3.1.4. A polynomial map $x + H$ over a commutative ring A is called *elementary* if $H_i \in A[x_1, \ldots, x_{i-1}, x_{i+1}, , \ldots x_n]$ for some i and $H_j = 0$ for all $j \neq i$. A polynomial map over a commutative ring A is called *tame* if it is a composition of elementary maps and invertible linear maps.

Example 3.1.5. Elementary maps are examples of quasi-translations. More precisely, elementary maps are quasi-translations $x+H$ such that $\mathrm{rk}\mathcal{J}H = 1$. In theorem 3.4.6, we shall show that $H(x + tH(y)) = H(x)$ for such quasi-translations.

Example 3.1.6. The map

$$x + H = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} bg(ax_1 - bx_2) \\ ag(ax_1 - bx_2) \end{pmatrix}$$

with $a, b \in A$, is a quasi-translation. We will show that these are all quasi-translations in dimension 2 in case A is a unique factorization domain containing $\mathbb{Q}$.

The following proposition shows that linear conjugations of quasi-translations are quasi-translations themselves.

Proposition 3.1.7. *If $x+H$ is a quasi-translation over a commutative ring $A \supseteq \mathbb{Q}$ and $T \in \mathrm{GL}_n(A)$, then $x + T^{-1}H(Tx)$ is a quasi-translation as well.*

Proof. Notice that

$$x + tT^{-1}H(Tx) = T^{-1}Tx + T^{-1}tH(Tx) = T^{-1} \circ (x + tH) \circ T$$

By substituting $t = -1$ and $t = +1$, we obtain

$$\big(x - T^{-1}H(Tx)\big) \circ \big(x + T^{-1}H(Tx)\big) = T^{-1} \circ (x - H) \circ T \circ T^{-1} \circ (x + H) \circ T = x$$

as desired. $\square$

Notice that with $T = \left(\begin{smallmatrix} d & b \\ c & a \end{smallmatrix}\right)$ and H as in example 3.1.6, we have

$$\begin{aligned} & T^{-1}H(Tx) \\ &= \frac{1}{ad - bc} \begin{pmatrix} ax_1 + -bx_2 \\ -cx_1 + dx_2 \end{pmatrix} \circ \begin{pmatrix} bg(ax_1 - bx_2) \\ ag(ax_1 - bx_2) \end{pmatrix} \circ \begin{pmatrix} dx_1 + bx_2 \\ cx_1 + ax_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ g\big((ad - bc)x_1\big) \end{pmatrix} \end{aligned}$$

and $x + T^{-1}H(Tx)$ is elementary as in example 3.1.5.

Definition 3.1.8. Let $x+H$ be a quasi-translation over a commutative ring A and $f \in A[x]$. Define the *exponent* $\nu(f)$ with respect to $x + H$ as

$$\nu(f) := \deg_t f(x + tH)$$

We call f an *invariant* of $x + H$ if $f(x + H) = f$.

Notice that

$$\nu(fg) \leq \nu(f) + \nu(g) \tag{3.2}$$

for polynomials $f, g \in A[x]$, *with equality* if A is an integral *domain*. Since $f(x+H) = f$ implies

$$f\big(x+(m+1)H\big) = f\big(x+mH+H(x+mH)\big) = f(x+mH)$$

one can prove (cf. i) $\Rightarrow$ ii) of proposition 3.1.2) that

$$f(x+H) = f \Longleftrightarrow f(x+tH) = f$$

Hence f is an invariant of $x+H$, if and only if $\nu(f) \le 0$. In case A is an integral *domain*, we have equality in (3.2), whence fg is an invariant of $x+H$, if and only if either $fg = 0$ or both f and g are invariants of $x+H$. Since $H = H(x+tH)$, we have

$$\frac{\partial}{\partial t} f(x+tH) = \mathcal{J} f|_{x=x+tH} \cdot H = \mathcal{J} f|_{x=x+tH} \cdot H|_{x=x+tH} = (Df)(x+tH)$$

if we define $Df := \mathcal{J} f \cdot H$. So if $\nu(f) \ge 1$, then

$$\nu(f) = \deg_t \left(\frac{\partial}{\partial t} f(x+tH) \right) + 1 = \nu(Df) + 1$$

The interested reader may verify that D is a so-called *locally nilpotent derivation* and that $\nu(f) = \nu_D(f) := \deg_t\big((\exp tD)f\big)$ is the exponent with respect to D as defined in [53, §1].

Let us go back to the article of by P. Gordan and M. Nöther. It is remarkable that P. Gordan and M. Nöther already juggled with nilpotent derivations before derivations were invented; at least they use different terms and notations. They write F_y instead of $(\sum_{i=1}^n y_i \frac{\partial}{\partial x_i})F(x)$, and the Kernel elements of $\sum_{i=1}^n \xi_i \frac{\partial}{\partial x_i}$ are called 'Functionen Φ'. The 'Functionen Φ' are solutions of the 'lineare Partiellle Differentialgleichung' $\Phi_\xi = 0$, where the subscript ξ corresponds to the subscript y in the definition of F_y.

P. Gordan and M. Nöther were interested in derivations D as above because of the following. Let $f \in A[x]$ with $\det \mathcal{H} f = 0$. Since

$$\mathcal{H} f = \mathcal{J} \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right)$$

it follows from proposition 1.2.9 that $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}$ are algebraically dependent over $\mathbb{Q}(A)$ in case A is an integral domain. So assume that there exists a nonzero polynomial $R \in A[y_1, y_2, \ldots, y_n]$ such that $R\big(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\big) = 0$.

Proposition 3.1.9. *Assume $R \in A[y]$ satisfies $R\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right) = 0$. If we define*

$$H_i := \frac{\partial R}{\partial y_i}\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right)$$

for all i, then $x + H$ is a quasi-translation.

Proof. Define H_i as above for all i. In order to show that $x + H$ is a quasi-translation, it suffices to prove that $\mathcal{J} H_i \cdot H = 0$ for all i. Since

$$H_i \in A\left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right]$$

we are done if we show that $\mathcal{J} h \cdot H = 0$ for all h in the algebra on the right hand side. Since $\mathcal{J}(g+h)\cdot H = \mathcal{J} g\cdot H + \mathcal{J} h\cdot H$ and $\mathcal{J} gh\cdot H = (g\mathcal{J} h + h\mathcal{J} g)\cdot H$, we can see that it suffices to restrict to the generators $\frac{\partial f}{\partial x_j}$ of the above algebra. For that purpose, we differentiate $R(f_{x_1}, f_{x_2}, \ldots, f_{x_n}) = 0$ with respect to x_j.

$$\begin{aligned}
0 &= \frac{\partial}{\partial x_j} R\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right) \\
&= \sum_{i=1}^{n} \frac{\partial R}{\partial y_i}\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right) \cdot \frac{\partial}{\partial x_j}\frac{\partial f}{\partial x_i} \\
&= \sum_{i=1}^{n} H_i \cdot \frac{\partial}{\partial x_i}\frac{\partial f}{\partial x_j} \\
&= \mathcal{J}\frac{\partial f}{\partial x_j} \cdot H
\end{aligned}$$

□

3.2 Homogeneous quasi-translations

Definition 3.2.1. We call a quasi-translation $x + H$ *homogeneous* if H is homogeneous and *irreducible* if $\gcd\{H_1, H_2, \ldots, H_n\} = 1$.

Homogeneous quasi-translations are interesting for several reasons. One reason is that they were studied by P. Gordan and M. Nöther in 1876. They found the following property of homogeneous quasi-translation $x + H$:

$$H(tH) = 0 \tag{3.3}$$

This equality can be obtained by looking at the leading coefficient of t in $H(x + tH) = H$.

Next, they used the connection between homogeneity and projective geometry to study homogeneous quasi-translations. Unfortunately, their arguments are written in an old-fashioned style and hard to understand. This chapter is partially an attempt to comprehend their arguments. On the other hand, we give substitutional arguments.

Another reason to study homogeneous quasi-translation is that non-homogeneous quasi-translations can be made homogeneous. A similar result applies for reducible quasi-translations.

Proposition 3.2.2. *Assume $x+H$ is a quasi-translation over a commutative ring $A \supseteq \mathbb{Q}$. Then*

$$(x, x_{n+1}) + x_{n+1}^d \big(H(x_{n+1}^{-1} x), 0\big)$$

is a homogeneous *quasi-translation over A, where $d \geq \deg H$.*

Proof. Notice that $x_{n+1}^d H(x_{n+1}^{-1} x)$ is a homogeneous polynomial map of degree d. Since x_{n+1} is an invariant of $(x, x_{n+1}) + x_{n+1}^d (H(x_{n+1}^{-1} x), 0)$, it suffices to show that $x + x_{n+1}^d H(x_{n+1}^{-1} x)$ is a quasi-translation over $A[x_{n+1}, x_{n+1}^{-1}]$.

Since $x+tH$ is a quasi-translation over $A[t]$, $x+x_{n+1}^{d-1} H$ is a quasi-translation over $A[x_{n+1}, x_{n+1}^{-1}]$. Now apply proposition 3.1.7 with $Tx = x_{n+1}^{-1} x$ and $T^{-1} x = x_{n+1} x$ to obtain the desired result. ☐

Proposition 3.2.3. *Assume $x + H$ is a quasi-translation over $\mathbb{C}$ and*

$$g := \gcd\{H_1, H_2, \ldots, H_n\}$$

Then $x + g^{-1} H$ is an irreducible quasi-translation.

Proof. Since $H_i(x+tH) = H_i$, $0 = \nu(H_i) = \nu(g) + \nu(g^{-1} H_i)$. It follows that $\nu(g^{-1} H_i) = 0$ and hence

$$g^{-1} H_i = (g^{-1} H_i) \circ (x + tH)|_{t=g^{-1}} = (g^{-1} H_i) \circ (x + g^{-1} H)$$

So $(x - g^{-1} H) \circ (x + g^{-1} H) = x + g^{-1} H - ((g^{-1} H) \circ (x + g^{-1} H)) = x + g^{-1} H - g^{-1} H = x$, as desired. ☐

The above proposition shows that we can make homogeneous quasi-translations over $\mathbb{C}$ irreducible. For homogeneous quasi-translations $x + H$ over $\mathbb{C}$, this process only changes the rank of $\mathcal{J}H$ if the obtained irreducible translation happens to be a proper translation. This follows from the proposition below. We use the term proper instead of real, because real also refers to some set of numbers.

Proposition 3.2.4. *Assume $H \in \mathbb{C}[x]^n$ is homogeneous of degree $d \geq 1$ and $g \in \mathbb{C}[x]$ is homogeneous and nonzero. Then* $\operatorname{rk}\mathcal{J}(gH) = \operatorname{rk}\mathcal{J}H$.

Proof. Since the algebraic relation between the components of a homogeneous map are generated by the homogeneous relations and $R(gH) = g^r R(H)$ if R is homogeneous of degree r,

$$\operatorname{trdeg}_{\mathbb{C}} \mathbb{C}(gH) = \operatorname{trdeg}_{\mathbb{C}} \mathbb{C}(H)$$

Now apply proposition 1.2.9 to get the desired result. □

Proposition 3.2.5. *A quasi-translation $x + H$ over $\mathbb{C}$ is reducible, if and only if* $\dim V(H) = n - 1$.

Proof. Assume $\dim V(H) = n - 1$. Then by Krull's principal ideal theorem, $V(H)$ contains an irreducible $(n-1)$-dimensional component $V(g')$. It follows that $g' \mid H_i$ for all i, whence $x + H$ is reducible. The converse is similar. □

Proposition 3.2.6. *If $x + H$ is a homogeneous quasi-translation over $\mathbb{C}$, then* $\operatorname{rk}\mathcal{J}H \leq \max\{n-2, 1\}$.

Proof. From proposition 3.2.4, it follows that making a quasi-translation $x+H$ irreducible may only change the rank from 1 to 0. So we may assume that $x + H$ is irreducible. Now proposition 3.2.5 tells us that $\dim V(H) \leq n - 2$ and from (3.3), we obtain that $\dim H(\mathbb{C}^n) \leq \dim V(H)$. So by propositions 1.2.9 and 1.2.10, $\operatorname{rk}\mathcal{J}H = \dim H(\mathbb{C}^n) \leq n - 2$, as desired. □

The main theorem of this section follows below. It is somewhat technical, but its benefit becomes clear in the next sections.

Theorem 3.2.7. *Assume $x + H$ is a homogeneous quasi-translation and $\operatorname{rk}\mathcal{J}H = s$. Assume that $p, q \in H(\mathbb{C}^n)$ are independent. Then*

$$\dim V_t(H(x + tp), H(x + tq)) \geq n - 2s + 2$$

where t is an indeterminate and V_t is the zero set of all coefficients with respect to t of its arguments.

Proof. Assume first that $s \geq n/2+1$. From (3.3), it follows that $H(tp) = 0 = H(tq)$, so $0 \in V_t(H(x+tp), H(x+tq))$. Consequently,

$$\dim V_t\big(H(x+tp), H(x+tq)\big) \geq 0 = n - 2\left(\frac{n}{2}+1\right) + 2 \geq n - 2s + 2$$

as desired. So assume from now on that $s < n/2+1$.

From [35, Ch. I, Exc. 2.10 (c)], it follows that the codimension of homogeneous varieties does not depend on whether we see them as affine varieties of dimension n (affine cones) or projective varieties in dimension $n-1$. By propositions 1.2.9 and 1.2.10, we obtain that $\dim H(\mathbb{C}^n) = s$. From the fiber theorem [45, Ch. I, §8, Th. 2], it follows that $\operatorname{codim} H^{-1}(\mathbb{C}^* p) \leq s-1$. Similarly, $\operatorname{codim} H^{-1}(\mathbb{C}^* q) \leq s-1$.

Let C_p and C_q be components of codimension $s-1$ at most of the Zariski closures of $H^{-1}(\mathbb{C}^* p)$ and $H^{-1}(\mathbb{C}^* q)$. From the projective intersection theorem [35, Ch. 1, Th. 7.2], it follows that every component of $C_p \cap C_q$ has codimension $2(s-1)$ at most. Since $0 \in C_p \cap C_q$, the affine intersection $C_p \cap C_q$ has dimension $n - 2(s-1) = n - 2s + 2$ at least.

So it suffices to show that $C_p \cap C_q \subseteq V_t(H(x+tp), H(x+tq))$. For that purpose, assume $r \in C_p \cap C_q$ and $r' \in \mathbb{C}^* H^{-1}(p)$. Then $H(r') = \lambda p$ for some $\lambda \in \mathbb{C}^*$. Since $r \in C_p$ is contained in the Zariski closure of $\mathbb{C}^* H^{-1}(p)$, we obtain by $H(r') \in \mathbb{C}p$ that $H(r) \in \mathbb{C}p$. Similarly, $H(r) \in \mathbb{C}q$. Since $H(x+tH) = H$,

$$H(r' + tp) = H\big(r' + (t\lambda^{-1})H(r')\big) = H(r')$$

and we obtain in a similar manner as above that $H(r+tp) = H(r)$. So $H(r+tp) = H(r) \in \mathbb{C}p \cap \mathbb{C}q$. Since p and q are independent, $\mathbb{C}p \cap \mathbb{C}q = \{0\}$. Consequently, $H(r+tp) = 0$. Similarly, $H(r+tq) = 0$, as desired. □

3.3 Quasi-translations in small dimensions

In dimension 1, all quasi-translation over an integral domain with Q are proper translations. This is because for such a quasi-translation $x + H$, we have $\mathcal{J}H = (\operatorname{tr} \mathcal{J}H) = (0)$ on account of the nilpotency of $\mathcal{J}H$. So let us advance with dimension 2. We need to restrict ourselves to unique factorization domains $A \supseteq \mathbb{Q}$ from now.

Theorem 3.3.1. *Assume $x + H$ is a quasi-translation in dimension $n = 2$ over a unique factorization domain $A \supseteq \mathbb{Q}$. Then H is of the form*

$$H = \big(bg(ax_1 - bx_2), ag(ax_1 - bx_2)\big)$$

In particular, the components of H are linearly dependent over A: $aH_1 - bH_2 = 0$.

Proof. Assume H has degree d. Looking at the coefficient of t^d in $H(x + tH) = H$, we obtain that $\bar{H}(H) = 0$, where $\bar{H}$ is the homogeneous part of maximal degree of H. Let $K \supseteq A$ be the algebraic closure of the field of fractions of A. Since $\bar{H} \in K[x_1, x_2]$ is homogeneous, $\bar{H}$ decomposes into linear factors over K, and one of these factors is already a relation between the components of H.

So $aH_1 - bH_2 = 0$ for some $a, b \in K$, not both zero. Multiplying a and b by either a^{-1} or b^{-1} (at least one of both exists), we obtain that either a or b equals 1 and the other is contained in the field of fractions of A. Next, multiply with the denominator at hand to obtain $a, b \in A$ and $\gcd\{a, b\} = 1$.

Assume without loss of generality that $a \neq 0$ (the case $b \neq 0$ is similar). Then we can write $a^{-1}H_2 = g(ax_1 - bx_2, x_2)$, where $g \in \mathbb{Q}(A)[y_1, y_2]$. Let $\nu(f) := \deg_t f(x + tH)$ be the exponent of f with respect to $x = H$. Since $\nu(g(ax_1 - bx_2, x_2)) = \nu(a^{-1}H_2) = \nu(H_2) - \nu(a) = 0$,

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} g\big(a(x_1 + tH_1) - b(x_2 + tH_2), x_2 + tH_2\big) \\ &= (aH_1 - bH_2) \cdot \left(\left(\frac{\partial}{\partial y_1} g\right)\Big(a(x_1 + tH_1) - b(x_2 + tH_2), x_2 + tH_2\Big)\right) + \\ &\qquad H_2 \cdot \left(\left(\frac{\partial}{\partial y_2} g\right)\Big(a(x_1 + tH_1) - b(x_2 + tH_2), x_2 + tH_2\Big)\right) \end{aligned}$$

Since $aH_1 - bH_2 = 0$ and $H_2 \neq 0$, it follows that $\big(\frac{\partial}{\partial y_2} g\big)\big(a(x_1 + tH_1) - b(x_2 + tH_2), x_2 + tH_2\big) = 0$. But since $(x_1 + tH_1, x_2 + tH_2)$ is invertible, the arguments of $\big(\frac{\partial}{\partial y_2} g\big)$ are algebraically independent. So $\big(\frac{\partial}{\partial y_2} g\big) = 0$ and hence $g \in \mathbb{Q}(A)[y_1]$.

So it remains to show that the coefficients of g are contained in A. Substituting $x_2 = 0$ in $H_2 = ag(ax_1 - bx_2)$, we obtain that the denominators of the coefficients of g are composed of factors of a. If $b = 0$, then a is a unit in A due to $\gcd\{a, b\} = 1$ and we are done, so assume $b \neq 0$. Substituting $x_1 = 0$ in $H_1 = bg(ax_1 - bx_2)$, we obtain that the denominators of the coefficients of g are composed of factors of b, and again by $\gcd\{a, b\} = 1$ we get the desired result. $\square$

For dimension 3, we assume that the base ring is $\mathbb{C}$ and we describe all quasi-translations up to linear conjugations. Let us start with the homogeneous ones:

Proposition 3.3.2. *Assume $x + H$ is a homogeneous quasi-translation in dimension $n = 3$ over $\mathbb{C}$. Then there exists a $T \in \mathrm{GL}_3(\mathbb{C})$ such that*

$$T^{-1}H(Tx) = \big(g(x_2, x_3), 0, 0\big)$$

where $g \in \mathbb{C}[y_2, y_3]$.

Proof. From proposition 3.2.6, it follows that $\mathrm{rk}\mathcal{J}H \leq 1$. So the components of H are pairwise linearly dependent over $\mathbb{C}$. It follows that there exists a $T \in \mathrm{GL}_3(\mathbb{C})$ such that $T^{-1}H(Tx) = (g(x_1, x_2, x_3), 0, 0)$. By proposition 3.1.2, $\mathcal{J}(T^{-1}H(Tx))$ is nilpotent, and the trace condition of the latter Jacobian gives the desired result. $\square$

In order to comprehend quasi-translations over $\mathbb{C}$ in dimension 3, we study *homogeneous* quasi-translations over $\mathbb{C}$ in dimension 4 and use the homogenization technique of proposition 3.2.2 to get to general quasi-translations in dimension 3.

We will not classify quasi-translations in dimension 4 over $\mathbb{C}$ other than the homogeneous ones. This is because the classification process presented here relies on the property of the quasi-translations $x + H$ we encounter that there are linear relations between the components of H. We shall see in section 3.6 that this is no longer the case for non-homogeneous quasi-translations in dimension 4 and up.

But we will do some investigation on quasi-translations in dimension 4 implicitly, namely by studying homogeneous quasi-translations in dimension 5. In section 3.5, we shall show there are 3 nonempty classes of homogeneous quasi-translations $x + H$ in dimension 5 with $\mathrm{rk}\mathcal{J}H = 3$.

Theorem 3.3.3. *Assume $x + H$ is a homogeneous quasi-translation in dimension $n = 4$ over $\mathbb{C}$. Then there exists a $T \in \mathrm{GL}_4(\mathbb{C})$ such that*

$$T^{-1}H(Tx) = \big(bg(ax_1 - bx_2), ag(ax_1 - bx_2), 0, 0\big)$$

where $a, b \in \mathbb{C}[x_3, x_4]$ and $g \in \mathbb{C}[x_3, x_4][y_1]$.

Proof. We first show that $H(\mathbb{C}^4)$ is contained in a two-dimensional subspace of $\mathbb{C}^4$. Let $g := \gcd\{H_1, H_2, H_3, H_4\}$. If $\deg g = \deg H$, then $H = gp$ for some $p \in \mathbb{C}^4$ and $H(\mathbb{C}^4) \subseteq \mathbb{C}p$. So assume $\deg g < \deg H$. Then $\deg g^{-1}H > 0$ and $H(\mathbb{C}^4) = (g^{-1}H)(\mathbb{C}^4)$ because g and H are homogeneous.

So we may assume that H is irreducible. The case that $\deg H \le 0$ is easy, so assume $\deg H \ge 1$. If $\dim H(\mathbb{C}^4) = 1$ and $H(\mathbb{C}^4)$ contains a pair of independent points, then

$$\dim V(H) \ge \dim V_t\big(H(x + tp), H(x + tq)\big) \ge n - 2 \cdot 1 + 2 = n$$

on account of theorem 3.2.7; contradiction. So $H(\mathbb{C}^4)$ is contained in a line through the origin in case $\dim H(\mathbb{C}^4) = 1$. So assume $\dim H(\mathbb{C}^4) \ge 2$. Then $H(\mathbb{C}^4)$ does contain a pair of independent points p, q. Furthermore, $\dim V(H) \le 4 - 2 = 2$ since $x + H$ is irreducible. From (3.3), $H(\mathbb{C}^4) \subseteq V(H)$ follows, so

$$\dim H(\mathbb{C}^4) = \dim V(H) = 2$$

Since $H(\mathbb{C}^4)$ is irreducible on account of [45, p. 48, Prop. 1] and $\dim H(\mathbb{C}^4) = \dim V(H)$, the interior of $H(\mathbb{C}^4)$ in $V(H)$ (with the induced Zariski topology) is non-empty. Assume without loss of generality that p is contained in this interior. From theorem 3.2.7, it follows that

$$\dim V_t(x + tp) \ge 4 - 2\mathrm{rk}\mathcal{J}H + 2 = 4 - 2\dim H(\mathbb{C}^4) + 2 = 2$$

So we can take $r \in V_t(x + tp)$ independent of p. Since $H(r + \mathbb{C}p) = 0$ and H is homogeneous, $H(\mathbb{C}r + \mathbb{C}p) = 0$. So $\mathbb{C}r + \mathbb{C}p \subseteq V(H)$. But since p is contained in the interior of $H(\mathbb{C}^4) \subseteq V(H)$, the whole line $\mathbb{C}r + \mathbb{C}p$ is contained in the Zariski closure of $H(\mathbb{C}^4)$. Since $H(\mathbb{C}^4)$ is irreducible and $\dim H(\mathbb{C}^4) = \dim(\mathbb{C}r + \mathbb{C}p)$, we obtain $H(\mathbb{C}^4) \subseteq \mathbb{C}r + \mathbb{C}p$, as desired.

So $H(\mathbb{C}^4)$ is contained in a two-dimensional subspace of $\mathbb{C}^4$, say that $H(\mathbb{C}^4) \subseteq \mathbb{C}p + \mathbb{C}r$, where p and r are independent. Now choose T of the form $T = (p \mid r \mid \cdot \mid \cdot)$. Then $T^{-1}p = e_1$ and $T^{-1}r = e_2$, whence $T^{-1}H(\mathbb{C}^4) \subseteq \mathbb{C}e_1 + \mathbb{C}e_2$.

It follows that $(T^{-1}H)_3 = (T^{-1}H)_4 = 0$. So assume without loss of generality that $H_3 = H_4 = 0$. Then x_3 and x_4 are invariants of $x + H$, whence $(x_1 + H_1, x_2 + H_2)$ is a quasi-translation over $\mathbb{C}[x_3, x_4]$. Now apply theorem 3.3.1 with $A = \mathbb{C}[x_3, x_4]$. □

The quasi-translations in dimension 3 have already been classified by Z. Wang in [53], but his techniques were different.

Theorem 3.3.4 (Z. Wang)**.** *Assume $x + H$ is a quasi-translation in dimension 3 over $\mathbb{C}$. Then there exists a $T \in \mathrm{GL}_3(\mathbb{C})$ such that*

$$T^{-1}H(Tx) = \big(bg(ax_1 - bx_2), ag(ax_1 - bx_2), 0\big)$$

where $a, b \in \mathbb{C}[x_3]$ and $g \in \mathbb{C}[x_3][y_1]$.

Proof. Let $d := \deg H$ and $\tilde{H} = x_4^d(H(x_4^{-1}x), 0)$. Then $\tilde{H}(\mathbb{C}^4)$ is contained in a subspace of dimension 2 of $\mathbb{C}^4$ on account of proposition 3.2.2 and theorem 3.3.3. So $H(\mathbb{C}^3) \cong \tilde{H}(\mathbb{C}^3 \times \{1\})$ is contained in a subspace of dimension 2 of $\mathbb{C}^3$. The rest of the proof is similar to the end of the proof of theorem 3.3.3. □

At this point, we know that for quasi-translations $x + H$ in dimension 3 over $\mathbb{C}$ and homogeneous quasi-translations $x + H$ in dimension 4 over $\mathbb{C}$, the image of H is contained in a 2-dimensional subspace of $\mathbb{C}^n$. With that knowledge, we will classify all singular Hessians in dimension 3 over $\mathbb{C}$ and all homogeneous singular Hessians in dimension 4 over $\mathbb{C}$ in chapter 5.

3.4 Homogeneous quasi-translations of rank 2

The irreducible homogeneous quasi-translations $x+H$ over $\mathbb{C}$ with $\mathrm{rk}\mathcal{J}H \le 2$ have been classified by Gordan and Nöther in [34]. In particular they showed that such quasi-translations have two independent linear relations between their components. This property is equivalent to the last sentence of the theorem below.

Theorem 3.4.1. *Assume $x+H$ is a homogeneous quasi-translation of degree $d \ge 1$ over $\mathbb{C}$, $\mathrm{rk}\mathcal{J}H = 2$ and $\dim V(H) \le n-2$. Then $\dim V(H) = n-2$ and H vanishes on the linear span of $H(\mathbb{C}^n)$. Furthermore, $H(\mathbb{C}^n)$ is contained in a linear subspace of $\mathbb{C}^n$ of dimension $n-2$.*

Proof. Since $\dim H(\mathbb{C}^n) = \operatorname{rk}\mathcal{J}H = 2$ by propositions 1.2.9 and 1.2.10, $H(\mathbb{C}^n)$ contains two independent points p and q and we have

$$\dim V_t(H(x+tp), H(x+tq)) \geq n - 2 \cdot 2 + 2 \geq \dim V(H)$$

on account of theorem 3.2.7. Since $V_t(H(x+tp), H(x+tq)) \subseteq V(H)$,

$$\dim V_t(H(x+tp), H(x+tq)) = n - 2 = \dim V(H)$$

So for each $q \in H(\mathbb{C}^n)$ that is independent of p, $V_t(H(x+tp), H(x+tq))$ contains an irreducible component of maximum dimension of $V(H)$. Since $V(H)$ has only finitely many irreducible components, it follows that $V(H)$ has an irreducible component W of maximum dimension that is independent of q, such that $W \subseteq V(H(x+tp), H(x+tq))$ for a 'substantial' part of the $q \in H(\mathbb{C}^n)$, i.e. (the Zariski closure of)

$$U := \{q \in H(\mathbb{C}^n) \mid W \subseteq V(H(x+tp), H(x+tq))\}$$

has the same dimension as $H(\mathbb{C}^n)$.

Since $\dim W = \dim V(H)$, the interior W° of W in $V(H)$ (with the induced Zariski topology) is non-empty. Now let y_0 be a point of W°. Assume $y_1, y_2, \ldots, y_m$ are points of U such that $y_1, y_2, \ldots, y_m$ is a basis of the linear span of U. Since $H(\mathbb{C}^n)$ is irreducible (the ideal $(R \in \mathbb{C}[y] \mid R(H) = 0)$ is prime), it follows from $U \subseteq H(\mathbb{C}^n)$ and $\dim U = \dim H(\mathbb{C}^n)$ that $y_1, y_2, \ldots, y_m$ is a basis of the linear span of $H(\mathbb{C}^n)$.

We will show by induction on i that generic points r of the linear span of $y_0, y_1, y_2, \ldots, y_i$ are contained in the interior W° of $W \subseteq V(H)$ for all $i \leq m$. Since $y_0 \in W^\circ$ and $\mathbb{C}y_0 \subseteq V(H)$, generic points of $\mathbb{C}y_0$ are contained in W° as well, and the case $i = 0$ follows. So assume $i \geq 1$ and assume that generic points r of the linear span of $y_0, y_1, y_2, \ldots, y_{i-1}$ are contained in W°. Since $r \in W \subseteq V_t(H(x+tp), H(x+ty_i))$, $H(r + ty_i) = 0$, i.e. $\mathbb{C}y_i + r \subseteq V(H)$. But r is contained in W°, so generic points of $\mathbb{C}y_i + r$ are contained in the W° as well, as desired.

So generic points r of the linear span of $y_0, y_1, y_2, \ldots, y_m$ are contained in the interior of $W \subseteq V(H)$. In particular, the linear span of $H(\mathbb{C}^n)$ is contained in $W \subseteq V(H)$, as desired. $\square$

Corollary 3.4.2. *If $x + H$ is a homogeneous quasi-translation over $\mathbb{C}$ and $\operatorname{rk}\mathcal{J}H \leq 2$, then $H(\mathbb{C}^n)$ is contained in a linear subspace of dimension $\max\{n-2, 1\}$ of $\mathbb{C}^n$.*

Proof. In a similar manner as in the proof of theorem 3.3.3, we may assume that $x + H$ is irreducible and $\operatorname{rk}\mathcal{J}H = \dim H(\mathbb{C}^n) \geq 2$. So $\operatorname{rk}\mathcal{J}H = 2$ and hence $\deg H \geq 1$. Since $x + H$ is irreducible, $\dim V(H) \neq n - 1$. Since $H \neq 0$, $\dim V(H) \leq n - 2$. Now apply theorem 3.4.1 to obtain that $H(\mathbb{C}^n)$ is contained in a linear subspace of $\mathbb{C}^n$ of dimension $n - 2$, as desired. □

Notice that irreducible homogeneous quasi-translations $x+H$ with $\operatorname{rk}\mathcal{J}H = 0$ are normal translations. Homogeneous quasi-translations $x + H$ with $\operatorname{rk}\mathcal{J}H = 1$ are always reducible and making one of them irreducible results in a normal translation.

In [34], P. Gordan and M. Nöther prove that for irreducible homogeneous quasi-translations $x + H$ with $\operatorname{rk}\mathcal{J}H \leq 2$,

$$H(x + tH(y)) = H(x) \tag{3.4}$$

This identity can be found on lines 5-7 on page 558 in [34]. Next, they prove assertions ii) and iii) of the proposition below.

Proposition 3.4.3. *The following assertions are equivalent over $\mathbb{C}$:*

i) $H(x + tH(y)) = H(x)$,

ii) $H(x + p) = H(x)$ *for all p in the linear span of $H(\mathbb{C}^n)$,*

iii) There exists an $s \in \mathbb{N}$ and a $T \in \mathrm{GL}_n(\mathbb{C})$, such that

$$T^{-1}H(Tx) = \big(0^1, \ldots, 0^s, h_{s+1}(x_1, \ldots, x_s), \ldots, h_n(x_1, \ldots, x_s)\big)$$

where $h_i \in \mathbb{C}[x_1, \ldots, x_s]$ for all $i \geq s + 1$.

Proof.

i) ⇒ ii) Notice that the linear span of $H(\mathbb{C}^n)$ is generated by some $q_1, q_2, \ldots, q_m \in H(\mathbb{C}^n)$. Write $p = \lambda_1 q_1 + \cdots + \lambda_{m-1} q_{m-1} + \lambda_m q_m$. Now substitute $t = \lambda_m$, $y = q_m$ and $x = x + \lambda_1 q_1 + \cdots + \lambda_{m-1} q_{m-1}$ in $H(x + tH(y)) = H(x)$ to obtain

$$H\big((x+\lambda_1 q_1+\cdots+\lambda_{m-1} q_{m-1})+\lambda_m q_m\big) = H(x+\lambda_1 q_1+\cdots+\lambda_{m-1} q_{m-1})$$

Next, apply $H(x + tH(y)) = H(y)$ another $m - 1$ times to obtain $H(x + p) = H$, as desired.

ii) $\Rightarrow$ iii) Replacing H by $T^{-1}H(Tx)$ for a suitable $T \in \mathrm{GL}_n(\mathbb{C})$, we obtain that the linear span of $H(\mathbb{C}^n)$ is $\{0\}^s \times \mathbb{C}^{n-s}$. Since $H(x+p) = H(x)$ for all $p \in \{0\}^s \times \mathbb{C}^{n-s}$, $H(r+p) = H(r)$ for all $r \in \mathbb{C}^s \times \{0\}^{n-s}$ and $p \in \{0\}^s \times \mathbb{C}^{n-s}$. It follows that $H = H|_{x_{s+1}=\cdots=x_n=0}$. This gives the desired result.

iii) $\Rightarrow$ i) We first assume that $T = I_n$, i.e. $H = (0^1, \ldots, 0^s, h_{s+1}(x_1, \ldots, x_s), \ldots, h_n(x_1, \ldots, x_s))$. Then $\mathbb{C}H(\mathbb{C}^n)$ is contained in $\{0\}^s \times \mathbb{C}^{n-s}$ and hence $H(x + tp) = H(x)$ for all $p \in H(\mathbb{C}^n)$. It follows that $H(x + tH(y)) = H(x)$.

In general, we can use the above argument for $T = I_n$ to obtain the result $\tilde{H}(x + t\tilde{H}(y)) = \tilde{H}(x)$ for $\tilde{H} = T^{-1}H(Tx)$ instead of $\tilde{H} = H$, i.e. $T^{-1}H(T(x + tT^{-1}H(Ty))) = T^{-1}H(Tx)$. Substituting $x = T^{-1}x$ and $y = T^{-1}y$, we obtain $T^{-1}H(x + tH(y)) = T^{-1}H(x)$. This gives the desired result. □

The proof of (3.4) for irreducible homogeneous quasi-translations $x + H$ with $\mathrm{rk}\mathcal{J}H \le 2$ in [34] has been reconstructed by A. van den Essen. But in section 4.5, we will give another proof, using a theorem of Lüroth that appeared earlier in the same year and in the same journal as [34]. So the following theorem applies to homogeneous quasi-translations $x + H$ over $\mathbb{C}$ with $\mathrm{rk}\mathcal{J}H \le 2$.

Theorem 3.4.4. *Assume $x + gH$ is a quasi-translation over a commutative ring $A \supseteq \mathbb{Q}$, such that $H(x + tH(y)) = H(x)$. Then $x + H$ and $(x, x_{n+1}) + (gH, 0)$ are tame.*

Proof. From iii) of proposition 3.4.3, it follows that H is linearly triangularizable. Consequently, $x + H$ is tame. The tameness of $(x, x_{n+1}) + (gH, 0)$ follows from iii) of proposition 3.4.3 and [50]. See also [24, Prop. 6.1.4]. □

Quasi-translations over $\mathbb{C}$ that satisfy (3.4) are called nice of order ≤ 2. Normal translations are nice of order ≤ 1 and translations of the form $x + e_i$ are nice of order 0. See [24, p. 158], for the definition of nice quasi-translation. All homogeneous quasi-translations $x + H$ over $\mathbb{C}$ with $\mathrm{rk}\mathcal{J}H \le 2$ are nice of order ≤ 3.

If we compare proposition 2.2.7 in chapter 2 and theorem 3.3.1, we see that the result of proposition 2.2.7 is a translation from the left on the result of theorem 3.3.1. In other words, the Jacobian of an unipotent map in dimension 2 is the Jacobian of a quasi-translation (but not vice versa). This holds more generally.

Theorem 3.4.5. *Assume A is an integral domain with $\mathbb{Q}$ and $H \in A[x]^n$ such that $\mathcal{J}H$ is nilpotent and $\mathrm{rk}\mathcal{J}H = 1$. Then $\mathcal{J}(x+H)$ is the Jacobian of a quasi-translation. More precisely, if $\tilde{H}$ is the non-constant part of H, then $x + \tilde{H}$ is such a quasi-translation and*

$$\tilde{H}(x + t\tilde{H}(y)) = \tilde{H} \tag{3.4$'$}$$

Proof. Let $\tilde{H}$ be the non-constant part of H. From proposition 3.1.2, it follows that it suffices to show that (3.4$'$) is satisfied. For that purpose, notice that A is contained in a field K of characteristic zero and that linear conjugations do not affect the properties of H, including (3.4$'$).

Replacing H by $T^{-1}H(Tx)$ for a suitable $T \in \mathrm{GL}_n(K)$, we can obtain that $\tilde{H}_1 = \tilde{H}_2 = \cdots = \tilde{H}_s = 0$ and $\tilde{H}_{s+1}, \tilde{H}_{s+2}, \ldots, \tilde{H}_n$ are linearly independent over K. From [24, Th. 7.1.7 i)] (the dependence over $\mathbb{C}$ of the rows of a nilpotent Jacobian of rank ≤ 1), it follows that $s \ge 1$. Furthermore, we obtain by induction on n (with $A = K[x_1, x_2, \ldots, x_s]$) that there exists polynomials $p_i \in K[x_1, x_2, \ldots, x_s]$, not all zero, such that

$$p_{s+1}H_{s+1} + p_{s+2}H_{s+2} + \cdots + p_nH_n \in K[x_1, x_2, \ldots, x_s] \tag{3.5}$$

If $\tilde{H}_{s+1}, \tilde{H}_{s+2}, \ldots, \tilde{H}_n \in \mathbb{C}[x_1, x_2, \ldots, x_s]$, then we are done on account of proposition 3.4.3, so assume $\tilde{H}_{s+i} \notin \mathbb{C}[x_1, x_2, \ldots, x_s]$ for some $i \ge 1$. Then $\tilde{H}_{s+i}$ is algebraically independent of $x_1, x_2, \ldots, x_s$ over K, so x_1 is algebraically independent of $x_2, \ldots, x_s, \tilde{H}_{s+i}$. Since $\mathrm{rk}\mathcal{J}\tilde{H} \le 1$, it follows from proposition 1.2.9 that each pair of components of $\tilde{H}$ is algebraically dependent over K, so x_1 is algebraically independent of $x_2, \ldots, x_s$, $\tilde{H}_{s+1}, \tilde{H}_{s+2}, \ldots, \tilde{H}_n$.

It follows that we can replace x_1 by y_1 in (3.5), to obtain

$$p_{s+1}|_{x_1=y_1}H_{s+1} + p_{s+2}|_{x_1=y_1}H_{s+2} + \cdots + p_n|_{x_1=y_1}H_n \in K[y_1, x_2, \ldots, x_s]$$

Reasoning on like this (with $x_2, \ldots, x_s$ instead of x_1), we obtain

$$p_{s+1}|_{x=y}H_{s+1} + p_{s+2}|_{x=y}H_{s+2} + \cdots + p_n|_{x=y}H_n \in K[y_1, y_2, \ldots, y_s]$$

Now a generic substitution in y gives a contradiction to the linear independence of $\tilde{H}_{s+1}, \tilde{H}_{s+2}, \ldots, \tilde{H}_n$. So $\tilde{H}_{s+1}, \tilde{H}_{s+2}, \ldots, \tilde{H}_n \in \mathbb{C}[x_1, x_2, \ldots, x_s]$, as desired. □

In the proof of [29, Th. 4], E. Formanek proves a similar result as above, but with the nilpotency of $\mathcal{J}H$ replaced by the conditions that $\det \mathcal{J}(x+H) = 1$ and that H has no terms of degree 1. Formanek does not use the term quasi-translation, but his description is that of iii) of proposition 3.4.3. [29, Th. 4] itself does not give the result of theorem 3.4.5, because H in theorem 3.4.5 may have linear terms.

In order to obtain [24, Th. 7.1.7 i)] (used in the above proof), Van den Essen uses theorem 4.3.5 in the next chapter. Formanek uses this theorem as well to obtain [29, Th. 4]: see [29, Lm. 2].

At last, we describe some cases in which quasi-translations satisfy (3.4), using the already announced result of section 4.5 that irreducible homogeneous quasi-translations $x + H$ over $\mathbb{C}$ with $\mathrm{rk}\mathcal{J}H \le 2$ satisfy (3.4).

Theorem 3.4.6. *Assume $x+H$ is a quasi-translation (not necessarily homogeneous) over $\mathbb{C}$ and $g := \gcd\{H_1, H_2, \ldots, H_n\}$. Then $H(x+tH(y)) = H(x)$ in each of the following cases:*

i) $\mathrm{rk}\mathcal{J}H \le 1$,

ii) $\deg H \le 1$,

iii) H is homogeneous, $\mathrm{rk}\mathcal{J}H \le 2$ *and* $\deg g \le 1$,

iv) H is homogeneous, $\mathrm{rk}\mathcal{J}H \le 2$ *and* $\deg H \le 2$.

Proof.

i) Assume $\mathrm{rk}\mathcal{J}H \le 1$. Let $\tilde{H}$ be the non-constant part of H. From theorem 3.4.5, it follows that $\tilde{H}(x + t\tilde{H}(y)) = \tilde{H}(x)$, so

$$H(x + t\tilde{H}(y)) = H(x) \tag{3.6}$$

No substitute $x = x + tH(x) - t\tilde{H}(x)$ in (3.6) to obtain $H(x+tH(y)) = H(x+tH(x)-t\tilde{H}(x))$, and $x = x - tH(x)$, $y = x$ and $t = -t$ successively in (3.6), to obtain $H(x+tH(x)-t\tilde{H}(x)) = H(x+tH(x)) = H(x)$. So $H(x + tH(y)) = H(x)$, as desired.

ii) Assume $\deg H \le 1$. Since $H(x + tH(y)) = H(x))$ is satisfied for $t = 0$, it suffices to show that the derivatives with respect to t of both sides are the same. This follows from $H(x + tH(x)) = H(x)$ and the fact that $\mathcal{J}H$ is constant:

$$(\mathcal{J}H)|_{x=x+tH(y)} \cdot H(y) = (\mathcal{J}H)|_{x=y+tH(y)} \cdot H(y) = \frac{\partial}{\partial t} H(y+tH(y))) = 0$$

iii) Assume H is homogeneous, $\operatorname{rk}\mathcal{J}H \le 2$ and $\deg g \le 1$. Put $\tilde{H} = g^{-1}H$. From theorem 4.5.2, it follows that $\tilde{H}(x + t\tilde{H}(y)) = \tilde{H}(x)$, and by substituting $t = g(y)t$, we obtain

$$\tilde{H}(x + tH(y)) = \tilde{H}(x)$$

So it suffices to show that $g(x + tH(y)) = g(x)$. This follows from $g(x + tH(x)) = g(x))$ and the fact that $\mathcal{J}g$ is constant, in a similar manner as $H(x + tH(y)) = H(x)$ follows in ii).

iv) Assume H is homogeneous, $\operatorname{rk}\mathcal{J}H \le 2$ and $\deg H \le 2$. Since $\deg H \le 2$, we have either $\deg g \le 1$ or $\operatorname{rk}\mathcal{J}H \le 1$. So i) and iii) together give the desired result. □

3.5 Homogeneous quasi-translations in dimension 5

In [34, §8], P. Gordan and M. Nöther investigate homogeneous quasi-translations $x + H$ in dimension 5 for which $\operatorname{rk}\mathcal{J}H = 3$. They classify these quasi-translations in three types, namely a1), a2) and b). We shall prove this classification. In addition, we shall show that all three types really exist.

Before we can formulate and prove the classification, we need some preparations.

Definition 3.5.1. Assume $W \subseteq \mathbb{C}^n$ is an irreducible variety. We call a nonzero $p \in W$ *Gonöric in W* if there exists an affine cone L_p of a line in $\mathbb{P}(\mathbb{C}^n)$ with $p \in L_p \subseteq W$, such that for *generic* $q \in W$, there exists an affine cone L_q of a line in $\mathbb{P}(\mathbb{C}^n)$ such that $q \in L_q \subseteq W$ and $L_q \cap L_p \ne \{0\}$.

Lemma 3.5.2. *Assume p is Gonöric in W. Then there exists an affine cone L_p of a line in $\mathbb{P}(\mathbb{C}^n)$ with $p \in L_p \subseteq W$, such that for* all *$q \in W$, there exists an affine cone L_q of a line in $\mathbb{P}(\mathbb{C}^n)$ such that $q \in L_q \subseteq W$ and $L_q \cap L_p \ne \{0\}$.*

Proof. Choose L_p as in the definition of Gonöric and assume that $p, r \in L_p$ are independent. Now for generic $q \in W$, the variety

$$V_{t,u}(I(W)|_{x=tq+u(y_1p+y_2r)}) \subseteq \mathbb{C}^2 \tag{3.7}$$

is nontrivial, where $V_{t,u}$ is the zero set of all coefficients with respect to t, u of its argument. Furthermore, the coefficients with respect to t and u of the argument of $V_{t,u}$ on the left hand side of (3.7) are homogeneous in y, so we can view

$$V_{t,u}I(W)|_{x=tx+u(y_1p+y_2r)}$$

as a subvariety of $\mathbb{C}^n \times \mathbb{P}(\mathbb{C}^2)$.

Since the projective line is complete on account of [45, p. 55, Th. 1], it follows that the projection onto $\mathbb{C}^n$ of $V_{t,u}I(W)|_{x=tx+u(y_1p+y_2r)}$ is closed. Consequently, the $q \in W$ for which the left hand side of (3.7) is nontrivial is a closed set. Since W is irreducible, the left hand side of (3.7) is nontrivial for all $q \in W$.

If $q \in L_p$, then $L_q = L_p$ does the job. In all other cases, we take for L_q the space generated by q and one of the nontrivial vectors contained in the left hand side of (3.7). □

Lemma 3.5.3. *Assume $x + H$ is a homogeneous quasi-translation and suppose that $\dim H(\mathbb{C}^n) = \dim V(H) = 3 \le \lceil n/2 \rceil$. Let W be the Zariski closure of $H(\mathbb{C}^n)$. Take $p \in W$ generic. Then either p is Gonöric in W, or there exist infinitely many affine cones L_p of lines in $\mathbb{P}(\mathbb{C}^n)$, such that $p \in L_p \subseteq W$.*

Proof. On account of [45, p. 48, Prop. 1], W is irreducible. From (3.3), it follows that $W \subseteq V(H)$. Since $\dim W = \dim H(\mathbb{C}^n) = \dim V(H)$, we obtain that the interior of W in $V(H)$ is nonempty. From i) of [45, p. 49, Th. 3], we obtain that there exists an open $U \neq \varnothing$ that is contained in $H(\mathbb{C}^n)$.

Assume that $p, q \in U$ are independent. By theorem 3.2.7, we obtain

$$\dim V_t(H(x + tp), H(x + tq)) \ge n - 2 \cdot 3 + 2 \ge n - 2\lceil n/2 \rceil + 2 \ge 1$$

So $V_t(H(x + tp), H(x + tq))$ contains a nonzero point r. It follows from $H(r + tp) = 0$ that $r + \mathbb{C}p \subseteq V(H)$. Since H is homogeneous, $\mathbb{C}p + \mathbb{C}r \subseteq V(H)$. Similarly $\mathbb{C}q + \mathbb{C}r \subseteq V(H)$. So there exists affine cones $L_p \ni p$ and $L_q \ni q$ of lines in $\mathbb{P}(\mathbb{C}^n)$, such that $L_p, L_q \subseteq V(H)$ and $L_p \cap L_q \neq \{0\}$. Let

W° be the interior of W in $V(H)$. By $H(\mathbb{C}^n) \subseteq W$, we obtain $p \in U \subseteq W^\circ$. Since L_p is irreducible, L_p is even contained in W. Similarly, L_q is even contained in W.

So for each $q \in U$, there are affine cones $L_q \ni q$ and $L_p \ni p$ of lines in $\mathbb{P}(\mathbb{C}^n)$, such that $L_q \cap L_p \ne \{0\}$. Assume that there are only finitely many affine cones L_p of lines in $\mathbb{P}(\mathbb{C}^n)$ such that $p \in L_p \subseteq W$. Then one of this cones, say L^*, intersects nontrivially with an affine cone L_q of a line in $\mathbb{P}(\mathbb{C}^n)$ such that $q \in L_q \subseteq W$, for a 'substantial' part of the $q \in U$.

So there exists an affine cone L^* of a line in $\mathbb{P}(\mathbb{C}^n)$ with $p \in L^* \subseteq W$, such that the dimension of (the Zariski closure of)

$$\left\{ q \in U \;\middle|\; \begin{array}{l} L_q \cap L^* \ne \{0\} \text{ for some affine cone } L_q \\ \text{of a line in } \mathbb{P}(\mathbb{C}^n) \text{ with } q \in L_q \subseteq W \end{array} \right\}$$

is equal to that of W. Since W is irreducible, the desired result follows. $\square$

The term Gönoric is a contraction of Gordan, Nöther and generic. From case 1) in the proof of theorem 3.5.4 below, it follows that generic $p \in W$ are Gönoric as well in case there are infinitely many affine cones L_p of lines in $\mathbb{P}(\mathbb{C}^n)$, such that $p \in L_p \subseteq W$.

Theorem 3.5.4. *Assume $x + H$ is a homogeneous quasi-translation and*

$$\dim H(\mathbb{C}^n) = \dim V(H) = 3 \le \lceil n/2 \rceil$$

Let W be the Zariski closure of $H(\mathbb{C}^n)$. Then either there exist a fixed point $p \ne 0$ such that W is a union of affine cones $L_p \ni p$ of lines in $\mathbb{P}(\mathbb{C}^n)$, or there exist affine cones L^* and L° of lines in $\mathbb{P}(\mathbb{C}^n)$ such that $W \subseteq L^* + L^\circ$.

Proof. We distinguish two cases:

1) For some nonzero $p \in W$, there exist infinity many affine cones L_p of lines in $\mathbb{P}(\mathbb{C}^n)$ such that $p \in L_p \subseteq W$.
   Let Z be the Zariksi closure of the union of these infinity many affine cones L_p and assume that $\dim Z = 2$. Then Z is the union of finitely many components of dimension 2 at most, but the components that are no affine cones of lines in $\mathbb{P}(\mathbb{C}^n)$ can be removed, since they contain only finitely many points of each affine cone of a line in $\mathbb{P}(\mathbb{C}^n)$ and hence do not harm the Zariski closure of a set of such affine cones.

So Z is the union of finitely many affine cones of lines in $\mathbb{P}(\mathbb{C}^n)$. Contradiction, so $\dim Z \geq 3$. Since $Z \subseteq W$ and W is irreducible, it follows that $W = Z$, as desired.

2) There does *not* exist a $p \in W$ for which there are infinity many affine cones L_p of lines in $\mathbb{P}(\mathbb{C}^n)$ such that $p \in L_p \subseteq W$.
Take $p \in W$. Since there are only finitely many affine cones L_p of lines in $\mathbb{P}(\mathbb{C}^n)$ such that $p \in L_p \subseteq W$ and $\dim W = 3 > 2$, we can choose $q \in W$ such that $q \notin L_p$ for every affine cone L_p of a line in $\mathbb{P}(\mathbb{C}^n)$ such that $p \in L_p \subseteq W$.

Now take $r \in W$ generic. Since r is Gönoric in W on account of lemma 3.5.3, it follows from lemma 3.5.2 that there exist affine cones $L_p \ni p$, $L_q \ni q$ and $L_r \ni r$ of lines in $\mathbb{P}(\mathbb{C}^n)$ that are contained in W, such that $L_r \cap L_p \neq 0 \neq L_r \cap L_q$. Now there are only finitely many pairs $(L_p, L_q) \subseteq W^2$ such that $L_p \ni p$ and $L_q \ni q$ are affine cones of lines in $\mathbb{P}(\mathbb{C}^n)$, so for one of these pairs, say (L^*, L°), we have that for a 'substantial' part of the $r \in W$, there exists an affine cone $L_r \ni r$ of a line in $\mathbb{P}(\mathbb{C}^n)$ that is contained in W, such that $L_r \cap L^* \neq 0 \neq L_r \cap L^\circ$.

So there exist affine cones L^*, L° of lines in $\mathbb{P}(\mathbb{C}^n)$ that are contained in W, such that the dimension of (the Zariski closure of)

$$\left\{ r \in U \;\middle|\; \begin{array}{l} L_r \cap L^* \neq \{0\} \neq L_r \cap L^\circ \text{ for some affine cone} \\ L_r \ni r \text{ of a line in } \mathbb{P}(\mathbb{C}^n) \text{ with } L_r \subseteq W \end{array} \right\}$$

is equal to that of W. Furthermore, if $L^* \cap L^\circ \neq \{0\}$, then there are only finitely many affine cones L_r of lines in $\mathbb{P}(\mathbb{C}^n)$ such that $L^* \cap L^\circ \subseteq L_r \subseteq W$, so the dimension of (the Zariski closure of)

$$\left\{ r \in U \;\middle|\; \begin{array}{l} \{0\} \neq L_r \cap L^* \neq L_r \cap L^\circ \neq \{0\} \text{ for some affine} \\ \text{cone } L_r \ni r \text{ of a line in } \mathbb{P}(\mathbb{C}^n) \text{ with } L_r \subseteq W \end{array} \right\}$$

is equal to that of W as well. Since W is irreducible, it follows that $W \subseteq L^* + L^\circ$. □

Some of the ideas of the above theorem and its proof can be found on [34, pp. 565-566]. The idea of using that r is Gonöric instead of p and q comes from H. Derksen.

Gordan and Nöther distinguish between two cases as well: 'Fall a)' and 'Fall b)'. But their cases do not correspond to the two cases 1) and 2) in the proof above. 'Fall b)' is entirely included in case 1) above, but 'Fall a)' subdivides into both case 1) and case 2). But Gordan and Nöther subdivides 'Fall a)' as well, namely in 'Im ersten Falle' and 'When zweitens', and this subdivision does correspond to that into the cases 1) and 2).

The following example shows that all of the three cases a1), a2), and b1) exist.

Example 3.5.5. The following quasi-translations are examples of each of the three cases a1), a2), and b) described above.

a1) $H = (x_4^2, x_4x_5, x_1x_5 - x_2x_4, 0, 0)$,

a2) $H = (x_5^2(ax_1 - x_5^2x_2), a(ax_1 - x_5^2x_2), x_5^2(ax_3 - x_5^2x_4), a(ax_3 - x_5^2x_4), 0)$ with $a = x_1x_4 - x_2x_3$,

b1) $H = (x_4^5, bx_4^3, b^2x_4, 0, -b^2x_1 + 2bx_2x_4^2 - x_3x_4^4)$ with $b = x_1x_3 - x_2^2 + x_4x_5$.

The first quasi-translation is nice of order 3. But the other two are not nice. This is due to a theorem of Z. Wang that says that the image of nice homogeneous quasi-translations over $\mathbb{C}$ is contained in an $(n-2)$-dimensional subspace of $\mathbb{C}^n$.

For quasi-translations $x + H$ of types a1) and a2), we do know that the components of H need to be linearly dependent. We do not know whether the components of H need to be linearly dependent in the remaining case b1). Notice first that for this dependence problem, we may restrict ourselves to irreducible quasi-translations by proposition 3.2.3. We shall give a structure theorem for irreducible quasi-translations $x + H$ of type b1) for which the components of H are *not* linearly dependent. Furthermore, we shall show that the degree d must be a product of an integer ≥ 3 and another integer ≥ 3 and therefore $d \geq 9$.

Theorem 3.5.6. *Assume $x+H$ is an irreducible homogeneous quasi-translation in dimension $n = 5$, say of degree d. If the components of H are not linearly dependent over $\mathbb{C}$, then there exists a $T \in \mathrm{GL}_5(\mathbb{C})$ such that $\tilde{H} := T^{-1} \circ H \circ T$ is of the form*

$$\tilde{H} = \begin{pmatrix} h_1(p,q) \\ h_2(p,q) \\ h_3(p,q) \\ h_4(p,q) \\ r \end{pmatrix}$$

and $x+\tilde{H}$ *is a quasi-translation as well. Furthermore,* $\deg h \geq 3$ *and* $\deg q \geq 3$*, so* $d \geq 9$*. More precisely,* $0 < \deg_{x_5} q < \deg q - 1$*.*

Proof. From proposition 3.1.7, it follows that $x+\tilde{H}$ is a quasi-translation as well. Assume the components of H are not linearly dependent over $\mathbb{C}$. Then $d \geq 2$. From theorem 3.5.4, we obtain that the Zariski closure of $H(\mathbb{C}^5)$ is a union of affine cones $\mathbb{C}\tilde{p}+\mathbb{C}\tilde{q}$ of projective lines through $\tilde{p}$, for some nonzero $\tilde{p} \in \mathbb{C}^5$. Take T such that $Te_5 = \tilde{p}$. Then $\tilde{H}(\mathbb{C}^5)$ is a union of affine cones $\mathbb{C}e_5 + \mathbb{C}\tilde{q}$ of projective lines through e_5.

i) Notice that $\operatorname{trdeg}_{\mathbb{C}}(\tilde{H}) = \dim W = 3$ on account of proposition 1.2.10. Furthermore, $\tilde{H}_5$ is algebraically independent over $\mathbb{C}$ of $\tilde{H}_1$, $\tilde{H}_2$, $\tilde{H}_3$, $\tilde{H}_4$. From proposition 1.2.9, it follows that

$$\operatorname{rk}\mathcal{J}(\tilde{H}_1, \tilde{H}_2, \tilde{H}_3, \tilde{H}_4) = \operatorname{trdeg}_{\mathbb{C}}(\tilde{H}_1, \tilde{H}_2, \tilde{H}_3, \tilde{H}_4) = 2$$

Now by theorem 4.3.1 in the next section, we obtain that $(\tilde{H}_1, \tilde{H}_2, \tilde{H}_3, \tilde{H}_4)$ is of the form $gh(p,q)$, where g, p and q are homogeneous polynomials and p and q are relatively prime.

Furthermore, we can choose T such that $gp \mid H_1$, $gpq \mid H_2$, $gpq \mid H_3$ and $gq \mid H_4$. So $gq \nmid H_1$ and $gp \nmid H_4$ by definition of g. Let $\nu(f) := \deg_t f(x + t\tilde{H})$ be the exponent with respect to $x + \tilde{H}$. From ii) of proposition 3.1.2, it follows $\nu(\tilde{H}_i) = 0$ for all i. Now by (3.2) with equality, we obtain $\nu(p) = \nu(q) = 0$.

ii) Notice that $x+\tilde{H}$ is irreducible and therefore $\gcd\{g, \tilde{H}_5\} = 1$. We first show that $g \in \mathbb{C}[x_1,x_2,x_3,x_4]$. For that purpose, assume that g has an irreducible divisor $g_1 \notin \mathbb{C}[x_1,x_2,x_3,x_4]$. Since $g_1 \mid \tilde{H}_1$, it follows from (3.2) with equality that $\nu(g_1) = 0$. So

$$0 = \frac{\partial}{\partial t} g_1(x + t\tilde{H})\bigg|_{t=0} = \mathcal{J}g_1 \cdot \tilde{H} \equiv \tilde{H}_5 \frac{\partial}{\partial x_5} g_1 \pmod{g_1}$$

Since $g_1 \nmid \tilde{H}_5$ and $\deg g_1 > \deg \frac{\partial}{\partial x_5} g_1$, we obtain $\frac{\partial}{\partial x_5} g_1 = 0$, i.e. $g_1 \in \mathbb{C}[x_1,x_2,x_3,x_4]$. Contradiction, so $g \in \mathbb{C}[x_1,x_2,x_3,x_4]$.

iii) Assume $f \in \mathbb{C}[x_1,x_2,x_3,x_4]$ is homogeneous such that $\nu(f) = 0$. Since

$$0 = \frac{\partial}{\partial t} f(x + t\tilde{H})\bigg|_{t=0} = \mathcal{J}f \cdot \tilde{H} \equiv \tilde{H}_1 \frac{\partial}{\partial x_1} f \pmod{gq}$$

and

$$0 = \frac{\partial}{\partial t} f(x + t\tilde{H})\bigg|_{t=0} = \mathcal{J}f \cdot \tilde{H} \equiv \tilde{H}_1 \frac{\partial}{\partial x_4} f \pmod{gp}$$

it follows that $p \mid \frac{\partial}{\partial x_4} f \neq 0$ or $q \mid \frac{\partial}{\partial x_1} f \neq 0$ or $f \in \mathbb{C}[x_2, x_3]$.
In case $f \in \mathbb{C}[x_2, x_3] \backslash \mathbb{C}$, f decomposes in linear factors that are relations between the components of $\tilde{H}$, which contradicts that the compoents of $\tilde{H}$ are linearly independent over $\mathbb{C}$. So $f \in \mathbb{C}$ or $p \mid \frac{\partial}{\partial x_4} f \neq 0$ or $q \mid \frac{\partial}{\partial x_1} f \neq 0$. If $p \in \mathbb{C}[x_1, x_2, x_3, x_4]$, then we get a contradiction by taking $f = p$, so $p \notin \mathbb{C}[x_1, x_2, x_3, x_4]$. Similarly, $q \notin \mathbb{C}[x_1, x_2, x_3, x_4]$. So $\deg_{x_5} q > 0$.

Since $f \in \mathbb{C}[x_1, x_2, x_3, x_4]$, it follows that both $p \mid \frac{\partial}{\partial x_4} f \neq 0$ and $q \mid \frac{\partial}{\partial x_1} f \neq 0$ are impossible. Contradiction, so $f \in \mathbb{C}$. In particular, $g \in \mathbb{C}$.

iv) Notice that $\deg h \geq 3$ because $\tilde{H}_1$, $\tilde{H}_2$, $\tilde{H}_3$, $\tilde{H}_4$ are linearly dependent. So it remains to show that $\deg_{x_5} q < \deg q - 1$. Since $\nu(q) = 0$, it follows that $q(\tilde{H}) = 0$. But $\tilde{H}_5$ is algebraically independent of $\tilde{H}_1, \tilde{H}_2, \tilde{H}_3, \tilde{H}_4$, so $q(\tilde{H}_1, \tilde{H}_2, \tilde{H}_3, \tilde{H}_4, y_5) = 0$. Now look at the leading coefficient with respect to y_5 of $q(y)$. Since $\tilde{H}_1, \tilde{H}_2, \tilde{H}_3, \tilde{H}_4$ are linearly independent, this coefficient has degree 2 at least. So $\deg q \geq \deg_{x_5} q + 2$, as desired. □

Notice that for homogeneous quasi-translations $x + H$ in dimension 5 with $\mathrm{rk}\mathcal{J}H \leq 2$, $H(\mathbb{C}^n)$ is contained in a 3-dimensional linear subspace of $\mathbb{C}^5$ on account of corollary 3.4.2. In chapter 5, we will use this fact and theorem 3.5.4 to show that for homogeneous quasi-translations $x + H$ in dimension 5 over $\mathbb{C}$ *that come from a homogeneous singular Hessian* (i.e. $H = \nabla R(\nabla f)$ and $\det \mathcal{H}f = 0$), $H(\mathbb{C}^5)$ is contained in a 3-dimensional linear subspace of $\mathbb{C}^5$. This result will subsequently be used to classify all homogeneous singular Hessians in dimension 5 over $\mathbb{C}$.

For quasi-translations $x + H$ in dimension 4 over $\mathbb{C}$, $H(\mathbb{C}^4)$ does not need to be contained in a proper linear subspace of $\mathbb{C}^4$. Take for instance

$$H = \big(ax_1 - x_2, a(ax_1 - x_2), ax_3 - x_4, a(ax_3 - x_4)\big)$$

with $a = x_1x_4 - x_2x_3$. Notice that the second quasi-translation in example 3.5.5 is the homogenization of $x + H$ with H as above. In the next section, we will discuss quasi-translations $x + H$ in dimension 4 over $\mathbb{C}$ for which $H(\mathbb{C}^4)$ is not contained in a proper linear subspace of $\mathbb{C}^4$.

In chapter 5, we will show that quasi-translations in dimension 4 over $\mathbb{C}$ *that come from a nilpotent Hessian* are contained in a proper linear subspace of $\mathbb{C}^4$. This result will subsequently be used to classify all nilpotent Hessians in dimension 4 over $\mathbb{C}$.

3.6 Quasi-translations in dimension 4 with linearly independent components

In proposition 3.1.7, we have seen that a quasi translation $x + H$ over a commutative ring $A \supseteq \mathbb{Q}$ remains a quasi-translation after a linear conjugation. This does not hold for conjugations with invertible polynomial maps in general. But the following theorem indicates in which situation the quasi-translation remains a quasi-translation.

Theorem 3.6.1. *Assume $x + H$ is a quasi-translation over a commutative ring $A \supseteq \mathbb{Q}$ and $F : A^n \to A^n$ is an invertible polynomial map with inverse G. Then the following statements are equivalent:*

i) *$G \circ (x + H) \circ F$ is a quasi-translation,*

ii) *$\nu(G_i) \leq 1$ for all i, where $\nu(f) := \deg_t f(x + tH)$ is the exponent with respect to $x + H$,*

iii) *$G \circ (x + tH) \circ F$ is a quasi-translation over $A[t]$.*

Proof. Notice that the inverse polynomial map of $G \circ (x + tH) \circ F$ is $G \circ (x - tH) \circ F$.

i) ⇒ ii) Assume $G \circ (x + H) \circ F$ is a quasi-translation $x + \tilde{H}$. Since $x + H$ is a quasi-translation, $H(x \pm 2^k H) = H$, whence

$$\big(G \circ (x \pm 2^k H) \circ F\big) \circ \big(G \circ (x \pm 2^k H) \circ F\big) = \big(G \circ (x \pm 2^{k+1} H) \circ F\big)$$

Since by ii) of proposition 3.1.2, $\tilde{H}(x \pm 2^k \tilde{H}) = \tilde{H}$, it follows that $(x \pm 2^k \tilde{H}) \circ (x \pm 2^k \tilde{H}) = (x \pm 2^{k+1} \tilde{H})$, and by induction on k we obtain

$$\big(G \circ (x \pm 2^k H) \circ F\big) = x \pm 2^k \tilde{H}$$

so $G \circ (x + 2^k H) \circ F$ is a quasi-translation (with inverse $G \circ (x - 2^k H) \circ F$) for all $k \in \mathbb{N}$.

Now the sum of a quasi-translation and its inverse is $2x$, so

$$\big(G \circ (x + 2^k H) \circ F\big) + \big(G \circ (x - 2^k H) \circ F\big) = 2x$$

Substituting $x = G(x + 2^k H)$ and using $2^k H(x + 2^k H) = 2^k H$, we obtain

$$G(x + 2^{k+1} H) + G(x) = 2G(x + 2^k H)$$

Adding $2(2^k - 1)G$ on both sides, we obtain

$$G(x + 2^{k+1} H) + (2^{k+1} - 1)G = 2\big(G(x + 2^k H) + (2^k - 1)G\big)$$

and by induction on k, we obtain $G(x + 2^k G) - (2^k - 1)G = 2^k(G(x + 2^0 H) + (2^0 - 1)G) = 2^k G(x + H)$ for all $k \in \mathbb{N}$.

Now let $d := \max\{\deg G, 1\}$ and write $G_i(x + tH) + (t-1)G_i - tG_i(x + H) = c_d t^d + c_{d-1} t^{d-1} + \cdots + c_1 t + c_0$. Since $G_i(x + mH) + (m-1)G_i - mG_i(x + H) = 0$ for all $m \in \{1, 2, 4, \ldots, 2^d\}$,

$$\begin{pmatrix} 1 & 1 & 1^2 & \cdots & 1^d \\ 1 & 2 & 2^2 & \cdots & 2^d \\ 1 & 4 & 4^2 & \cdots & 4^d \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2^d & 2^{d\cdot 2} & \cdots & 2^{d \cdot d} \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Since $\mathbb{Q} \subseteq A$, the matrix on the left hand side is invertible, and $c_0 = c_1 = c_2 = \cdots = c_d = 0$ follows. So $G_i(x + tH) + (t-1)G_i = tG_i(x + H)$ for all i. It follows that $\deg_t G_i(x + tH) \le 1$ for all i, as desired.

ii) $\Rightarrow$ iii) Assume $\deg_t G(x + tH) \le 1$ for all i. Then we can write

$$G(x + tH) = G^{(0)} + tG^{(1)}$$

Notice that $G^{(0)} = G(x + tH)|_{t=0} = G$. So

$$G \circ (x + tH) \circ F = G^{(0)}(F) + tG^{(1)}(F) = G(F) + tG^{(1)}(F) = x + tG^{(1)}(F)$$

Now by substituting $t = -t$ on the left hand side, and hence on the right hand side as well, we obtain the inverse map. This gives the desired result.

iii) $\Rightarrow$ i) This follows by substituting $t = 1$. $\square$

The below example shows that the result of theorem 3.3.4 that for a quasi-translation $x + H$ over $\mathbb{C}$, the image of H is contained in a proper subspace of $\mathbb{C}^n$ for $n = 3$, does not extend to $\mathrm{rk}\mathcal{J}H \le 2$.

Example 3.6.2. Let $n \ge 4$ and take $c = x_1 + x_2x_4 - x_3^2$, $F = (c, x_2, x_3, \ldots, x_n)$ and $G = (2x_1 - c, x_2, x_3, \ldots, x_n)$. Then G is the inverse of F. Take $H = (0, x_1^k, x_1^{k+1}, \ldots, x_1^{k+n-2})$. Then $x + H$ is a quasi-translation (that satisfies (3.4)). Furthermore, $\deg_t c(x + tH) \le \deg c = 2$, and the coefficient of t^2 of $c(x + tH)$ is equal to

$$H_2H_4 - H_3^2 = x_1^k x_1^{k+2} - x_1^{k+1}x_1^{k+1} = 0$$

So $\deg_t c(x + tH) \le 1$. It follows from the above theorem that

$$x + \tilde{H} := G \circ (x + H) \circ F$$

is a quasi-translation. Now

$$\begin{aligned}
x_1 + \tilde{H}_1 &= G_1\big(F + H(F)\big) \\
&= G_1(c, x_2 + c^k, x_3 + c^{k+1}, \ldots, x_n + c^{k+n-2}) \\
&= 2c - c(c, x_2 + c^k, x_3 + c^{k+1}, x_4 + c^{k+2}) \\
&= c - (x_2x_4 + x_2c^{k+2} + x_4c^k + c^{2k+2}) + (x_3^2 + 2c^{k+1}x_3 + c^{2k+2}) \\
&= x_1 - (c^{k+2}x_2 - 2c^{k+1}x_3 + c^kx_4)
\end{aligned}$$

and

$$\tilde{H} = \big(-(c^{k+2}x_2 - 2c^{k+1}x_3 + c^kx_4), c^k, c^{k+1}, \ldots, c^{k+n-2}\big)$$

Since the degrees of the components of $\tilde{H}$ are all different, there does not exist a linear relation between them. So $\tilde{H}(\mathbb{C}^n)$ is not contained in a proper linear subspace of $\mathbb{C}^n$.

The third map in example 3.5.5 is essentially the homogenization of $\tilde{H}|_{k=0}$ above with $n = 4$. The image of

$$\big(ax_1 - x_2, a(ax_1 - x_2), ax_3 - x_4, a(ax_3 - x_4)\big) \tag{3.8}$$

with $a = x_1x_4 - x_2x_3$, is not contained in a proper linear subspace of $\mathbb{C}^4$ either. We will describe the construction of (3.8) in the next section.

3.7 Homogeneous quasi-translations with linearly independent components

Assume n is *even* and put

$$Q_{A,B} := \begin{pmatrix} B(Ax_1 - Bx_2) \\ A(Ax_1 - Bx_2) \\ B(Ax_3 - Bx_4) \\ A(Ax_3 - Bx_4) \\ \vdots \\ B(Ax_{n-1} - Bx_n) \\ A(Ax_{n-1} - Bx_n) \end{pmatrix} \tag{3.9}$$

Then $Q_{A,B}$ is a polynomial map over the polynomial ring $\mathbb{C}[A, B]$. Since

$$\mathcal{J}Q_{A,B} = \begin{pmatrix} AB & -B^2 & & & \emptyset \\ A^2 & -AB & & & \\ & & \ddots & & \\ & & & AB & -B^2 \\ \emptyset & & & A^2 & -AB \end{pmatrix}$$

it follows that $\mathcal{J}Q_{A,B}^2 = 0$. Notice that $Q_{A,B}$ is homogeneous of degree 1 with respect to x. So by Euler's formula, we obtain

$$\mathcal{J}Q_{A,B} \cdot Q_{A,B} = \mathcal{J}Q_{A,B} \cdot \mathcal{J}Q_{A,B} \cdot x = 0$$

So $x + Q_{A,B}$ is a quasi-translation.

Theorem 3.7.1. *If $x + H$ is a quasi-translation over the polynomial ring $\mathbb{C}[A, B]$, and $a, b \in \mathbb{C}[x]$ are invariants of $x + H$, then $x + H|_{A=a,B=b}$ is a quasi-translation over $\mathbb{C}$.*

Proof. Notice that

$$a(x + H|_{A=a,B=b}) = a(x + H)|_{A=a,B=b} = a$$

and

$$b(x + H|_{A=a,B=b}) = b(x + H)|_{A=a,B=b} = b$$

So

$$\begin{aligned}
&H|_{A=a,B=b}(x + H|_{A=a,B=b}) \\
&\quad= H\Big|_{A = a(x+H|_{A=a,B=b}),\, B = b(x+H|_{A=a,B=b}),\, x = x + H|_{A=a,B=b}} \\
&\quad= H\Big|_{A = a,\, B = b,\, x = x + H|_{A=a,B=b}} \\
&\quad= H(x + H)|_{A=a,B=b} \\
&\quad= H|_{A=a,B=b}
\end{aligned}$$

So $(x - H|_{A=a,B=b}) \circ (x + H|_{A=a,B=b}) = x + H|_{A=a,B=b} - H|_{A=a,B=b}(x + H|_{A=a,B=b}) = x + H|_{A=a,B=b} - H|_{A=a,B=b} = x$, as desired. □

If $i, j \le n/2$, then

$$\begin{aligned}
&x_{2i-1}x_{2j} \circ (x + Q_{A,B}) \\
&\quad= \big(x_{2i-1} + B(Ax_{2i-1} - Bx_{2i})\big)\big(x_{2j} + A(Ax_{2j-1} - Bx_{2j})\big) \\
&\quad= x_{2i-1}x_{2j} + B(Ax_{2i-1} - Bx_{2i})A(Ax_{2j-1} - Bx_{2j}) + \\
&\qquad (BAx_{2i-1}x_{2j} - B^2x_{2i}x_{2j} + A^2x_{2i-1}x_{2j-1} - ABx_{2i-1}x_{2j}) \\
&\quad= x_{2i-1}x_{2j} + B(Ax_{2i-1} - Bx_{2i})A(Ax_{2j-1} - Bx_{2j}) + \\
&\qquad (A^2x_{2i-1}x_{2j-1} - B^2x_{2i}x_{2j})
\end{aligned}$$

Since the terms $B(Ax_{2i-1} - Bx_{2i})A(Ax_{2j-1} - Bx_{2j})$ and $(A^2x_{2i-1}x_{2j-1} - B^2x_{2i}x_{2j})$ are symmetric with respect to i, j, we obtain a similar formula for $x_{2j-1}x_{2i}$ instead of $x_{2i-1}x_{2j}$, whence $x_{2i-1}x_{2j} - x_{2i}x_{2j-1}$ is an invariant of $x + Q_{A,B}$.

Example 3.7.2. Assume $n = 4$ and take

$$a := x_1x_4 - x_2x_3$$

Then a and 1 are invariants of $x + Q_{A,B}$. On account of the above theorem

$$x + Q_{a,1} = x + \begin{pmatrix} ax_1 - x_2 \\ a(ax_1 - x_2) \\ ax_3 - x_4 \\ a(ax_3 - x_4) \end{pmatrix}$$

is a quasi-translation: the quasi-translation in (3.8). Since $\mathrm{rk}\mathcal{J}Q_{a,1} = 3$ and a is irreducible, it follows from proposition 1.2.9 that a is in fact the only relation between the components of $Q_{a,1}$ and that the components of $Q_{a,1}$ are linearly independent.

One can compute that for the map $Q_{a,1}$ in the above example, $\mathcal{J}Q_{a,1}^{n-1} \cdot x = 0$ and $\mathrm{rk}\mathcal{J}Q_{a,1} = n-1$. In appendix A, we will show that for nilpotent Jacobians $\mathcal{J}H$ of size n over $\mathbb{C}$ that satisfy both $\mathcal{J}H^{n-1} \cdot x = 0$ and $\mathrm{rk}\mathcal{J}H = n-1$, the rows are automagically linearly independent over $\mathbb{C}$.

Example 3.7.3. Assume that $n \geq 6$ even and take $a := x_1x_4 - x_2x_3$ and

$$b := x_3x_6 - x_4x_5$$

Then a and b are invariants of $x + Q_{A,B}$, so by the above theorem

$$x + Q_{a,b} = x + \begin{pmatrix} b(ax_1 - bx_2) \\ a(ax_1 - bx_2) \\ b(ax_3 - bx_4) \\ a(ax_3 - bx_4) \\ \vdots \\ b(ax_{n-1} - bx_n) \\ a(ax_{n-1} - bx_n) \end{pmatrix}$$

is a quasi-translation. Since

$$a(t, t^2 - 1, t^3, t^4, t^5 - 1, t^6, t^7, \ldots, t^n) = t^3$$

and

$$b(t, t^2 - 1, t^3, t^4, t^5 - 1, t^6, t^7, \ldots, t^n) = t^4$$

we see that

$$(Q_{a,b})_{2i-1}(t, t^2 - 1, t^3, t^4, t^5 - 1, t^6, t^7, \ldots, t^n) = -t^{2i+8} + O(t^{2i+7})$$

and

$$(Q_{a,b})_{2i}(t, t^2 - 1, t^3, t^4, t^5 - 1, t^6, t^7, \ldots, t^n) = -t^{2i+7} + O(t^{2i+6})$$

from which it follows that the components of $Q_{a,b}$ are linearly independent.

Example 3.7.4. Assume that $n \geq 6$ even and take $a := x_1x_4 - x_2x_3$, $b := x_3x_6 - x_4x_5$ and

$$c := x_1x_6 - x_2x_5$$

Then c is an invariant of the quasi-translation $x + Q_{a,b}$ and $x_1 + C$ is a translation. So $(x, x_{n+1}) + (Q_{a,b}, C)$ is the Cartesian product of two quasi-translations and hence a quasi-translation itself. Furthermore,

$$\begin{aligned}
&c(ax_{n-1} - bx_n) \circ \big((x, x_{n+1}) + (Q_{a,b}, C)\big) \\
&\quad = \quad c(ax_{n-1} - bx_n) \circ (x + Q_{a,b}) \\
&\quad = \quad c\Big(a\big(x_{n-1} + b(ax_{n-1} - bx_n)\big) - b\big(x_n + a(ax_{n-1} - bx_n)\big)\Big) \\
&\quad = \quad c\big(ax_{n-1} - bx_n + (ab - ba)(ax_{n-1} - bx_n)\big) \\
&\quad = \quad c(ax_{n-1} - bx_n)
\end{aligned}$$

So $c(ax_{n-1} - bx_n)$ is an invariant of $(x, x_{n+1}) + (Q_{a,b}, C)$ (this can also be shown by reasoning with invariants). It follows from the above theorem that

$$(x, x_{n+1}) + \big(Q_{a,b}, c(ax_{n-1} - bx_n)\big) = (x, x_{n+1}) + \begin{pmatrix} b(ax_1 - bx_2) \\ a(ax_1 - bx_2) \\ \vdots \\ b(ax_{n-1} - bx_n) \\ a(ax_{n-1} - bx_n) \\ c(ax_{n-1} - bx_n) \end{pmatrix}$$

is a quasi-translation. Since

$$c(t, t^2 - 1, t^3, t^4, t^5 - 1, t^6, t^7, \ldots, t^n, 0) = t^5 + t^2 - 1$$

we see that

$$c(ax_{n-1} - bx_n) \circ (t, t^2 - 1, t^3, t^4, t^5 - 1, t^6, t^7, \ldots, t^n, 0) = -t^{n+9} + O(t^{n+8})$$

It follows from the example above that the components of $\big(Q_{a,b}, c(ax_{n-1} - bx_n)\big)$ are linearly independent.

So in dimension 6 and up, there are homogeneous quasi-translations $x + H$ over $\mathbb{C}$ without linear relations between the components of H. Since there are no such quasi-translations in dimension 4 and below, dimension 5 remains. As we already mentioned, it is not known yet whether for homogeneous quasi-translations $x + H$ in dimension 5 the components of H need to be linearly dependent. In the spirit of C. Olech, I promise a bottle of Joustra Beerenburg (Frisian spirit) for the one who first solves the problem whether for quasi-translations $x + H$ in dimension 5 with H homogeneous, the components of H need to be linearly dependent or not.

Chapter 4

The homogeneous dependence problem

4.1 Introduction

In the previous chapter, we constructed counterexamples to the homogeneous dependence problem in dimension 6 and up. In this chapter, we shall construct other such examples, including one in dimension 5 and one of degree 3.

Furthermore, we shall show that the homogeneous dependence problem has an affirmative answer in dimension $n \leq 3$. The case $n \leq 2$ has already been proved by H. Bass, E. Connell and D. Wright in [4]. More generally, the following can be proved easily.

Proposition 4.1.1. *Assume $H \in \mathbb{C}[x]^n$ is homogeneous, $n \geq 2$ and $\operatorname{rk} \mathcal{J} H \leq 1$. Then H_1 and H_2 are linearly dependent over $\mathbb{C}$.*

Proof. Since $\operatorname{rk} \mathcal{J}(H_1, H_2) \leq \operatorname{rk} \mathcal{J} H \leq 1$, it follows that H_1 and H_2 are algebraically dependent over $\mathbb{C}$. Since H_1 and H_2 are homogeneous of the same degree, it follows that there exists a nonzero homogeneous $R \in \mathbb{C}[y_1, y_2]$ such that $R(H_1, H_2) = 0$. Since R is homogeneous and bivariate, R decomposes in linear factors $\alpha y_1 + \beta y_2$. Consequently, $\alpha H_1 + \beta H_2 = 0$ for some nonzero $(\alpha, \beta) \in \mathbb{C}^2$, as desired. $\square$

So the homogeneous dependence problem has an affirmative answer in dimensions $n = 1$ and $n = 2$. In dimension $n = 3$, this dependence problem has already been proved for cubic maps by D. Wright. We shall generalize this result to arbitrary degree. More precisely, we will prove the following:

Theorem 4.1.2. *Assume $H \in \mathbb{C}[x]^n$ is homogeneous, $2 \le n \le 4$, $\mathrm{rk}\mathcal{J}H \le 2$ and at least $n-1$ of the n eigenvalues of $\mathcal{J}H$ are zero. Then the components of H are linearly dependent over $\mathbb{C}$.*

Notice that the above theorem, more or less the main theorem of this chapter, is not true if we omit $2 \le n$: the components of the map $H = x_1^d$ are not linearly independent over $\mathbb{C}$.

Corollary 4.1.3. *Assume $H \in \mathbb{C}[x]^n$ is homogeneous, $n = 3$, and $\mathcal{J}H$ is nilpotent. Then the components of H are linearly dependent over $\mathbb{C}$.*

Proof. Since $\mathcal{J}H$ is nilpotent, $\mathrm{rk}\mathcal{J}H \le 2$ and all eigenvalues of $\mathcal{J}H$ are zero. So H satisfies the properties of theorem 4.1.2. □

Theorem 4.1.4. *Assume $H = (H_1, H_2, H_3)$ is a homogeneous polynomial map in x over $\mathbb{C}$ and $\mathcal{J}H$ is nilpotent. Then H is linearly triangularizable.*

Proof. From corollary 4.1.3, it follows that there exists a nonzero $\lambda \in \mathbb{C}^3$ such that $\lambda^{\mathrm{t}} H = 0$. Now choose $T \in \mathrm{GL}_n(\mathbb{C})$ such that the first row of T^{-1} equals λ. Then the first component of $T^{-1}H(Tx)$ is zero. So we may assume that $H_1 = 0$.

Since $H_1 = 0$, the nilpotency of $\mathcal{J}H$ is equivalent to that of $\mathcal{J}_{x_2,x_3}(H_2, H_3)$. In other words, (H_2, H_3) is a two-dimensional map with nilpotent Jacobian over the unique factorization domain $\mathbb{C}[x_1]$. It follows from theorem 2.2.7 that H is of the form

$$H = \begin{pmatrix} 0 \\ b(x_1)g\big(a(x_1)x_2 - b(x_1)x_3, x_1\big) + d(x_1) \\ a(x_1)g\big(a(x_1)x_2 - b(x_1)x_3, x_1\big) + c(x_1) \end{pmatrix}$$

where $g \in \mathbb{C}[y_1, y_2]$. If $\deg_{y_1} g \le 0$, then H is lower triangular, so assume $\deg_{y_1} g \ge 1$. Assume for a similar reason that $(a(x_1), b(x_1)) \ne 0$ in addition. Since H and hence also $(\frac{\partial}{\partial x_2} H_1, \frac{\partial}{\partial x_2} H_2)$ is homogeneous, it follows that $(b(x_1), a(x_1))$ is homogeneous as well. Say that $a(x_1) = \alpha x_1^r$ and

$b(x_1) = \beta x_1^r$. Since $(a(x_1), b(x_1)) \neq 0$, it follows that $(\alpha, \beta) \neq 0$ as well. So we can take T such that T^{-1} is of the form

$$T^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & -\beta \\ * & * & * \end{pmatrix}$$

Put $f(x) = x_1^r g\big(a(x_1)x_2 - b(x_1)x_3, x_1\big)$. Since the first row of T is equal to e_1^{t},

$$\begin{aligned} (T^{-1})_2 H(Tx) &= \alpha\big(\beta f(Tx) + c(T_1x)\big) - \beta\big(\alpha f(Tx) + d(T_1x)\big) \\ &= \alpha c(x_1) - \beta d(x_1) \end{aligned}$$

It follows that $\mathcal{J}T^{-1}H(Tx)$ is lower triangular, as desired. □

The above theorem is not true if only two eigenvalues of $\mathcal{J}H$ are zero instead of three. Take for instance $H = (0, x_1^2x_2x_3, x_2^2x_3^2)$. Then the vector space over $\mathbb{C}$ spanned by the entries of $\mathcal{J}H$ has dimension 6. This dimension does not change by linear conjugations. But a triangular matrix in dimension 3 with determinant zero can only have five nonzero entries.

The proof of theorem 4.1.2 consists of two cases: the case that the vectors x, $\mathcal{J}H \cdot x$ and $\mathcal{J}H^2 \cdot x$ are dependent and the case that they are not. For the first case, we do not require the restriction $n \leq 4$. The following lemma says something about the first case.

Lemma 4.1.5. *Assume $H \in \mathbb{C}[x]^n$ is homogeneous,* $\mathrm{rk}\mathcal{J}H \leq 2$ *and at least $n-1$ eigenvalues of $\mathcal{J}H$ are zero. If x, $\mathcal{J}H \cdot x$ and $\mathcal{J}H^2 \cdot x$ are dependent, then $\mathcal{J}H \cdot H = \mathrm{tr}\mathcal{J}H \cdot H$. This holds in particular if* $\mathrm{rk}\mathcal{J}H \leq 1$.

Proof. If $\mathcal{J}H = 0$, then $\mathcal{J}H \cdot H = \mathrm{tr}\mathcal{J}H \cdot H$. So assume $\mathcal{J}H \neq 0$. Then $\mathrm{rk}\mathcal{J}H \geq 1$ and $d := \deg H \geq 1$. We distinguish two cases:

- x and $\mathcal{J}H \cdot x$ are dependent.
  Since $d \geq 1$ and $\mathcal{J}H \cdot x = dH$, it follows that

  $$H = g(x) \cdot x$$

  for some polynomial g. Consequently, $n \mid \mathrm{trdeg}_{\mathbb{C}}\mathbb{C}(H) = \mathrm{rk}\mathcal{J}H$. Since $\mathcal{J}H \neq 0$, we obtain $n = \mathrm{rk}\mathcal{J}H$. So all eigenvalues of $\mathcal{J}H$ are nonzero. It follows that $n = 1$ and hence $\mathcal{J}H = (\mathrm{tr}\mathcal{J}H)$, which gives the desired result.

- x and $\mathcal{J}H \cdot x$ are independent and $\mathcal{J}H^2 \cdot x$ is dependent of x and $\mathcal{J}H \cdot x$.
  Say that
  $$\mathcal{J}H^2 \cdot x = a(x) \cdot x + b(x) \cdot \mathcal{J}H \cdot x$$
  where $a(x), b(x) \in \mathbb{C}(x)$. Now take $v_1 \in \mathbb{C}^n$ generic. Then the denominators of $a(x)$ and $b(x)$ do not vanish at $x = v_1$. Furthermore, $a(v_1) = 0$, if and only if $a(x) = 0$. Define $v_2 := dH(v_1)$. Since x and $\mathcal{J}H \cdot x$ are independent and v_1 is generic, it follows that $v_1 = x|_{x=v_1}$ and $v_2 = (\mathcal{J}H \cdot x)|_{x=v_1}$ are independent. Choose $v_3, v_4, \ldots, v_n \in \mathbb{C}^n$ such that
  $$T := (\, v_1 \mid v_2 \mid v_3 \mid v_4 \mid \cdots \mid v_n \,)$$
  is invertible. Then
  $$\begin{aligned} \mathcal{J}\big(T^{-1}H(Tx)\big)|_{x=e_1} &= T^{-1}\mathcal{J}H|_{x=Te_1}T \\ &= T^{-1}\mathcal{J}H|_{x=v_1}(\, v_1 \mid v_2 \mid v_3 \mid v_4 \mid \cdots \mid v_n \,) \\ &= T^{-1}(\, v_2 \mid a(v_1)v_1 + b(v_1)v_2 \mid * \mid * \mid \cdots \mid * \,) \\ &= (\, e_2 \mid a(v_1)e_1 + b(v_1)e_2 \mid * \mid * \mid \cdots \mid * \,) \end{aligned}$$
  Assume first that $a(v_1) \neq 0$. Since $\mathrm{rk}\mathcal{J}\big(T^{-1}H(Tx)\big) = \mathrm{rk}\mathcal{J}H \leq 2$, only the first and second row of $\mathcal{J}\big(T^{-1}H(Tx)\big)|_{x=e_1}$ are nonzero. It follows that the sum of the determinants of the principal minors of size 2 of $\mathcal{J}\big(T^{-1}H(Tx)\big)|_{x=e_1}$ is equal to $-a(v_1)$. This contradicts the assumption that $n-1$ of the n eigenvalues are zero. So $a(v_1) = 0$. Since v_1 is generic, we obtain $a(x) = 0$.

  So $\mathcal{J}H^2 \cdot x = b(x) \cdot \mathcal{J}H \cdot x$. Since $d \geq 1$ and $\mathcal{J}H \cdot x = dH$, we obtain $\mathcal{J}H \cdot H = b(x) \cdot H$. If $b(x) = 0$, then $x + H$ is a quasi-translation, and by the nilpotency of $\mathcal{J}H$ we have $\mathrm{tr}\mathcal{J}H = 0$ and obtain
  $$\mathcal{J}H \cdot H = b(x) \cdot H = 0 \cdot H = \mathrm{tr}\mathcal{J}H \cdot H$$
  as desired. So assume $b(x) \neq 0$. Then $b(x)$ is an eigenvalue of $\mathcal{J}H$ with eigenvector H. Since at most one eigenvalue of $\mathcal{J}H$ is nonzero and the sum of all eigenvalues is equal to $\mathrm{tr}\mathcal{J}H$, $b(x) = \mathrm{tr}\mathcal{J}H$ follows, as desired.

If $\mathrm{rk}\mathcal{J}H = 1$, then the image of $\mathcal{J}H$ is one-dimensional, whence $\mathcal{J}H \cdot x$ and $\mathcal{J}H \cdot y$ are dependent. Now substitute $y = \mathcal{J}H \cdot x$ to obtain that x, $\mathcal{J}H \cdot x$ and $\mathcal{J}H^2 \cdot x$ are dependent, as desired. □

Corollary 4.1.6. *Assume $H \in \mathbb{C}[x]^n$ is homogeneous, $\mathcal{J}H$ is nilpotent and* $\mathrm{rk}\mathcal{J}H \leq 2 \leq n$*. Assume that either x, $\mathcal{J}H \cdot x$ and $\mathcal{J}H^2 \cdot x$ are dependent or* $\mathrm{rk}\mathcal{J}H \leq 1$*. Then the components of H are linearly dependent.*

Proof. Since $\mathcal{J}H$ is nilpotent, $\mathrm{tr}\mathcal{J}H = 0$. From the above lemma, we obtain

$$\mathcal{J}H \cdot H = \mathrm{tr}\mathcal{J}H \cdot H = 0$$

so $x + H$ is a homogeneous quasi-translation with $\mathrm{rk}\mathcal{J}H \leq 2$. Now apply corollary 3.4.2 to get the desired result. □

The above corollary is not true in dimension 1: the components of $H = (1)$ are not linearly dependent. Lemma 4.1.5 can also be proved by means of investigating Jordan normal forms of $\mathcal{J}H$, but the method with the generic vector v_1, which is due to David Wright, will be used again in the proof of theorem 4.1.2.

In the next section, we construct counterexamples to the homogeneous dependence problem in dimension 5 and up, and cubic ones in dimension 9 and up. After that, the proof of theorem 4.1.2 follows. In section 4.3, we formulate a structure theorem for homogeneous Jacobians of rank ≤ 2. After that we prove the case that x, $\mathcal{J}H \cdot x$ and $\mathcal{J}H^2 \cdot x$ are *independent* of theorem 4.1.2 in section 4.4.

At that point, we will have proved theorem 4.1.2 with the extra condition that $\mathrm{tr}\mathcal{J}H = 0$, and hence corollary 4.1.3 and theorem 4.1.4 as well. This is because in case x, $\mathcal{J}H \cdot x$ and $\mathcal{J}H^2 \cdot x$ are *dependent*, we have $\mathcal{J}H \cdot H = \mathrm{tr}\mathcal{J}H \cdot H$ on account of lemma 4.1.5. Consequently, $x + H$ is a homogeneous quasi-translation with $\mathrm{rk}\mathcal{J} = 2$ if we assume $\mathrm{tr}\mathcal{J}H = 0$, and theorem 3.4.2 applies.

But we will remove the trace condition for the case x, $\mathcal{J}H \cdot x$ and $\mathcal{J}H^2 \cdot x$ are dependent in section 4.5. Furthermore, we omit the condition $n \leq 4$ for this case, just as in the above paragraph.

In the last section, we will show that there are essentially two cubic homogeneous maps with nilpotent Jacobians in dimension 4, namely the so-called linearly triangularizable ones and a slightly generalized version of Anick's example. Furthermore, we study quadratic homogeneous maps in dimension 5.

4.2 Homogeneous maps with nilpotent Jacobians and linearly independent components

In this section, we construct counterexamples to the homogeneous dependence problem that are not quasi-translations themselves, but are composed of quasi-translations and other invertible maps. The main theorem of this section is the following:

Theorem 4.2.1. *Let $F = (F_1, F_2, \ldots, F_s, F_{s+1}, F_{s+2}, \ldots, F_n)$ be a polynomial map in x over $\mathbb{C}$ such that*

1. *$(F_1, F_2, \ldots, F_s)$ is invertible over $\mathbb{C}[x_{s+1}, x_{s+2}, \ldots, x_n]$,*
2. *$(F_{s+1}, F_{s+2}, \ldots, F_n)$ is invertible over $\mathbb{C}[x_1, x_2, \ldots, x_s]$,*
3. *$F_{s+i}(F_1, F_2, \ldots, F_s, x_{s+1}, x_{s+2}, \ldots, x_n) = F_{s+i}$ for all $i \geq 1$.*

Then F is invertible over $\mathbb{C}$. In particular, if $F = x + H$ and H is homogeneous of degree ≥ 2, then $\mathcal{J}H$ is nilpotent.

Proof. Since

$$F = (x_1, x_2, \ldots, x_s, F_{s+1}, F_{s+2}, \ldots, F_n) \circ (F_1, F_2, \ldots, F_s, x_{s+1}, x_{s+2}, \ldots, x_n)$$

it follows that F is a composition of two invertible maps over $\mathbb{C}$, and hence invertible itself.
Assume $F = x + H$. Then $\det(I_n + \mathcal{J}H) = \det \mathcal{J}(x + H) \in \mathbb{C}^*$. From the proof of iii) of proposition 1.2.6, we obtain that $\mathcal{J}H$ is nilpotent in case H is homogeneous of degree ≥ 2. □

Corollary 4.2.2. *Let $H = (H_1, H_2, \ldots, H_s, \ldots, H_n) \in \mathbb{C}[x]^n$ and assume that*

$$H_i(x_1 + H_1, x_2 + H_2, \ldots, x_s + H_s, x_{s+1}, x_{s+2}, \ldots, x_n) = H_i \tag{4.1}$$

for all i and $H_{s+i} \in \mathbb{C}[x_1, x_2, \ldots, x_{s+i-1}]$ for all $i \geq 1$. Then $x + H$ is an invertible polynomial map over $\mathbb{C}$. In particular, $\mathcal{J}H$ is nilpotent if H is homogeneous of degree ≥ 2.

Proof. Since (4.1) is satisfied for all $i \leq s$, it follows that $(x_1 + H_1, x_2 + H_2, \ldots, x_s + H_s)$ is a quasi-translation over $\mathbb{C}[x_{s+1}, x_{s+2}, \ldots, x_n]$. Since (4.1) is satisfied for all $i \geq s+1$, we obtain that $x_{s+i} + H_{s+i}$ is an invariant of the above quasi-translation for all $i \geq 1$.

At last, $(x_{s+1}+H_{s+1}, x_{s+2}+H_{s+2}, \ldots, x_n+H_n)$ is invertible over $\mathbb{C}[x_1, x_2, \ldots, x_s]$, because its Jacobian with respect to $x_{s+1}, x_{s+2}, \ldots, x_n$ is lower triangular with ones on the diagonal. So theorem 4.2.1 gives the desired result. □

We use corollary 4.2.2 first to construct counterexamples to the homogeneous dependence problem. Just as in section 3.7 we take $a := x_1x_4 - x_2x_3$ and $b := x_3x_6 - x_4x_5$. Furthermore, we define $Q_{A,B}$ as in (3.9) with $n = 6$.

Corollary 4.2.3. *Put*

$$H := \big(x_5(Q_{a,x_5^2})_1, x_5(Q_{a,x_5^2})_2, x_5(Q_{a,x_5^2})_3, x_5(Q_{a,x_5^2})_4, a^3\big)$$

Then H is a counterexample in dimension 5 to the homogeneous dependence problem. Observe that H has degree 6.

Proof. We will apply corollary 4.2.2 with $s = 4$ and $n = 5$. For that purpose, put $\tilde{H} = x_5^{-1}(H_1, H_2, H_3, H_4, 0)$. Then $x + \tilde{H}$ is a quasi-translation and a is an invariant of $x+\tilde{H}$. It follows that $t\tilde{H}(x+t\tilde{H}) = t\tilde{H}$. Substituting $t = x_5$, we can derive that $x + (H_1, H_2, H_3, H_4, 0)$ is a quasi-translation.

Since a is an invariant of $x + \tilde{H}$, it follows that $a(x + t\tilde{H}) = a$. Again by substituting $t = x_5$, we obtain that $a(x_1+H_1, x_2+H_2, x_3+H_3, x_4+H_4) = a$. So $H_i(x_1 + H_1, x_2 + H_2, x_3 + H_3, x_4 + H_4, x_5) = H_i$ for all i. Now apply corollary 4.2.2 with $s = 4$ to obtain that $x + H$ is invertible and $\mathcal{J}H$ is nilpotent.

To show the linear independence of the components of H, notice that

$$a(t+1, t^2, t^3, t^4, 1) = t^4$$

and hence

$$H(t+1, t^2, t^3, t^4, 1) = \big(\Theta(t^5), \Theta(t^9), \Theta(t^7), \Theta(t^{11}), \Theta(t^{12})\big)$$

whence the components of H are linearly independent. □

Corollary 4.2.4. *Let $n \geq 6$ and put*

$$H := \big(x_5(Q_{x_5,x_6})_1, x_5(Q_{x_5,x_6})_2, x_5(Q_{x_5,x_6})_3, x_5(Q_{x_5,x_6})_4, a^2, x_5^4, \ldots, x_{n-1}^4\big)$$

Then H is a counterexample of degree 4 to the homogeneous dependence problem.

Proof. Again we apply corollary 4.2.2 with $s = 4$. The invertibility of $x + H$ and the nilpotency of $\mathcal{J}H$ follow in a similar manner as in the previous corollary.

To show the linear independence of the components of H, notice that

$$a(t, t^2, t^3, t^4 + 1, 1, t^6, t^7, \dots, t^n) = t$$

and hence

$$\begin{aligned} H(t, t^2, t^3, t^4 + 1, 1, t^6, t^7, \dots, t^n) &= \big(\Theta(t^{14}), \Theta(t^8), \Theta(t^{16}), \Theta(t^{10}), \Theta(t^2), \\ &\quad \Theta(1), \Theta(t^{24}), \dots, \Theta(t^{4(n-1)})\big) \end{aligned}$$

whence the components of H are linearly independent. $\square$

Corollary 4.2.5. *Let $n \geq 10$ and put*

$$\begin{aligned} H &:= \big((Q_{x_9,x_{10}})_1, (Q_{x_9,x_{10}})_2, (Q_{x_9,x_{10}})_3, (Q_{x_9,x_{10}})_4, (Q_{x_9,x_{10}})_5, \\ &\quad (Q_{x_9,x_{10}})_6, bx_9, ax_9, ax_7 - bx_8, x_9^3, x_{10}^3, \dots, x_{n-1}^3\big) \end{aligned}$$

Then H is a cubic counterexample to the homogeneous dependence problem.

Proof. We apply corollary 4.2.2 with $s = 8$. Notice that H_i is an invariant of $(x_1, x_2, \dots, x_6) + (H_1, H_2, \dots, H_6)$ for all $i \leq 8$. Since $\frac{\partial}{\partial x_7} H_i = \frac{\partial}{\partial x_8} H_i = 0$ for all $i \leq 8$, it follows that H_i is an invariant of $(x_1, x_2, \dots, x_8) + (H_1, H_2, \dots, H_8)$ for all $i \leq 8$. So $(x_1, x_2, \dots, x_8) + (H_1, H_2, \dots, H_8)$ is a quasi-translation.

One can easily verify that H_9 is an invariant of $(x_1, x_2, \dots, x_8) + (H_1, H_2, \dots, H_8)$ and that the conditions of corollary 4.2.2 are fulfilled for $s = 8$. So $x + H$ is invertible and $\mathcal{J}H$ is nilpotent.

To show that the components of H are linearly independent over $\mathbb{C}$, notice that

$$a(t, t^2, t^3 + 1, t^4, t^5, t^6, t^7, t^8, t^9, t^{10}, t^{11}, \dots, t^n) = -t^2$$

and

$$b(t, t^2, t^3 + 1, t^4, t^5, t^6, t^7, t^8, t^9, t^{10}, t^{11}, \dots, t^n) = t^6$$

Hence

$$\begin{aligned}
&H(t, t^2, t^3+1, t^4, t^5, t^6, t^7, t^8, t^9, t^{10}, t^{11}, \ldots, t^n) \\
&\quad = \big(\Theta(t^{22}), \Theta(t^{21}), \Theta(t^{24}), \Theta(t^{23}), \Theta(t^{26}), \Theta(t^{25}), \\
&\qquad \Theta(t^{15}), \Theta(t^{11}), \Theta(t^{14}), \Theta(t^{27}), \Theta(t^{30}), \ldots, \Theta(t^{3(n-1)})\big)
\end{aligned}$$

So the components of H are linearly independent. □

Corollary 4.2.6. *Let $n = 11$ and put*

$$\begin{aligned}
H \quad := \quad & \big((Q_{x_{10},x_{11}})_1, (Q_{x_{10},x_{11}})_2, (Q_{x_{10},x_{11}})_3, (Q_{x_{10},x_{11}})_4, (Q_{x_{10},x_{11}})_5, \\
& (Q_{x_{10},x_{11}})_6, cx_{10}, bx_{10}, ax_{10}, bx_7 - cx_8, ax_8 - bx_9\big)
\end{aligned}$$

where $c := x_1x_6 - x_2x_5$. Then H is a cubic counterexample to the homogeneous dependence problem. Observe that $x+H$ is a composition of two quasi-translations, namely $x + (0^1, 0^2, \ldots, 0^9, H_{10}, H_{11})$ and $x + (H_1, H_2, \ldots, H_9, 0, 0)$.

Proof. We apply corollary 4.2.2 with $s = 9$. The proof that $x+H$ is invertible and $\mathcal{J}H$ is nilpotent is similar to that of the previous corollary.
To show that the components of H are linearly independent over $\mathbb{C}$, notice that

$$\begin{aligned}
a(t, t^2, t^3+1, t^4, t^5, t^6+1, t^7, t^8, t^9, t^{10}, t^{11}, \ldots, t^n) &= -t^2 \\
b(t, t^2, t^3+1, t^4, t^5, t^6+1, t^7, t^8, t^9, t^{10}, t^{11}, \ldots, t^n) &= t^6 + t^3 + 1 \\
c(t, t^2, t^3+1, t^4, t^5, t^6+1, t^7, t^8, t^9, t^{10}, t^{11}, \ldots, t^n) &= t
\end{aligned}$$

whence

$$\begin{aligned}
&H(t, t^2, t^3+1, t^4, t^5, t^6+1, t^7, t^8, t^9, t^{10}, t^{11}) \\
&\quad = \big(\Theta(t^{24}), \Theta(t^{23}), \Theta(t^{26}), \Theta(t^{25}), \Theta(t^{28}), \Theta(t^{27}), \\
&\qquad \Theta(t^{11}), \Theta(t^{16}), \Theta(t^{12}), \Theta(t^{13}), \Theta(t^{15})\big)
\end{aligned}$$

So the components of H are linearly independent. □

We shall give two cubic homogeneous maps with nilpotent Jacobians in dimension 9, with linearly independent components. In order to prove the nilpotency of their Jacobians, we need the following lemma.

Lemma 4.2.7. *Let*

$$F := \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_d \\ x_{d+1} \\ x_{d+2} \end{pmatrix} + A \begin{pmatrix} 0 \\ \binom{d}{0} x_1^d \\ \binom{d}{1} x_1^{d-1} x_2 \\ \binom{d}{2} x_1^{d-2} x_2^2 \\ \vdots \\ \binom{d}{d-2} x_1^2 x_2^{d-2} \\ \binom{d}{d-1} x_1 x_2^{d-1} \\ \binom{d}{d} x_2^d \end{pmatrix} + B \begin{pmatrix} x_2 \\ -x_3 \\ -x_4 \\ -x_5 \\ \vdots \\ -x_{d+1} \\ -x_{d+2} \\ 0 \end{pmatrix}$$

Then F is invertible over $\mathbb{C}[A,B]$, even tame. More precisely, F decomposes into $d+2$ elementary maps.

Proof. Notice that

$$AF_1^d + x_2 = F_2 + BF_3 + B^2 F_4 + \cdots + B^{d-1} F_{d+1} + B^d F_{d+2}$$

It follows that there exists an elementary map E such that

$$E(F) = (F_1, x_2, F_3, F_4, \ldots, F_{d+2})$$

Now take $Px = (x_2, x_1, x_{d+2}, x_{d+1}, \ldots, x_4, x_3)$. Then one can easily verify that the Jacobian of $P^{-1}E(F(Px))$ is lower triangular. This gives the desired result. $\square$

The following map was obtained as a variation of the cubic homogeneous map in corollary 4.2.5 by G. Zampieri in [60]. But it does not fit into an invertibility proof with corollary 4.2.2. For that reason, I was forced to generalize corollary 4.2.2, with theorem 4.2.1 as the result.

Corollary 4.2.8 (G. Zampieri)**.** *Let $n \geq 9$ and put*

$$\begin{aligned} H \;=\; & \big((Q_{x_7,x_8})_1, (Q_{x_7,x_8})_2, (Q_{x_7,x_8})_3, (Q_{x_7,x_8})_4, (Q_{x_7,x_8})_5, (Q_{x_7,x_8})_6, \\ & bx_8, ax_7 - bx_9, ax_8, x_9^3, \ldots, x_{n-1}^3\big) \end{aligned}$$

Then H is a cubic counterexample to the homogeneous dependence problem.

Proof. We apply theorem 4.2.1 with $s = 6$. Notice that $(x_1, x_2, \ldots, x_6) + (H_1, H_2, \ldots, H_6)$ is a quasi-translation over $\mathbb{C}[x_7, x_8, \ldots, x_n]$ with invariants

a and b and that $(x_7, x_8, x_9) + (H_7, H_8, H_9)$ is invertible over $\mathbb{C}[a, b] \subseteq \mathbb{C}[x_1, x_2, x_3, x_4, x_5, x_6]$ on account of lemma 4.2.7 with $d = 1$, $A = a$ and $B = b$.

One can easily verify that $x + H$ satisfies the conditions of F in theorem 4.2.1 with $s = 6$. So $x + H$ is invertible. Since H is homogeneous of degree 3, it follows that $\mathcal{J}H$ is nilpotent.

To show that the components of H are linearly independent over $\mathbb{C}$, notice that

$$a(t, t^2, t^3 + 1, t^4, t^5, t^6, t^7, t^8, t^9, t^{10}, \dots, t^n) = -t^2$$

and

$$b(t, t^2, t^3 + 1, t^4, t^5, t^6, t^7, t^8, t^9, t^{10}, \dots, t^n) = t^6$$

Hence

$$\begin{aligned} & H(t, t^2, t^3 + 1, t^4, t^5, t^6, t^7, t^8, t^9, t^{10}, t^{11}, \dots, t^n) \\ & \quad = \big(\Theta(t^{18}), \Theta(t^{17}), \Theta(t^{20}), \Theta(t^{19}), \Theta(t^{22}), \Theta(t^{21}), \\ & \qquad \Theta(t^{14}), \Theta(t^{15}), \Theta(t^{10}), \Theta(t^{27}), \dots, \Theta(t^{3(n-1)})\big) \end{aligned}$$

So the components of H are linearly independent. □

Below is another cubic counterexample to the homogeneous dependence problem in dimension 9. The construction differs in that no homogeneous invariants of degree 3 are used.

Corollary 4.2.9. *Let $n \geq 9$ and put*

$$\begin{aligned} H \;:=\; & \big((Q_{x_5,x_6})_1, (Q_{x_5,x_6})_2, (Q_{x_5,x_6})_3, (Q_{x_5,x_6})_4, ax_6, x_5^3 - ax_7, \\ & 3x_5^2x_6 - ax_8, 3x_5x_6^2 - ax_9, x_6^3, x_7^3, \dots, x_{n-3}^3\big) \end{aligned}$$

Then H is a cubic counterexample to the homogeneous dependence problem.

Proof. We apply theorem 4.2.1 with $s = 4$. Notice that $(x_1, x_2, x_3, x_4) + (H_1, H_2, H_3, H_4)$ is a quasi-translation over $\mathbb{C}[x_7, x_8, \dots, x_n]$ with invariants a and b and that $(x_5, x_6, x_7, x_8, x_9) + (H_5, H_6, H_7, H_8, H_9)$ is invertible over $\mathbb{C}[a, b] \subseteq \mathbb{C}[x_1, x_2, x_3, x_4]$ on account of lemma 4.2.7 with $d = 3$, $A = a$ and $B = b$.

One can easily verify that $x + H$ satisfies the conditions of F in theorem 4.2.1. So $x + H$ is invertible. Since H is homogeneous of degree 3, it follows that $\mathcal{J}H$ is nilpotent.

To show that the components of H are linearly independent over $\mathbb{C}$, notice that

$$a(t,-2,t^3,-t^2,t^5,t^6,t^7,t^8,t^9,t^{10},\ldots,t^n) = t^3$$

whence

$$\begin{aligned}
&H(t,-2,t^3,-t^2,t^5,t^6,t^7,t^8,t^9,t^{10},t^{11},\ldots,t^n)\\
&\quad= \big(\Theta(t^{12}),\Theta(t^{11}),\Theta(t^{14}),\Theta(t^{13}),\Theta(t^{9}),\Theta(t^{15}),\\
&\qquad\Theta(t^{16}),\Theta(t^{17}),\Theta(t^{18}),\Theta(t^{21}),\ldots,\Theta(t^{3(n-3)})\big)
\end{aligned}$$

So the components of H are linearly independent. $\square$

Notice that $a = x_1x_4 - x_2x_3$ is a homogeneous invariant of the latter map $x + H$. Other homogeneous invariants of degree 3 at most are λ and λa, where $\lambda \in \mathbb{C}$. By way of computer calculations, one can verify that there are no other homogeneous invariants of degree 3 at most. Below I will describe how.

Assume H is homogeneous of degree d and let f be a homogeneous invariant of $x+H$, say of degree r. Then by substituting $x = t^{1/(d-1)}x$ in f, we obtain

$$\begin{aligned}
f(x+tH) &= t^{-r/(d-1)}f\big(t^{1/(d-1)}x + t^{1/(d-1)}tH\big)\\
&= t^{-r/(d-1)}f\big(t^{1/(d-1)}x + H(t^{1/(d-1)}x)\big)\\
&= t^{-r/(d-1)}f\big(t^{1/(d-1)}x\big)\\
&= f(x)
\end{aligned}$$

so f is an invariant of $x + tH$. By differentiating $f(x+tH) - f(x)$ to t, we obtain

$$(\mathcal{J}f)_{x=x+tH} \cdot H = 0 \tag{4.2}$$

Substituting $t = -t$ and $x = x + tH$ after that in (4.2), we obtain

$$\mathcal{J}f \cdot H(x+tH) = 0 \tag{4.3}$$

Since $(x+tH)|_{t=s^{1-d}}$ is homogeneous of degree 1, it follows that on the other hand, both (4.2) and (4.3) imply for general $f \in \mathbb{C}[x]$ that the homogeneous parts of f are invariants of $x + H$. If H is a quasi-translation, then (4.3) simplifies to

$$\mathcal{J}f \cdot H = 0 \tag{4.4}$$

and (4.2) simplifies to this in case f is linear, for a substitution in a constant row $\mathcal{J}f$ has no effect. In general, (4.4) can be obtained by substituting $t = 0$ in either (4.2) or (4.3), and therefore must be satisfied.

So we can use (4.4) to compute candidate invariants. We take for f the general polynomial of degree 3 at most with $\binom{12}{3} = 220$ terms, and then we solve the coefficients with respect to x. Since these coefficients are linear equations, they are solved fast by a computer algebra system. For the map of corollary 4.2.9, the space of candidate invariants of degree 3 at most is $\mathbb{C} + \mathbb{C}a$. So the candidate invariants of degree 3 at most of this map are already invariants.

In [47], the authors give a cubic homogeneous map H in dimension 11 with $\mathcal{J}H$ nilpotent, such that space of the invariants of $x + tH$ of degree 3 at most is $\mathbb{C} + \mathbb{C}\tilde{d}$, where

$$\tilde{d} := \det \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix}$$

namely a symmetric conjugation (conjugation with a permutation) of the map

$$H = (bx_2, ax_1 - bx_3, ax_2, bx_5, ax_4 - bx_6, ax_5, bx_8, ax_7 - bx_9, ax_8, \tilde{d}, x_{10}^3)$$

where $a = x_{10}^2$ and $b = x_{11}^2$. For that map, the candidate invariants of degree 3 at most are already invariants as well.

4.3 A structure theorem for homogeneous maps of transcendence degree 2

Let K be an arbitrary field and $H = (H_1, H_2, \ldots, H_n)$ be a polynomial map over K.

Theorem 4.3.1. *Assume $H \in K[x]^n$ is homogeneous and $\operatorname{trdeg}_K K(H) \leq 2$ and let $g := \gcd\{H_1, H_2, \ldots, H_n\}$. Then there exist a homogeneous $h \in K[y_1, y_2]^n$, and irreducible p and q in $K[x]$ that are homogeneous of the same degree r, such that $H = g \cdot h(p, q)$.*

The proof of theorem 4.3.1 is based on a version of Lüroth's theorem. H. Derksen pointed me this result.

Theorem 4.3.2. *If $L \supseteq K$ is a subfield of $K(x)$ and* $\operatorname{trdeg}_K(L) \le 1$, *then $L = K(p/q)$ for some relatively prime $p, q \in K[x]$.*

Notice that the case $\operatorname{trdeg}_K(L) = 0$ is trivial, so do not bother if the theorem is formulated with $\operatorname{trdeg}_K(L) = 1$. This theorem has been proved first in this form by Igusa in [38]. The case $K = \mathbb{C}$ and $n = 1$ was done by Lüroth. Next, it was extended to arbitrary n by P. Gordan in [33] and to arbitrary fields K (with $n = 1$) by E. Steinitz in 1910. See also [48, §3, 4] and the introduction of [5].

Proof of theorem 4.3.1. Assume without loss of generality that $\gcd\{H_1, H_2, \ldots, H_n\} = 1$ and $H_1 \ne 0$, and let

$$L := K\left(\frac{H_2}{H_1}, \frac{H_3}{H_1}, \ldots, \frac{H_n}{H_1}\right)$$

Since H_1 is transcendental over L and $K(H) = L(H_1)$, it follows from the condition $\operatorname{trdeg}_K K(H) \le 2$ that $\operatorname{trdeg}_K L \le 1$. By Lüroth's theorem above, we obtain $L = K(p/q)$ for some relatively prime $p, q \in K[x]$.

By multiplication of the numerator and the denominator by a power of q, we obtain that for each $i \ge 2$, we have

$$\frac{H_i}{H_1} = \frac{h_i(p,q)}{\tilde{h}_i(p,q)}$$

for certain homogeneous bivariate $h_i, \tilde{h}_i$ of the same degree. Now replace, for each $i \ge 2$, $\tilde{h}_i$ by $h_1 := \operatorname{lcm}\{\tilde{h}_2, \tilde{h}_3, \ldots, \tilde{h}_n\}$ and alter h_i accordingly. Then

$$\frac{H_i}{H_1} = \frac{h_i(p,q)}{h_1(p,q)}$$

for each $i \ge 1$ and all h_i are homogeneous of the same degree, say s.

Since $\gcd\{H_1, H_2, \ldots, H_n\} = 1$, it follows that $h(p,q) = gH$ for some $g \in K[x]$. Assume $g \ne 1$. Since $g \mid h_1(p,q)$, there exist a linear combination $\lambda p + \mu q$ over $\mathbb{C}$ of p and q such that $\gcd\{g, \lambda p + \mu q\} \ne 1$. Since p and q are relatively prime, it follows that (λ, μ) is unique up to scalar multiplication and that $\lambda y_1 + \mu y_2 \mid h_i$ for all i. So we can divide $\lambda y_1 + \mu y_2$ out of h. Going on like this, we obtain that $h(p,q) = H$.

We show that p and q are homogeneous of the same degree. Assume the contrary. Let

$$p = p_e + \cdots + p_f \quad \text{and} \quad q = q_e + \cdots + q_f$$

be the decompositions of p and q in homogeneous parts, with $(p_e, q_e) \neq 0$ and $(p_f, q_f) \neq 0$. Then $e < f$ and

$$h_i(p, q) = h_i(p_e, q_e) + \cdots + h_i(p_f, q_f)$$

where the dots are terms of degrees in $\{se + 1, se + 2, \ldots, sf - 1\}$. Since all $h_i(p, q)$ are homogeneous of the same degree, it follows that either $h_i(p_e, q_e) = 0$ for all i or $h_i(p_f, q_f) = 0$ for all i. So say $h_i(p_e, q_e) = 0$ for all i (the other case is similar).

Since $h_i \neq 0$ for some i, it follows from the fact that h_i is homogeneous and bivariate that p_e and q_e are linearly dependent over K, say $\alpha p_e + \beta q_e = 0$. It follows from $h(p_e, q_e) = 0$ that $\alpha y_1 + \beta y_2 \mid h_j$ for all j. So $\alpha p + \beta q \mid h_j(p, q)$ for all j. This contradicts $\gcd\{H_1, H_2, \ldots, H_n\} = 1$. $\square$

In case K is algebraically closed, theorem 4.3.1 can be proved by means of theorem 4.3.3 below as well. See [I.7] for the case $K = \mathbb{C}$. Moreover, the derivation of theorem 4.3.1 from theorem 4.3.2 has a converse, so we can proof theorem 4.3.2 for the case that K is algebraically closed by way of theorem 4.3.3 below as well.

Theorem 4.3.3. *Let K be an algebraically closed field and $F(x, y) \in K[x, y]$. Assume that F is irreducible over $K(y)$ and $\deg_y F = 1$. If $F(x, \lambda)$ is reducible for all $\lambda \in K^n$, then there exist an $s \geq 2$, $p, q \in K[x]$ and $a_i(y) \in K[y]$ such that either*

$$F(x, y) = \sum_{i=0}^{s} a_i(y) p^i q^{s-i}$$

or $F(x, y) \in K[x^{*c}, y]$, where $c > 0$ is the characteristic of K.

The case $c = 0$ was proved by Bertini in 1882 and the general case was done by Krull in 1937. See [48, §11].

Corollary 4.3.4. *Assume K is algebraically closed in theorem 4.3.2 or theorem 4.3.1. Then we can choose p and q irreducible.*

Proof. Notice that in case $\gcd\{p, q\} = 1$ and $\lambda \neq \mu$,

$$K\left(\frac{p}{q}\right) = K\left(\frac{p + \mu q}{q}\right) = K\left(\frac{q}{p + \mu q}\right) = K\left(\frac{q + \lambda p}{p + \mu q}\right) = K\left(\frac{p + \mu q}{q + \lambda p}\right)$$

so we may replace p and q by linear combinations of p and q in theorem 4.3.2, provided p and q remain independent in case they are not constant. This is the case in theorem 4.3.1 as well.

If two independent linear combinations of p and q are irreducible, then we can replace p and q by these linear combinations. So assume that at most one independent linear combination of p and q is irreducible. Assume without loss of generality that q might be irreducible and $p + \lambda q$ is reducible for all $\lambda \in \mathbb{C}$.

Since $\gcd\{p, q\} = 1$, it follows that $p + tq$ is irreducible in $K[x, t]$ and hence in $K(t)[x]$. It follows from theorem 4.3.3 that

$$p + tq = \sum_{i=1}^{s} a_i(t)(\tilde{p})^i(\tilde{q})^{s-i}$$

Taking coefficients with respect to t, we obtain $(p, q) = \tilde{h}(\tilde{p}, \tilde{q})$, where $\tilde{h}$ is homogeneous of degree $s \geq 2$. Replacing p by $\tilde{p}$ and q by $\tilde{q}$, we obtain that the degree of (p, q) decreases by a factor s. Assuming that the degree of (p, q) was chosen minimal in advance, we obtain a contradiction. So there are two independent linear combinations of p and q that are irreducible over K, as desired. □

An interesting question is whether corollary 4.3.4 remains valid if K is not algebraically closed. Below we formulate a polynomial variant of Lüroth's theorem 4.3.2 about rational functions.

Theorem 4.3.5. *Let K be a field and $A \supseteq K$ be a subalgebra of $K[x]$ such that* $\operatorname{trdeg}_K(\mathbb{Q}(A)) \leq 1$. *Then $A \subseteq K[p]$ for some $p \in K[x]$.*

Proof. Assume $f, g \in A \backslash K$ such that the degree of f is minimal and g is arbitrary. Then f and g are algebraically dependent over K, so there exists a *homogeneous* $R \in \mathbb{C}[y_1, y_2, y_3]$ such that $R(f, g, 1) = 0$. Let $d := \deg g$ and define

$$H = t^d\big(f(t^{-1}x), g(t^{-1}x), 1\big)$$

then H is homogeneous as a polynomial map in $\mathbb{C}[x, t]$ and $R(H) = 0$. Since $H_3 = t^d$ and $d = \deg g$, $\gcd\{H_1, H_2, H_3\} = 1$ follows.

It follows from theorem 4.3.1 that $H = h(p, q)$ for a homogeneous $h \in \mathbb{C}[y_1, y_2]^3$ and certain $p, q \in K[t][x]$ that are homogeneous of the same degree. Since $H_3 = t^d$ and H_3 decomposes into factors $\lambda p + \mu q$ as well, it follows that

$\lambda p + \mu q$ is a power of t for some $\lambda, \mu \in K$. Now take $T \in \mathrm{GL}_2(K)$ such that the last row of T equals $(\lambda\ \mu)$ and replace h by $h(T^{-1}(y_1, y_2))$ and (p, q) by $T(p, q)$ to obtain that q is a power of t.

Substituting $t = 1$, we obtain that f and g are polynomials in $p|_{t=1}$. Since $\deg f$ was chosen minimal, it follows that $\deg f = \deg p|_{t=1}$ and that g is a polynomial in f as well. This gives the desired result. $\square$

As E. Formanek remarks in [29, Lm. 2], a result that implies theorem 4.3.5 is that in the formulation of theorem 4.3.2, $L = K(p)$ for some polynomial p, in case L contains a nonconstant polynomial. This result was first proved by E. Nöther for characteristic zero and the general case was done by A. Schinzel in 1963. See [48, p. 10] for a proof.

Another proof of theorem 4.3.5 can be found in [6]. Zaks in [59] and Eakin in [22] prove the same result, except that they assume that A is algebraically closed (in $\mathbb{Q}(A)$). But this condition can easily be removed. Assume $\bar{A}$ is the integral closure of A in $K(x)$. Since $A \subseteq K[x]$ and $K[x]$ is integrally closed (in $K(x)$), we obtain that $\bar{A} \subseteq K[x]$. Since $\mathbb{Q}(\bar{A}) \subseteq K(x)$, it follows that $\bar{A}$ is integrally closed (in $\mathbb{Q}(\bar{A})$). So $A \subseteq \bar{A} \subseteq K[p]$.

Now you see that the condition that A is integrally closed can be removed so easily, you might wonder why it is included in [59] and [22]. This is because even $A = K[p]$ is proved instead of $A \subseteq K[p]$, just as in theorem 4.3.2 about rational functions. The example $A = K[x_1^2, x_1^3]$ is not equal to $K[p]$ for some $p \in K[x]$, so the condition that A is integrally closed is necessary to obtain equality. But this is not the case for theorem 4.3.2 about rational functions.

4.4 Theorem 4.1.2: the case that x, $\mathcal{J}H \cdot x$ and $\mathcal{J}H^2 \cdot x$ are independent

Let $H \in \mathbb{C}[x]^n$ be homogeneous of degree d such that $\mathrm{rk}\mathcal{J}H \le 2$. Assume x, $\mathcal{J}H \cdot x$ and $\mathcal{J}H^2 \cdot x$ are independent, and at least $n-1$ of the n eigenvalues of $\mathcal{J}H$ are zero. We do not assume that $n \le 4$ yet. In order to get some insight for $n \ge 5$ as well, we just reason on until we get stuck. Only then, we assume that $n \le 4$.

Since $\mathrm{rk}\mathcal{J}H \le 2$, it follows from theorem 4.3.1 that H is of the form $gh(p, q)$, where h and (p, q) are homogeneous. We do not assume that p and q are irreducible. This enables us to replace p and q by linear combinations of p

and q that are not irreducible. We do assume that $\gcd\{p, q\} = 1$, however.

We first start with an idea of Wright in [55] to get H in a more convenient form.

Lemma 4.4.1. *Let v_1 be a generic vector. Define $v_2 := \mathcal{J}H|_{x=v_1} \cdot v_1$ and $v_3 := \mathcal{J}H|_{x=v_1}^2 \cdot v_1$. Choose $v_4, v_5, \ldots, v_n \in \mathbb{C}^n$ independent of v_1, v_2, v_3 and define*

$$T := (\, v_1 \mid v_2 \mid v_3 \mid v_4 \mid \cdots \mid v_n \,)$$

Then T is invertible and

$$\mathcal{J}\big(T^{-1}H(Tx)\big)|_{x=e_1} = (\, e_2 \mid e_3 \mid * \mid * \mid \cdots \mid * \,)$$

Furthermore, the multiplicities of the eigenvalue zero are the same for $\mathcal{J}H$ and $\mathcal{J}(T^{-1}H(Tx))$, and the vectors x, $\mathcal{J}(T^{-1}H(Tx)) \cdot x$ and $\mathcal{J}(T^{-1}H(Tx))^2 \cdot x$ are independent over $\mathbb{C}(x)$, just as the vectors in the title of this section.

Proof. Since x, $\mathcal{J}H \cdot x$ and $\mathcal{J}H^2 \cdot x$ are independent and v_1 is generic, it follows that $v_1 = x|_{x=v_1}$, $v_2 = (\mathcal{J}H \cdot x)|_{x=v_1}$ and $v_3 = (\mathcal{J}H^2 \cdot x)|_{x=v_1}$ are independent. So the columns of T are independent and T^{-1} exists.

Furthermore,

$$\begin{aligned}
\mathcal{J}\big(T^{-1}H(Tx)\big)|_{x=e_1} &= T^{-1}\mathcal{J}H|_{x=Te_1}T \\
&= T^{-1}\mathcal{J}H|_{x=v_1}(\, v_1 \mid v_2 \mid v_3 \mid v_4 \mid \cdots \mid v_n \,) \\
&= T^{-1}(\, v_2 \mid v_3 \mid * \mid * \mid \cdots \mid * \,) \\
&= (\, e_2 \mid e_3 \mid * \mid * \mid \cdots \mid * \,)
\end{aligned}$$

and by

$$\begin{aligned}
&\det\Big(tI_n - \mathcal{J}\big(T^{-1}H(Tx)\big)\Big) \\
&\quad= \det(T) \cdot \det\Big(tI_n - \mathcal{J}\big(T^{-1}H(Tx)\big)\Big) \cdot \det(T^{-1}) \\
&\quad= \det(tI_n - \mathcal{J}H|_{x=Tx})
\end{aligned}$$

we see that the characteristic polynomial of $\mathcal{J}(T^{-1}H(Tx))$ is obtained from that of $\mathcal{J}H$ by substituting $x = Tx$. This gives the result about the eigenvalue zero.

Furthermore, we obtain by substituting $x = Tx$ and multiplication by T^{-1} from the left that the vectors $T^{-1}Tx$, $T^{-1}\mathcal{J}H|_{x=Tx}Tx$ and $T^{-1}\mathcal{J}H|_{x=Tx}^2 Tx$ are independent. But these are exactly the vectors x, $\mathcal{J}(T^{-1}H(Tx)) \cdot x$ and $\mathcal{J}(T^{-1}H(Tx))^2 \cdot x$, as desired. $\square$

Put $\tilde{H} = T^{-1}H(Tx)$. By lemma 4.4.1, we obtain that the first column of $\mathcal{J}\tilde{H}|_{x=e_1}$ is equal to e_2 and the second column of $\mathcal{J}\tilde{H}|_{x=e_1}$ is equal to e_3. So we get the following if we write each component of $\tilde{H}$ as a sum of monomials ordered by $x_1 > x_2 > \cdots > x_n$, starting with the monomial of the highest order:

$$\tilde{H} = \begin{pmatrix} 0 \cdot x_1^d & + & 0 \cdot x_1^{d-1}x_2 & + & \cdots \\ \frac{1}{d} \cdot x_1^d & + & 0 \cdot x_1^{d-1}x_2 & + & \cdots \\ 0 \cdot x_1^d & + & 1 \cdot x_1^{d-1}x_2 & + & \cdots \\ 0 \cdot x_1^d & + & 0 \cdot x_1^{d-1}x_2 & + & \cdots \\ & & \vdots & & \\ 0 \cdot x_1^d & + & 0 \cdot x_1^{d-1}x_2 & + & \cdots \end{pmatrix}$$

where the dots are terms lower in the lexicographical ordering.

Lemma 4.4.2. *Let T as is lemma 4.4.1 and put $\tilde{H} = T^{-1}H(Tx)$. Then we can write $\tilde{H}$ in the form $\tilde{H} = gh(p,q)$, with h homogeneous of degree s, g homogeneous of degree t and (p,q) homogeneous of degree r, such that*

$$g = x_1^t + \cdots \qquad p = x_1^{r-1}x_2 + \cdots \qquad \textit{and} \qquad q = x_1^r + \cdots$$

and

$$\begin{aligned} h_1(p,q) &\equiv 0 \pmod{p^2} \\ h_2(p,q) &\equiv \tfrac{1}{d}q^s \pmod{p} \\ h_3(p,q) &\equiv pq^{s-1} \pmod{p^2} \\ h_4(p,q) &\equiv 0 \pmod{p^2} \\ &\vdots \\ h_n(p,q) &\equiv 0 \pmod{p^2} \end{aligned}$$

Proof. Since $gh_2(p,q) = \tilde{H}_2 = \frac{1}{d}x_1^d + \cdots$, it follows that g is of the form $g = x_1^t + \cdots$ and that we may assume that $q = x_1^r + \cdots$. Since $gh_3(p,q) = \tilde{H}_3 = x_1^{d-1}x_2 + \cdots$, we obtain that we may assume that $p = x_1^{r-1}x_2 + \cdots$. Looking at the equation $gh_i(p,q) = \alpha x_1^d + \beta x_1^{d-1}x_2 + \cdots$ and using that $h_i(y_1,y_2)$ is homogeneous, we see that $h_i(p,q) \equiv \frac{\alpha}{d}q^s \pmod{p}$ if $\alpha \neq 0$ and $h_i(p,q) \equiv \beta pq^{s-1} \pmod{p^2}$ if $\alpha = 0$. This gives the desired result. $\square$

From lemma 4.4.1, it follows that we may replace $\tilde{H}$ by H. Assume g, h and (p,q) are as in lemma 4.4.2 such that $H = gh(p,q)$. Notice that $\gcd\{p,q\} = 1$, but $\gcd\{p,g\}$ does not need to be equal to 1.

Assume $1 \le i < j \le n$ such that $\{i,j\} \ne \{2,3\}$. Then either H_i or H_j is divisible by gp^2. Say that H_j is divisible by gp^2 (the other case of (4.5) below is similar). From the product rule of differentiation, it follows that

$$\tilde{g} \mid \tilde{h} \implies \frac{\partial \tilde{h}}{\partial x_k} = \tilde{g}\frac{\partial}{\partial x_k}\frac{\tilde{h}}{\tilde{g}} + \frac{\tilde{h}}{\tilde{g}}\frac{\partial}{\partial x_k}\tilde{g} \equiv \frac{\tilde{h}}{\tilde{g}}\frac{\partial}{\partial x_k}\tilde{g} \pmod{\tilde{g}}$$

for all k, so $\frac{\partial H_i}{\partial x_k} \equiv \frac{H_i}{g}\frac{\partial}{\partial x_k} g \pmod{g}$ and

$$\begin{aligned}
\frac{\partial H_j}{\partial x_k} &\equiv \frac{H_j}{gp^2}\frac{\partial}{\partial x_k}(gp^2) \pmod{gp} \\
&\equiv \frac{H_j}{gp^2}\left(p^2\frac{\partial}{\partial x_k}g + 2pg\frac{\partial}{\partial x_k}p\right) \pmod{gp} \\
&\equiv p\frac{H_j}{gp}\frac{\partial}{\partial x_k}g \pmod{gp}
\end{aligned}$$

for all k. Consequently, $\frac{1}{p}\frac{\partial H_j}{\partial x_k} \equiv \frac{H_j}{gp}\frac{\partial}{\partial x_k} g \pmod{g}$ for all k and

$$\begin{aligned}
&\frac{1}{p}\det \mathcal{J}_{x_i,x_j}(H_i,H_j) \\
&= \left(\frac{\partial H_i}{\partial x_i}\right)\cdot\left(\frac{1}{p}\frac{\partial H_j}{\partial x_j}\right) - \left(\frac{\partial H_i}{\partial x_j}\right)\cdot\left(\frac{1}{p}\frac{\partial H_j}{\partial x_i}\right) \\
&\equiv \left(\frac{H_i}{g}\frac{\partial}{\partial x_i}g\right)\cdot\left(\frac{H_j}{gp}\frac{\partial}{\partial x_j}g\right) - \left(\frac{H_i}{g}\frac{\partial}{\partial x_j}g\right)\cdot\left(\frac{H_j}{gp}\frac{\partial}{\partial x_i}g\right) \pmod{g} \\
&\equiv 0 \pmod{g}
\end{aligned} \tag{4.5}$$

so $gp \mid \det \mathcal{J}_{x_i,x_j}(H_i,H_j)$. Since $n-1$ of the n eigenvalues of $\mathcal{J}H$ are zero, we obtain

$$\sum_{1\le i<j\le n} \det \mathcal{J}_{x_i,x_j}(H_i,H_j) = 0$$

It follows that $gp \mid \det \mathcal{J}_{x_2,x_3}(H_2,H_3)$ as well. Since $p \mid H_3$, we obtain again by the product rule of differentiation that

$$gp \,\Big|\, \det \mathcal{J}_{x_2,x_3}(H_2,H_3) = \frac{H_3}{p}\det \mathcal{J}_{x_2,x_3}(H_2,p) + p\det \mathcal{J}_{x_2,x_3}\left(H_2,\frac{H_3}{p}\right) \tag{4.6}$$

Since $\frac{\partial}{\partial x_k}H_2 \equiv \frac{H_2}{g}\frac{\partial}{\partial x_k}g \pmod{g}$ and $\frac{\partial}{\partial x_k}\frac{H_3}{p} \equiv \frac{H_3}{gp}\frac{\partial}{\partial x_k}g \pmod{g}$, we can derive that $g \mid \det \mathcal{J}_{x_2,x_3}(H_2, \frac{H_3}{p})$, so all terms of (4.6) are divisible by gp. Since p divides H_3 only once,

$$p \mid \mathcal{J}_{x_2,x_3}(H_2,p) \tag{4.7}$$

The following lemma shows that there exists a polynomial f such that $p \mid f(H_2, x_1, x_4, \dots, x_n)$:

Lemma 4.4.3. *Assume A is a unique factorization domain with $\mathbb{Q}$ and $p, \tilde{g} \in A[x_2, x_3]$ such that*

$$p \mid \det \mathcal{J}_{x_2,x_3}(\tilde{g}, p)$$

Then there exists a nonzero $f \in A[y_1]$ such that $p \mid f(\tilde{g})$.

Proof.

i) If $p = p_1^r p_2$ such that $\gcd\{p_1, p_2\} = 1$, then it follows from the product rule of differentiating that

$$\det \mathcal{J}_{x_2,x_3}(\tilde{g}, p) = p_1^r \det \mathcal{J}_{x_2,x_3}(\tilde{g}, p_2) + r p_1^{r-1} p_2 \det \mathcal{J}_{x_2,x_3}(\tilde{g}, p_1)$$

Reducing modulo p_1^r, we obtain that $p_1 \mid \mathcal{J}_{x_2,x_3}(\tilde{g}, p_1)$. So we may assume that p is irreducible.

ii) Assume without loss of generality that $\frac{\partial p}{\partial x_3} \neq 0$. Since p, $\tilde{g}$ and x_2 are algebraically dependent over $\mathbb{Q}(A)$, there exists a nonzero $R \in A[y_1, y_2, y_3]$ such that $R(\tilde{g}, x_2, p) = 0$. It follows that $p \mid R(\tilde{g}, x_2, 0)$. Assume that

$$p \mid f(\tilde{g}, x_2) \Longrightarrow p \,\Bigg|\, \left(\frac{\partial f}{\partial y_2}\right)(\tilde{g}, x_2) \tag{4.8}$$

for all $f \in A[y_1, y_2]$. Then we can start with $f = R(y_1, y_2, 0)$ and then differentiate with respect to y_2 until $f \in A[y_1]$, and we have $p \mid f(\tilde{g})$ for some $f \neq 0$, as desired.

iii) So it remains to prove the claim (4.8). For that purpose, define the operator D by

$$D(a) = \det \mathcal{J}_{x_2,x_3}(a, p)$$

for all $a \in A[x_2, x_3]$. Notice that $D(x_2) = \frac{\partial p}{\partial x_3}$. Since by assumption $p \mid D(\tilde{g})$, we obtain

$$\begin{aligned} D(f(\tilde{g}, x_2)) &= \left(\frac{\partial f}{\partial y_1}\right)(\tilde{g}, x_2) D(\tilde{g}) + \left(\frac{\partial f}{\partial y_2}\right)(\tilde{g}, x_2) D(x_2) \\ &\equiv \left(\frac{\partial f}{\partial y_2}\right)(\tilde{g}, x_2) \frac{\partial p}{\partial x_3} \pmod{p} \end{aligned} \tag{4.9}$$

Assume $p \mid f(\tilde{g}, x_2)$. One can easily prove that $p \mid a \Rightarrow p \mid D(a)$. As a consequence, the left hand side of (4.9) is divisible by p. So the right hand side of (4.9) is divisible by p as well. Since p is irreducible, $\frac{\partial p}{\partial x_3} \neq 0$ and $\deg \frac{\partial p}{\partial x_3} < \deg p$, it follows that $p \mid (\frac{\partial f}{\partial y_2})(\tilde{g}, x_2)$, as desired. $\square$

Since $\gcd\{p, q\} = 1$, it follows from the projective intersection theorem [35, Ch. 1, Th. 7.2], that $V(p, q)$ is a pure variety of codimension 2. Hence $\mathfrak{r}(p, q)$ is the intersection of prime ideals $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_m$ of height 2.

Corollary 4.4.4. *Assume $\mathfrak{r}(p, q) = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_m$ for prime ideals $\mathfrak{p}_i$ of height 2. Then the intersection $\mathfrak{p}_i \cap \mathbb{C}[x_1, x_4, \ldots, x_n]$ is non-trivial for some i.*

Proof. Since $p = x_1^{r-1} x_2 + \cdots$, it follows that p has an irreducible factor $\tilde{p} = x_1^{\tilde{r}-1} x_2 + \cdots$. Notice that dH_2 is monic in x_1 and therefore $\gcd\{\tilde{p}, H_2\} = 1$. On account of lemma 4.4.3, there exists a polynomial $f \in A[y_1]$ such that

$$\tilde{p} \mid p \mid f(H_2)$$

where $A := \mathbb{C}[x_1, x_4, \ldots, x_n]$. If $y_1 \mid f$, then by $\gcd\{\tilde{p}, H_2\} = 1$, we obtain that we can replace f by f/y_1 without affecting $\tilde{p} \mid f(H_2)$. So we may assume that $y_1 \nmid f$. Since $y_1 \nmid f$ and $H_2 \in (\tilde{p}, q)$, $0 \neq f(0) \in (\tilde{p}, q)$.

Let $\mathfrak{q}$ be a prime ideal of height 2 that contains $(\tilde{p}, q)$. Since $\mathfrak{r}(p, q) \subseteq \mathfrak{r}(\tilde{p}, q)$, it follows that $q \supseteq \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_m$. So $\mathfrak{q} \supseteq \mathfrak{p}_i$ for some i. But since both $\mathfrak{q}$ and $\mathfrak{p}_i$ have height 2, we obtain $\mathfrak{q} = \mathfrak{p}_i$. It follows that $0 \neq f(0) \in \mathfrak{p}_i$, as desired. $\square$

Assume $T \in \mathrm{GL}_n(\mathbb{C})$ such that T^{-1} is of the form $(e_1 \mid e_2 \mid e_3 \mid * \mid \cdots \mid *)$. Then T is of this form as well. Furthermore, $e_2 = \mathcal{J}H|_{x=e_1} \cdot e_1$ and $e_3 = \mathcal{J}H|_{x=e_1}^2 \cdot e_1$, so T is again of the form of lemma 4.4.1. It follows that $\tilde{H} = T^{-1} H(Tx)$ satisfies the properties of lemma 4.4.2, i.e. $\tilde{H} = \tilde{g}\tilde{h}(\tilde{p}, \tilde{q})$ with $\tilde{g}$, $\tilde{h}$ and $(\tilde{p}, \tilde{q})$ as g, h and (p, q) respectively in lemma 4.4.2.

But we can choose $\tilde{g}$, $\tilde{h}$ and $(\tilde{p}, \tilde{q})$ in a special form. Since

$$p(Tx) = p(x_1 + \cdots, x_2 + \cdots, x_3 + \cdots, 0 \cdot x_3 + \cdots, \ldots) = x_1^{r-1} x_2 + \cdots$$

and

$$q(Tx) = q(x_1 + \cdots, x_2 + \cdots, x_3 + \cdots, 0 \cdot x_3 + \cdots, \ldots) = x_1^r + \cdots$$

we obtain by $\tilde{H} = T^{-1}H(Tx) = (T^{-1}h)(p(Tx), q(Tx))$ that we can chose $\tilde{h} = T^{-1}h$, $\tilde{p} = p(Tx)$ and $\tilde{q} = q(Tx)$.

Assume now that the components of h are linearly independent over $\mathbb{C}$. Since $\deg_{y_2} h_2 > \deg_{y_2} h_3 > \deg_{y_2} h_i$ for all $i \notin \{2, 3\}$, we can choose (the last $n-3$ columns of) T^{-1} in such a way that the components of $T^{-1}h$ have all different degrees with respect to y_2.

From lemma 4.4.1, it follows that we may replace H by $\tilde{H}$. Replacing g, h and (p, q) by $\tilde{g}$, $\tilde{h}$ and $(\tilde{p}, \tilde{q})$ respectively, we obtain that $H = gh(p, q)$ and that g, h and (p, q) satisfy the properties of lemma 4.4.2. Furthermore, the components of h have all different degrees with respect to y_2. Let $s - l_i$ be the degree of h_i with respect to y_2 and λ_i be the nonzero coefficient of $y_1^{l_i} y_2^{s-l_i}$ of h_i, for all i.

Now take

$$v_1 = (1, \epsilon, 0, 0, \ldots, 0) \qquad v_2 = \mathcal{J}H|_{x=v_1} \cdot v_1 \qquad v_3 = \mathcal{J}H|_{x=v_1}^2 \cdot v_1$$

Since $v_2 = dH(v_1)$, we get for all i that

$$(v_2)_i = d\lambda_i \epsilon^{l_i} + \mathrm{O}(\epsilon^{l_i+1})$$

and the second coordinate $d\frac{1}{d}\epsilon^0 + \mathrm{O}(\epsilon^1) = 1 + \mathrm{O}(\epsilon)$ of v_2 is the only coordinate of v_2 that is not $\mathrm{O}(\epsilon)$. Since all coordinates of $\mathcal{J}H_i|_{x=v_1}$ are $\mathrm{O}(\epsilon^{l_i-1})$ and the second coordinate of $\mathcal{J}H_i|_{x=v_1}$ is $\lambda_i l_i \epsilon^{l_i-1} + \mathrm{O}(\epsilon^{l_i})$, we obtain

$$(v_3)_i = \mathcal{J}H_i|_{x=v_1} \cdot dH|_{x=v_1} = \lambda_i l_i \epsilon^{l_i-1} + \mathrm{O}(\epsilon^{l_i})$$

Notice that $l_2 = 0$ does not affect the validity of the above for $i = 2$. In vector notation, we obtain

$$v_3 = \frac{1}{\epsilon d}(l * v_2) + \Big(\mathrm{O}(\epsilon^{l_1}), \mathrm{O}(\epsilon^{l_2}), \ldots, \mathrm{O}(\epsilon^{l_n})\Big)$$

Take ϵ close to zero and

$$T := (\, v_1 \mid v_2 \mid v_3 \mid e_4 \mid e_5 \mid \cdots \mid e_n \,) \tag{4.10}$$

We do not take $\epsilon = 0$, because in that case, T would be the identity and we do not get very far. Since $\det T = 1 + \mathrm{O}(\epsilon)$, T is invertible, so T is again of the form of 4.4.1. It follows that $\tilde{H} = T^{-1}H(Tx)$ satisfies the properties of lemma 4.4.2, i.e.

$$T^{-1}H(Tx) = \tilde{h}(\tilde{p}, \tilde{q})$$

where $\tilde{h}$, $\tilde{p}$ and $\tilde{q}$ satisfy the properties of h, p and q in lemma 4.4.2. Since $\tilde{p}$ and $\tilde{q}$ are linear combinations of $p(Tx)$ and $q(Tx)$, we obtain

$$\mathfrak{r}(\tilde{p},\tilde{q}) = \mathfrak{r}\big(p(Tx), q(Tx)\big) = \mathfrak{p}_1(Tx) \cap \mathfrak{p}_2(Tx) \cap \cdots \cap \mathfrak{p}_m(Tx)$$

Lemma 4.4.5. *Assume $\mu, \nu \in \mathbb{C}^n$ are independent and let T as in (4.10), with for all i*

$$\begin{aligned} (v_2)_i &= d\lambda_i \epsilon^{l_i} + \mathrm{O}(\epsilon^{l_i+1}) \\ (v_3)_i &= \lambda_i l_i \epsilon^{l_i-1} + \mathrm{O}(\epsilon^{l_i}) \end{aligned}$$

such that the components of $l \in \mathbb{N}^n$ are all different and $\lambda_1\lambda_2\cdots\lambda_n \neq 0$. Then for ϵ close to zero, there are no linear combinations of $\mu^{\mathrm{t}}Tx$ and $\nu^{\mathrm{t}}Tx$ that are contained in $\mathbb{C}[x_1, x_4, \ldots, x_n]$.

Proof. Since μ and ν are independent, there exists an i such that $\mu_i \neq 0$ and a j such that $\nu_j \neq 0$. Choose i such that $\mu_i \neq 0$ and l_i is minimal and choose j such $\nu_j \neq 0$ and l_j is minimal.

If $l_i > l_j$, then we can obtain $l_i < l_j$ by interchanging μ and ν. If $l_i = l_j$, then $i = j$ because the components of l are all different, and we can obtain $l_i < l_j$ by subtracting μ ν_i/μ_i times from ν. So we can obtain $l_i < l_j$ in such a manner that the space $\mathbb{C}\mu + \mathbb{C}\nu$ is not affected. For that reason, we may assume that $l_i < l_j$.

Now take ϵ close to zero and assume that

$$\alpha\mu^{\mathrm{t}}Tx + \beta\nu^{\mathrm{t}}Tx \in \mathbb{C}[x_1, x_4, \ldots, x_n]$$

for some nonzero $(\alpha, \beta) \in \mathbb{C}^2$. Notice that by definition of i and j, the coefficients with respect to both x_2 and x_3 of $\alpha\mu^{\mathrm{t}}Tx$ and $\beta\nu^{\mathrm{t}}Tx$ are dominated by those of $\alpha\mu_i T_i x$ and $\beta\nu_j T_j x$ respectively. Since the coefficients of x_2 in $\alpha\mu^{\mathrm{t}}Tx$ and $\beta\nu^{\mathrm{t}}Tx$ must cancel out, we obtain that $\alpha\beta \neq 0$.

So assume without loss of generality that $\alpha = 1$. Again by looking at the coefficients of x_2 in $\alpha\mu^{\mathrm{t}}Tx$ and $\beta\nu^{\mathrm{t}}Tx$ and their dominating parts, we obtain that

$$\beta = -\frac{\mu_i d\lambda_i}{\nu_j d\lambda_j}\epsilon^{l_i-l_j} + O(\epsilon^{l_i-l_j+1})$$

but by looking at the coefficient of x_3, we get

$$\beta = -\frac{\mu_i l_i\lambda_i}{\nu_j l_j\lambda_j}\epsilon^{l_i-l_j} + O(\epsilon^{l_i-l_j+1})$$

It follows that $l_i/l_j = d/d$, i.e. $l_i = l_j$. Contradiction. □

Assume from now on that $n \le 4$. Since $\mathfrak{p}_i$ has height 2, it follows that $\mathfrak{p}_i$ contains at most two independent linear forms, say $\mu^{\mathrm{t}}x$ and $\nu^{\mathrm{t}}x$, for each i. From lemma 4.4.5, it follows that $\mathfrak{p}_i(Tx)$ does not contain a linear form in x_1 and x_4 for any i if we choose ϵ sufficiently close to zero. We shall derive a contradiction.

From corollary 4.4.4, it follows that $\mathfrak{p}_i(Tx)$ contains a non-trivial polynomial in x_1 and x_4 for some $i \le m$. Since $\mathfrak{r}(p,q)$ is homogeneous, $\mathfrak{p}_i(x)$ is homogeneous as well, whence $\mathfrak{p}_i(Tx)$ contains a nontrivial homogeneous polynomial in x_1 and x_4. But such a polynomial decomposes into linear factors, and by the definition of prime ideal, one of these linear factors is already contained in $\mathfrak{p}_i(Tx)$. Contradiction.

So the components of H are linearly dependent over $\mathbb{C}$ in case $n \le 4$, as desired.

4.5 Theorem 4.1.2: the case that x, $\mathcal{J}H \cdot x$ and $\mathcal{J}H^2 \cdot x$ are dependent

Let H be homogeneous such that $\mathrm{rk}\mathcal{J}H \le 2$. Assume x, $\mathcal{J}H \cdot x$ and $\mathcal{J}H^2 \cdot x$ are dependent, and at least $n-1$ of the n eigenvalues of $\mathcal{J}H$ are zero. From lemma 4.1.5, we obtain $\mathcal{J}H \cdot H = \mathrm{tr}\mathcal{J}H \cdot H$. The following proposition shows that maps with this property form a $\mathbb{C}[x]$-module.

Proposition 4.5.1. *Assume $H = g\tilde{H}$ for some polynomial $g \ne 0$. Then*

$$\mathcal{J}\tilde{H} \cdot \tilde{H} = \mathrm{tr}\mathcal{J}\tilde{H} \cdot \tilde{H} \iff \mathcal{J}H \cdot H = \mathrm{tr}\mathcal{J}H \cdot H$$

Proof. From $H = g\tilde{H}$, it follows that

$$H(x + t\tilde{H}) = g(x + t\tilde{H}) \cdot \tilde{H}(x + t\tilde{H})$$

Now differentiate with respect to t and substitute $t = 0$ to obtain

$$\begin{aligned}
\mathcal{J}H \cdot \tilde{H} &= \mathcal{J}g \cdot \tilde{H} \cdot \tilde{H} + g \cdot \mathcal{J}\tilde{H} \cdot \tilde{H} \\
&= (\mathcal{J}g \cdot \tilde{H} + g \cdot \mathrm{tr}\mathcal{J}\tilde{H}) \cdot \tilde{H} + g \cdot (\mathcal{J}\tilde{H} - \mathrm{tr}\mathcal{J}\tilde{H}) \cdot \tilde{H} \\
&= \mathrm{tr}\mathcal{J}H \cdot \tilde{H} + g \cdot (\mathcal{J}\tilde{H} - \mathrm{tr}\mathcal{J}\tilde{H}) \cdot \tilde{H}
\end{aligned}$$

Next move $\mathrm{tr}\mathcal{J}H \cdot \tilde{H}$ to the left to obtain

$$\frac{1}{g}(\mathcal{J}H - \mathrm{tr}\mathcal{J}H) \cdot H = g(\mathcal{J}\tilde{H} - \mathrm{tr}\mathcal{J}\tilde{H}) \cdot \tilde{H}$$

This gives the desired result. □

We show that in case H is homogeneous and $\mathcal{J}H \leq 2$, all maps that satisfy $\mathcal{J}H \cdot H = \mathrm{tr}\mathcal{J}H \cdot H$ are of the form $h = g\tilde{H}$, where $x + \tilde{H}$ is a quasi-translation:

Theorem 4.5.2. *Assume that $H = g\tilde{H}$ is homogeneous over $\mathbb{C}$, $\mathrm{rk}\mathcal{J}H = 2$ and $g = \gcd\{H_1, H_2, \dots, H_n\}$. Then the following statements are equivalent:*

i) $\mathcal{J}H \cdot H = \mathrm{tr}\mathcal{J}H \cdot H$,

ii) $x + \tilde{H}$ *satisfies (3.4), i.e.*

$$\tilde{H}(x + t\tilde{H}(y)) = \tilde{H}(x)$$

and hence the other properties of proposition 3.4.3 as well.

Proof. Assume first that ii) is satisfied. Then $x + \tilde{H}$ is a quasi-translation, whence $\mathcal{J}\tilde{H} \cdot \tilde{H} = 0$ and $\mathcal{J}\tilde{H}$ is nilpotent. In particular $\mathrm{tr}\mathcal{J}\tilde{H} = 0$, so $\mathcal{J}\tilde{H} \cdot \tilde{H} = \mathrm{tr}\mathcal{J}\tilde{H} \cdot \tilde{H}$. Now apply proposition 4.5.1 to obtain i).

Assume next that i) is satisfied. From theorem 4.3.1, it follows that H is of the form $g \cdot h(p, q)$, where $g = \gcd\{H_1, H_2, \dots, H_n\}$ and p and q are homogeneous of the same degree. Furthermore, they are relatively prime by definition of g. From proposition 4.5.1, it follows that we may assume that $g = 1$. So $H = h(p, q)$.

Replacing H by $T^{-1}H(Tx)$, p and q respectively by $p(Tx)$ and $q(Tx)$ respectively, and $\tilde{h}$ by $T^{-1}\tilde{h}$, for a suitable $T \in \mathrm{GL}_n(\mathbb{C})$, we can obtain that for each $i \geq 2$, either $h_i = 0$ or y_2 divides h_i more often than y_2 divides h_{i-1}. Since $\gcd\{h_1, h_2, \dots, h_n\} = 1$, we obtain that

$$\begin{aligned}
H_1 &= p^s \bmod q \\
H_2 &= p^{s-t}q^t \bmod q^{t+1} \\
H_3 &= 0 \bmod q^{t+1} \\
&\vdots \\
H_n &= 0 \bmod q^{t+1}
\end{aligned} \tag{4.11}$$

for some $t \geq 1$.

Notice that by $\mathcal{J}H \cdot H = \mathrm{tr}\mathcal{J}H \cdot H$ and $q \mid H_2$, we obtain

$$\begin{aligned}
0 &= H_2 \cdot \mathrm{tr}\mathcal{J}H - \mathrm{tr}\mathcal{J}H \cdot H_2 \\
&= H_2 \cdot \mathrm{tr}\mathcal{J}H - \mathcal{J}H_2 \cdot H \\
&\equiv H_2 \frac{\partial}{\partial x_1} H_1 + H_2 \frac{\partial}{\partial x_2} H_2 - H_1 \frac{\partial}{\partial x_1} H_2 - H_2 \frac{\partial}{\partial x_2} H_2 \pmod{q^{t+1}} \\
&\equiv H_2 \frac{\partial}{\partial x_1} H_1 - H_1 \frac{\partial}{\partial x_1} H_2 \pmod{q^{t+1}}
\end{aligned}$$

Since $q^t \mid H_2$, we obtain $q^t \mid H_1 \frac{\partial}{\partial x_1} H_2$. But $H_1 \frac{\partial}{\partial x_1} H_2 \equiv t p^{2s-t} q^{t-1} \frac{\partial}{\partial x_1} q \pmod{q^t}$ and $\gcd\{p, q\} = 1$, so $q \mid \frac{\partial}{\partial x_1} q$. Comparing degrees, we obtain that $\frac{\partial}{\partial x_1} q = 0$. It follows that

$$H_2 \frac{\partial}{\partial x_1} H_1 - H_1 \frac{\partial}{\partial x_1} H_2 \equiv s p^{2s-t-1} q^t \frac{\partial}{\partial x_1} p - (s-t) p^{2s-t-1} q^t \frac{\partial}{\partial x_1} p \pmod{q^{t+1}}$$

whence by $\gcd\{p, q\} = 1$, we get $q \mid \frac{\partial}{\partial x_1} p$. So $\frac{\partial}{\partial x_1} p = 0$ as well.
We show that $p(x + tH(y)) = p$ and $q(x + tH(y)) = q$. For that purpose, we distinguish two cases:

- $H_3 = H_4 = \cdots = 0$.
  From $\gcd\{H_1, H_2\} = 1$, it follows that we can replace p by H_1 and q by H_2. In a similar manner as we obtained $\frac{\partial q}{\partial x_1} = \frac{\partial p}{\partial x_1} = 0$, we can show that $\frac{\partial H_1}{\partial x_1} = \frac{\partial H_2}{\partial x_1} = 0$ and $\frac{\partial H_1}{\partial x_2} = \frac{\partial H_2}{\partial x_2} = 0$. So $H_2 \in \mathbb{C}[x_3, x_4, \ldots, x_n]$. Since $q \mid H_2$, $q \in \mathbb{C}[x_3, x_4, \ldots, x_n]$ as well. So $\frac{\partial q}{\partial x_2} = 0$ and

  $$0 = \frac{\partial}{\partial x_2} H_1 = \frac{\partial h_1}{\partial y_1}(p, q) \cdot \frac{\partial p}{\partial x_2} = (s p^{s-1} + \cdots) \cdot \frac{\partial p}{\partial x_2}$$

  whence $\frac{\partial p}{\partial x_2} = 0$.
  It follows that

  $$p(x + tH(y)) = p(*, *, x_3, \ldots, x_n) = p$$

  and

  $$q(x + tH(y)) = q(*, *, x_3, \ldots, x_n) = q$$

  as desired.

- $H_i \neq 0$ for some $i \geq 3$.
  Let $\tilde{x} = x_2, x_3, \ldots, x_n$ and $\tilde{H} = (H_2, H_3, \ldots, H_n)$ Since $\frac{\partial}{\partial x_1} p = 0 = \frac{\partial}{\partial x_1} q$ and $H = h(p, q)$, it follows that $\frac{\partial}{\partial x_1} H_i = 0$ for all i and hence

$$\begin{aligned}(*, \mathcal{J}_{\tilde{x}} \tilde{H} \cdot \tilde{H}) = \mathcal{J}_{\tilde{x}} H \cdot \tilde{H} = \mathcal{J} H \cdot H = \\ \operatorname{tr} \mathcal{J} H \cdot H = \operatorname{tr} \mathcal{J}_{\tilde{x}} \tilde{H} \cdot H = (*, \operatorname{tr} \mathcal{J}_{\tilde{x}} \tilde{H} \cdot \tilde{H})\end{aligned}$$

  So $\mathcal{J}_{\tilde{x}} \tilde{H} \cdot \tilde{H} = \operatorname{tr} \mathcal{J}_{\tilde{x}} \tilde{H} \cdot \tilde{H}$. Furthermore $H_i \nmid H_2$, so $H_i \nmid \tilde{g} := \gcd\{H_2, H_3, \ldots, H_n\}$. So $\deg \tilde{g} < \deg \tilde{H}$ and $\tilde{g}^{-1} \tilde{H} = \tilde{h}(p, q)$ for some homogeneous $\tilde{h}$ of degree 1 at least. By induction, it follows that

$$p\big(0, \tilde{x} + t\tilde{h}(p(y), q(y))\big) = p(0, \tilde{x})$$

  and

$$q\big(0, \tilde{x} + t\tilde{h}(p(y), q(y))\big) = q(0, \tilde{x})$$

  Substituting $t = \tilde{g}(y)t$, and using that $\frac{\partial}{\partial x_1} p = \frac{\partial}{\partial x_1} q = 0$, we obtain

$$p\big(x + tH(y)\big) = p\big(0, \tilde{x} + t\tilde{H}(y)\big) = p(0, \tilde{x}) = p$$

  and

$$q\big(x + tH(y)\big) = q\big(0, \tilde{x} + t\tilde{H}(y)\big) = q(0, \tilde{x}) = q$$

  as desired.

So $p(x + tH(y)) = p(x)$ and $q(x + tH(y)) = q(x)$. Since $H_i \in \mathbb{C}[p, q]$ for all i, we get $H(x + tH(y)) = H(x)$, as desired. □

We have now proved theorem 4.1.2. The case that x, $\mathcal{J}H \cdot x$ and $\mathcal{J}H^2 \cdot x$ are independent as vectors has been done in section 4.4. In case that x, $\mathcal{J}H \cdot x$ and $\mathcal{J}H^2 \cdot x$ are dependent, we have seen in lemma 4.1.5 that $\mathcal{J}H \cdot H = \operatorname{tr}\mathcal{J}H \cdot H$ and hence, H satisfies iii) of proposition 3.4.3 on account of the above theorem. The first row of T^{-1} in iii) of proposition 3.4.3 indicates a linear dependence of both the components of H and the rows of $\mathcal{J}H$.

Theorem 4.1.2 can be generalized somewhat, since it is the case $s = n \leq 4$ of the corollary below.

Corollary 4.5.3. *Assume $H \in \mathbb{C}[x]^n$ is homogeneous over $\mathbb{C}$, $\operatorname{rk}\mathcal{J}H \leq 2$ and $n - 1$ of the n eigenvalues of $\mathcal{J}H$ are zero. Then H is of the form $H = gh(p, q)$ with $\gcd\{p, q\} = 1$.*

Assume that there exists an $s \leq 4$ and a $T \in \mathrm{GL}_n(\mathbb{C})$ such that the degrees of $p(Tx)$ and $q(Tx)$ with respect to $x_{s+1}, x_{s+2}, \ldots, x_n$ are equal, i.e. the lowest degree homogeneous parts with respect to $x_1, x_2, \ldots, x_s$ of $p(Tx)$ and $q(Tx)$ have the same degree. Assume in addition that the lowest degree homogeneous parts of $p(Tx)$ and $q(Tx)$ with respect $x_1, x_2, \ldots, x_s$ are linearly independent over $\mathbb{C}[x_{s+1}, x_{s+2}, \ldots, x_n]$. Then the components of H are linearly dependent over $\mathbb{C}$.

Proof. Notice that H is of the form $H = gh(p,q)$ with $\gcd\{p,q\} = 1$ on account of theorem 4.3.1. By replacing H by $T^{-1}H(Tx)$, we may assume that $T = I_n$. Furthermore, we may assume that x, $\mathcal{J}H \cdot x$ and $\mathcal{J}H^2 \cdot x$ are independent, because theorem 4.5.2 above does not have a restriction on the dimension n.

Let $\hat{g}$, $\hat{p}$ and $\hat{q}$ be the lowest degree homogeneous parts of g, p, and q with respect to $x_1, x_2, \ldots, x_s$, $\hat{h} = (h_1, h_2, \ldots, h_s)$ and $\hat{H} = \hat{g}\hat{h}(\hat{p}, \hat{q})$. By assumption, $(\hat{p}, \hat{q})$ is homogeneous with respect to both $x_{s+1}, x_{s+2}, \ldots, x_n$ and $x_1, x_2, \ldots, x_s$, and linearly independent over $\mathbb{C}[x_{s+1}, x_{s+2}, \ldots, x_n]$.

Since the Jordan normal form of $\mathcal{J}H$ is contained in

$$\left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & & & \\ \vdots & \cdots & \vdots & & \emptyset & \\ 0 & 0 & 0 & & & \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \tau & 0 & \cdots & 0 \\ 0 & 0 & 0 & & & \\ \vdots & \cdots & \vdots & & \emptyset & \\ 0 & 0 & 0 & & & \end{pmatrix} \right\}$$

where $\tau = \mathrm{tr}\mathcal{J}H$, we obtain that $\mathcal{J}H^3 = \mathrm{tr}\mathcal{J}H \cdot \mathcal{J}H^2$. Consequently, $\mathcal{J}H^2 \cdot H - \mathrm{tr}\mathcal{J}H \cdot \mathcal{J}H \cdot H = \frac{0}{d} \cdot x = 0$. We show that

$$\mathcal{J}_{x_1,x_2,\ldots,x_s}\hat{H}^2 \cdot \hat{H} - \mathrm{tr}\mathcal{J}_{x_1,x_2,\ldots,x_s}\hat{H} \cdot \mathcal{J}_{x_1,x_2,\ldots,x_s}\hat{H} \cdot \hat{H} = 0 \tag{4.12}$$

So assume the opposite. Then the left hand side is the lowest degree homogeneous part with respect to $x_1, x_2, \ldots, x_s$ of the first s components of $\mathcal{J}H^2 \cdot H - \mathrm{tr}\mathcal{J}H \cdot \mathcal{J}H \cdot H = 0$, because $\frac{\partial}{\partial x_i}$ decreases the degree with respect to $x_1, x_2, \ldots, x_s$, if and only if $i \leq s$. Contradiction.

If $\mathcal{J}_{x_1,x_2,\ldots,x_s}\hat{H} \cdot \hat{H} \neq 0$, then it is an eigenvector of $\mathcal{J}_{x_1,x_2,\ldots,x_s}\hat{H}$ with eigenvalue $\mathrm{tr}\mathcal{J}_{x_1,x_2,\ldots,x_s}\hat{H}$ by (4.12). If $\mathcal{J}_{x_1,x_2,\ldots,x_s}\hat{H} \cdot \hat{H} = 0$, then $(x_1, x_2, \ldots, x_s) + \hat{H}$ is a quasi-translation in dimension s on account of iii) of proposition 3.1.2.

By corollary 3.1.3, $\mathrm{tr}\mathcal{J}_{x_1,x_2,\ldots,x_s}\hat{H}$ is an eigenvalue of $\mathcal{J}_{x_1,x_2,\ldots,x_s}\hat{H}$ in both cases.

Since $\mathrm{rk}\mathcal{J}_{x_1,x_2,\ldots,x_s}\hat{H} \le 2$ and $\mathrm{tr}\mathcal{J}_{x_1,x_2,\ldots,x_s}\hat{H}$ is an eigenvalue of $\mathcal{J}_{x_1,x_2,\ldots,x_s}\hat{H}$, it follows that at least $s-1$ of the s eigenvalues of $\mathcal{J}_{x_1,x_2,\ldots,x_s}\hat{H}$ are zero. Let K be the algebraic closure of $\mathbb{C}(x_{s+1}, x_{s+2}, \ldots, x_n)$. Notice that $\hat{H}$ is a homogeneous map of dimension s over K. By way of Lefschetz' principle, we obtain by theorem 4.1.2 that the component of $\hat{H}$ are linearly dependent over K.

Since $\hat{p}$ and $\hat{q}$ are homogeneous of the same degree over K and linearly independent over $\mathbb{C}[x_{s+1}, x_{s+2}, \ldots, x_n]$, they are algebraically independent over K. It follows that the components of $\hat{h}$ are already linearly dependent over K. Since $\hat{h} \in \mathbb{C}[y_1, y_2]^s$, the components of $\hat{h}$ are even linearly dependent over $\mathbb{C}$. This gives the desired result. □

Corollary 4.5.4. *Assume $H \in \mathbb{C}[x]^n$ is homogeneous over $\mathbb{C}$, $\mathrm{rk}\mathcal{J}H \le 2$ and $n-1$ of the n eigenvalues of $\mathcal{J}H$ are zero. Write $H = gh(p,q)$ with $\gcd\{p,q\} = 1$.*

Assume that there exists an $\omega \in V(p,q)$ such that $(\nabla p)(\omega)$ and $(\nabla q)(\omega)$ are independent vectors. Then the components of H are linearly dependent over $\mathbb{C}$.

Proof. By way of linear conjugation, we can obtain $\omega = e_3$. It follows that $\deg_{x_3} p < \deg p$ and $\deg_{x_3} q < \deg q$. So $\frac{\partial}{\partial x_3} p$ and $\frac{\partial}{\partial x_3} q$ vanish at ω as well. It follows that e_3 is independent of $(\nabla p)(\omega)$ and $(\nabla q)(\omega)$. Since $(\nabla p)(\omega)$ and $(\nabla q)(\omega)$ are independent, we may assume that $(\nabla p)(\omega) = e_1$ and $(\nabla q)(\omega) = e_2$.

Now define $\hat{p}$, $\hat{q}$ as in corollary 4.5.3 with $s = 2$. Then $\hat{p} = x_1 x_3^{r-1} + \cdots$ and $\hat{q} = x_2 x_3^{r-1} + \cdots$, where $r = \deg(p,q)$. If we assume that $\hat{p}$ and $\hat{q}$ are linearly dependent over $\mathbb{C}[x_3, x_4, \ldots, x_n]$, then we get a contradiction by looking at terms of maximal degree with respect to x_3. Now apply corollary 4.5.3 to get the desired result. □

4.6 Some computable cases of the dependence problem

Definition 4.6.1. We say that H satisfies DP, if $\lambda^{\mathrm{t}} H \in \mathbb{C}$ for some $\lambda \in \mathbb{C}^n \backslash \{0\}$. We say that H satisfies DP+ if in addition, $\mu^{\mathrm{t}} H = p(\lambda^{\mathrm{t}} x)$ for some

$p \in \mathbb{C}[u]$ and a $\mu \in \mathbb{C}^n$ that is independent of λ.

The idea behind DP+ is the following. Assume H is homogeneous and satisfies DP. Then we can obtain $H_n = 0$ by way of a linear conjugation. But in that case, H is the homogenization of $\tilde{H} := (H_1, H_2, \ldots, H_{n-1})|_{x_n=1}$. Next, one can ask the question whether $\tilde{H}$ satisfies DP. Now this last question is equivalent to the question whether H satisfies DP+.

Proposition 4.6.2. *In the definition of DP+, the existence of μ does not depend on the choice of λ.*

Proof. In case there is really a second choice for λ, i.e. there exists a λ' that is independent of λ such that $(\lambda')^{\mathrm{t}} H \in \mathbb{C}$, then $\mu' = \lambda$ satisfies the desired properties. $\square$

Proposition 4.6.3. *H satisfies DP(+), if and only if there exists a $T \in \mathrm{GL}_n(\mathbb{C})$ such that $\tilde{H} := T^{-1}H(Tx)$ satisfies $\tilde{H}_1 \in \mathbb{C}$ (and $\tilde{H}_2 \in \mathbb{C}[x_1]$). In particular, property DP(+) is invariant under linear conjugations.*

Proof. The backward implication follows by taking $\lambda^{\mathrm{t}} = (T^{-1})_1$ and (in case of DP+) $\mu^{\mathrm{t}} = (T^{-1})_2$.

So assume $\lambda^{\mathrm{t}} H \in \mathbb{C}$ and suppose that μ be independent of λ. Now take $T \in \mathrm{GL}_n(\mathbb{C})$ such that $T_1^{-1} = \lambda^{\mathrm{t}}$ and $T_2^{-1} = \mu^{\mathrm{t}}$ and put $\tilde{H} = T^{-1}H(Tx)$. Then $\tilde{H}_1 = e_1^{\mathrm{t}} T^{-1} H(Tx) = \lambda^{\mathrm{t}} H(Tx) \in \mathbb{C}$.

If, in addition, $\mu^{\mathrm{t}} H = p(\lambda^{\mathrm{t}} x)$ for some $p \in \mathbb{C}[u]$, then $\tilde{H}_2 = e_2^{\mathrm{t}} T^{-1} H(Tx) = \mu^{\mathrm{t}} H(Tx) = p(\tilde{\lambda}^{\mathrm{t}} Tx) = p(e_1^{\mathrm{t}} x) = p(x_1)$, as desired. $\square$

The following theorem can also be seen by looking at the list of 8 solutions in [36] or [24, Th. 7.1.2]: a theorem of E. Hubbers. This is because in each solution of that list, either $H_1 = 0$ and $H_2 \in \mathbb{C}[x_1]$ or two components of H are zero. That is why the below theorem is considered to be from Hubbers.

Theorem 4.6.4 (Hubbers)**.** *Assume $H = (H_1, H_2, H_3, H_4)$ is a homogeneous polynomial map of degree ≤ 3 in x over $\mathbb{C}$, $n = 4$ and $\mathcal{J}H$ is nilpotent. Then H satisfies DP+.*

Proof. We only prove the case $\mathrm{rk}\mathcal{J}H \leq 2$ here, because we have a computational proof for the case $\mathrm{rk}\mathcal{J}H = 3$. The case $\mathrm{rk}\mathcal{J}H = 3$ will be done in appendix A, together with some other computational results.

Assume $\mathrm{rk}\mathcal{J}H \leq 2$. Then H is of the form $gh(p,q)$ as in theorem 4.3.1. Assume first that $\deg h \leq 1$. The case $\deg h \leq 0$ is easy, so assume $\deg h = 1$. Then each component of H is linearly dependent of gp and gq, whence the first two components of $T^{-1}H(Tx)$ are zero for a suitable $T \in \mathrm{GL}_4(\mathbb{C})$. It follows from proposition 4.6.3 that H satisfies DP+. The cases $\deg h = 0$ and $\deg(p,q) = 0$ follow in a similar manner.

So assume that $\deg h \geq 2$ and $\deg(p,q) \geq 1$. Since $\deg H \leq 3$, it follows that p and q are linear and g is either linear or constant. So H can be expressed as a polynomial map in at most three linear forms. Consequently, there exists a $T \in \mathrm{GL}_n(\mathbb{C})$ such that $H_i(Tx) \in \mathbb{C}[x_1, x_2, x_3]$ for all i.

Let $\tilde{H}$ be the first three components of $T^{-1}H(Tx)$. Since $\mathcal{J}T^{-1}H(Tx)$ is nilpotent and $\tilde{H}_i \in \mathbb{C}[x_1, x_2, x_3]$ for all i, it follows that $\mathcal{J}_{x_1,x_2,x_3}\tilde{H}$ is nilpotent. Since $\tilde{H}$ is homogeneous (with respect to x_1, x_2, x_3), it follows from theorem 4.1.4 that $\tilde{H}$ is linearly triangularizable. So H is linearly triangularizable as well. From proposition 4.6.3, we obtain that H satisfies DP+, as desired. $\square$

Hubbers proved the above theorem by working out many cases. Only two of these cases satisfy $\mathrm{rk}\mathcal{J}H = 3$, and these cases are computed in the appendix A.

Theorem 4.6.5. *Assume $H = (H_1, H_2, H_3, H_4)$ is a homogeneous polynomial map of degree ≤ 3 in x over $\mathbb{C}$, $n = 4$ and $\mathcal{J}H$ is nilpotent. If $\mathcal{J}H$ is* not *linearly triangularizable, then $\deg H = 3$ and there exists a $T \in \mathrm{GL}_n(\mathbb{C})$ such that*

$$T^{-1}H(Tx) = \begin{pmatrix} 0 \\ \lambda x_1^3 \\ x_2(x_1x_3 - x_2x_4) + p(x_1, x_2) \\ x_1(x_1x_3 - x_2x_4) + q(x_1, x_2) \end{pmatrix}$$

for certain polynomials p, q, where $\lambda = \mathrm{rk}\mathcal{J}H - 2 \in \{0, 1\}$.

Proof. Since H satisfies DP+, it follows from proposition 4.6.3 that we may assume that $H_1 = 0$ and $H_2 = \lambda x_1^3$ for some $\lambda \in \mathbb{C}$. Consequently, $\mathcal{J}_{x_3,x_4}(H_3, H_4)$ is nilpotent, whence it follows from theorem 2.2.7 that $H_3 = bg(ax_3 - bx_4) + p$ and $H_4 = ag(ax_3 - bx_4) + q$ for some $g \in \mathbb{C}[x_1, x_2][t]$ and $a, b, p, q \in \mathbb{C}[x_1, x_2]$.

If $\deg_t g = 0$, then H is lower triangular. If $a, b \in \mathbb{C}$, then $T^{-1}H(Tx)$ is lower triangular if we take $T \in \mathrm{GL}_4(\mathbb{C})$ of the form

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & d & b \\ 0 & 0 & c & a \end{pmatrix}$$

If a and b are linearly dependent over $\mathbb{C}$, then we can reduce to the case that $a, b \in \mathbb{C}$ by dividing a and b by a suitable polynomial in x_1 and x_2 and adapting g accordingly.

So assume that a and b are linearly independent over $\mathbb{C}$ and $\deg_t g \geq 1$. Since $\deg H_3 = 3$, it follows that $\deg_t g = 1$ and $\deg a = \deg b = 1$. Furthermore, the coefficient of t^1 of g is contained in $\mathbb{C}$, so we may assume that g is monic with respect to t.

Notice that $(\mathcal{J}_{x_3,x_4}(H_3, H_4))|_{x=e_2}$ is a nilpotent matrix over $\mathbb{C}$. If this matrix is zero, then we can derive that $x_1 \mid a$ and $x_1 \mid b$, which contradicts that a and b are linearly independent over $\mathbb{C}$. So for a suitable $S \in \mathrm{GL}_2(\mathbb{C})$, we have

$$S^{-1}\big(\mathcal{J}_{x_3,x_4}(H_3, H_4)\big)\big|_{x=e_2} S = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$$

Now take $Tx = (x_1, x_2, S_1(x_3, x_4), S_2(x_3, x_4))$. Then

$$\begin{aligned} \big(\mathcal{J}T^{-1}H(Tx)\big)|_{x=e_2} &= T^{-1}\mathcal{J}H|_{x=Tx}|_{x=e_2}T \\ &= T^{-1}\mathcal{J}H|_{x=e_2}T \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ * & * & 0 & -1 \\ * & * & 0 & 0 \end{pmatrix} \end{aligned}$$

Next replace H by $T^{-1}H(Tx)$. After recomputing a and b, we obtain from the structure of $(\mathcal{J}H)|_{x=e_2}$ given above that $a = \alpha x_1$ and $b = \beta x_1 + x_2$ for some $\alpha, \beta \in \mathbb{C}$. By replacing H by $T^{-1}H(Tx)$, where

$$T = \begin{pmatrix} \alpha^{-1} & 0 & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

we obtain that $a = x_1$ and $b = x_2$.

Furthermore, $H_2 = \lambda x_1^3$ for some $\lambda \in \mathbb{C}$. If $\lambda = 0$, then H is of the desired form. So assume $\lambda \neq 0$. By taking $T = \operatorname{diag}(\lambda^{-1/4}, \lambda^{1/4}, \lambda^{1/4}, \lambda^{-1/4})$ and replacing H by $T^{-1}H(Tx)$, we obtain that $\lambda = 1$, as desired.

Since $\mathcal{J}H|_{x=e_1} \cdot \mathcal{J}H|_{x=e_2}$ is not nilpotent, it follows that $\mathcal{J}H$ is not linearly triangularizable. Furthermore, the coefficient of $x_1^3 x_3$ of $\det \mathcal{J}_{x_2,x_3}(H_3, H_4)$ is equal to 1, whence $\lambda = \operatorname{rk}\mathcal{J}H - 2$, as desired. □

Corollary 4.6.6. *Assume $H = (H_1, H_2, H_3)$ is a polynomial map in x over $\mathbb{C}$, $n = 3$ and $\mathcal{J}H$ is nilpotent. If $\deg H \leq 3$, then H satisfies DP. If $\deg H \leq 2$, then H is linearly triangularizable.*

Proof. Let $\tilde{H} = (x_4^3 H(x_4^{-1}x), 0)$. Then $\tilde{H}$ is homogeneous of degree 3. From theorem 4.6.4, it follows that $\tilde{H}$ satisfies DP+. So there exists a $\mu \in \mathbb{C}^4$ that is independent of e_4, such that $\mu^{\mathrm{t}}\tilde{H} \in \mathbb{C}[x_4]$. It follow that $\mu_1 H_1 + \mu_2 H_2 + \mu_3 H_3 \in \mathbb{C}$. So H satisfies DP, as desired.

If $\deg H \leq 2$, then each component of $\tilde{H}$ is divisible by x_4. We show that this behavior of $\tilde{H}$ does not match the map H of theorem 4.6.5. The only linear form that might divide $x_2(x_1x_3 - x_2x_4) + p(x_1, x_2)$ is x_2. But $x_1(x_1x_3 - x_2x_4) + q(x_1, x_2)$ is not divisible by any linear form except maybe x_1.

It follows that $\tilde{H}$ is linearly triangularizable in case $\deg H \leq 2$. Consequently, H is linearly triangularizable in this case as well, as desired. □

Hubbers computed all quadratic homogeneous maps H in dimension 5 with $\mathcal{J}H$ nilpotent and H satisfying DP+ in his Ph.D. thesis [37]. The following theorem is essentially [37, Th.7.11], which is proved without computations here.

Theorem 4.6.7 (Hubbers)**.** *Assume H is homogeneous over $\mathbb{C}$, $n = 5$ and $\deg H = 2$. Assume furthermore that $\mathcal{J}H$ is nilpotent and H satisfies DP+. If H is* not *linearly triangularizable, then there exists a $T \in \mathrm{GL}_n(\mathbb{C})$ such that*

$$T^{-1}H(Tx) = \begin{pmatrix} 0 \\ \lambda x_1^2 \\ x_2x_4 + p(x_1, x_2) \\ x_1x_3 - x_2x_5 + q(x_1, x_2) \\ x_1x_4 + r(x_1, x_2) \end{pmatrix}$$

for certain polynomials p, q, r, where $\lambda = \operatorname{rk}\mathcal{J}H - 3 \in \{0, 1\}$.

Proof. Since H satisfies DP+, we may assume that $H_1 = 0$ and $H_2 = \lambda x_1^2$ for some $\lambda \in \mathbb{C}$. Hence $M := \mathcal{J}_{x_3,x_4,x_5}(H_3, H_4, H_5)$ is nilpotent. Let $K = \mathbb{C}(x_1, x_2)$ and $\bar{K}$ be the algebraic closure of K. We first formulate two cases and show that H is linearly triangularizable in these cases:

- The rows of M are dependent over $\mathbb{C}$.
  Then we may assume that the first row of M is zero and hence that $\mathcal{J}_{x_4,x_5}(H_4, H_5)$ is nilpotent. Consequently, (H_4, H_5) is of the form $(bg(ax_4 - bx_5) + p, ag(ax_4 - bx_5) + q)$, where $a, b, p, q \in \mathbb{C}[x_1, x_2, x_3]$ and $g \in \mathbb{C}[x_1, x_2, x_3][t]$. It follows that either $\deg g \le 0$ or $\deg(a, b) \le 0$. Now one can easily show that H is linearly triangularizable over $\mathbb{C}$.

- The columns of M are dependent over $\mathbb{C}$.
  Then we may assume that the last column of M is zero and hence $\mathcal{J}_{x_3,x_4}(H_3, H_4)$ is nilpotent. It follows in a similar manner as above that H is linearly triangularizable over $\mathbb{C}$.

The rows or columns of M do not need to be dependent over $\mathbb{C}$. But M is nilpotent, so its determinant is zero. Consequently, its rows are dependent over $\bar{K}$. But since all entries of M are contained in K and $\bar{K}$ is a vector space over K, we obtain that the rows of M are dependent over K and hence over $\mathbb{C}[x_1, x_2]$ as well. The same holds for the columns of M.

In order to investigate these dependences in M more closely, we distinguish two cases:

- $\deg_{\underline{x_3},x_4,\underline{x_5}}(H_3, H_4, H_5) = 2$.
  Let $\bar{H}_3$, $\bar{H}_4$, $\bar{H}_5$ be the quadratic homogeneous parts of H_3, H_4, H_5 with respect to x_3, x_4, x_5. Since the rows of M are dependent over $\mathbb{C}[x_1, x_2]$, $\bar{H}_3$, $\bar{H}_4$ and $\bar{H}_5$ are linearly dependent over $\mathbb{C}[x_1, x_2]$. But $\bar{H}_3$, $\bar{H}_4$ and $\bar{H}_5$ are contained in $\mathbb{C}[x_3, x_4, x_5]$, so they are linearly dependent over $\mathbb{C}$. So we may assume that $\bar{H}_5 = 0$.

  Assume that the rows of M are *not* dependent over $\mathbb{C}$. Then $H_5 \notin \mathbb{C}[x_1, x_2]$. Since the rows of M are dependent over $\mathbb{C}[x_1, x_2]$, it follows that $\bar{H}_3$ and $\bar{H}_4$ are linearly dependent over $\mathbb{C}[x_1, x_2]$ and hence over $\mathbb{C}$. So we may assume that $\bar{H}_4 = 0$ as well.

  Since the rows of M are dependent over $\mathbb{C}[x_1, x_2]$, it follows from $\bar{H}_4 = \bar{H}_5 = 0$ that the rows of $\mathcal{J}_{x_3,x_4,x_5}(H_4, H_5)$ are dependent over $\mathbb{C}[x_1, x_2]$. So $aH_4 - bH_5 \in \mathbb{C}[x_1, x_2]$ for certain $a, b \in \mathbb{C}[x_1, x_2]$ with $\gcd\{a, b\} = 1$.

If $\frac{\partial}{\partial x_3} H_4 = 0$, then $\frac{\partial}{\partial x_3} H_5 = b^{-1} a \cdot 0 = 0$ as well, whence e_1 is an eigenvector of M. Consequently, $\frac{\partial}{\partial x_3} H_3 = 0$ in addition, whence the columns of M are dependent over $\mathbb{C}$.

So assume $\frac{\partial}{\partial x_3} H_4 \neq 0$. Since $a \frac{\partial}{\partial x_3} H_4 - b \frac{\partial}{\partial x_3} H_5 = 0$, it follows from $\gcd\{a, b\} = 1$ that $a \mid \frac{\partial}{\partial x_3} H_5$ and $b \mid \frac{\partial}{\partial x_3} H_4$. Replacing H by $T^{-1} H(Tx)$ for a suitable diagonal matrix T, we obtain $H_4 = b x_3 + q(x_1, x_2)$ and $H_5 = a x_3 + r(x_1, x_2)$.

Since (H_3, H_4, H_5) is linearly triangularizable over $\bar{K}$, it satisfies DP+ over $\bar{K}$. So there exists a $\mu \in \bar{K}^3$ that is independent of $(0, a, -b)$ such that $\mu^{\mathrm{t}}(H_3, H_4, H_5) \in \bar{K}(a x_4 - b x_5)$. The only possibility is that $H_3 \in \bar{K}(a x_4 - b x_5)$. But since $\deg_{x_3, x_4, x_5} H_3 = 2$, we obtain that $a, b \in \mathbb{C}$ and hence the rows of M are dependent over $\mathbb{C}$. Contradiction, so H is linearly triangularizable.

- $\deg_{x_3, x_4, x_5}(H_3, H_4, H_5) \leq 1$.
  If $\deg_{x_3, x_4, x_5}(H_3, H_4, H_5) \leq 0$, then $\mathcal{J} H$ is lower triangular. So assume $\deg_{x_3, x_4, x_5}(H_3, H_4, H_5) = 1$. Notice that $M|_{x = e_1}$ is a nilpotent matrix over $\mathbb{C}$.

  Assume first that the rank of $M|_{x = e_1}$ is 0. Then the entries of M are contained in $\mathbb{C}[x_2]$ and hence the rows of M are dependent over $\mathbb{C}[x_2]$. But since M is homogeneous of degree 1, it follows that the rows of M are dependent over $\mathbb{C}$ and hence H is linearly triangularizable over $\mathbb{C}$.

  So assume that the rank of $M|_{x = e_1}$ is $\mu + 1 \in \{1, 2\}$. Then there exists an $S \in \mathrm{GL}_3(\mathbb{C})$ such that

$$S^{-1} M|_{x = e_1} S = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \mu & 0 \end{pmatrix}$$

  Now we take $Tx = (x_1, x_2, S_1(x_3, x_4, x_5), S_2(x_3, x_4, x_5), S_3(x_3, x_4, x_5))$. Then

$$\begin{aligned} \big(\mathcal{J} T^{-1} H(Tx)\big)|_{x = e_1} &= T^{-1} \mathcal{J} H|_{x = Tx}|_{x = e_1} T \\ &= T^{-1} \mathcal{J} H|_{x = Te_1} T \\ &= T^{-1} \mathcal{J} H|_{x = e_1} T \end{aligned}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2\lambda & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & 1 & 0 & 0 \\ * & * & 0 & \mu & 0 \end{pmatrix} \tag{4.13}$$

Replacing H by $T^{-1}H(Tx)$, we obtain

$$\begin{aligned} H_3 &= x_2 \cdot (\cdots) + p(x_1, x_2) \\ H_4 &= x_1 x_3 + x_2 \cdot (\cdots) + q(x_1, x_2) \\ H_5 &= \mu x_1 x_4 + x_2 \cdot (\cdots) + r(x_1, x_2) \end{aligned}$$

for certain $p, q, r \in \mathbb{C}[x_1, x_2]$, where the dots can be any linear form. We distinguish two subcases:

- $\mu = 0$.
  Since the sum of the determinants of the principal minors of size 2 of M is zero, all its coefficients with respect to x_1 are zero as well. In particular, the coefficient of x_1^1 of this sum is zero, whence $\frac{\partial}{\partial x_4} H_3 = 0$. Looking at the coefficient of x_1^1 of $\det M = 0$, seen as polynomial in x_1 as well, we obtain that either $\frac{\partial}{\partial x_5} H_3 = 0$ or $\frac{\partial}{\partial x_4} H_5 = 0$. In the first case, one can derive that the first row of M is zero. In the second case, one can derive that the second column of M is zero. So H is linearly triangularizable over $\mathbb{C}$.

- $\mu = 1$.
  Looking at the coefficient of x_1^2 of $\det M$, seen as polynomial in x_1, we obtain $\frac{\partial}{\partial x_5} H_3 = 0$. If $\frac{\partial}{\partial x_4} H_3 = 0$ as well, then one can derive that the first row of M is zero, whence H is linearly triangularizable. So assume $\frac{\partial}{\partial x_4} H_3 \neq 0$. Notice that $\frac{\partial}{\partial x_3} H_3$ is linearly dependent of $\frac{\partial}{\partial x_4} H_3$. Replacing H by $T^{-1}H(Tx)$ for a suitable $T \in \mathrm{GL}_5(\mathbb{C})$, we can obtain that $\frac{\partial}{\partial x_3} H_3 = 0$ without affecting (4.13).

  So $H_3 = bx_4 + p(x_1, x_2)$ for some $b \in \mathbb{C}x_2$. Replacing H by $T^{-1}H(Tx)$ for a suitable diagonal matrix T, we obtain that $b = x_2$. Looking at the coefficient of x_1^1 of $\det M$, seen as polynomial in x_1, we obtain $\frac{\partial}{\partial x_5} H_5 = 0$. If $\frac{\partial}{\partial x_5} H_4 = 0$, then on can derive that the last column of M is zero, whence H is linearly triangularizable.

So assume $\frac{\partial}{\partial x_5} H_4 \neq 0$. Looking at the coefficient of x_1^0 of $\det M$, seen as polynomial in x_1, we obtain $\frac{\partial}{\partial x_3} H_5 = 0$. Notice that H_5 is linearly dependent of $H_3 = x_2x_4 + p(x_1, x_2)$, x_1x_4 and some polynomial in x_1 and x_2. Replacing H by $T^{-1}H(Tx)$ for a suitable $T \in \mathrm{GL}_5(\mathbb{C})$, we obtain that H_5 is of the form $x_1x_4 + r(x_1, x_2)$, without affecting (4.13).

So $x_1H_3 - x_2H_5 \in \mathbb{C}[x_1, x_2]$. It follows from the fact that (H_3, H_4, H_5) satisfies DP+ over $\bar{K}$ that there exists a $\mu \in \bar{K}^3$ such that $\mu^{\mathrm{t}}(H_3, H_4, H_5) \in \bar{K}(x_1x_3 - x_2x_5)$. The only possibility is that $H_4 \in K(x_1x_3 - x_2x_5)$. Since H_4 is of the form $x_1x_3 + x_2 \cdot (\cdots) + q(x_1, x_2)$, we obtain $H_4 = x_1x_3 - x_2x_5 + q(x_1, x_2)$, as desired.

Furthermore, $H_2 = \lambda x_1^2$ for some $\lambda \in \mathbb{C}$. If $\lambda = 0$, then H is of the desired form. So assume $\lambda \neq 0$. By taking $T = \mathrm{diag}(\lambda^{-1/3}, \lambda^{1/3}, \lambda^{1/3}, 1, \lambda^{-1/3})$ and replacing H by $T^{-1}H(Tx)$, we obtain that $\lambda = 1$, as desired.

Since $\mathcal{J}H|_{x=e_1} \cdot \mathcal{J}H|_{x=e_2}$ is not nilpotent, it follows that $\mathcal{J}H$ is not linearly triangularizable. Furthermore, the coefficient of $x_1^2x_4$ of $\det \mathcal{J}_{x_2,x_3,x_4}(H_3, H_4, H_5)$ is equal to 1, whence $\lambda = \mathrm{rk}\mathcal{J}H - 3$, as desired. □

Notice that the map H above with $\lambda = 0$ is in fact the map of lemma 4.2.7 with $d = 1$, $A = x_1$ and $B = x_2$. The map

$$H = (0, x_1x_3, x_2^2 - x_1x_4, 2x_2x_3 - x_1x_5, x_3^2) \tag{4.14}$$

is in fact the map of lemma 4.2.7 with $d = 2$, $A = 1$ and $B = x_1$, and does *not* satisfy DP+. A linear conjugation of this map and its variant with $A = x_1^2$ and $B = x_1$ can be found in [24, §8.4] ([24, Th. 8.4.3] and [24, Th. 8.4.1] respectively). The map of lemma 4.2.7 with $d = 2$ and $A = B = 1$ can be found in section 7.4 of [37], where it is disguised by a lower triangular linear conjugation. (4.14) is in fact the homogenization of this map.

Notice that $x + H$ with H as in (4.14) remains a quadratic homogeneous Keller map when it is composed with $x + \lambda x_1^2$ in the right order, where $\lambda \in \mathbb{C}^5$ such that $\lambda_1 = 0$.

The following theorem shows that (4.14) is more or less the only quadratic homogeneous map in dimension 5 with a nilpotent Jacobian, that does *not* satisfy DP+. In particular, quadratic homogeneous maps in dimension 5 with nilpotent Jacobians satisfy DP.

Theorem 4.6.8. *Assume H is homogeneous over $\mathbb{C}$ of degree 2, $n = 5$ and $\mathcal{J}H$ is nilpotent. Then $x + H$ is tame and H satisfies DP. If* $\mathrm{rk}\mathcal{J}H \le 2$*, then H is linearly triangularizable.*

If H does not *satisfy DP+, then* $\mathrm{rk}\mathcal{J}H = 4$ *and there exists a* $T \in \mathrm{GL}_n(\mathbb{C})$ *such that*

$$T^{-1}H(Tx) \equiv (0, x_1x_3, x_2^2 - x_1x_4, 2x_2x_3 - x_1x_5, x_3^2) \pmod{x_1^2}$$

i.e. H is essentially the map of (4.14).

Proof. The tameness of $x + H$ and the assertion that H satisfies DP follow from lemma 4.2.7, theorem 4.6.7 and the last claim. We will prove the case $\mathrm{rk}\mathcal{J}H \ge 3$ of the last claim by way of computations in appendix A.

So assume $\mathrm{rk}\mathcal{J}H \le 2$. Since H is quadratic homogeneous and $\mathrm{rk}\mathcal{J}H = 0$, it follows from theorem 4.3.1 that each component of H is linearly dependent over $\mathbb{C}$ of either

$$gp^2, gpq, gq^2$$

or

$$gp, gq$$

or

$$g$$

Consequently, there are at least two independent linear relations between the components of H, so H satisfies DP+. Now apply theorem 4.6.7 to obtain that H is linearly triangularizable, as desired. $\square$

At last, we look at removing the trace condition from the nilpotency condition for homogeneous maps in dimension 4. Assume H is homogeneous over $\mathbb{C}$, $n = 4$ and three of the four eigenvalues of $\mathcal{J}H$ are zero. As we have proved, the components of H are linearly dependent over $\mathbb{C}$ in case $\mathrm{rk}\mathcal{J}H \le 2$.

We shall show in some appendix A that the components of H are also linearly dependent over $\mathbb{C}$ in case H is quadratic. This is however not necessarily the case if $\mathrm{rk}\mathcal{J}H = 3$ and $\deg H \ge 3$. Take for instance

$$H = \begin{pmatrix} (x_2x_4 - x_1^2)x_4^2 \\ x_3x_4^3 + 2x_1(x_2x_4 - x_1^2)x_4 \\ -(x_2x_4 - x_1^2)^2 \\ x_4^4 \end{pmatrix} \tag{4.15}$$

The first three components of H are the homogenization of the map of [24, Prop. 7.1.9], with $r = 2$, $n = 3$ and $a = x_1^2$.

Notice that the map in (4.15) above has degree 4. By way of multiplication by a power of x_4, we obtain maps of larger degree. For cubic homogeneous maps in dimension 4, we do not know whether a counterexample as in (4.15) exists.

Chapter 5

Nilpotent Hessians

5.1 Introduction

In this chapter, we will classify nilpotent Hessians over $\mathbb{C}$ for small dimensions n. In particular, we will show that nilpotent Hessians over $\mathbb{C}$ of dimension $n \leq 4$ are triangular up to linear conjugations over $\mathbb{C}$ and that for each $n \geq 5$, there are nilpotent Hessians over $\mathbb{C}$ of dimension n that are *not* triangular up to linear conjugations over $\mathbb{C}$.

For homogeneous nilpotent Hessians over $\mathbb{C}$, we will show that those of dimension $n \leq 5$ are triangular up to linear conjugations over $\mathbb{C}$ and those of dimension $n \geq 7$ are not always triangular up to linear conjugations over $\mathbb{C}$. For homogeneous Hessians of dimension $n = 6$, we show that those for which the rows are dependent over $\mathbb{C}$ are always triangular up to linear conjugations over $\mathbb{C}$. Notice that square Jacobians of any dimension n, and in particular homogeneous Hessians of dimension $n = 6$, for which the rows are *not* dependent over $\mathbb{C}$, are never triangular up to linear conjugations over $\mathbb{C}$.

Observe that nilpotent matrices over reduced commutative rings are singular, i.e. have determinant zero. For that reason and the fact that singular Hessians can be studied by way of quasi-translations associated with it, as pointed out in chapter 3 (proposition 3.1.9), we will first study singular Hessians. We start with dimensions 1 and 2.

Proposition 5.1.1. *Assume $h \in \mathbb{C}[x]$ and $n \leq 2$ such that $\det \mathcal{H}h = 0$. Then the rows of $\mathcal{H}h$ are dependent over $\mathbb{C}$.*

Proof. Since $\det \mathcal{H}h = 0$, the components of $H := \nabla h$ are algebraically dependent over $\mathbb{C}$. In case $n = 1$, we have $H_1 \in \mathbb{C}$ and $\mathcal{H}h = (0)$. So assume $n = 2$. Then $R(H_1, H_2) = 0$ for some nonzero $R \in \mathbb{C}[y_1, y_2]$. Put

$$Q := (\nabla R)(H_1, H_2)$$

then $x + Q$ is a quasi-translation on account of proposition 3.1.9, whence $\lambda^{\mathrm{t}} Q = 0$ for some nonzero $\lambda \in \mathbb{C}^2$ on account of theorem 3.3.1. So $\lambda^{\mathrm{t}} \nabla R$ is an algebraic relation between the components of H as well. Since $\deg \lambda^{\mathrm{t}} \nabla R < \deg R$, we obtain that $\lambda^{\mathrm{t}} \nabla R = 0$ if we choose R of minimal degree in advance.

Assume without loss of generality that $\lambda_1 \neq 0$ (the case $\lambda_2 \neq 0$ is similar). Then $R \in \mathbb{C}[y_1, \lambda_1 y_2 - \lambda_2 y_1]$. Since $\lambda^{\mathrm{t}} \nabla R = 0$, it follows that $R \in \mathbb{C}[\lambda_1 y_2 - \lambda_2 y_1]$.

Since the degree of R was chosen minimal, we obtain that R is a polynomial of degree 1 in $\lambda_1 y_2 - \lambda_2 y_1$. So $\lambda_1 H_2 - \lambda_2 H_1 \in \mathbb{C}$. It follows that the rows of $\mathcal{H}h = \mathcal{J}H$ are dependent over $\mathbb{C}$. $\square$

Notice that in case of a relation R of degree 1, ∇R has degree 0, so the corresponding quasi-translation $x + (\nabla R)(H) = x + \nabla R$ is a real translation.

Corollary 5.1.2. *Assume A is a unique factorization domain and $h \in A[x_1, x_2]$ such that $\det \mathcal{H}h = 0$. Then h is of the form*

$$g(ax_1 - bx_2) + (cx_1 - dx_2)$$

where $g \in A[y_1]$ and $a, b, c, d \in A$. Furthermore, g is constant in case $\operatorname{rk} \mathcal{H}g = 0$.

Proof. Let K be the algebraic closure of $\mathbb{Q}(A)$. From the above theorem, it follows by way of Lefschetz' principle that $b(\mathcal{H}h)_1 + a(\mathcal{H}h)_2 = 0$ for some $a, b \in K$, not both zero. Since K is a vector space over $\mathbb{Q}(A)$, we obtain that we can take $a, b \in \mathbb{Q}(A)$ and hence in A. Furthermore, we can get $\gcd\{a, b\} = 1$.

Assume without loss of generality that $a \neq 0$ (the case $b \neq 0$ is similar). Then we can write $h = g(ax_1 - bx_2, x_2)$, where $g \in \mathbb{Q}(A)[y_1, y_2]$. Let $n = 2$. Since

$$\mathcal{H}h = \left.\begin{pmatrix} a^2 \frac{\partial^2}{\partial y_1^2} g & -ab \frac{\partial^2}{\partial y_1^2} g + a \frac{\partial}{\partial y_1} \frac{\partial}{\partial y_2} g \\ -ab \frac{\partial^2}{\partial y_1^2} g + a \frac{\partial}{\partial y_1} \frac{\partial}{\partial y_2} g & b^2 \frac{\partial^2}{\partial y_1^2} g - 2b \frac{\partial}{\partial y_1} \frac{\partial}{\partial y_2} g + \frac{\partial^2}{\partial y_2^2} g \end{pmatrix}\right|_{\substack{y_1 = ax_1 - bx_2 \\ y_2 = x_2}}$$

and $ax_1 - bx_2$ and x_2 are algebraically independent over K, it follows from $b(\mathcal{H}h)_1 + a(\mathcal{H}h)_2 = 0$ that $\frac{\partial}{\partial y_1}\frac{\partial}{\partial y_2} g = \frac{\partial^2}{\partial y_2^2} g = 0$.

It follows that we can write $h = g(ax_1 - bx_2) - dx_2$, where $g \in \mathbb{Q}(A)[y_1]$ and $d \in \mathbb{Q}(A)$. Let λ be the coefficient of y_1^1 of g. Replacing g by $g - \lambda y_1$, we obtain that $h = g(ax_1 - bx_2) + \lambda a x_1 - (d + \lambda b)x_2$ and that the coefficient of y_1^1 of g becomes zero. So if we put $c = \lambda a$ and replace d by $d + \lambda b$, we obtain that $h = g(ax_1 - bx_2) + (cx_1 - dx_2)$. Since the coefficient of y_1^1 of g is zero, it follows that c and $-d$ are the coefficients of x_1 and x_2 of h. So $c, d \in A$.

Looking at the coefficients of x_1^i of h, we see that the denominators of the coefficients of g are composed of factors of a. If $b = 0$, then a is a unit in A due to $\gcd\{a, b\} = 1$ and we are done, so assume $b \neq 0$. Looking at the coefficients of x_2^i of h, we obtain that the denominators of the coefficients of g are composed of factors of b, and again by $\gcd\{a, b\} = 1$ we get the desired result. □

The following result about homogeneous singular Hessians of dimension $n \leq 4$ was obtained in 1876 already in [34].

Proposition 5.1.3 (Gordan and Nöther)**.** *Assume $h \in \mathbb{C}[x]$ is homogeneous, $n \leq 4$ and $\det \mathcal{H}h = 0$. Then the rows of $\mathcal{H}h$ are dependent over $\mathbb{C}$.*

Proof. The case $n \leq 2$ follows from the above theorem, so assume $3 \leq n \leq 4$. Since $\det \mathcal{H}h = 0$, the components of $H = \nabla h$ are algebraically dependent over $\mathbb{C}$, say $R(H_1, H_2, \ldots, H_n) = 0$ for some nonzero homogeneous $R \in \mathbb{C}[y_1, y_2, \ldots, y_n]$. Choose R of minimal degree and put

$$Q := (\nabla R)(H_1, H_2, \ldots, H_n)$$

then $x + Q$ is a homogeneous quasi-translation on account of proposition 3.1.9. From proposition 3.3.2 and theorem 3.3.3, it follows that there exists a $T \in \mathrm{GL}_n(\mathbb{C})$ such that the first 2 components of $T^{-1}Q(Tx)$ are zero.

Put $\tilde{h} = h(Tx)$. Since $\mathcal{J}\tilde{h} = \mathcal{J}h|_{x=Tx}T$, it follows that

$$\tilde{H} := \nabla\tilde{h} = T^{\mathrm{t}} H(Tx) \tag{5.1}$$

and

$$\mathcal{H}\tilde{h} = \mathcal{J}\nabla\tilde{h} = T^{\mathrm{t}}\mathcal{H}h|_{x=Tx}T \tag{5.2}$$

Furthermore, it follows from (5.1) that $\tilde{R} := R((T^{\mathrm{t}})^{-1}y)$ is an algebraic relation between the components of $\nabla h(Tx)$. It follows that

$$\nabla \tilde{R} = T^{-1}(\nabla R)((T^{\mathrm{t}})^{-1}y)$$

and

$$\tilde{Q} := (\nabla \tilde{R})(\tilde{H}) = (T^{-1}\nabla R)(H(Tx)) = T^{-1}Q(Tx) \tag{5.3}$$

Since T is invertible, we can reverse the above transformation. It follows that $\tilde{R}$ is a relation of minimal degree between the components of $\tilde{H}$. From the definition of T, $\tilde{Q}_1 = \tilde{Q}_2 = 0$ follows, whence $\frac{\partial}{\partial y_1}\tilde{R} = \frac{\partial}{\partial y_2}\tilde{R} = 0$. So $\tilde{R} \in \mathbb{C}[y_3, y_4]$. Since $\tilde{R}$ is of minimal degree, $\tilde{R}$ is irreducible. Since $\tilde{R}$ is homogeneous and bivariate at most, it follows that $\tilde{R}$ is linear.

So R is linear as well. It follows that the rows of $\mathcal{H}h = \mathcal{J}H$ are dependent over $\mathbb{C}$. $\square$

In all other situations ($n \geq 3$ or h homogeneous and $n \geq 5$), the condition $\det \mathcal{H}h = 0$ is not sufficient to obtain that the rows of $\mathcal{H}h$ are dependent over $\mathbb{C}$. But this is not the whole story. Actually, it is a matter of rank and homogeneity only. A singular Hessian of dimension 2 has rank ≤ 1 and a homogeneous singular Hessian of dimension ≤ 4 has rank ≤ 3.

Theorem 5.1.4. *Assume $h \in \mathbb{C}[x]$ and $\mathrm{rk}\mathcal{H}h \leq 1$. Then every pair of rows of $\mathcal{H}h$ is dependent over $\mathbb{C}$.*

Theorem 5.1.5. *Assume $h \in \mathbb{C}[x]$ is homogeneous and $\mathrm{rk}\mathcal{H}h \leq 3$. Then the dependences between the rows of $\mathcal{H}h$ are generated by such dependences over $\mathbb{C}$.*

The polynomials

$$h = x_1^2x_2 + x_1^3x_3 + \cdots + x_1^nx_n$$

and

$$h = x_1^nx_2^3x_3 + x_1^{n-1}x_2^4x_4 + \cdots + x_1^3x_2^nx_n$$

show that theorems 5.1.4 are 5.1.5 cannot be extended to Hessians of larger rank.

Proof of theorem 5.1.4. We show that the first two rows of $\mathcal{H}h$ are dependent over $\mathbb{C}$. This is trivially the case if one of both rows is zero, so assume the opposite. Since $\mathrm{rk}\mathcal{H}h = 1$, we can clean the second and subsequent rows of

$\mathcal{H}h$ by row operations with the first row. But if $\frac{\partial^2}{\partial x_1^2}h = 0$, then the first coordinate of the first row of $\mathcal{H}h$ is unable to clean, and therefore the first column of $\mathcal{H}h$ must be 0 already. But the first row of $\mathcal{H}h$ is nonzero by assumption and $\mathcal{H}h$ is symmetric. It follows that $\frac{\partial^2}{\partial x_1^2}h \neq 0$.

So $\mathrm{rk}\mathcal{H}_{x_1,x_2}(H_1, H_2) = 1$. By proposition 5.1.1, we obtain that the rows of $\mathcal{H}_{x_1,x_2}(H_1, H_2)$ are dependent over the algebraic closure of $\mathbb{C}(x_3, \ldots, x_n)$ and hence over $\mathbb{C}[x_3, \ldots, x_n]$. But since $\mathrm{rk}\mathcal{H}_{x_1,x_2}(H_1, H_2) = 1$, there is essentially only one relation between the rows of $\mathcal{H}_{x_1,x_2}(H_1, H_2)$. This relation must be the same as the relation between the first two rows of $\mathcal{H}h$, so the first two rows of $\mathcal{H}h$ are dependent over $\mathbb{C}[x_3, x_4, \ldots, x_n]$.

We show that we can get rid of every variable of $x_3, x_4, \ldots, x_n$ by showing that the first two rows of $\mathcal{H}h$ are dependent over $\mathbb{C}[x_1, x_2, x_4, \ldots, x_n]$ as well. This is sufficient, because there is essentially only one relation between the first two rows of $\mathcal{H}h$. If the third row of $\mathcal{H}h$ is zero, then $\mathcal{H}h$ does not contain terms with x_3 and hence its first two rows are dependent over $\mathbb{C}[x_4, \ldots, x_n]$.

So assume the third row of $\mathcal{H}h$ is nonzero. Then the third row of $\mathcal{H}h$ is dependent over $\mathbb{C}[x_2, x_4, \ldots, x_n]$ of the first row $\mathcal{H}h$ and the second row of $\mathcal{H}h$ is dependent over $\mathbb{C}[x_1, x_4, \ldots, x_n]$ of the third row of $\mathcal{H}h$. This gives the desired result. □

Corollary 5.1.6. *Assume A is a unique factorization domain with $\mathbb{Q}$ and $h \in A[x]$ such that $\mathrm{rk}\mathcal{H}h \leq 1$. Then*

$$h = g(a_1x_1 + a_2x_2 + \cdots + a_nx_n) + (b_1x_1 + b_2x_2 + \cdots + b_nx_n)$$

where $g \in A[y_1]$ and $a_i, b_i \in A$ for all i. Furthermore, g is constant in case $\mathrm{rk}\mathcal{H}h = 0$.

Proof. The proof is similar to that of corollary 5.1.2. □

We prove theorem 5.1.5 (in the equivalent form of theorem 5.3.10) in section 5.3.

5.2 Degenerate gradient relations

Definition 5.2.1. Let $g, h \in \mathbb{C}[x]$. We call g and h *linearly equivalent* if there exists a $T \in \mathrm{GL}_n(\mathbb{C})$ such that $g = h(Tx)$. We call h *degenerate* if there exists a $g \in \mathbb{C}[x_1, x_2, \ldots, x_{n-1}]$ that is linearly equivalent to h. We call h *degenerate of order* s if there exists a $g \in \mathbb{C}[x_1, x_2, \ldots, x_{n-s}]$ that is linearly equivalent to h but no $g \in \mathbb{C}[x_1, x_2, \ldots, x_{n-s-1}]$ that is linearly equivalent to h.

Notice that linear equivalence is an equivalence relation. Furthermore, it follows from (5.2) that $\mathrm{rk}\mathcal{H}g = \mathrm{rk}\mathcal{H}h$ if g is linearly equivalent to h. From theorem 5.1.4, it follows that $h \in \mathbb{C}[x]$ is degenerate of order $\geq n-2$ if $\mathrm{rk}\mathcal{H}h = 1$.

Definition 5.2.2. Let $h \in \mathbb{C}[x]^n$ with $\det \mathcal{H}h = 0$. Then there exists a nonzero $R \in \mathbb{C}[y]$ such that $R(\nabla h) = 0$. We define s_h as the maximal degeneracy order a relation R as above can have.

Proposition 5.2.3. *Assume $h \in \mathbb{C}[x]$ and $\det \mathcal{H}h = 0$. Then there exists a g that is equivalent to h such that*

$$\frac{\partial}{\partial x_{s_h+1}} g, \frac{\partial}{\partial x_{s_h+2}} g, \ldots, \frac{\partial}{\partial x_n} g$$

are algebraically dependent over $\mathbb{C}$.

Proof. Let $H = \nabla h$ and assume $R(H) = 0$ for some nonzero $R \in \mathbb{C}[y]$ that is degenerate of order s_h. Then there exists a $T \in \mathrm{GL}_n(\mathbb{C})$ such that

$$R\big((T^{\mathrm{t}})^{-1} y\big) \in \mathbb{C}[y_{s_h+1}, y_{s_h+2}, \ldots, y_n]$$

Put $g := h(Tx)$. Then $\nabla g = T^{\mathrm{t}} H(Tx)$. Since $R(H(Tx)) = 0$, the desired result follows. $\square$

Proposition 5.2.4. *Assume $h \in \mathbb{C}[x]$ such that h does not have linear terms. Then $s_h = n-1$, if and only if h is degenerate.*

Proof. If h is degenerate, then there is a $T \in \mathrm{GL}_n(\mathbb{C})$ such that $g = h(Tx) \in \mathbb{C}[x_1, x_2, \ldots, x_{n-1}]$. So $\frac{\partial}{\partial x_n} g = 0$ and $R = y_n$ is a relation of ∇g that is degenerate of order $n-1$.

So assume that $s_h = n-1$. Then there is a $T \in \mathrm{GL}_n(\mathbb{C})$ such that $g = h(Tx)$ satisfies $R(\frac{\partial}{\partial x_n} g) = 0$ for some nonzero $R \in \mathbb{C}[y_n]$. So $\frac{\partial}{\partial x_n} g \in \mathbb{C}$. Since h and hence g does not have linear terms, $\frac{\partial}{\partial x_n} g = 0$, whence $g \in \mathbb{C}[x_1, x_2, \ldots, x_{n-1}]$. So h is degenerate. $\square$

Let $h \in \mathbb{C}[x]$ such that $\det \mathcal{H}h = 0$. Let g be equivalent to h such that $R(\nabla g) = 0$ for some nonzero $R \in \mathbb{C}[y_{s+1}, \dots, y_n]$. Then the last $n - s$ rows of $\mathcal{H}g$ are dependent on account of proposition 1.2.9. In particular

$$\mathrm{rk}\mathcal{H}_{x_{s+1},x_{s+2},\dots,x_n} g \le n - s - 1 \tag{5.4}$$

Notice that $s_h = s_g \ge s$. Proposition 5.2.5 below tells us that in case of equality in (5.4), it can be shown that $s_h = s_g > s$ in some cases.

Proposition 5.2.5. *Assume that $R(\nabla g) = 0$ for some nonzero $R \in \mathbb{C}[y_{s+1}, \dots, y_n]$. Assume furthermore that $s_h \ge 1$ for all $h \in \mathbb{C}[x_1, x_2, \dots, x_{n-s}]$ such that $\det \mathcal{H}_{x_1,x_2,\dots,x_{n-s}} h = 0$, where h should be seen as a polynomial in $n - s$ variables. Notice that R is degenerate of order $\ge s$. If R is irreducible and*

$$\mathrm{rk}\mathcal{H}_{x_{s+1},\dots,x_n} g = n - s - 1$$

then R is degenerate of order $\ge s + 1$.

Proof. Put $K := \mathbb{C}(x_1, x_2, \dots, x_s)$ and let $\bar{K}$ be the algebraic closure of $\mathbb{C}(x_1, x_2, \dots, x_s)$.

i) Since $\det \mathcal{H}_{x_{s+1},\dots,x_n} g = 0$, there exists an $\tilde{R} \in K[y_{s+1}, y_{s+2}, \dots, y_n]$ such that $\tilde{R}(G_{s+1}, G_{s+2}, \dots, G_n) = 0$, where $G = \nabla g$. Choose $\tilde{R}$ of minimal degree.

   Since $s_h \ge 1$ for all $h \in \mathbb{C}[x_1, x_2, \dots, x_{n-s}]$, it follows from Lefschetz' principle that $\tilde{R}$ is degenerate over $\bar{K}$. So there exists a nonzero $\lambda \in \bar{K}^{n-s}$ such that $\lambda^{\mathrm{t}} \nabla y_{s+1}, y_{s+1}, \dots, y_n \tilde{R} = 0$.

ii) Assume without loss of generality that $\tilde{R}$ is irreducible over K. Assume $\mathrm{rk}\mathcal{H}_{x_{s+1},\dots,x_n} g = n - s - 1$. Then the ideal

$$\mathfrak{p} := \big(S \in K[y_{s+1}, y_{s+2}, \dots, y_n] \,\big|\, S(G_{s+1}, G_{s+2}, \dots, G_n) = 0\big)$$

   is principal. Since $\tilde{R}$ is irreducible and contained in $\mathfrak{p}$, we obtain $\mathfrak{p} = (\tilde{R})$. But $R \in \mathbb{C}[y]$ is irreducible over K as well, so we can derive that $\mathfrak{p} = (R)$. It follows that $\tilde{R} = \mu R$ for some $\mu \in K$. Consequently, $\nabla_y \tilde{R} = \mu \nabla_y R$ and

$$\lambda^{\mathrm{t}} \nabla_{y_{s+1},y_{s+1},\dots,y_n} R = \lambda^{\mathrm{t}} \nabla_{y_{s+1},y_{s+1},\dots,y_n} \tilde{R} = 0$$

   Since $\bar{K}$ is a vector space over $\mathbb{C}$, we obtain that we can take $\lambda \in \mathbb{C}^{n-s}$. It follows that R is degenerate as a polynomial over $\mathbb{C}$ in $n-s$ variables. So R is degenerate of order $\ge s + 1$. □

Corollary 5.2.6. *If $s = s_g$ in proposition 5.2.5, then*

$$\mathrm{rk}\mathcal{H}_{x_{s+1},\ldots,x_n} g \leq n - s - 2$$

The following result was proved in 1876 as well.

Proposition 5.2.7 (Gordan and Nöther)**.** *Assume $h \in \mathbb{C}[x]$ and $R(\nabla h) = 0$ for some nonzero $R \in \mathbb{C}[y]$. Choose R of minimal degree and let $Q := (\nabla R)(\nabla h)$. Then the order of degeneracy of R is equal to the codimension of the linear span of $Q(\mathbb{C}^n)$.*

Proof. Assume the linear span of $Q(\mathbb{C}^n)$ has codimension s. Then the space

$$V := \{\lambda \in \mathbb{C}^n \mid \lambda^{\mathrm{t}} Q = 0\}$$

has dimension s. Assume $\lambda \in V$. Since $(\lambda^{\mathrm{t}} \nabla R)(\nabla h) = \lambda^{\mathrm{t}} Q = 0$ and R was chosen of minimal degree, it follows that $\lambda^{\mathrm{t}} \nabla R = 0$. So R is degenerate of order $\geq s$.

On the other hand, if $\lambda^{\mathrm{t}} \nabla R = 0$, then $\lambda^{\mathrm{t}} Q = 0$ as well, so R is degenerate of order $\leq s$, as desired. $\square$

Corollary 5.2.8. *Assume $h \in \mathbb{C}[x]$, $n \leq 3$ and $\det \mathcal{H}h = 0$. Then $s_h \geq 1$.*

Proof. Take R and Q as in proposition 5.2.7. From proposition 3.1.9, it follows that $x + Q$ is a quasi-translation. Since $n \leq 3$, it follows from theorems 3.3.1 and 3.3.4 that the linear span of $Q(\mathbb{C}^n)$ has dimension $n - 1$ at most. Now apply the above proposition. $\square$

We shall investigate $h \in \mathbb{C}[x]$ with $\det \mathcal{H}h = 0$ and $s_h \geq n-3$, using Lüroth's theorem 4.3.2 about rational functions.

Lemma 5.2.9. *Assume*

$$R(a_1 t + b_1, a_2 t + b_2, a_3 t + b_3) = 0$$

for some nonzero $R \in \mathbb{C}[y_1, y_2, y_3]$ of minimal degree and rational functions $a_i, b_i \in \mathbb{C}(x)$ with $a_3 \neq 0$. If $\deg R \geq 2$, then

$$a = a_3 h(q) \qquad \text{and} \qquad b = b_3 h(q) + g(q)$$

for certain $g, h \in \mathbb{C}(y_1)^3$ and a $q \in \mathbb{C}(x)$, where $h_3 = 1$ and $g_3 = 0$.

Proof.

i) Assume first that $a_3 = 1$ and $b_3 = 0$. We show the desired result, i.e. that $a = h(q)$ and $b = g(q)$ for some $q \in \mathbb{C}(x)$ and $g, h \in \mathbb{C}(y_1)^3$.

   Let $\bar{R}$ be the largest degree homogeneous part of R. Looking at the leading coefficient with respect to t of $R(a_1t+b_1, a_2t+b_2, a_3t+b_3)$, we obtain $\bar{R}(a_1, a_2, 1) = 0$. So by theorem 4.3.2, $a_1, a_2 \in \mathbb{C}(q)$ for some $q \in \mathbb{C}(x)$.

   Notice that R is irreducible because its degree is minimal. Since $y_3 \nmid R$, we obtain by substituting $t = 0$ that b_1 and b_2 are algebraically dependent over $\mathbb{C}$. So $b_1, b_2 \in \mathbb{C}(p)$ for some $p \in \mathbb{C}(x)$. If p and q are algebraically dependent over $\mathbb{C}$, then $a = h(r)$ and $b = g(r)$ for some $r \in \mathbb{C}(x)$ and $g, h \in \mathbb{C}(y_1)^3$ on account of theorem 4.3.2, as desired.

   So assume that p and q are algebraically independent over $\mathbb{C}$. Since the degree of R is minimal, it follows that $(\mathcal{J}_y R)(a_1t+b_1, a_2t+b_2, a_3t+b_3) \neq 0$. By computing $\mathcal{J}_{t,q,p} R(a_1t+b_1, a_2t+b_2, a_3t+b_3) = 0$ with the chain rule, we obtain that the rows of

$$\mathcal{J}_{t,q,p}\begin{pmatrix} a_1t+b_1 \\ a_2t+b_2 \\ a_3t+b_3 \end{pmatrix} = \mathcal{J}_{t,q,p}\begin{pmatrix} a_1t+b_1 \\ a_2t+b_2 \\ t \end{pmatrix}$$

   are dependent. So its determinant

$$t\left(\frac{\partial}{\partial q}a_1 \cdot \frac{\partial}{\partial p}b_2 - \frac{\partial}{\partial q}a_2 \cdot \frac{\partial}{\partial p}b_1\right)$$

   is zero. Since $\deg R \geq 2$, it follows that $(a_2, b_2) \notin \mathbb{C}^2$. If $(a_1, a_2) \in \mathbb{C}^2$, then we can take $q = 1$, so we may assume that $(a_1, a_2) \notin \mathbb{C}^2$. Due to the vanishing of the above determinant, $a_2 \notin \mathbb{C}$. Similarly, we may assume that $(b_1, b_2) \notin \mathbb{C}^2$ and we obtain that $b_2 \notin \mathbb{C}$ as well.

   Since p and q are algebraically independent, it follows that

$$\lambda := \frac{\frac{\partial}{\partial q}a_1}{\frac{\partial}{\partial q}a_2} = \frac{\frac{\partial}{\partial p}b_1}{\frac{\partial}{\partial p}b_2} \in \mathbb{C}(q) \cap \mathbb{C}(p) = \mathbb{C}$$

   So $(a_1t - b_1) - \lambda(a_2t - b_2)$ does not survive $\frac{\partial}{\partial q}$ nor $\frac{\partial}{\partial p}$. Consequently $(a_1t - b_1) - \lambda(a_2t - b_2) \in \mathbb{C}t + \mathbb{C}$. This contradicts $\deg R \geq 2$.

ii) In the general case, we substitute $t = a_3^{-1}(t - b_3)$ to reduce to i). We get $a_3^{-1}a = h(q)$ and $b - a_3^{-1}b_3 a = g(q)$. So $a = a_3 h(q)$ and $b = b_3 h(q) + g(q)$, as desired. $\square$

At last, the main theorem of this section follows here.

Theorem 5.2.10. *Assume $h \in \mathbb{C}[x]$ such that $\det \mathcal{H}h = 0$. Take g linearly equivalent to h such that*

$$\frac{\partial}{\partial x_{s_h+1}} g, \frac{\partial}{\partial x_{s_h+2}} g, \ldots, \frac{\partial}{\partial x_n} g$$

are algebraically dependent. Put $A := \mathbb{C}[x_1, x_2, \ldots, x_{s_h}]$.

i) If $s_h = n - 1$, then we can choose

$$g = a_1 + \lambda x_n$$

for some $a_1 \in A$ and a $\lambda \in \{0, 1\}$.

ii) If $s_h = n - 2$, then we can choose

$$g = a_1 + a_2 x_{n-1} + a_3 x_n$$

for some $a_i \in A$, where $a_2, a_3 \in \mathbb{C}[p]$ for some $p \in A$.

iii) If $s_h = n - 3$, then we can choose

$$g = f(a_1 x_{n-2} + a_2 x_{n-1} + a_3 x_n) + b_1 x_{n-2} + b_2 x_{n-1} + b_3 x_n$$

for some $a, b \in A^3$ and an $f \in A[t]$ without a linear term, such that

- *If $a = 0$ or $\deg_t f \leq 0$, then b_1, b_2, b_3 are algebraically dependent over $\mathbb{C}$,*
- *If $a \neq 0$ and $\mathrm{rk}\mathcal{J}_x(a_1, a_2, a_3) \leq 1$ and $\deg_t f \geq 2$, then $a \in \mathbb{C}[p]^3$ for some $p \in A$, and*
$$b = \beta a + \tilde{b}$$
where $\beta \in \mathbb{Q}(A)$ and $\tilde{b} \in \mathbb{C}(p)$,
- *If $\mathrm{rk}\mathcal{J}_x(a_1, a_2, a_3) \geq 2$ and $\deg_t f \geq 2$, then $a \in \mathbb{C}[p, q]^3$ is homogeneous in p and q for some $p, q \in A$, and*
$$b = \beta a + \tilde{b}$$
where $\beta \in \mathbb{Q}(A)$ and $\tilde{b} \in \mathbb{C}(p)$.

Furthermore, we have the following if h is homogeneous.

iv) *If $s_h \geq n-2$, then $s_h = n-1$. If $s_h = n-1 \neq 0$, then $g \in A$.*

v) *If $s_h = n-3$, then*

$$g = f(a_1 x_{n-2} + a_2 x_{n-1} + a_3 x_n)$$

for some $a \in A^3$ and an $f \in A[t]$, where $a \in \mathbb{C}[p,q]^3$ is homogeneous in p and q for some $p, q \in A$.

Proof.

i) Assume $s_h = n-1$. Assume first that h does not have a linear term. Then it follows from proposition 5.2.4 that h is degenerate. So we can take $g \in A$.

Assume next that h does have a linear term and let $\tilde{h}$ be the part of h without linear term. Then $\tilde{h}$ is linearly equivalent to some $\tilde{g} \in A$, and h is linearly equivalent to $a_1 + \lambda x_n$ for some $a_1 \in A$ and a $\lambda \in \mathbb{C}$. If $\lambda \neq 0$, then we can substitute $x_n = \lambda^{-1} x_n$ to obtain that $\lambda = 1$, as desired.

ii) Assume $s_h = n-2$. We take g such that $\frac{\partial}{\partial x_{n-1}} g$ and $\frac{\partial}{\partial x_n} g$ are algebraically dependent over $\mathbb{C}$. The formula for g follows from proposition 5.2.5 and corollary 5.2.8, except that A should be replaced by the algebraic closure K of $\mathbb{Q}(A)$. But from $g \in \mathbb{C}[x]$, it follows immediately that $a \in A^3$. Since $a_2 = \frac{\partial}{\partial x_{n-1}} g$ and $a_3 = \frac{\partial}{\partial x_n} g$, it follows from theorem 4.3.5 that $a_2, a_3 \in \mathbb{C}[p]$ for some $p \in A$.

iii) Assume $s_h = n-3$. We take g such that the last three components of ∇g are algebraically dependent over $\mathbb{C}$. We obtain the formula for g from proposition 5.2.5, corollary 5.2.8 and corollary 5.1.6, except that A should be replaced by the algebraic closure K of $\mathbb{Q}(A)$.

Since the linear part of f can be added to $b_1 x_{n-2} + b_2 x_{n-1} + b_3 x_n$, we can obtain that f does not have a linear part. Since $g \in \mathbb{C}[x]$, we obtain immediately that $b \in A^3$. Furthermore, the constant part of f is contained in A as well.

So assume $\deg f \geq 2$. Notice that we may assume that $a_1 \in \mathbb{Q}(A)$. Looking at the coefficients of $x_{n-2}^i x_{n-1}^0 x_n^0$ for all i, we obtain that

$f \in \mathbb{Q}(A)[t]$. It follows that $a \in \mathbb{Q}(A)^3$. So we may assume that $a \in A^3$.

Choose a such that $\gcd\{a_1, a_2, a_3\} = 1$. Again by looking at the coefficients of $x_{n-2}^i x_{n-1}^0 x_n^0$, we obtain that the denominators of the coefficients of f are composed of factors of a_1. By way of similar assertions for a_2 and a_3 and $\gcd\{a_1, a_2, a_3\} = 1$, we obtain $f \in A[t]$.

Assume first that $a = 0$ or $\deg_t f \le 0$. Then the last three components of ∇g are b_1, b_2, b_3, so b_1, b_2, b_3 are algebraically dependent over $\mathbb{C}$.

So assume $a \ne 0$ and $\deg_t f > 0$. Then $\deg_t f \ge 2$. Notice that the last three components of g are

$$(at + b)\big|_{t=\left(\frac{\partial}{\partial t} f\right)(a_1 x_{n-2} + a_2 x_{n-1} + a_3 x_n)}$$

Since $\deg_t f \ge 2$, it follows that the substitution for t above is not contained in A and hence algebraically independent of the components of a and b. Without loss of generality, we assume that $a_3 \ne 0$. It follows from lemma 5.2.9 that $a_3^{-1} a \in \mathbb{C}(p/q)^3$ and $b - a_3^{-1} b_3 a \in \mathbb{C}(p/q)^3$ for some $p, q \in A$ with $\gcd\{p, q\} = 1$.

Consequently,

$$a_3^{-1} a = \frac{1}{\alpha(p, q)} h(p, q)$$

for some homogeneous $\alpha \in \mathbb{C}[y_1, y_2]$ and a homogeneous $h \in \mathbb{C}[y_1, y_2]^3$. It follows from $a \in A^3$ and $\gcd\{a_1, a_2, a_3\} = 1$ that $a \in \mathbb{C}[p, q]^3$ is homogeneous in p and q. If $\mathrm{rk}\mathcal{J}_x a \le 1$, then $p, q \in \mathbb{C}[r]$ for some $r \in \mathbb{C}[x]$ on account of theorem 4.3.5. This gives the desired result.

iv) Assume h is homogeneous and $s_h \ge n-2$. If h is linear, then $s_h = n-1$ and we can choose $g \in \mathbb{C}x_1 \subseteq A$, provided $n - 1 \ne 0$. So assume h is not linear. If $s_h = n - 1$, then h does not have a linear term, and we obtain the desired result by i).

So assume $s_h = n - 2$. Then the last two components of ∇g are algebraically dependent. But since h is homogeneous and hence g also, we obtain that the dependence relation R can be taken homogeneous as well. It follows that R decomposes into linear factors, and hence one of these factors is already a relation between the last two components of ∇g. So h is degenerate and $s_h = n - 1$.

v) Assume h is homogeneous and $s_h = n - 3$. Choose g and R as in iii), such that R is homogeneous, and take f, a and b as in iii) as well. If $a = 0$ or $\deg_t f = 0$, then b_1, b_2, b_3 are the last three components of ∇g, and since (b_1, b_2, b_3) is homogeneous, it follows from theorem 4.3.1 that g is of the desired form.

So assume $a \neq 0$ and $\deg_t f \geq 2$. Following iii), we have $R(at+b) = 0$. Now let c_i be the irreducible divisor of $a_i t + b_i$ with $\deg_t c_i = 1$, for all i. Notice that since $R(y_1, y_2, 0)$ is homogeneous and bivariate, it decomposes in linear factors over $\mathbb{C}$. Since $c_3 \mid b_3$, it follows from $R(a_1 t + b_1, a_2 t + b_2, a_3 t + b_3) = 0$ that

$$c_3 \mid \alpha(a_1 t + b_1) + \beta(a_2 t + b_2)$$

for some factor $\alpha y_1 + \beta y_2$ of $R(y_1, y_2, 0)$. Assume without loss of generality that c_3 divides $a_2 t + b_2$. Since A is a unique factorization domain from which the units are contained in $\mathbb{C}$, it follows that c_2 and c_3 are linearly dependent over $\mathbb{C}$. In a similar manner as above, we obtain that c_1 divides a linear combination of $a_2 t + b_2$ and $a_3 t + b_3$, so c_1 and c_2 are linearly dependent over $\mathbb{C}$ as well.

So $\deg_t \gcd\{a_1 t + b_1, a_2 t + b_2, a_3 t + b_3\} = 1$. Since $\gcd\{a_1, a_2, a_3\} = 1$, we may assume that $\gcd\{a_1 t + b_1, a_2 t + b_2, a_3 t + b_3\}$ is monic in t, say it is equal to $t + \mu$. Then $b = \mu a$, so b can be eaten up by a by replacing g by $g + \mu t$.

If $\mathrm{rk} \mathcal{J}_x a \leq 1$, then it follows in a similar manner as $s_h \neq n - 2$ that a_1 and a_2 are linearly dependent over $\mathbb{C}$. This contradicts $s_h = n - 3$, so $\mathrm{rk} \mathcal{J}_x a = 2$. Now theorem 4.3.1 gives the desired result. □

5.3 Hessians of small rank

Assume $h \in \mathbb{C}[x]$ such that $\mathcal{H}h$ has rank r. Then the last $r + 1$ components of ∇h are algebraically dependent, whence $s_h \geq n - r - 1$. With proposition 5.2.5, we can boost this to $s_h \geq n - r$ in some occasions.

With the tools of the previous section, we are going to classify Hessians of rank 2 and homogeneous Hessians of rank 3. Furthermore, we classify all homogeneous singular Hessians in dimension 5. But first, we classify all singular Hessians $\mathcal{H}h$ in dimension 4 that satisfy $s_h \geq 1$.

Some of the results for homogeneous Hessians were already proved by P. Gordan and M. Nöther is 1876 in [34].

Proposition 5.3.1. *Assume $h \in \mathbb{C}[x]$, $n = 3$ and $\det \mathcal{H}h = 0$. Then there exists a g that is linearly equivalent to h such that either*

$$g = a_1(x_1, x_2) + \lambda x_3$$

for some $a_1 \in \mathbb{C}[x_1, x_2]$ and a $\lambda \in \{0, 1\}$, or

$$g = a_1(x_1) + a_2(x_1)x_2 + a_3(x_1)x_3$$

for some $a \in \mathbb{C}[x_1]^3$.

Proof. From corollary 5.2.8, it follows that $s_h \geq 1$. If $s_h = 2 = n - 1$, then we get the first form of g by theorem 5.2.10. If $s_h = 1 = n - 2$, then we get the second form of g by the same theorem. □

Proposition 5.3.2 (Gordan and Nöther)**.** *Assume $h \in \mathbb{C}[x]$ is homogeneous, $2 \leq n \leq 4$ and $\det \mathcal{H}h = 0$. Then h is degenerate.*

Proof. From theorem 5.2.10, it follows that it suffices to show that $s_h \geq n-2$. This is trivial if $n = 2$. If $n = 3$, then $s_h \geq n - 2$ follows from corollary 5.2.8. If $n = 4$, then apply proposition 5.2.7 and theorem 3.3.3 to obtain $s_h \geq n - 2$. □

Theorem 5.3.3. *Assume $h \in \mathbb{C}[x]$, $n = 4$ and $\det \mathcal{H}h = 0$. If $s_h \geq 1$, then there exists a g that is linearly equivalent to h, such that either*

$$g = a_1(x_1, x_2, x_3) + \lambda x_4$$

for some $a_1 \in \mathbb{C}[x_1, x_2, x_3]$ and a $\lambda \in \{0, 1\}$, or

$$g = a_1(x_1, x_2) + a_2(x_1, x_2)x_3 + a_3(x_1, x_2)x_4$$

for some $a \in \mathbb{C}[x_1, x_2]^3$ such that $a_2, a_3 \in \mathbb{C}[p]$ for some $p \in \mathbb{C}[x_1, x_2]$, or

$$g = f\big(x_1, a_1(x_1)x_2 + a_2(x_1)x_3 + a_3(x_1)x_4\big) + \big(b_1(x_1)x_2 + b_2(x_1)x_3 + b_3(x_1)x_4\big)$$

for some $a, b \in \mathbb{C}[x_1]^3$ and an $f \in \mathbb{C}[x_1, t]$.

Proof. Since $s_h \geq 1$, it follows that $s_h \in \{1, 2, 3\} = \{n - 3, n - 2, n - 1\}$. Now apply theorem 5.2.10 to get the forms of g. □

The following lemma is needed to understand Hessians of rank 2.

Lemma 5.3.4. *Let A be an integral domain and assume $M \in \mathrm{Mat}_n(A)$ is symmetric with respect to the main diagonal. If $\mathrm{rk} M = r$, then M has a* principal *minor of size r with (full) rank r. Furthermore, if some set of r rows of M are independent, then the corresponding principal minor of size r has (full) rank r.*

Proof. Since M is symmetric, we can write

$$M = T^{\mathrm{t}} \mathrm{diag}(1^1, 1^2, \ldots, 1^r, 0^{r+1}, \ldots, 0^n) T$$

with $T \in \mathrm{GL}_n(\mathbb{Q}(A))$. Notice that the first r rows of T have a minor of size r with rank r. One can easily verify that the corresponding *principal* minor of M has rank r as well.

Say that the first r rows of M are independent. Let $\tilde{T}$ be the upper left principal minor of size r of T. Then the matrix of the first r rows of M is equal to $\tilde{T}^{\mathrm{t}}(T_1, T_2, \ldots, T_r)$. If this matrix has full rank r, then $\mathrm{rk}\tilde{T}^{\mathrm{t}} = r$ and hence $\mathrm{rk}\tilde{T}^{\mathrm{t}}\tilde{T} = r$. But the latter matrix is exactly the upper left principal minor of size r of M. $\square$

Theorem 5.3.5. *Assume $h \in \mathbb{C}[x]$ such that $\mathrm{rk}\mathcal{H}h = 2$. Then there exists a g that is linearly equivalent to h such that either*

$$g = a_1(x_1, x_2) + \lambda x_3$$

for some $a_1 \in \mathbb{C}[x_1, x_2]$ and a $\lambda \in \{0, 1\}$, or

$$g = a_1(x_1) + a_2(x_1)x_2 + a_3(x_1)x_3 + \cdots + a_n(x_1)x_n$$

for some $a \in \mathbb{C}[x_1]^n$.

Proof. Notice that the second formula for g can absorb linear terms directly and the first formula can absorb linear terms by way of a linear transformation and possibly changing λ into $1 - \lambda$. Since linear terms do not influence the Hessian, we may assume that h does not have a linear term.

i) Assume $s_h = n-1$. Since h does not have a linear term, h is degenerate on account of lemma 5.2.4 and $g \in \mathbb{C}[x_1, x_2, \ldots, x_{n-1}]$. It follows by induction on n that g is linearly equivalent to one of the forms above.

ii) Assume $s_h \leq n-2$. If $n = 2$, then h is of the form $a_1(x_1, x_2)$. So assume $n \geq 3$. From lemma 5.3.4, we obtain that we may assume that $\mathrm{rk}\mathcal{H}_{x_1,x_2,x_3} g = 2$. Since $\mathrm{rk}\mathcal{H} g = 2$, there exists an irreducible $R \in \mathbb{C}[y_1, y_2, y_3]$ such that

$$R\left(\frac{\partial}{\partial x_1} g, \frac{\partial}{\partial x_2} g, \frac{\partial}{\partial x_3} g\right) = 0$$

Since $\mathrm{rk}\mathcal{H}_{x_1,x_2,x_3} g = 2$, it follows from proposition 5.2.5 and corollary 5.2.8 that R is degenerate as a polynomial in $\mathbb{C}[y_1, y_2, y_3]$. It follows that we may assume that $R \in \mathbb{C}[y_2, y_3]$. Since $s_h \leq n-2$, it follows from again proposition 5.2.5 and corollary 5.2.8 that $\mathrm{rk}\mathcal{H}_{x_2,x_3} g = 0$.

From lemma 5.3.4, it follows that one of the three principal minors of size two of $\mathcal{H}_{x_1,x_2,x_3} g$ has rank 2. Since $\mathrm{rk}\mathcal{H}_{x_2,x_3} g = 0$, only $\mathcal{H}_{x_1,x_2} g$ and $\mathcal{H}_{x_1,x_3} g$ can be that principal minor. Assume without loss of generality that $\mathrm{rk}\mathcal{H}_{x_1,x_2} g = 2$, i.e. $\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} g \neq 0$.

The second column of $\mathcal{H} g$ reveals that the first row of $\mathcal{H} g$ is independent of the second and the third. Assume that $\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_3} g = \mu \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} g$ for some $\mu \in \mathbb{C}$. Then we can replace g by $g|_{x_2 = x_2 - \mu x_3}$ to obtain that $\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_3} g = 0$. But since $\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} g \neq 0$ and $\mathrm{rk}\mathcal{H}_{x_2,x_3} g = 0$ are preserved, it follows that the third row of $\mathcal{H} g$ has become zero. This contradicts $s_h \leq n-2$, so $\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_3} g$ is linearly independent over $\mathbb{C}$ of $\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} g$.

Consequently, if $\frac{\partial^2}{\partial x_1^2} g$ is linearly dependent over $\mathbb{C}$ of $\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} g$, then we can destruct that by replacing g by $g|_{x_3 = x_3 + x_1}$, because substituting $x_3 = x_3 + x_1$ in

$$\begin{pmatrix} \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}\right)^2 g \\ \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}\right)\frac{\partial}{\partial x_2} g \\ \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}\right)\frac{\partial}{\partial x_3} g \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x_1}\left(\frac{\partial}{\partial x_1} + 2\frac{\partial}{\partial x_3}\right) g \\ \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} g \\ \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_3} g \end{pmatrix}$$

gives $(\frac{\partial}{\partial x_1}^2 (g|_{x_3 = x_3 + x_1}), \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}(g|_{x_3 = x_3 + x_1}), \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_3}(g|_{x_3 = x_3 + x_1}))$.

iii) So assume that $\frac{\partial^2}{\partial x_1^2} g$ is linearly independent over $\mathbb{C}$ of $\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} g$ and that $\mathrm{rk}\mathcal{H}_{x_2,x_3,\ldots,x_{m-1}} g = 0$ for some $m \geq 4$.

From ii), it follows that for some $T \in \mathrm{GL}_3(\mathbb{C})$,

$$T^{\mathrm{t}}(\mathcal{H}_{x_1,x_2,x_m} g)T = \begin{pmatrix} * & * & * \\ * & 0 & 0 \\ * & 0 & 0 \end{pmatrix}$$

Let λ be a nonzero linear combination of Te_2 and Te_3 such that $\lambda_2 = 0$. Then $\lambda^{\mathrm{t}} \cdot \mathcal{H}_{x_1,x_2,x_m} g \cdot \lambda = 0$. Since $\frac{\partial^2}{\partial x_1^2} g \neq 0$, it follows that $\lambda_3 \neq 0$ Consequently, we can choose λ such that $\lambda_3 = 1$.

Notice that there exists another nonzero linear combination μ of Te_2 and Te_3 such that $\mu_3 = 0$. Since $\frac{\partial^2}{\partial x_1^2} g$ and $\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} g$ are linearly independent over $\mathbb{C}$ and $\frac{\partial^2}{\partial x_2^2} g = 0$, it follows that $\mu_1 = 0$ as well. So $\mathbb{C}e_2 + \mathbb{C}\lambda = \mathbb{C}\mu + \mathbb{C}\lambda = \mathbb{C}Te_2 + \mathbb{C}Te_3$. It follows that

$$S^{\mathrm{t}}\mathcal{H}_{x_1,x_2,x_m} g S = \begin{pmatrix} * & * & * \\ * & 0 & 0 \\ * & 0 & 0 \end{pmatrix}$$

as well, where $S := (e_1 \mid e_2 \mid \lambda)$.

So replace g by $g|_{x_1 = x_1 + \lambda_1 x_m} = g|_{x = Sx}$ to obtain that $\mathcal{H}_{x_1,x_2,x_m} g$ is of the form of the right hand side above. Notice that $\frac{\partial^2}{\partial x_1^2} g$ and $\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} g$ are only affected by an invertible linear substitution and therefore remain linearly independent over $\mathbb{C}$. Furthermore, $\mathrm{rk}\mathcal{H}_{x_2,x_3,\ldots,x_{m-1}} g = 0$ is preserved as well.

It follows that $\mathrm{rk}\mathcal{H}_{x_2,x_m} g = 0$. Take i such that $3 \leq i \leq m-1$. Looking at $\det \mathcal{H}_{x_1,x_2,x_i,x_m} g = 0$, we obtain

$$\left(\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} g\right)^2 \cdot \left(\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_m} g\right)^2 = 0$$

Since $\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} g \neq 0$, we obtain $\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_m} g = 0$. So $\mathrm{rk}\mathcal{H}_{x_2,x_3,\ldots,x_m} g = 0$.

By induction on m, we obtain $\mathrm{rk}\mathcal{H}_{x_2,x_3,\ldots,x_n} g = 0$. So g is of the desired form. $\square$

The following lemma is from 1876, at least the homogeneous version of it, where the operator E, and hence $f \mapsto Ef - f$ as well, comes down to an easy multiplication.

Lemma 5.3.6 (Gordan and Nöther)**.** *Assume $h \in \mathbb{C}[x]$ and $\det \mathcal{H}h = 0$. Then there exists a nonzero $R \in \mathbb{C}[y]$ such that $R(\nabla h) = 0$. Put $Q := (\nabla R)(\nabla h)$ and let $E := \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}$. Then $(Eh - h)(x + tQ) = (Eh - h)$. In particular, if h is homogeneous of degree ≥ 2, then $h(x + tQ) = h$.*

Proof. From proposition 1.2.9, it follows that an R as above exists. From proposition 3.1.9, we obtain that $x + Q$ is a quasi-translation. If $\mathcal{J}f \cdot Q = 0$, then by ii) of proposition 3.1.2

$$\begin{aligned}
\frac{\partial}{\partial t} f(x + tQ) &= \mathcal{J}f|_{x=x+tQ} \cdot Q \\
&= \mathcal{J}f|_{x=x+tQ} \cdot Q(x + tQ) \\
&= (\mathcal{J}f \cdot Q)|_{x=x+tQ} \\
&= 0
\end{aligned}$$

and hence $f(x+tQ) = f$. Taking $f = \frac{\partial}{\partial x_i} h$, we obtain $(\frac{\partial}{\partial x_i} h)(x+tQ) = \frac{\partial}{\partial x_i} h$ for all i. Now by the product rule of differentiating,

$$\mathcal{J}(Eh) = \mathcal{J}(x^{\mathrm{t}} \cdot \nabla h) = \mathcal{J}(\mathcal{J}h \cdot x) = x^{\mathrm{t}} \cdot \mathcal{J}\nabla h + \mathcal{J}h \cdot \mathcal{J}x = x^{\mathrm{t}}\mathcal{H}h + \mathcal{J}h$$

so $\mathcal{J}(Eh - h) = x^{\mathrm{t}}\mathcal{H}h$. By differentiating $(\nabla h)(x + tQ) = \nabla h$ with respect to t, we obtain $\mathcal{H}h \cdot Q = 0$. It follows that $\mathcal{J}(Eh - h) \cdot Q = x^{\mathrm{t}}\mathcal{H}h \cdot Q = 0$. Now take $f = Eh - h$ above to obtain the desired result. ☐

Theorem 5.3.7 (Gordan and Nöther)**.** *Assume $h \in \mathbb{C}[x]$ is homogeneous, $n = 5$ and $\det \mathcal{H}h = 0$. If h is not degenerate, then there exists a g that is linearly equivalent to h, such that*

$$g = f(x_1, x_2, a_1(x_1, x_2)x_3 + a_2(x_1, x_2)x_4 + a_3(x_1, x_2)x_5)$$

for some $f \in \mathbb{C}[x_1, x_2, t]$ and $a \in \mathbb{C}[x_1, x_2]^3$.

Proof. From theorem 5.2.10, it follows that it suffices to show that $s_h \geq 2$. Let $R \in \mathbb{C}[y]$ be homogeneous and nonzero, such that $R(\nabla h) = 0$. Choose R of minimal degree and put $Q := \nabla R(\nabla h)$.

If $\deg h \leq 1$, then $s_h = 4 \geq 2$, so assume that $\deg h \geq 2$. Then by the above lemma, $h(x + tQ) = h$. We distinguish two cases:

- The components of Q are linearly independent over $\mathbb{C}$.
  From theorem 3.5.4, it follows that $Q(\mathbb{C}^5)$ contains a point $p \neq 0$

such that the Zariski closure W of $Q(\mathbb{C}^5)$ is a union of projective lines through p. Assume without loss of generality that $p = e_5$. For each $q \in W$, we have $q + \lambda p \in W$ for all $\lambda \in \mathbb{C}$. It follows that Q_5 is algebraically independent of Q_1, Q_2, Q_3, Q_4.

Let A be the leading coefficient to x_5 of h, i.e.

$$h = Ax_5^m + \mathrm{O}(x_5^{m-1})$$

and $A \in \mathbb{C}[x_1, x_2, x_3, x_4] \backslash \{0\}$. Looking at the leading coefficient with respect to t in $h(x + tQ) = h$, we obtain $h(Q) = 0$. Since Q_5 is algebraically independent of Q_1, Q_2, Q_3, Q_4, $h(Q_1, Q_2, Q_3, Q_4, t) = 0$ follows. Again by looking at the leading coefficient with respect to t, we obtain $A(Q) = 0$.

Since $n = 5$, the fifth component of ∇A is zero. The fifth component of ∇h has degree $m - 1 < m$, so $x_5^m(\nabla A)$ is the leading part of ∇h with respect to x_5. So the leading coefficient with respect to x_5 of $R(\nabla h)$ is $R(\nabla A)$, whence $R(\nabla A) = 0$. From lemma 5.3.8 below, it follows that Q_1, Q_2, Q_3, Q_4 are linearly dependent over $\mathbb{C}$. Contradiction.

- The components of Q are linearly dependent over $\mathbb{C}$.
  Without loss of generality, we assume that $Q_5 = 0$. Now write

  $$h = Ax_5^m + Bx_5^{m+1}$$

  such that $A \in \mathbb{C}[x_1, x_2, x_3, x_4] \backslash \{0\}$ and $B \in \mathbb{C}[x]$. If $m = \deg h$, then trivially $s_h = 4 \geq 2$, so assume $\deg h - m \geq 1$. Since $Q_5 = 0$, it follows that $\deg_t h(x + tQ) \leq \deg h - m$. Looking at the corresponding leading coefficient with respect to t in $h(x + tQ) = h$, we obtain $x_5^m A(Q) = 0$, because $\deg h - m \geq 1$. So $A(Q) = 0$.

  Furthermore, we have $\frac{\partial}{\partial y_5} R = 0$, because we chose the degree of R minimal and $Q_5 = 0$. It follows that the bottom coefficient with respect to x_5 of $R(\nabla h) = 0$ equals $R(\nabla A) = 0$. From lemma 5.3.8 below, it follows that Q_1, Q_2, Q_3, Q_4 are linearly dependent. So there are two independent linear relations between the components of Q. Now apply proposition 5.2.7 to obtain $s_h \geq 2$. □

Lemma 5.3.8. *Let $n = 5$ and $h \in \mathbb{C}[x_1, x_2, x_3, x_4, x_5]$ be homogeneous and $R \in \mathbb{C}[y_1, y_2, y_3, y_4, y_5]$ be homogeneous and irreducible, such that $R(\nabla h) = 0$. Assume further that $A \in \mathbb{C}[x_1, x_2, x_3, x_4] \backslash \{0\}$ is homogeneous such that*

$R(\nabla A) = 0$ ($n = 5$, so the fifth component of ∇A is zero) and $A(Q) = 0$, where $Q = \nabla R(\nabla h)$. Then Q_1, Q_2, Q_3, Q_4 are linearly dependent over $\mathbb{C}$.

Proof. If $y_5 \mid R$, then it follows from the irreducibility of R and $Q = \nabla R(\nabla h)$ that $Q_1 = Q_2 = Q_3 = Q_4 = 0$. So assume that $y_5 \nmid R$. Since A is homogeneous and $R(\nabla A) = 0$, it follows from proposition 5.1.3 that the first four components of ∇A are linearly dependent over $\mathbb{C}$, say that $L(\nabla A) = 0$ for some linear form $L \in \mathbb{C}[y_1, y_2, y_3, y_4]$. Assume first that $\mathrm{rk}\mathcal{H}A = 3$. Then the relations between the components of ∇A form a prime ideal of height one, which is a principal ideal. Since L is irreducible, (L) must be that principal ideal, and $L \mid R$. Since R is irreducible it follows that R is linear. Consequently, Q is constant. In particular, Q_1, Q_2, Q_3, Q_4 are linearly dependent over $\mathbb{C}$.

So assume that $\mathrm{rk}\mathcal{H}A \le 2$. Since there exists a linear relation L between the components of ∇A, A can be expressed as a polynomial in three homogeneous linear coordinates. The rank of the (3×3)-Hessian with respect to these three linear coordinates of A cannot be larger than 2, since the rank of the original Hessian is at most 2. So this (3×3)-Hessian is singular as well. It follows from proposition 5.1.3 again that A can be expressed as a polynomial in two linear coordinates. Since A is homogeneous and bivariate, A decomposes into linear factors, and one of these factors is already a relation between Q_1, Q_2, Q_3, Q_4. $\square$

Now let $n = 5$ and assume that $h := f(x_1, x_2, a_1(x_1, x_2)x_3 + a_2(x_1, x_2)x_4 + a_3(x_1, x_2)x_5)$ is homogeneous, where $f \in \mathbb{C}[x_1, x_2, t]$, and define g by $g := (\frac{\partial}{\partial t} f)(x_1, x_2, a_1(x_1, x_2)x_3 + a_2(x_1, x_2)x_4 + a_3(x_1, x_2)x_5)$. Since $\mathrm{trdeg}_{\mathbb{C}}(a_1, a_2, a_3) \le 2$ and (a_1, a_2, a_3) is homogeneous, there exists a nonzero homogeneous $R \in \mathbb{C}[y_3, y_4, y_5]$ such that $R(a_1, a_2, a_3) = 0$. It follows that

$$R\left(\frac{\partial}{\partial x_3}h, \frac{\partial}{\partial x_4}h, \frac{\partial}{\partial x_5}h\right) = R\big(g \cdot (a_1(x_1, x_2), a_2(x_1, x_2), a_3(x_1, x_2))\big) = 0$$

Let r be the degree of R and $q := (\nabla_y R)(a_1, a_2, a_3)$. Then the Jacobian of $Q := (\nabla_y R)(\nabla h) = g^{r-1} q(x_1, x_2)$ has rank 2 at most. This was proved in 1876 already by P. Gordan and M. Nöther in [34, §8] in order to obtain theorem 5.3.7.

At last we will classify all homogeneous Hessians of rank 3. We start with investigating dimension 5.

Lemma 5.3.9. *Let $h \in \mathbb{C}[x]$ be homogeneous, $n = 5$ and $r := \operatorname{rk}\mathcal{H}h \le 3$. Then h is degenerate.*

Proof. Assume that h is not degenerate. Then we can choose

$$g = f(x_1, x_2, a_1(x_1, x_2)x_3 + a_2(x_1, x_2)x_4 + a_3(x_1, x_2)x_5)$$

for some $f \in \mathbb{C}[x_1, x_2, t]$, where a_1, a_2, a_3 are linearly independent over $\mathbb{C}$ and homogeneous of the same degree. Assume without loss of generality that $\deg_{x_2} a_2 > \deg_{x_2} a_1 > \deg_{x_2} a_3$. Then $\deg_{x_2} a_2 - \deg_{x_2} a_3 \ge 2$.
Let $d := \deg g$. Since $\operatorname{rk}\mathcal{H}h \le 3$, it follows that $g + x_3^d$ has a singular Hessian. Since $\deg_{x_3} g \le d - 2$, it follows that $\frac{\partial}{\partial x_3}(g + x_3^d)$ is linearly independent over $\mathbb{C}$ of the other components of $\nabla(g + x_3^d)$. So $g + x_3^d$ is not degenerate. It follows that

$$g + x_3^d = \tilde{f}(z_1, z_2, b_1(z_1, z_2)z_3 + b_2(z_1, z_2)z_4 + b_3(z_1, z_2)z_5]$$

where $z := Tx$ for some $T \in \mathrm{GL}_5(\mathbb{C})$. Now write $T^{-1} = LP\tilde{L}$, where L and $\tilde{L}$ are lower triangular and P is a permutation. This is possible, because one can get an invertible matrix on permutation form by row operations to below and column operations to the left only.
If we replace g by $g(Lx)$, then g keeps the form $g = f(x_1, x_2, a_1(x_1, x_2)x_3 + a_2(x_1, x_2)x_4 + a_3(x_1, x_2)x_5)$ and the property $\deg_{x_2} a_2 - \deg_{x_2} a_3 \ge 2$ in maintained as well. But $g + x_3^d$ *is* affected and becomes $\tilde{g} := g + (L_3x)^d$, where $L_3x = L_{31}x_1 + L_{32}x_2 + L_{33}x_3$ and $L_{33} \ne 0$.
Due to the above replacement of g by $g(Lx)$, T becomes $TL = \tilde{L}^{-1}P^{-1}$ and z becomes $\tilde{L}^{-1}P^{-1}x$. Furthermore,

$$\tilde{g} = \tilde{f}(z_1, z_2, b_1(z_1, z_2)z_3 + b_2(z_1, z_2)z_4 + b_3(z_1, z_2)z_5)$$

Since $\tilde{L}z = P^{-1}x$ and the form of $\tilde{g}$ above is not affected by substituting $z = \tilde{L}^{-1}z$, it follows that we may assume that $T = P^{-1}$. So z is a permutation of x.
Since $\deg_{x_2} a_2 - \deg_{x_2} a_3 \ge 2$, it follows that $\deg_{x_2} a_2 \ge \deg_{x_2} a_3 + 2 \ge 2$, and by $\deg a_3 = \deg a_2 \ge \deg_{x_2} a_2$, we obtain $\deg_{x_1} a_3 \ge \deg a_3 - \deg_{x_2} a_3 \ge \deg_{x_2} a_2 - \deg_{x_2} a_3 \ge 2$. Consequently,

$$\frac{1}{2}\deg_{x_2} \tilde{g} \ge \deg_{x_4} \tilde{g} = \deg_{x_5} \tilde{g} \le \frac{1}{2}\deg_{x_1} \tilde{g} \tag{5.5}$$

Notice that
$$\deg_{x_3} g = \deg_{x_4} g = \deg_{x_5} g < d$$
and similarly
$$\deg_{z_3} \tilde{g} = \deg_{z_4} \tilde{g} = \deg_{z_5} \tilde{g} < d$$
Since $\deg_{x_3} \tilde{g} = d$ it follows that $x_3 \in \{z_1, z_2\}$. So the degree with respect to three of the variables $\{x_1, x_2, x_4, x_5\}$ of $\tilde{g}$ is equal to each other. This contradicts (5.5), so h is degenerate, as desired. ☐

Notice that the below theorem, which classifies all homogeneous Hessians of rank 3 at most, is equivalent to theorem 5.1.5.

Theorem 5.3.10. *Let $h \in \mathbb{C}[x]$ be homogeneous and $r := \mathrm{rk}\mathcal{H}h \leq 3$. Then there exists a g that is linearly equivalent to h such that $g \in \mathbb{C}[x_1, x_2, \ldots, x_r]$.*

Proof. The case $n = r$ is trivial and the case $n = r + 1$ follows from proposition 5.3.2. So the cases $n = r + 2$ and $n \geq r + 3$ remain.

i) Assume that $n = r+2$. The case $n \leq 4$ follows by applying proposition 5.3.2 twice, so let $n = 5$. The above lemma tells us that h is degenerate. So we can choose g of the form $f(x_1, x_2, x_3, x_4)$. But since $\mathrm{rk}\mathcal{H}h \leq 3$, we obtain $\det \mathcal{H}_{x_1,x_2,x_3,x_4} f = 0$. It follows from proposition 5.3.2 that h is degenerate of order 2, as desired.

ii) Assume that $n \geq r + 3$. From lemma 5.3.4, it follows that we may assume that $\mathrm{rk}\mathcal{H}_{x_1,x_2,\ldots,x_r} g = r$. Notice that g is homogeneous as a polynomial in $K[x_1, x_2, \ldots, x_r, x_{r+1}, x_{r+2}]$, where K is the algebraic closure of $\mathbb{C}(x_{r+2}^{-1}x_{r+3}, x_{r+2}^{-1}x_{r+4}, \ldots, x_{r+2}^{-1}x_n)$, and that its corresponding Hessian $\mathcal{H}_{z_1,z_2,\ldots,z_r,z_{r+1},z_{r+2}} g$ has rank r as well, because $\frac{\partial}{\partial z_i} = \frac{\partial}{\partial x_i}$ for all $i \leq r + 1$ and
$$\frac{\partial}{\partial z_{r+2}} = \frac{1}{x_{r+2}} \left(x_{r+2} \frac{\partial}{\partial x_{r+2}} + x_{r+3} \frac{\partial}{\partial x_{r+3}} + \cdots + x_n \frac{\partial}{\partial x_n} \right)$$
So the row spaces of $\mathcal{H}_{x_1,x_2,\ldots,x_r,x_{r+1},z_{r+2}} g = \mathcal{H}_{z_1,z_2,\ldots,z_r,z_{r+1},z_{r+2}} g$ and $\mathcal{H}_x g$ are equal.

It follows from case i) that $\frac{\partial}{\partial x_{r+1}} g$ is linearly dependent over K of
$$\frac{\partial}{\partial x_1} g, \frac{\partial}{\partial x_2} g, \ldots, \frac{\partial}{\partial x_r} g$$

But since $\operatorname{rk}\mathcal{H}g = r = \operatorname{rk}\mathcal{H}_{x_1,x_2,\ldots,x_r}g$, it follows that $\frac{\partial}{\partial x_{r+1}}g$ is algebraically dependent over $\mathbb{C}$ of the above components of ∇g. Now reason as in ii) of the proof of proposition 5.2.5 to obtain that the above algebraic relation R over $\mathbb{C}$ and the above linear relation $\tilde{R}$ over K are essentially the same. So $\frac{\partial}{\partial x_{r+1}}g$ is linearly dependent over $\mathbb{C}$ of the first r components of ∇g. In a similar manner, it follows that $\frac{\partial}{\partial x_{r+i}}g$ is linearly dependent over $\mathbb{C}$ of the first r components of ∇g for all $i \geq 1$. This gives the desired result. □

5.4 Nilpotent Hessians of corank one

Definition 5.4.1. We call a vector v *isotropic* if $v^{\mathrm{t}}v = 0$. We call a row of a matrix isotropic if its transpose is isotropic.

If $h \in \mathbb{C}[x]$ such that $\operatorname{rk}\mathcal{H}h = n-1$, then there is essentially one dependence between the rows of $\mathcal{H}h$ on account of proposition 1.2.9. By applying the chain rule on $\mathcal{J}(R(\nabla h))$ as in the proof of lemma 1.2.8, we see that this dependence is $Q^{\mathrm{t}}\mathcal{H}h = 0$, where $Q = (\nabla R)(\nabla h)$ and $R \in \mathbb{C}[y]$ is (any polynomial that is divisible by) the essentially unique irreducible gradient relation of h.

This fact can be used to show that Q is isotropic as a vector in case the quasi-translation $x + Q$ comes from a nilpotent Hessian of corank one. In addition, one can show that there are two essentially different homogeneous relations between the components of such a Q in case $n \geq 3$.

Theorem 5.4.2. *Let $h \in \mathbb{C}[x]$ such that $\deg h \geq 2$ and $\operatorname{rk}\mathcal{H}h \leq n-1$. Then there exists a nonzero $R \in \mathbb{C}[y]$ such that $R(\nabla h) = 0$. Let $Q := (\nabla R)(\nabla h)$. Then $\bar{h}(tQ) = 0$, where $\bar{h}$ is the leading homogeneous part of h.*

If in addition $\mathcal{H}h$ is nilpotent and $\operatorname{rk}\mathcal{H}h = n-1$, then Q is isotropic as a vector and $\operatorname{trdeg}_{\mathbb{C}} tQ \leq \max\{n-2, 1\}$.

Proof.

i) From lemma 5.3.6, it follows that $E(h) - h$ is an invariant of $x + tQ$. Looking at the highest degree part with respect to t of $(E(h) - h)(x + tQ) - (E(h) - h)$, we obtain $\bar{h}(tQ) = 0$.

ii) Assume $\mathcal{H}h$ is nilpotent and $\mathrm{rk}\mathcal{H}h = n-1$. We show that Q is isotropic as a vector. Since $\mathrm{rk}\mathcal{H}h = n-1$ and $\mathcal{H}h \cdot Q = 0$, it follows that Q is dependent of every $v \neq 0$ for which $\mathcal{H}h \cdot v = 0$.

Let $M := (\mathcal{H}h)^m$, where $M \neq 0$ and $(\mathcal{H}h)^{m+1} = 0$. Choose a nonzero column v of M. Since $\mathcal{H}h \cdot M = 0$ it follows that $\mathcal{H}h \cdot v = 0$. Since M is symmetric and $M^2 = (\mathcal{H}h)^{2m} = 0$, we obtain that v is isotropic. So Q is isotropic as well.

iii) Assume $\mathrm{trdeg}_{\mathbb{C}} tQ \geq n-1$ and $n \geq 3$. Then the ideal of relations between components of tQ is principal and homogeneous, whence it is generated by an irreducible divisor of $\bar{h}$. Due to $n \geq 3$ we have that $\sum_{i=1}^n y_i^2$ is irreducible. Since Q is isotropic, it follows that the ideal of relations between components of tQ is generated by $\sum_{i=1}^n y_i^2$. We obtain that $\sum_{i=1}^n x_i^2 \mid \bar{h}$.

iv) We will derive a contradiction. Since $\mathcal{H}h$ is nilpotent, it follows that $\mathrm{tr}\mathcal{H}\bar{h} = 0$. We show that this is not the case. For that purpose, we write $\bar{h} = (\sum_{i=1}^n x_i^2)^r \cdot \tilde{h}$, where $\sum_{i=1}^n x_i^2 \nmid \tilde{h}$. Now

$$\frac{\partial}{\partial x_i}\bar{h} = 2rx_i\left(\sum_{i=1}^n x_i^2\right)^{r-1}\tilde{h} + \left(\sum_{i=1}^n x_i^2\right)^r \frac{\partial}{\partial x_i}\tilde{h}$$

whence

$$\begin{aligned}\frac{\partial^2}{\partial x_i^2}\bar{h} &= 2r\left(\sum_{i=1}^n x_i^2\right)^{r-1}\tilde{h} + 4(r-1)rx_i^2\left(\sum_{i=1}^n x_i^2\right)^{r-2}\tilde{h} + \\ &\quad 2\cdot 2rx_i\left(\sum_{i=1}^n x_i^2\right)^{r-1}\frac{\partial}{\partial x_i}\tilde{h} + \left(\sum_{i=1}^n x_i^2\right)^r\frac{\partial^2}{\partial x_i^2}\tilde{h}\end{aligned}$$

Notice that $\mathrm{tr}\mathcal{H}\bar{h} = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} h$. Consequently

$$\begin{aligned}\mathrm{tr}\mathcal{H}\bar{h} &= \big(2nr + 4r(r-1)\big)\left(\sum_{i=1}^n x_i^2\right)^{r-1}\tilde{h} + \\ &\quad 4r\left(\sum_{i=1}^n x_i^2\right)^{r-1}E\tilde{h} + \left(\sum_{i=1}^n x_i^2\right)^r \mathrm{tr}\mathcal{H}\tilde{h} \end{aligned} \tag{5.6}$$

where $E := \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}$. Since $E\tilde{h} = \deg\tilde{h}\cdot\tilde{h}$ and $\mathrm{tr}\mathcal{H}\bar{h} = 0$, it follows that $\sum_{i=1}^n x_i^2 \mid \tilde{h}$. Contradiction, so $\mathrm{trdeg}_{\mathbb{C}} tQ \leq n-2$. □

The operator $f \mapsto \mathrm{tr}\mathcal{H}f$ is called the Laplace operator. In the above proof, it is shown that polynomials that are divisible by $\sum_{i=1}^{n} x_i^2$ cannot be killed by the Laplace operator. A more general result can be found in [3, 5.5]. This result is over $\mathbb{R}$ instead of $\mathbb{C}$, but by decompositions in real and imaginary parts as in (2.4), one can get the same result over $\mathbb{C}$.

By the above theorem, we can immediately show that the rows of nilpotent Hessians in dimension 3 are dependent over $\mathbb{C}$. Furthermore, it provides a crucial step for a similar result in dimension 4.

Corollary 5.4.3. *Assume $h \in \mathbb{C}[x]$, and $\mathcal{H}h$ is nilpotent. If $n = 3$, then the rows of $\mathcal{H}h$ are dependent over $\mathbb{C}$. If $n = 4$, then $s_h \geq 1$.*

Proof. If $\mathrm{rk}\mathcal{H}h \leq n-2$, then $s_h \geq 1$ follows by elimination and we can apply theorem 5.1.4 to obtain the desired result for $n = 3$ as well. So assume $\mathrm{rk}\mathcal{H}h = n-1$. Then the above theorem gives $\mathrm{trdeg}_{\mathbb{C}} tQ \leq n-2$. Say that Q has degree r and define

$$\tilde{Q} := x_{n+1}^r Q|_{x = x_{n+1}^{-1} x}$$

Then $\tilde{Q}$ is the homogenization of Q and

$$\mathrm{rk}\mathcal{J}_{x,x_{n+1}} \tilde{Q} = \mathrm{trdeg}_{\mathbb{C}} \tilde{Q} \leq \mathrm{trdeg}_{\mathbb{C}} tQ \leq n-2$$

We distinguish the cases $n = 3$ and $n = 4$ to prove the corresponding claims.

i) Assume $n = 3$. Then $\mathrm{rk}\mathcal{J}\tilde{Q} = 1$ and hence the linear span of the image of $\tilde{Q}$ has dimension 1. So the linear span of the image of $(Q, 0) = \tilde{Q}|_{x_{n+1}=1}$ has dimension 1 as well. Now apply proposition 5.2.7 to obtain that we can choose the gradient relation R degenerate of order $n-1$. So there exists a linear combination of the components of ∇h that is dependent over $\mathbb{C}$ and hence contained in $\mathbb{C}$.

ii) Assume $n = 4$. Then $\mathrm{rk}\mathcal{J}\tilde{Q} = 2$ and hence by theorem 3.4.1, the linear span of the image of $\tilde{Q}$ has dimension $5-2 = 3 = n-1$ at most. Now apply proposition 5.2.7 to obtain that we can choose the gradient relation R degenerate. So $s_h \geq 1$. □

Definition 5.4.4. Let $h \in \mathbb{C}[x]$. We call g *orthogonally equivalent* to h if there exists a $T \in \mathrm{GO}_n(\mathbb{C})$ such that $g = h(Tx)$. We call g *symmetrically equivalent* to h if there exists a permutation P such that $g = h(Px)$.

The below theorem exploits the fact that Q is isotropic as a vector. We shall use this theorem later. Although the use of the theorem below is not clear yet, observe that the right hand side of the displayed formula is an algebra of $n - s$ linear forms. The theorem says that this number of linear forms is larger than necessary in case less than three of these linear forms are in the second line of the algebra.

Theorem 5.4.5. *Assume $\mathcal{H}h$ is nilpotent of corank 1 and $R(\nabla g) = 0$ for some g that is orthogonally equivalent to h and a nonzero*

$$R \in \mathbb{C}[y_{r+1} + \mathrm{i}y_{s+1}, y_{r+2} + \mathrm{i}y_{s+2}, \ldots, y_s + \mathrm{i}y_{2s-r}, \\ y_{2s-r+1}, y_{2s-r+2}, \ldots, y_n]$$

If either $n = 2s - r + 1$ or $n = 2s - r + 2$ (i.e. the line above has 1 or 2 linear forms), then $s_h > s$.

Proof. Choose R of minimal degree. Put $Q := (\nabla R)(\nabla g)$. From theorem 5.4.2, it follows that Q is isotropic as a vector. But from the form of R, it follows that $(\nabla R)_j = 0$ for all $j \le r$ and $(\nabla R)_{r+j} = \mathrm{i}(\nabla R)_{s+j}$ for all j with $1 \le j \le s - r$. So $Q_j = 0$ for all $j \le r$ and $Q_{r+j} = \mathrm{i}Q_{s+j}$ for all j with $1 \le j \le s - r$. It follows that $(Q_{2s-r+1}, Q_{2s-r+2}, \ldots, Q_n)$ is isotropic as well.

Assume first that $n = 2s - r + 1$. Then (Q_{2s-r+1}) is isotropic, so $Q_{2s-r+1} = 0$. Since R was chosen of minimal degree, it follows that $\frac{\partial}{\partial y_{2s-r+1}} R = 0$. So $s_h > s$, as desired.

Assume next that $n = 2s - r + 2$. Then (Q_{2s-r+1}, Q_{2s-r+2}) is isotropic, so $Q_{2s-r+1} \pm \mathrm{i}Q_{2s-r+2} = 0$. Since R was chosen of minimal degree, it follows that $(\frac{\partial}{\partial y_{2s-r+1}} \pm \mathrm{i}\frac{\partial}{\partial y_{2s-r+2}})R = 0$. So $s_h > s$, as desired. □

5.5 Orthogonal transformations

Assume $h \in \mathbb{C}[x]$ such that $\mathcal{H}h$ is nilpotent. If $T \in \mathrm{GL}_n(\mathbb{C})$, then $g := h(Tx)$ does not need to have a nilpotent Hessian. Take for instance $h = (x_1 + \mathrm{i}x_2)^2$ and $g = x_1^2$. But from (5.2), it follows that g does have a nilpotent Hessian if $T \in \mathrm{GO}_n(\mathbb{C})$. So we can transform, but not as freely as with singular Hessians.

In order to understand what we can do with orthogonal transformations only, we formulate a proposition.

Proposition 5.5.1. *Assume b is a symmetric bilinear form over a finite dimensional vector space V over $\mathbb{C}$. Let $v_1, v_2, \ldots, v_n$ be a basis of V. Then b is uniquely determined by the symmetric matrix B defined by $B_{ij} := b(v_i, v_j)$. Let $r := \mathrm{rk}B$. Then (V, b) is isomorphic to*

$$\mathbb{C}e_1, \mathbb{C}e_2, \ldots, \mathbb{C}e_r, \mathbb{C}(e_{r+1} + \mathrm{i}e_{n+1}), \mathbb{C}(e_{r+2} + \mathrm{i}e_{n+2}), \ldots, \mathbb{C}(e_n + \mathrm{i}e_{2n-r})$$

with the standard symmetric bilinear form $\cdot^{\mathrm{t}}\cdot$. Furthermore, (V, b) can be embedded in $(\mathbb{C}^m, \cdot^{\mathrm{t}}\cdot)$ for any $m \geq 2n - r$. In addition, two such embeddings ϕ and ψ are orthogonally equivalent, i.e. there exists a $T \in \mathrm{GO}_m(\mathbb{C})$ such that $\psi(v_i) = T\phi(v_i)$ for all i.

Proof. Since B is symmetric and $\mathrm{rk}B = r$, we can write

$$SBS^{\mathrm{t}} = \mathrm{diag}(1^1, 1^2, \ldots, 1^r, 0^{r+1}, 0^{r+2}, \ldots, 0^n)$$

for some $S \in \mathrm{GL}_n(\mathbb{C})$. It follows that

$$b\left(\sum_{k=1}^{n} S_{ik}v_k, \sum_{k=1}^{n} S_{jk}v_k\right) = \sum_{k=1}^{n}\sum_{l=1}^{n} S_{ik}S_{jl}B_{kl} = (SBS^{\mathrm{t}})_{ij}$$

so by changing the basis v by

$$\sum_{k=1}^{n} S_{1k}v_k, \sum_{k=1}^{n} S_{2k}v_k, \ldots, \sum_{k=1}^{n} S_{nk}v_k$$

we obtain $B = \mathrm{diag}(1^1, 1^2, \ldots, 1^r, 0^{r+1}, 0^{r+2}, \ldots, 0^n)$.

i) Take $\phi(v_i) = e_i$ for all $i \leq r$ and $\phi(v_{r+i}) = e_{r+i} + \mathrm{i}e_{n+i}$ for all $i \geq 1$. Then one can easily show that ϕ induces an isomorphism of (V, b) and

$$\mathbb{C}e_1, \mathbb{C}e_2, \ldots, \mathbb{C}e_r, \mathbb{C}(e_{r+1} + \mathrm{i}e_{n+1}), \mathbb{C}(e_{r+2} + \mathrm{i}e_{n+2}), \ldots, \mathbb{C}(e_n + \mathrm{i}e_{2n-r})$$

with the standard bilinear form. Assume $m \geq 2n - r$ and let ϕ be the corresponding embedding in $\mathbb{C}^{2n-r} \subseteq \mathbb{C}^m$.

ii) Assume ψ is another embedding of (V, b) in $\mathbb{C}^m$. Let $w_i := \psi(v_i)$. We must show that $w_i = Te_i$ for all $i \leq r$ and $w_{r+i} = T(e_{r+i} + \mathrm{i}e_{n+i})$ for all $i \geq 1$, for some $T \in \mathrm{GO}_m(\mathbb{C})$. Now $w_i = Te_i$ for all $i \leq r$ determines the first r columns of T. Notice that these r columns are orthonormal and that they are orthogonal to w_{r+i} for all $i \geq 1$.

Now assume by induction that we have columns

$$1, 2, \dots, r, r+1, n+1, r+2, n+2, \dots, r+i-1, n+i-1$$

of T, i.e. these columns are orthonormal and orthogonal to w_{r+j} for all $j \geq i$, and $w_{r+j} = T(e_{r+j} + \mathrm{i} e_{n+j})$ for all $j < i$. Notice that since w_i is independent of $w_{i+1}, w_{i+2}, \dots, w_n$, the dimension of the space that is orthogonal to $w_{i+1}, w_{i+2}, \dots, w_n$ (a subspace of $\mathbb{C}^m$) is one bigger than that to $w_i, w_{i+1}, w_{i+2}, \dots, w_n$. Now let t_i be a vector of the first space that is not contained in the second space. Define

$$\tilde{t}_i := t_i - \sum_{j=1}^{r+i-1} \left((Te_j)^{\mathrm{t}} t_i\right) Te_j - \sum_{j=n+1}^{n+i-1} \left((Te_j)^{\mathrm{t}} t_i\right) Te_j$$

then $\tilde{t}_i$ is orthogonal to the $r + 2(i-1)$ columns of T we have thus far. Furthermore, $\tilde{t}_i$ is contained in the space that is orthogonal to w_{i+1}, $w_{i+2}, \dots, w_n$, but not in the space that is orthogonal to w_i, w_{i+1}, w_{i+2}, $\dots, w_n$. Since $w_i^{\mathrm{t}} \tilde{t}_i \neq 0$, we define the $(r+i)$-th column of T by

$$Te_{r+i} := \frac{1}{2 w_i^{\mathrm{t}} \tilde{t}_i} \left(2\tilde{t}_i + \left(\left(w_i - \frac{1}{w_i^{\mathrm{t}} \tilde{t}_i} \tilde{t}_i \right)^{\mathrm{t}} \tilde{t}_i \right) w_i \right)$$

to obtain by $\tilde{t}_i^{\mathrm{t}} w_i = w_i^{\mathrm{t}} \tilde{t}_i$ and $w_i^{\mathrm{t}} w_i = 0$ that

$$(Te_{r+i})^{\mathrm{t}} Te_{r+i} = \frac{1}{4 (w_i^{\mathrm{t}} \tilde{t}_i)^2} \left(4 \tilde{t}_i^{\mathrm{t}} \tilde{t}_i + 4 \left(\left(w_i - \frac{1}{w_i^{\mathrm{t}} \tilde{t}_i} \tilde{t}_i \right)^{\mathrm{t}} \tilde{t}_i \right) w_i^{\mathrm{t}} \tilde{t}_i \right) = 1$$

and

$$Te_{n+i} := -\mathrm{i}(w_i - Te_{r+i})$$

to obtain that $w_i = T(e_{r+i} + \mathrm{i} e_{n+i})$.

One can easily show that the induction hypotheses are preserved, so there exists a $T \in \mathrm{Mat}_{m,2n-r}(\mathbb{C})$ with orthonormal columns such that $\psi(v_i) = T\phi(v_i)$ for all i. Now extend T to an orthogonal matrix to get the desired result. □

Corollary 5.5.2. *Assume $M \in \mathrm{Mat}_{s,n}$ has rank s. Then M decomposes as*

$$M = SUT$$

where S is a square matrix, T is orthogonal and

$$U = \left(\begin{array}{c|c|c} I_s & \begin{matrix} \emptyset \\ \mathrm{i} I_r \end{matrix} & \emptyset \end{array} \right)$$

where $r \leq \min\{s, n-s\}$.

Proof. let V be the row space of M. Then by the above proposition, we obtain that $(V, \cdot\cdot^{\mathrm{t}})$ is isomorphic to the row space of U with the same bilinear form. So there exists a $T \in \mathrm{GO}_n(\mathbb{C})$ such that

$$V = \mathbb{C}U_1T + \mathbb{C}U_2T + \cdots + \mathbb{C}U_sT$$

whence $M_i = S_iUT$ for all i for a suitable square matrix S, as desired. $\square$

The following theorem shows a connection between linear and orthogonal equivalence. In fact, the connection is that each 'formula of linear reduction' has $s+1$ 'orthogonal cases', namely $r = 0, 1, \dots, s-1, s$.

Theorem 5.5.3. *Assume h is linearly equivalent to a polynomial of the form*

$$f\Big(x_1, x_2, \dots, x_s, \textstyle\sum_{i=1}^{n-s} a_i(x_1, x_2, \dots, x_s) \cdot x_{s+i},$$
$$\textstyle\sum_{i=1}^{n-s} b_i(x_1, x_2, \dots, x_s) \cdot x_{s+i}\Big)$$

Then h is orthogonally equivalent to a polynomial of the form

$$\tilde{f}\Big(x_1, \dots, x_r, x_{r+1} + \mathrm{i}x_{s+1}, \dots, x_s + \mathrm{i}x_{2s-r},$$
$$\textstyle\sum_{i=1}^{n-s} \tilde{a}_i(x_1, \dots, x_r, x_{r+1} + \mathrm{i}x_{s+1}, \dots, x_s + \mathrm{i}x_{2s-r}) \cdot x_{s+i},$$
$$\textstyle\sum_{i=1}^{n-s} \tilde{b}_i(x_1, \dots, x_r, x_{r+1} + \mathrm{i}x_{s+1}, \dots, x_s + \mathrm{i}x_{2s-r}) \cdot x_{s+i}\Big)$$

Furthermore, the degrees of f and $\tilde{f}$ with respect to their last two arguments are the same.

Proof. Assume

$$h = f\Big(M_1x, M_2x, \dots, M_sx, \textstyle\sum_{i=1}^{n-s} a_i(M_1x, M_2x, \dots, M_sx) \cdot M_{s+i}x,$$
$$\textstyle\sum_{i=1}^{n-s} b_i(M_1x, M_2x, \dots, M_sx) \cdot M_{s+i}x\Big)$$

where $M \in \mathrm{GL}_n(\mathbb{C})$. Let $\tilde{M}$ be the matrix consisting of the first s rows of M. From corollary 5.5.2, we obtain that $\tilde{M} = SUT$, where S, U and T are as in that corollary. Now replace h by $h(T^{-1}x)$ and f by $f(S\tilde{x}, x_{s+1}, x_{s+2}, \ldots, x_n)$, where $\tilde{x} = x_1, x_2, \ldots, x_s$, to obtain that

$$h = f\left(Ux, \sum_{i=1}^{n-s} a_i(Ux) \cdot M_{s+i}T^{-1}x, \sum_{i=1}^{n-s} b_i(Ux) \cdot M_{s+i}T^{-1}x\right)$$

Since for all $i \geq 1$,

$$M_{s+i}T^{-1}x \in \mathbb{C}[x_1, \ldots, x_r, x_{r+1} + \mathrm{i}x_{s+1}, \ldots, x_s + \mathrm{i}x_{2s-r}] + \mathbb{C}x_{s+1} + \cdots + \mathbb{C}x_n$$

we can write h in the desired form. □

5.6 Invertibility properties of degenerate gradient maps

In this section, we shall show that the invertibility of gradient maps for which the non-linear part is degenerate is equivalent to the invertibility of another map in lower dimension. Since every symmetric matrix of full rank can be expressed as $S^{\mathrm{t}}S$, it follows that we may assume that the linear part of the gradient map is equal to x.

Proposition 5.6.1. *Assume $\tilde{f}$ is linearly equivalent to f. Then $\nabla\tilde{f}$ is invertible, if and only if ∇f is invertible.*
Furthermore, if $\tilde{f} = f(Tx)$ and the linear part of ∇f is x, then the linear part of $\nabla\tilde{f}$ is x as well, if and only if T is orthogonal.

Proof. Assume $\tilde{f} = f(Tx)$. The first assertion follows from (5.1): $\nabla\tilde{f} = T^{\mathrm{t}}(\nabla f)(Tx)$. The second assertions follows from (5.2): $\mathcal{H}\tilde{f} = T^{\mathrm{t}}\mathcal{H}f|_{x=Tx}$ and the fact the the Jacobian of x is the identity matrix. □

Assume $h \in \mathbb{C}[x]$ such that ∇h is degenerate, i.e. the rows of $\mathcal{J}(\nabla h) = \mathcal{H}h$ are dependent over $\mathbb{C}$, say $\lambda^{\mathrm{t}}\mathcal{H}h = 0$. Then ∇h is not invertible, because its Jacobian determinant is zero. So let us look at the invertibility of $\nabla f = x + \nabla h$, where $f = h + \frac{1}{2}\sum_{i=1}^{n} x_i^2$. In order to preserve the nilpotency of $\mathcal{H}h$, we may only use orthogonal transformations in general. But in case we do not assume that $\mathcal{H}h$ is nilpotent, we need orthogonality in order to preserve the linear part x of ∇f.

Theorem 5.6.2. *Assume $\lambda^{\mathrm{t}}\mathcal{H}h = 0$ for some $h \in \mathbb{C}[x]$. If λ is not isotropic, then there exists a g that is orthogonally equivalent to h such that*

$$e_n^{\mathrm{t}}\mathcal{H}g = 0$$

Furthermore, $x + \nabla g$ is a(n invertible) Keller map, if and only if $\tilde{x} + \nabla_{\tilde{x}} g$ is a(n invertible) Keller map over $\mathbb{C}$, where $\tilde{x} = x_1, x_2, \ldots, x_{n-1}$.

If λ is isotropic and nonzero, then there exists a g that is orthogonally equivalent to h such that

$$(e_1 + \mathrm{i}e_n)^{\mathrm{t}}\mathcal{H}g = 0$$

Furthermore, $x + \nabla g$ is a(n invertible) Keller map, if and only if $\hat{x} + \nabla_{\hat{x}} g$ is a(n invertible) Keller map over $\mathbb{C}[x_1 + \mathrm{i}x_n]$, where $\hat{x} = x_2, x_3, \ldots, x_{n-1}$.

Proof. Take $T \in \mathrm{GO}_n(\mathbb{C})$ such that $T^{-1}\lambda$ is either dependent of e_n or equal to $e_1 + \mathrm{i}e_n$, and put $g := h(Tx)$. If λ is not isotropic, then $e_n^{\mathrm{t}}\mathcal{H}g = e_n^{\mathrm{t}}T^{\mathrm{t}}\mathcal{H}h|_{x=Tx}T = 0$. If λ is isotropic, then $(e_1 + \mathrm{i}e_n)^{\mathrm{t}}T^{\mathrm{t}} = \lambda^{\mathrm{t}}$ and hence $(e_1 + \mathrm{i}e_n)^{\mathrm{t}}\mathcal{H}g = 0$.

If $e_n^{\mathrm{t}}\mathcal{H}g = 0$, then $x + \nabla g$ is in fact an $(n-1)$-dimensional polynomial map up to an elementary translation on the last coordinate. This gives the desired result for the case that λ is not isotropic.

So assume that $(e_1 + \mathrm{i}e_n)^{\mathrm{t}}\mathcal{H}g = 0$. Let

$$S := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & \mathrm{i} \\ 0 & & & & & \\ \vdots & & & I_{n-1} & & \\ 0 & & & & & \end{pmatrix} \tag{5.7}$$

Since $\nabla g \in \mathbb{C}[x_1 + \mathrm{i}x_n, x_2, \ldots, x_{n-2}, x_{n-1}]^n$, it follows that $(\nabla g)|_{x=S^{-1}x} = (\nabla g)|_{x_n=0}$. Furthermore, $\mu := (\frac{\partial}{\partial x_1} + \mathrm{i}\frac{\partial}{\partial x_n})g \in \mathbb{C}$, whence

$$S(\nabla g)|_{x=S^{-1}x} = \begin{pmatrix} \mu \\ (\nabla_{\hat{x},x_n} g)|_{x_n=0} \end{pmatrix}$$

The first component can be cleared by applying an elementary translation from the left to $x + S(\nabla g)|_{x=S^{-1}x}$ and the last component of $S\nabla g|_{x=S^{-1}x}$ by applying an elementary map from the right to $x + S(\nabla g)|_{x=S^{-1}x}$. This gives the desired result. $\square$

The notion of strong nilpotency was introduced by G.H. Meisters and C. Olech. In [25], another definition of strong nilpotency is given by A.R.P. van den Essen and E.-M.G.M. Hubbers: see also [24, §7.4]. In order to make the disorder complete, here is a third definition.

Definition 5.6.3. Let A be a commutative ring and $H \in A[x]^n$ be a polynomial map with nilpotent Jacobian. We call $\mathcal{J}H$ *strongly nilpotent* if there exists an $m \in \mathbb{N}$ such that

$$\mathcal{J}H|_{x=y^{(1)}} \cdot \mathcal{J}H|_{x=y^{(2)}} \cdot \cdots \cdot \mathcal{J}H|_{x=y^{(m)}} = 0$$

where $y^{(i)} = y_1^{(i)}, y_2^{(i)}, \ldots, y_n^{(i)}$ is a tuple of n indeterminates for each i.

The definition of Hubbers and Van den Essen differs from the above in that m must be n. The below proposition shows that this does not matter as long as the base ring is reduced.

In the definition of G.H. Meisters and C. Olech, m must be n as well, but in addition, the $y^{(i)}$ must be vectors over the base ring. So this definition is equivalent to that of Hubbers and van den Essen in case the base ring contains $\mathbb{Q}$. Since m does not need to be n in the above definition, the above definition is more similar to the definition of nilpotent.

Proposition 5.6.4. *If A is a reduced commutative ring and $H \in A[x]^n$ such that $\mathcal{J}H$ is (strongly) nilpotent, then one can take $n = m$ in the definition of (strongly) nilpotent, i.e. $\mathcal{J}H^n = 0$ (and*

$$\mathcal{J}H|_{x=y^{(1)}} \cdot \mathcal{J}H|_{x=y^{(2)}} \cdot \cdots \cdot \mathcal{J}H|_{x=y^{(n)}} = 0$$

where $y^{(i)} = y_1^{(i)}, y_2^{(i)}, \ldots, y_n^{(i)}$ is a tuple of n indeterminates for each i).

Proof. Since A is reduced, it suffices to show the assertion for $A/\mathfrak{p}$, where $\mathfrak{p}$ runs through the prime spectrum of A. So we may assume that A is an integral domain, i.e. A is contained in a field.

Now the result follows from the following fact: if adding a factor $\mathcal{J}H$ or $\mathcal{J}H|_{x=y^{(m+1)}}$ to the right of a product $\mathcal{J}H^m$ or

$$\mathcal{J}H|_{x=y^{(1)}} \cdot \mathcal{J}H|_{x=y^{(1)}} \cdot \cdots \cdot \mathcal{J}H|_{x=y^{(m)}}$$

does not decrease the rank of the row space of the product, then the row space is stationary under adding such factors to the right. $\square$

Proposition 5.6.5. *Assume $\mathcal{J}H$ is a square matrix with only zeros in the submatrix of the leading m rows and the rightmost $n-m$ columns. Then $\mathcal{J}H$ is (strongly) nilpotent, if and only if the leading principal minor of size m and the complementary principal minor of size $n-m$ of $\mathcal{J}H$ are (strongly) nilpotent.*

Proof. We only prove the nilpotent case, since the strongly nilpotent case is similar except that the formulas are annoyingly complicated. Let M be the leading principal minor of size m of $\mathcal{J}H$ and N be the complementary principal minor of $\mathcal{J}H$. Then

$$\mathcal{J}H^r = \begin{pmatrix} M^r & \emptyset \\ * & N^r \end{pmatrix}$$

Now the forward implication is trivial The backward implication follows from the fact that $M^r = N^r = 0$ implies that $\mathcal{J}H^r \cdot \mathcal{J}H^r = 0$. □

The following theorem shows that for (strong) nilpotency, there are similar results as for invertibility.

Theorem 5.6.6. *Let $g \in \mathbb{C}[x]$. If $e_n^{\mathrm{t}}\mathcal{H}g = 0$, then $\mathcal{H}g$ is (strongly) nilpotent, if and only if $\mathcal{H}_{\tilde{x}}g$ is (strongly) nilpotent, where $\tilde{x} = x_1, x_2, \ldots, x_{n-1}$.*
If $(e_1 + \mathrm{i}e_n)^{\mathrm{t}}\mathcal{H}g = 0$, then $\mathcal{H}g$ is (strongly) nilpotent, if and only if $\mathcal{H}_{\hat{x}}g$ is (strongly) nilpotent (over $\mathbb{C}(x_1 + \mathrm{i}x_n)$), where $\hat{x} = x_2, x_3, \ldots, x_{n-1}$.

Proof. The first assertion follows from the above proposition with $m = n-1$. For the second proposition, notice that the nilpotency of $\mathcal{H}g$ is equivalent to that of $M := S\mathcal{J}((\nabla g)|_{x=S^{-1}x})$, where S is as in (5.7) on page 145. Next apply the above proposition, with $m = n-1$ on M and $m = 1$ on the leading principal minor of size $n-1$ of M. □

If $g \in \mathbb{C}[x]$ and $(e_{n-1}+\mathrm{i}e_n)^{\mathrm{t}}\mathcal{H}g = 0$, then the component of ∇g are contained in $\mathbb{C}[x_1, x_2, \ldots, x_{n-2}, x_{n-1}+\mathrm{i}x_n]$ and hence, the substitution $x_{n-1}+\mathrm{i}x_n = 1$ in ∇g or $\mathcal{H}g$ makes sense. Such a substitution in g directly might be ambiguous for the linear part of g, but why worry when you delete this linear part by taking the Hessian, as in the corollary below?

Corollary 5.6.7. *Let $g \in \mathbb{C}[x]$. If $(e_{n-1} + \mathrm{i}e_n)^{\mathrm{t}}\mathcal{H}g = 0$ and g is homogeneous, then $\mathcal{H}g$ is (strongly) nilpotent, if and only if $\mathcal{H}(g|_{x_{n-1}+\mathrm{i}x_n=1})$ is (strongly) nilpotent.*

Proof. Again, we only prove the nilpotent case. Notice that $\mathcal{H}(g|_{x_{n-1}+\mathrm{i}x_n=1})$ can be obtained from $\mathcal{H}_{\hat{x}}g$ by substitutions and adding zero rows and zero columns, where $\hat{x} = x_1, x_2, \ldots, x_{n-2}$. This and a permuted version of the above theorem give the forward implication.

The backward implication follows by substituting $t = (x_{n-1} + \mathrm{i}x_n)^{d-2}$ and $\hat{x} = (x_{n-1} + \mathrm{i}x_n)^{-1}\hat{x}$ in $t\mathcal{H}(g|_{x_{n-1}+\mathrm{i}x_n=1})$, where $d = \deg g$. □

Example 5.6.8. Notice that $(x_4 + \mathrm{i}x_5)^2 - 1, 2(x_4 + \mathrm{i}x_5), (x_4 + \mathrm{i}x_5)^2 + 1$ is a Pythagorean triple. Therefore the polynomial

$$g_1 := \Big(\big((x_4 + \mathrm{i}x_5)^2 - 1\big)x_1 + 2(x_4 + \mathrm{i}x_5)x_2 + \mathrm{i}\big((x_4 + \mathrm{i}x_5)^2 + 1\big)x_3\Big)^2$$

has a nilpotent Hessian, for it is orthogonally equivalent to $(x_1 + \mathrm{i}x_2)^2$ over $\mathbb{C}(x_4 + \mathrm{i}x_5)$. The polynomial

$$\begin{aligned} g_2 := {} & \Big(\big((x_4 + \mathrm{i}x_5)^2 - 1\big)x_1 + 2(x_4 + \mathrm{i}x_5)x_2 + \mathrm{i}\big((x_4 + \mathrm{i}x_5)^2 + 1\big)x_3\Big) \cdot \\ & \big(2(x_4 + \mathrm{i}x_5)x_1 - \big((x_4 + \mathrm{i}x_5)^2 - 1\big)x_2\big) \end{aligned}$$

is orthogonally equivalent to $(x_1 + \mathrm{i}x_2)x_3 = ((x_4 + \mathrm{i}x_5)^2 + 1)^{-1}(x_1 + \mathrm{i}x_2) \cdot ((x_4 + \mathrm{i}x_5)^2 + 1)x_3$ over $\mathbb{C}(x_4 + \mathrm{i}x_5)$. In the context of corollary 5.6.7, the homogenizations of these polynomials are

$$\begin{aligned} g_3 := {} & \Big(\big((x_4 + \mathrm{i}x_5)^2 - (x_6 + \mathrm{i}x_7)^2\big)x_1 + 2(x_4 + \mathrm{i}x_5)(x_6 + \mathrm{i}x_7)x_2 + \\ & \mathrm{i}\big((x_4 + \mathrm{i}x_5)^2 + (x_6 + \mathrm{i}x_7)^2\big)x_3\Big)^2 \end{aligned}$$

and

$$\begin{aligned} g_4 := {} & \Big(\big((x_4 + \mathrm{i}x_5)^2 - (x_6 + \mathrm{i}x_7)^2\big)x_1 + 2(x_4 + \mathrm{i}x_5)(x_6 + \mathrm{i}x_7)x_2 + \\ & \mathrm{i}\big((x_4 + \mathrm{i}x_5)^2 + (x_6 + \mathrm{i}x_7)^2\big)x_3\Big) \cdot \\ & \big(2(x_4 + \mathrm{i}x_5)(x_6 + \mathrm{i}x_7)x_1 - \big((x_4 + \mathrm{i}x_5)^2 - (x_6 + \mathrm{i}x_7)^2\big)x_2\big) \end{aligned}$$

Next

$$g_5 := g_3 + (x_6 + \mathrm{i}x_7)^5 x_4 \qquad \text{and} \qquad g_6 := g_4 + (x_6 + \mathrm{i}x_7)^5 x_4$$

have nilpotent Hessians as well, since they are orthogonally equivalent to $g_1 + x_4$ and $g_2 + x_4$ respectively over $\mathbb{C}(x_6 + \mathrm{i}x_7)$.

By generic substitutions, one can see that $\mathrm{rk}\mathcal{H}g_i \geq 3 + \lfloor i/2 \rfloor$ for all i. Checking the gradient relations $y_1^2 + y_2^2 + y_3^2$, $y_4^2 + y_5^2$ and $y_6^2 + y_7^2$, for each of the g_i ($y_6^2 + y_7^2$ only for $i \geq 3$), we obtain $\mathrm{trdeg}_{\mathbb{C}} \nabla g_i \leq 3 + \lfloor i/2 \rfloor$, so by proposition 1.2.9,

$$\mathrm{rk}\mathcal{H}g_i = 3 + \lfloor i/2 \rfloor$$

for all i. In section 5.8 (in the proof of theorem 5.8.6 to be precise), we shall show that the Jacobians of these maps are not strongly nilpotent.

5.7 The dependence problem for Hessians

In chapter 2 (corollary 2.2.6), we have seen that the unipotent and homogeneous dependence problem for Hessians in dimension $2n$ has a negative answer in case the unipotent and homogeneous dependence problem has a negative answer in dimension n respectively. With the knowledge that the unipotent and homogeneous dependence problem have been debunked in dimension 3 and 5 respectively, we obtain that the unipotent and homogeneous dependence problem for Hessians have a negative answer in dimension 6 and 10 respectively.

In this section, we will show that the unipotent and homogeneous dependence problem for Hessians has an affirmative answer in some cases. These cases are summarized in the theorem below.

Theorem 5.7.1. *Assume $h \in \mathbb{C}[x]$ such that $\mathcal{H}h$ is nilpotent. Then the rows of $\mathcal{H}h$ are linearly dependent over $\mathbb{C}$ if either $n \leq 4$ or $\mathrm{rk}\mathcal{H}h \leq 2$.*

If h is homogeneous in addition, then the rows of $\mathcal{H}h$ are linearly dependent over $\mathbb{C}$ if either $n \leq 5$ or $\mathrm{rk}\mathcal{H}h \leq 3$.

Corollary 5.7.2. *Assume $h \in \mathbb{C}[x]$ such that $\mathcal{H}h$ is nilpotent and $n \geq 2$. If the rows of $\mathcal{H}h$ are dependent over $\mathbb{C}$, and either $n \leq 5$ or $\mathrm{rk}\mathcal{H}h \leq 2$, then there exists a nonzero isotropic $\lambda \in \mathbb{C}^n$ such that $\lambda^{\mathrm{t}}\mathcal{H}h = 0$.*

If h is homogeneous and the rows of $\mathcal{H}h$ are dependent over $\mathbb{C}$, and either $n \leq 6$ or $\mathrm{rk}\mathcal{H}h \leq 3$, then there exists a nonzero isotropic $\lambda \in \mathbb{C}^n$ such that $\lambda^{\mathrm{t}}\mathcal{H}h = 0$ as well.

Proof. Since the rows of $\mathcal{H}h$ are linearly dependent over $\mathbb{C}$ there exists a $\lambda \in \mathbb{C}^n$ such that $\lambda^{\mathrm{t}}\mathcal{H}h = 0$. Assume λ is not isotropic. Then h is orthogonally equivalent to some $g \in \mathbb{C}[\tilde{x}]$ up to its linear part, where $\tilde{x} =$

$(x_1, x_2, \dots, x_{n-1})$. From theorem 5.7.1, it follows that the rows of $\mathcal{H}_{\tilde{x}} g$ are dependent over $\mathbb{C}$, say $\mu^{\mathrm{t}} \mathcal{H} g = 0$ for some $\mu \in \mathbb{C}^n$ with $(\mu_1, \mu_2, \dots, \mu_{n-1}) \neq 0$. Since changing μ_n does not affect $\mu^{\mathrm{t}} \mathcal{H} g = 0$, we can choose μ_n such that μ is isotropic. Take $T \in \mathrm{GO}_n(\mathbb{C})$ such that $h(Tx) - g$ is linear. Then $(T\mu)^{\mathrm{t}} \mathcal{H} h = \mu^{\mathrm{t}} T^{\mathrm{t}} \mathcal{H} g(T^{-1}x) = 0$. Since μ is isotropic and T is orthogonal, it follows that $T\mu$ is isotropic, as desired. □

The rest of this section is devoted to the proof of theorem 5.7.1. The case $n \leq 3$ follows from corollary 5.4.3. The case that h is homogeneous and $\mathrm{rk} \mathcal{H} h \leq 3$ follows from theorem 5.1.5. So three cases remain: $\mathrm{rk} \mathcal{H} h \leq 2$, $n = 5$ with h homogeneous, and $n = 4$. If we distinguish between $s_h \geq 2$ and $s_h \leq 1$ in the last case, then we get four cases.

Case 1: $\mathrm{rk} \mathcal{H} h \leq 2$. The case $\mathrm{rk} \mathcal{H} h \leq 1$ follows from theorem 5.1.4. So assume $\mathrm{rk} \mathcal{H} h = 2$. Since $\mathcal{H} h$ is nilpotent, it follows that $n \geq 3$. From theorem 5.3.5 and theorem 5.5.3 with $s = 1$, it follows that h is orthogonally equivalent to either

$$g := a_1(x_1) + a_2(x_1)x_2 + a_3(x_1)x_3 + \cdots + a_n(x_1)x_n \tag{5.8}$$

or

$$g := a_1(x_1 + \mathrm{i}x_2) + a_2(x_1 + \mathrm{i}x_2)x_2 + a_3(x_1 + \mathrm{i}x_2)x_3 + \cdots + a_n(x_1 + \mathrm{i}x_2)x_n \tag{5.9}$$

or the rows of $\mathcal{H} h$ are dependent over $\mathbb{C}$. Assume first that (5.8) is satisfied. From $\mathrm{tr} \mathcal{H} g = 0$ we obtain that $\frac{\partial^2}{\partial x_1^2} g = 0$, whence $\deg a_i \leq 1$ for all i. This gives that the second and third row of $\mathcal{H} g$ are dependent over $\mathbb{C}$, for $n \geq 3$. Assume next that (5.9) is satisfied. Since the Laplace operator $f \mapsto \mathrm{tr} \mathcal{H} f$ is additive and the only term of g that survives this operator is $a_2(x_1 + \mathrm{i}x_2)x_2$, we obtain that $\mathrm{tr} \mathcal{H}(a_2(x_1 + \mathrm{i}x_2)x_2) = 0$, whence a_2 is constant. It follows that $(1, \mathrm{i}, 0, \dots, 0)^{\mathrm{t}} \mathcal{H} h = 0$, as desired. Notice that this is an alternate proof for the case $n = 3$ in addition.

Case 2: $n = 5$ with h homogeneous. From theorem 5.3.7 and theorem 5.5.3 with $s = 2$, it follows that h is orthogonally equivalent to either

$$g := \begin{array}{l} f(x_1, x_2, a_1(x_1, x_2)x_3 + \\ a_2(x_1, x_2)x_4 + a_3(x_1, x_2)x_5) \end{array} \tag{5.10}$$

or

$$g := \begin{array}{l} f(x_1, x_2 + \mathrm{i}x_3, a_1(x_1, x_2 + \mathrm{i}x_3)x_3 + \\ a_2(x_1, x_2 + \mathrm{i}x_3)x_4 + a_3(x_1, x_2 + \mathrm{i}x_3)x_5) \end{array} \tag{5.11}$$

or

$$g := \begin{array}{l} f(x_1 + \mathrm{i}x_3, x_2 + \mathrm{i}x_4, a_1(x_1 + \mathrm{i}x_3, x_2 + \mathrm{i}x_4)x_3 + \\ a_2(x_1 + \mathrm{i}x_3, x_2 + \mathrm{i}x_4)x_4 + a_3(x_1 + \mathrm{i}x_3, x_2 + \mathrm{i}x_4)x_5) \end{array} \tag{5.12}$$

or h is degenerate.

First assume that (5.10) is satisfied. Since h is homogeneous, the three polynomials a_1, a_2, a_3 are homogeneous of the same degree, say d. If $d = 0$ then f is trivially degenerate. So assume $d \geq 1$. Write $f = \gamma_r(y_1, y_2)y_3^r + \gamma_{r-1}(y_1, y_2)y_3^{r-1} + \cdots$. Then $\tilde{g} := \frac{\partial^{r-1}}{\partial x_5^{r-1}} g$ is of the form

$$\tilde{f} = b_0(x_1, x_2) + b_1(x_1, x_2)x_3 + b_2(x_1, x_2)x_4 + b_3(x_1, x_2)x_5$$

with $b_j = r!a_3^{r-1}\gamma_r a_j$ for all $j \geq 1$. Since $\mathrm{tr}\mathcal{H}g = 0$ and the operator $\frac{\partial}{\partial x_5}$ commutes with the Laplace operator $\mathrm{tr}\mathcal{H}$, we get that $\mathrm{tr}\mathcal{H}\tilde{g} = 0$. It follows from the form of $\tilde{g}$ that for all $j \geq 1$, $x_{j+2}\mathrm{tr}\mathcal{H}b_j(x_1, x_2)$ is the leading term with respect to x_{j+2} of $\mathrm{tr}\mathcal{H}\tilde{g}$. So $\mathrm{tr}\mathcal{H}b_j(x_1, x_2) = 0$ for all $j \geq 1$.

So $b_j(x_1, x_2)$ is of the form $\lambda_j(x_1 + \mathrm{i}x_2)^{rd} + \mu_j(x_1 - \mathrm{i}x_2)^{rd}$ for some $\lambda_j, \mu_j \in \mathbb{C}$. It follows that the polynomials b_2, b_3, b_4 belong to a 2-dimensional $\mathbb{C}$-vector space and hence are linearly dependent over $\mathbb{C}$. Since $b_j = r!a_3^{r-1}\gamma_r a_j$ for all $j \geq 1$, also the polynomials a_1, a_2, a_3 are linearly dependent over $\mathbb{C}$.

It follows that $(\frac{\partial}{\partial x_3}g, \frac{\partial}{\partial x_4}g, \frac{\partial}{\partial x_5}g)$ and $(a_1(x_1, x_2), a_2(x_1, x_2), a_3(x_1, x_2))$ are dependent as vectors. So the last three rows of $\mathcal{H}g$ are dependent over $\mathbb{C}$, as desired.

Next assume that (5.11) is satisfied. If $\mathrm{rk}\mathcal{H}h \leq 3$, then h is degenerate, so assume that $\mathcal{H}h$ has corank one. Then the components of

$$\begin{pmatrix} \left(\frac{\partial}{\partial x_2} + \mathrm{i}\frac{\partial}{\partial x_3}\right) g \\ \frac{\partial}{\partial x_4} g \\ \frac{\partial}{\partial x_5} g \end{pmatrix} = \left(\frac{\partial}{\partial x_3} f\right)\Big|_{x=\cdots} \cdot \begin{pmatrix} \mathrm{i}\, a_1(x_1, x_2 + \mathrm{i}x_3) \\ a_2(x_1, x_2 + \mathrm{i}x_3) \\ a_3(x_1, x_2 + \mathrm{i}x_3) \end{pmatrix}$$

are algebraically dependent over $\mathbb{C}$, because they are homogeneous of the same degree. From theorem 5.4.5 with $n = 5$, $s = 2$, and $r = 1$ respectively, it follows that $s_h > 2$. Now apply iv) of theorem 5.2.10 to obtain the desired result. The case (5.12) is similar, except that $r = 0$ in theorem 5.4.5, because

$$\begin{pmatrix} \left(\frac{\partial}{\partial x_1} + \mathrm{i}\frac{\partial}{\partial x_3}\right) g \\ \left(\frac{\partial}{\partial x_2} + \mathrm{i}\frac{\partial}{\partial x_4}\right) g \\ \frac{\partial}{\partial x_5} g \end{pmatrix} = \left(\frac{\partial}{\partial x_3} f\right)\Big|_{x=\cdots} \cdot \begin{pmatrix} \mathrm{i}\, a_1(x_1 + \mathrm{i}x_3, x_2 + \mathrm{i}x_4) \\ \mathrm{i}\, a_2(x_1 + \mathrm{i}x_3, x_2 + \mathrm{i}x_4) \\ a_3(x_1 + \mathrm{i}x_3, x_2 + \mathrm{i}x_4) \end{pmatrix}$$

Case 3: $n = 4$ and $s_h \geq 2$. If $s_h = 3$, then the rows of $\mathcal{H}h$ are dependent over $\mathbb{C}$ due to proposition 5.2.4. So assume $s_h = 2$. From ii) of theorem 5.2.10 and theorem 5.5.3 with $s = 2$, it follows that h is equivalent to either

$$g := a_1(x_1, x_2) + a_2(x_1, x_2)x_3 + a_3(x_1, x_2)x_4 \tag{5.13}$$

or

$$g := a_1(x_1, x_2 + \mathrm{i}x_3) + a_2(x_1, x_2 + \mathrm{i}x_3)x_3 + a_3(x_1, x_2 + \mathrm{i}x_3)x_4 \tag{5.14}$$

or

$$g := a_1(x_1+\mathrm{i}x_3, x_2+\mathrm{i}x_4)+a_2(x_1+\mathrm{i}x_3, x_2+\mathrm{i}x_4)x_3+a_3(x_1+\mathrm{i}x_3, x_2+\mathrm{i}x_4)x_4 \tag{5.15}$$

where $a_2, a_3 \in \mathbb{C}[p]$ for some $p \in \mathbb{C}[x_1, x_2]$.

Assume first that either (5.13) or (5.14) is satisfied. If $\mathrm{rk}\mathcal{H}h \leq 2$, then the rows of $\mathcal{H}h$ are dependent over $\mathbb{C}$, so assume that $\mathcal{H}h$ has corank one. Just as in case 2, we apply theorem 5.4.5, this time with $n = 4$, $s = 2$, and $r = 2$ for (5.13) and $r = 1$ for (5.14). We obtain $s_h > 2$, so the rows of $\mathcal{H}h$ are dependent over $\mathbb{C}$.

Next assume that (5.15) is satisfied. Put

$$T := \begin{pmatrix} 1 & 0 & \mathrm{i} & 0 \\ 0 & 1 & 0 & \mathrm{i} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Notice that

$$g(T^{-1}x) = a_1(x_1, x_2) + a_2(x_1, x_2)x_3 + a_3(x_1, x_2)x_4$$

and

$$TT^{\mathrm{t}} = \begin{pmatrix} 0 & 0 & \mathrm{i} & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

Consequently,

$$T(\nabla g)(T^{-1}x) = TT^{\mathrm{t}}(T^{-1})^{\mathrm{t}}(\nabla g)(T^{-1}x) = TT^{\mathrm{t}}\nabla\big(g(T^{-1}x)\big)$$

so

$$G := T(\nabla g)(T^{-1}x) = \begin{pmatrix} \mathrm{i}\frac{\partial}{\partial x_3} g(T^{-1}x) \\ \mathrm{i}\frac{\partial}{\partial x_4} g(T^{-1}x) \\ * \\ * \end{pmatrix} = \begin{pmatrix} \mathrm{i}a_2(x_1,x_2) \\ \mathrm{i}a_3(x_1,x_2) \\ * \\ * \end{pmatrix}$$

Since G is a linear conjugation of ∇g and $\mathcal{H}g$ is nilpotent, it follows that $\mathcal{J}G$ is nilpotent. Looking at the right hand side above, one can see by proposition 5.6.5 that the leading principal minor of size 2 of $\mathcal{J}G$, which is $\mathcal{J}_{x_1,x_2}(\mathrm{i}a_1, \mathrm{i}a_2)$, is nilpotent. Now apply theorem 2.2.7 to obtain that the rows of $\mathcal{J}_{x_1,x_2}(\mathrm{i}a_1, \mathrm{i}a_2)$ are dependent over $\mathbb{C}$. Hence the first two rows of $\mathcal{J}G$ are dependent over $\mathbb{C}$. This gives the dependence over $\mathbb{C}$ of the rows of $\mathcal{H}h$.

Case 4: $n = 4$ and $s_h \le 1$. From corollary 5.4.3, it follows that the case $s_h = 0$ does not occur, so let $s_h = 1$. From iii) of theorem 5.2.10 and theorem 5.5.3 with $s = 1$, it follows that h is equivalent to either

$$\begin{aligned} g = f\bigl(x_1, a_1(x_1)x_2 + a_2(x_1)x_3 + a_3(x_1)x_4\bigr) +& \\ b_1(x_1)x_2 + b_2(x_1)x_3 + b_3(x_1)x_4& \end{aligned} \tag{5.16}$$

or

$$\begin{aligned} g = f\bigl(x_1 + \mathrm{i}x_2, a_1(x_1+\mathrm{i}x_2)x_2 + a_2(x_1+\mathrm{i}x_2)x_3 + a_3(x_1+\mathrm{i}x_2)x_4\bigr) +& \\ b_1(x_1+\mathrm{i}x_2)x_2 + b_2(x_1+\mathrm{i}x_2)x_3 + b_3(x_1+\mathrm{i}x_2)x_4& \end{aligned} \tag{5.17}$$

If $\deg_{y_2} f \le 1$, then $\mathrm{rk}\mathcal{H}g \le 2$ and hence the rows of $\mathcal{H}h$ are dependent over $\mathbb{C}$. So assume $r := \deg_{y_2} f \ge 2$ and let $\gamma_r(y_1)$ be the leading coefficient of f with respect to y_2.

Assume first that (5.16) is satisfied. Since $r \ge 2$ and the leading coefficient with respect to x_{i+1} of $\mathrm{tr}\mathcal{H}g$ equals $\frac{\partial^2}{\partial x_1^2}(\gamma_r(x_1)a_i(x_1)^r)$ for all $i \ge 1$, it follows that $a_i \in \mathbb{C}$ for all i. If $a_i = 0$ for all i then $\mathrm{rk}\mathcal{H}g \le 2$, so assume $a_1 \ne 0$. Then $(a_2\frac{\partial}{\partial x_2} - a_1\frac{\partial}{\partial x_3})g = a_2b_1(x_1) - a_1b_2(x_1)$ and $(a_3\frac{\partial}{\partial x_2} - a_1\frac{\partial}{\partial x_4})g = a_3b_1(x_1) - a_1b_3(x_1)$ are algebraically dependent, whence $s_h \ge 2$. Contradiction.

Assume next that (5.17) is satisfied. Then the components of

$$\Bigl(\Bigl(\frac{\partial}{\partial x_1} + \mathrm{i}\frac{\partial}{\partial x_2}\Bigr)g, \frac{\partial}{\partial x_3}g, \frac{\partial}{\partial x_4}g\Bigr)$$
$$= \Bigl(\frac{\partial}{\partial x_3}f\Bigr)\Big|_{x=\cdots} \cdot \begin{pmatrix} \mathrm{i}\,a_1(x_1+\mathrm{i}x_2) \\ a_2(x_1+\mathrm{i}x_2) \\ a_3(x_1+\mathrm{i}x_2) \end{pmatrix} + \begin{pmatrix} \mathrm{i}\,b_1(x_1+\mathrm{i}x_2) \\ b_2(x_1+\mathrm{i}x_2) \\ b_3(x_1+\mathrm{i}x_2) \end{pmatrix}$$

are contained in $\mathbb{C}[(\frac{\partial}{\partial x_3} f)|_{x=\cdots}, x_1 + \mathrm{i}x_2]$ and hence algebraically dependent. From theorem 5.4.5 with $n = 4$, $s = 1$ and $r = 0$, it follows that $s_h > 1$. Contradiction.

5.8 Linear triangularizable gradient maps

Assume $h \in \mathbb{C}[x]$ such that ∇h is linearly triangularizable: say that the Jacobian of $M(\nabla h)(M^{-1}x)$ is lower triangular. Now ∇h can be conjugated with an orthogonal matrix without affecting the symmetry of its Jacobian, and $M(\nabla h)(M^{-1}x)$ can be conjugated with a lower triangular matrix without affecting lower triangularity of its Jacobian. This can be used to simplify M.

Lemma 5.8.1. *Assume $M \in \mathrm{Mat}_{m,n}(\mathbb{C})$. Then M can be decomposed as*

$$M = LPST \tag{5.18}$$

where $L \in \mathrm{Mat}_{m,n}(\mathbb{C})$ is lower triangular, P is a permutation, T is orthogonal and S is of the form

$$S = \begin{pmatrix} I_s & \emptyset & \mathrm{i}I_s^{\mathrm{r}} \\ \emptyset & I_{n-2s} & \emptyset \\ \mathrm{i}I_s^{\mathrm{r}} & \emptyset & I_s \end{pmatrix}$$

where $s \leq n/2$.

Proof. By adding zero rows below to M or removing rows below from M, we obtain that M is square. Since PST is invertible, removed rows can be restored afterwards.

i) By row operations to below and column operations to the left, we can get M of the from DP, where D is diagonal and P is a permutation. So M decomposes as $M = LDP\tilde{L}$, where L and $\tilde{L}$ are lower triangular with ones on the diagonal. It follows that we may assume that $M = P\tilde{L}$. So assume from now on that M is invertible.

ii) Assume first that for all i with $1 \leq i \leq n$, there exist at most one j with $1 \leq j \leq n$ such that $M_i M_j^{\mathrm{t}} \neq 0$. Since M is invertible, it follows that for all i with $1 \leq i \leq n$, there exist exactly one j with $1 \leq j \leq n$ such that $M_i M_j^{\mathrm{t}} \neq 0$.

By way of a multiplication with an invertible diagonal matrix from the left, we can obtain that $M_i M_j^{\mathrm{t}} = 1$ in case $M_i M_j^{\mathrm{t}} \neq 0$ and $i = j$, and $M_i M_j^{\mathrm{t}} = 2\mathrm{i}$ in case $M_i M_j^{\mathrm{t}} \neq 0$ and $i \neq j$. It follows that M is of the desired form $M = LPST$, where L is a diagonal matrix.

iii) Assume next the opposite of ii), i.e. that there are i, j with $1 \le i \le j \le n$ such that $M_i M_j^{\mathrm{t}} \neq 0$, and either $M_i M_k^{\mathrm{t}} \neq 0$ for a $k \neq j$, or $M_k M_j^{\mathrm{t}} \neq 0$ for a $k \neq i$. Choose i as small as possible first, and j as small as possible after that. Then $i \le j$.

If $i < j$, then $M_i M_i^{\mathrm{t}} = 0$, and by adding a multiple of M_i to M_j, we can obtain $M_j M_i^{\mathrm{t}} = 0$ as well. If $M_i M_k^{\mathrm{t}} \neq 0$ for some $k \neq j$, then $k > j$ and we can obtain $M_i M_k^{\mathrm{t}} = 0$ by adding a multiple of M_j to M_k. Notice that this works both in case $i = j$ and in case $i < j$. If $M_k M_j^{\mathrm{t}} \neq 0$ for some $k \neq i$, then $k > i$ and we can obtain $M_k M_j^{\mathrm{t}} = 0$ by adding a multiple of M_i to M_k.

All of the above row operations can be eaten up by the lower triangular matrix L in the decomposition scheme 5.18. Advancing in this direction, i increases until it does not exist any more, in which case ii) applies. The diagonal matrix in ii) can be eaten up by the lower triangular matrix L as well, as desired. $\square$

Notice that

$$\begin{pmatrix} I_s & \emptyset & \mathrm{i}I_s^{\mathrm{r}} \\ \emptyset & I_{n-2s} & \emptyset \\ \mathrm{i}I_s^{\mathrm{r}} & \emptyset & I_s \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2}I_s & \emptyset & -\frac{1}{2}\mathrm{i}I_s^{\mathrm{r}} \\ \emptyset & I_{n-2s} & \emptyset \\ -\frac{1}{2}\mathrm{i}I_s^{\mathrm{r}} & \emptyset & \frac{1}{2}I_s \end{pmatrix}$$

The following lemma shows how we can improve the permutation P in (5.18). In fact, the permutation P can be eaten up by the orthogonal transformation, because it can pass through S.

Lemma 5.8.2. *Assume $M \in \mathrm{Mat}_n(\mathbb{C})$ is symmetric. If $PSMS^{-1}P^{-1}$ is lower triangular for some permutation P and an S of the form*

$$S = \begin{pmatrix} I_s & \emptyset & \mathrm{i}I_s^{\mathrm{r}} \\ \emptyset & I_{n-2s} & \emptyset \\ \mathrm{i}I_s^{\mathrm{r}} & \emptyset & I_s \end{pmatrix}$$

then we can choose P such that it commutes with S.

Proof. Assume $PSMS^{-1}P^{-1}$ is lower triangular. If $s = 0$, then $S = I_n$ and hence $PS = SP$. So assume from now on that $s \ge 1$. We distinguish four cases.

- $P_1 = e_1^{\mathrm{t}}$ and $P_n = e_n^{\mathrm{t}}$.
  Since the first and last column of PS are dependent of e_1 and e_n only, and the first and last row of $S^{-1}P^{-1}$ are dependent of e_1^{t} and e_n^{t} only, it follows that

  $$\begin{aligned}(P_{(n,n)})_{(1,1)}(S_{(n,n)})_{(1,1)}(M_{(n,n)})_{(1,1)}(S_{(n,n)})_{(1,1)}^{-1}(P_{(n,n)})_{(1,1)}^{-1} \\ = ((PS)_{(n,n)})_{(1,1)}(M_{(n,n)})_{(1,1)}((PS)_{(n,n)})_{(1,1)}^{-1}\end{aligned}$$

  is lower triangular. So by induction on n, we can replace $(P_{(n,n)})_{(1,1)}$ by a permutation that commutes with $(S_{(n,n)})_{(1,1)}$ without affecting the lower triangularity of $(M_{(n,n)})_{(1,1)}$.

  Notice that after this maneuver, P commutes with S. Since the column space of the first row of $PSMS^{-1}P^{-1}$ without the corners is not affected and neither is the row space of the last column of $PSMS^{-1}P^{-1}$ without the corners, it follows that $PSMS^{-1}P^{-1}$ is lower triangular, as desired.

- $P_1 = e_1^{\mathrm{t}}$ and $P_n \ne e_n^{\mathrm{t}}$.
  Looking at the first row of $PSMS^{-1}P^{-1}$, we get $(e_1^{\mathrm{t}} + \mathrm{i}e_n^{\mathrm{t}})MS^{-1}P^{-1}$ is dependent of e_1^{t}. So $(e_1^{\mathrm{t}} + \mathrm{i}e_n^{\mathrm{t}})M$ is dependent of $e_1^{\mathrm{t}}PS = e_1^{\mathrm{t}} + \mathrm{i}e_n^{\mathrm{t}}$.

  Since M is symmetric, we obtain by transposition that $2\mathrm{i}MS^{-1}e_n = \mathrm{i}M(e_n - \mathrm{i}e_1) = M(e_1 + \mathrm{i}e_n)$ is dependent of $e_1 + \mathrm{i}e_n = 2\mathrm{i}S^{-1}e_n$. By multiplication by PS from the left, we see that $PSMS^{-1}e_n$ is dependent of Pe_n.

  Take r such that $Pe_n = e_r$. Then $P^{-1}e_r = e_n$ and we obtain that $PSMS^{-1}P^{-1}e_r$ is dependent of e_r. Therefore we can cycle the r-th row of $PSMS^{-1}P^{-1}$ to the bottom and the r-th column of $PSMS^{-1}P^{-1}$ to the right side without affecting the lower triangularity of $PSMS^{-1}P^{-1}$. This way, we obtain $P_n = e_n^{\mathrm{t}}$, and the previous case gives the desired result.

- $P_1 = e_i^{\mathrm{t}}$ for some $i \notin \{1, s+1, s+2, \ldots, n-s\}$.
  Let Q be the permutation that flips x_1 and x_i on one hand and, in case $i < n$, flips x_n and x_{n+1-i} on the other hand. Then Q commutes

with S. So

$$(PQ)S(Q^{-1}MQ)S^{-1}(PQ)^{-1}$$

is lower triangular. Furthermore, the first row of PQ is equal to e_1^{t}, so from the above cases, it follows that $\tilde{P}S(Q^{-1}MQ)S^{-1}\tilde{P}^{-1}$ is lower triangular for some permutation $\tilde{P}$ that commutes with S. So $\tilde{P}Q^{-1}SMS^{-1}Q\tilde{P}^{-1}$ is lower triangular and $\tilde{P}Q^{-1}$ commutes with S, as desired.

- $P_1 = e_i^{\mathrm{t}}$ for some $i \in \{s+1, s+2, \dots, n-s\}$.
  Looking at the first row of $PSMS^{-1}P^{-1}$, we get $P_1SMS^{-1}P^{-1}e_j = 0$ for all $j \neq 1$. Since $P^{-1} = P^{\mathrm{t}}$, it follows from $e_1^{\mathrm{t}}P = P_1 = e_i^{\mathrm{t}}$ that $P^{-1}e_1 = e_i$ and that all other columns of P^{-1} are a unit vector $\neq e_i$. Consequently, $e_i^{\mathrm{t}}SMS^{-1}e_j = 0$ for all $j \neq i$.

  Since $i \in \{s+1, s+2, \dots, n-s\}$, we have $e_i^{\mathrm{t}}S = e_i^{\mathrm{t}}$. Furthermore, the column space of all but the i-th column of S^{-1} is equal to that of I_n, so by $e_i^{\mathrm{t}}MS^{-1}e_j = e_i^{\mathrm{t}}SMS^{-1}e_j = 0$ for all $j \neq i$, we obtain that $M_{ij} = e_i^{\mathrm{t}}MI_ne_j = 0$ for all $j \neq i$. So the i-th row of M is dependent of e_i^{t}. Since M is symmetric, it follows that the i-th column of M is dependent of e_i. Since $S^{-1}P_1^{-1} = S^{-1}e_i = e_i$, we obtain that the first column of $PSMS^{-1}P^{-1}$ is dependent of e_1.

  Consequently, we can cycle the first row of $PSMS^{-1}P^{-1}$ to the bottom and the first column of $PSMS^{-1}P^{-1}$ to the right side without affecting the lower triangularity of $PSMS^{-1}P^{-1}$. After this maneuver, P_1 might still be of the form e_i^{t} with $i \in \{s+1, s+2, \dots, n-s\}$, but if we repeat this maneuver over and over again, we obtain finally that $P_1 = e_i^{\mathrm{t}}$ for some $i \notin \{s+1, s+2, \dots, n-s\}$, and the above cases give the desired result. □

So after a suitable orthogonal transformation, the conjugation with S in the above theorem changes the Hessian in a lower triangular Jacobian, for some s with $0 \leq s \leq n/2$. In case the Hessian is nilpotent, we can replace S by $I_n + \mathrm{i}I_n^{\mathrm{r}}$, a matrix that does not depend of s.

Lemma 5.8.3. *Assume $M \in \mathrm{Mat}_n(\mathbb{C})$ is symmetric and nilpotent. If SMS^{-1} is lower triangular, where*

$$S := \begin{pmatrix} I_s & \emptyset & \mathrm{i}I_s^{\mathrm{r}} \\ \emptyset & I_{n-2s} & \emptyset \\ \mathrm{i}I_s^{\mathrm{r}} & \emptyset & I_s \end{pmatrix}$$

for some $s \le n/2$, then

$$(I_n + \mathrm{i}I_n^{\mathrm{r}})M(I_n + \mathrm{i}I_n^{\mathrm{r}})^{-1} = \frac{1}{2}(I_n + \mathrm{i}I_n^{\mathrm{r}})M(I_n - \mathrm{i}I_n^{\mathrm{r}})$$

is lower triangular and symmetric with respect to the anti-diagonal.

Proof. The proof is somewhat similar to the first case in the proof above. From the structure of S, it follows that the central principal minor $\tilde{M}$ of size $n-2s$ of M is the same as that of SMS^{-1}. Since $\tilde{M}$ is symmetric and lower triangular, it follows that $\tilde{M}$ is diagonal. But SMS^{-1} is lower triangular and nilpotent, so the diagonal of $\tilde{M}$ is zero.

So $\tilde{M} = 0$. It follows that we can do row operations on the central $n-2s$ rows of SMS^{-1} and column operations on the central $n-2s$ columns of SMS^{-1} without affecting the lower triangularity of SMS^{-1}. This gives the lower triangularity of

$$\begin{aligned}
&\begin{pmatrix} I_s & \emptyset & \emptyset \\ \emptyset & I_{n-2s} + \mathrm{i}I_{n-2s}^{\mathrm{r}} & \emptyset \\ \emptyset & \emptyset & I_s \end{pmatrix} SMS^{-1} \begin{pmatrix} I_s & \emptyset & \emptyset \\ \emptyset & I_{n-2s} + \mathrm{i}I_{n-2s}^{\mathrm{r}} & \emptyset \\ \emptyset & \emptyset & I_s \end{pmatrix}^{-1} \\
&= (I_n + \mathrm{i}I_n^{\mathrm{r}})M(I_n + \mathrm{i}I_n^{\mathrm{r}})^{-1} \\
&= \frac{1}{2}(I_n + \mathrm{i}I_n^{\mathrm{r}})M(I_n - \mathrm{i}I_n^{\mathrm{r}}) \qquad (5.19)\\
&= -\frac{1}{2}\mathrm{i}\left((I_n + \mathrm{i}I_n^{\mathrm{r}})^{\mathrm{t}}M(I_n + \mathrm{i}I_n^{\mathrm{r}})\right)^{\mathrm{r}}
\end{aligned}$$

The symmetry with respect to the anti-diagonal follows from the formula on the right hand side. □

Theorem 5.8.4. *Assume $h \in \mathbb{C}[x]$. Then ∇h is linearly triangularizable, if and only if there exists a g that is orthogonally equivalent to h such that*

$$\begin{aligned}
g \;=\; &\sum_{j=1}^{s} a_j\big(x_1 + \mathrm{i}x_n, x_2 + \mathrm{i}x_{n-1}, \ldots, x_j + \mathrm{i}x_{n+1-j}\big) \cdot x_j + \\
&\sum_{j=s+1}^{n-s} a_j\big(x_1 + \mathrm{i}x_n, x_2 + \mathrm{i}x_{n-1}, \ldots, x_s + \mathrm{i}x_{n+1-s}, x_j\big) \cdot x_j + \\
&a_\infty\big(x_1 + \mathrm{i}x_n, x_2 + \mathrm{i}x_{n-1}, \ldots, x_s + \mathrm{i}x_{n+1-s}\big) \qquad (5.20)
\end{aligned}$$

for some $s \le \lfloor n/2 \rfloor$.

If $\mathcal{H}H$ is nilpotent, then ∇h is linearly triangularizable, if and only if there exists a g that is orthogonally equivalent to h such that

$$\begin{aligned} g \quad = \quad & \sum_{j=1}^{\lceil n/2 \rceil} a_j\big(x_1 + \mathrm{i}x_n, x_2 + \mathrm{i}x_{n-1}, \ldots, x_{j-1} + \mathrm{i}x_{n+2-j}\big) \cdot x_j + \\ & a_\infty\big(x_1 + \mathrm{i}x_n, x_2 + \mathrm{i}x_{n-1}, \ldots, x_{\lfloor n/2 \rfloor} + \mathrm{i}x_{n+1-\lfloor n/2 \rfloor}\big) \end{aligned} \tag{5.21}$$

where a_1 is implicitly constant.

Proof. Assume that ∇h is linearly triangularizable. From lemma 5.8.1, it follows that we can choose g such that the Jacobian of $PS(\nabla g)(S^{-1}P^{-1}x)$ is lower triangular, where

$$S := \begin{pmatrix} I_s & \emptyset & \mathrm{i}I_s^{\mathrm{r}} \\ \emptyset & I_{n-2s} & \emptyset \\ \mathrm{i}I_s^{\mathrm{r}} & \emptyset & I_s \end{pmatrix}$$

for some $s \le n/2$. From lemma 5.8.2, it follows that we can choose g such that $S(\nabla g)(S^{-1}x)$ has a lower triangular Jacobian. Notice that $g \in \mathbb{C}[Sx]$. So we can write

$$\begin{aligned} g \quad = \quad & \sum_{j=1}^{s} a_j\big(S_1x, S_2x, \ldots, S_{n+1-j}x)\big) \cdot S_{n+1-j}x + \\ & \sum_{j=s+1}^{n-s} a_j\big(S_1x, S_2x, \ldots, S_sx, S_jx, S_{j+1}x, \ldots, S_{n-s}x)\big) \cdot S_jx + \\ & \tilde{a}_\infty\big(S_1x, S_2x, \ldots, S_sx\big) \end{aligned}$$

We show that $a_j \in \mathbb{C}[x_1, x_2, \ldots, x_j]$ for all $j \le s$. For that purpose, let $j \le s$ and assume by induction that $a_i \in \mathbb{C}[x_1, x_2, \ldots, x_i]$ for all $i < j$. Then every term of g except that with a_j is contained in $\mathbb{C}[x_j + \mathrm{i}x_{n+1-j}, x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n-j}, x_{n+2-j}, \ldots, x_n]$ and therefore killed by $\frac{\partial}{\partial x_j} + \mathrm{i}\frac{\partial}{\partial x_{n+1-j}}$. It follows that

$$\begin{aligned} S_j \nabla g \quad & = \quad \left(\frac{\partial}{\partial x_j} + \mathrm{i}\frac{\partial}{\partial x_{n+1-j}}\right)\big(a_j(S_1x, S_2x, \ldots, S_{n+1-j}x) \cdot S_{n+1-j}x\big) \\ & = \quad 2\mathrm{i}\frac{\partial}{\partial x_{n+1-j}}\big(a_j \cdot x_{n+1-j}\big)\Big|_{x=Sx} \end{aligned}$$

So the j-th component of $S(\nabla g)(S^{-1}x)$ is equal to

$$S_j(\nabla g)(S^{-1}x) = 2\mathrm{i}\frac{\partial}{\partial x_{n+1-j}}\big(a_j \cdot x_{n+1-j}\big) \tag{5.22}$$

But since the Jacobian of $S(\nabla g)(S^{-1}x)$ is lower triangular, it follows that $a_j \in \mathbb{C}[x_1, x_2, \ldots, x_j]$, as desired.

In a similar manner, one can show by induction on j that for j with $s+1 \le j \le n-s$, $a_j \in \mathbb{C}[x_1, x_2, \ldots, x_s, x_j]$ and that

$$S_j(\nabla g)(S^{-1}x) = \frac{\partial}{\partial x_j}\big(a_j \cdot x_j\big) \tag{5.23}$$

So we have

$$\begin{aligned} g \;=\; & \sum_{j=1}^{s} a_j\big(S_1x, S_2x, \ldots, S_jx)\big) \cdot S_{n+1-j}x \,+ \\ & \sum_{j=s+1}^{n-s} a_j\big(S_1x, S_2x, \ldots, S_sx, S_jx\big) \cdot S_jx \,+ \\ & \tilde{a}_\infty\big(S_1x, S_2x, \ldots, S_sx\big) \end{aligned}$$

Since $S_{n+1-j}x = 2\mathrm{i}x_j - \mathrm{i}x_j + x_{n+1-j} = 2\mathrm{i}x_j - \mathrm{i}S_jx$ for all $j \le s$, it follows that g is of the desired form.

In the nilpotent case, we can take $\tilde{S} := I_n + \mathrm{i}I_n^{\mathrm{r}}$ and on account of lemma 5.8.3, the Jacobian of $\tilde{S}(\nabla g)(\tilde{S}^{-1}x)$ is lower triangular. But $\tilde{S}$ and S above with $s = \lfloor n/2 \rfloor$ differ at most an elementary diagonal matrix as a factor, so we can take $s = \lfloor n/2 \rfloor$. The extra condition that the diagonal of the Jacobian of $S(\nabla g)(S^{-1}x)$ is zero gives the desired result.

For the converse implication, notice that the first $n-s$ rows of $\mathcal{J}(S(\nabla g)(S^{-1} \cdot x))$ are of the desired form by way of (5.22) and (5.23). For the last s rows of $\mathcal{J}(S(\nabla g)(S^{-1}x))$, it suffices to show that the lower right principal minor of size s is lower triangular. We do this by showing that this principal minor is the reflection of the upper left principal minor of size s of $\mathcal{J}(S(\nabla g)(S^{-1}x))$. Set $M := \mathcal{H}g|_{x=S^{-1}x}$. By conjugating $\mathcal{J}(S(\nabla g)(S^{-1}x)) = SMS^{-1}$ as in (5.19), we get a matrix that is symmetric with respect to the anti-diagonal. But the upper left and lower right principal minors of size s of M are only affected by the substitution corresponding to this conjugation. This gives the desired result in case $\mathcal{H}h$ is not nilpotent. In the nilpotent case, one can see that $\mathcal{J}(S(\nabla g)(S^{-1}x))$ has zeros on the diagonal, if g is as in (5.21). $\square$

If h is homogeneous of degree 2, then its Hessian has constant entries. So there is a correspondence between quadratic forms and symmetric matrices over $\mathbb{C}$ (more generally over rings with $\frac{1}{2}$). More is known about orthogonal conjugations of symmetric matrices over $\mathbb{C}$. We refer to [31] for more information. In terms of quadratic forms, one can prove the following with the techniques in [31].

Proposition 5.8.5. *Define a* simple *quadratic form as one whose Hessian has only one eigenvector up to scalar multiplication.*

i) After a suitable orthogonal transformation, a simple quadratic form h for which $\mathcal{H}h$ has eigenvalue λ satisfies

$$(I_n + \mathrm{i}I_n^{\mathrm{r}})\mathcal{H}h(I_n + \mathrm{i}I_n^{\mathrm{r}})^{-1} = \lambda I_n + N_n$$

where N_n is the nilpotent matrix of size n with ones on the subdiagonal and zeros elsewhere.

ii) After a suitable orthogonal transformation, a quadratic form can be written as the sum of simple quadratic forms such that no pair of them shares any of the variables $x_1, x_2, \ldots, x_n$.

The proof is left as an exercise to the reader. In [16], it is shown that quadratic forms can be transformed orthogonally such that their Hessian becomes *tridiagonal*, i.e. only entries on or adjacent to the diagonal may be nonzero. Furthermore, tridiagonal equivalents of symmetric Jordan blocks are constructed. The symmetric Jordan blocks

$$(I_n + \mathrm{i}I_n^{\mathrm{r}})(\lambda I_n + N_n)(I_n + \mathrm{i}I_n^{\mathrm{r}})^{-1} = \lambda I_n + \frac{1}{2}\big((N_n + N_n^{\mathrm{t}}) + \mathrm{i}(N_n - N_n^{\mathrm{t}})^{\mathrm{r}}\big)$$

of [31] are not tridiagonal.

The last theorem of this chapter classifies for which dimensions n and ranks r, gradient maps with nilpotent Jacobians are always linearly triangularizable.

Theorem 5.8.6. *Assume $h \in \mathbb{C}[x]$ such that $\mathcal{H}h$ is nilpotent. Then ∇h is linearly triangularizable in each the following cases:*

i) $n \leq 4$,

ii) $n \leq 5$ and h is homogeneous, *iii)* $n \le 6$, *h is homogeneous and the rows of $\mathcal{H}h$ are dependent over $\mathbb{C}$,*

iv) $\mathrm{rk}\mathcal{H}h \le 2$,

v) $\mathrm{rk}\mathcal{H}h \le 3$ *and h is homogeneous.*

Furthermore, there exist $g \in \mathbb{C}[x]$ with nilpotent Hessians of rank r such that ∇g is not *linearly triangularizable in case:*

vi) $n \ge 5$ *and* $3 \le r \le n-1$,

and even $g \in \mathbb{C}[x]$ that are homogeneous *in addition if*

vii) $n \ge 7$ *and* $4 \le r \le n-1$.

Proof.

i) Assume $n \le 4$. The case $n = 1$ is trivial, so assume $2 \le n \le 4$. From corollary 5.7.2, it follows that there exists an isotropic $\lambda \in \mathbb{C}^n$ such that $\lambda^{\mathrm{t}}\mathcal{H}h = 0$. Replacing h by $h(Tx)$ for a suitable $T \in \mathrm{GO}_n(\mathbb{C})$, we obtain $\lambda = e_1 + \mathrm{i}e_n$. It follows that h is of the form of g in (5.21) in case $n = 2$.

   So let $n \ge 3$. From theorem 5.6.2, it follows that $\mathcal{H}_{x_2,\dots,x_{n-1}}h$ is nilpotent. It follows that h is of the form of g in (5.21) in case $n = 3$.

   So let $n = 4$. Then there exists an isotropic $\mu \in \mathbb{C}(x_1 + \mathrm{i}x_4)^2$ such that $\mu^{\mathrm{t}}\mathcal{H}_{x_2,x_3}h = 0$. But since μ has only two coordinates, it follows that μ is dependent of either $(1, \mathrm{i})$ or $(1, -\mathrm{i})$. Again by replacing h by $h(Tx)$ for a suitable $T \in \mathrm{GO}_n(\mathbb{C})$, we obtain $\mu = (1, \mathrm{i})$ and that h is of the form of g in (5.21).

ii) Assume $n \le 5$ and h is homogeneous. The case $n \le 4$ follows from i), so let $n = 5$. From corollary 5.7.2, it follows that there exists an isotropic $\lambda \in \mathbb{C}^n$ such that $\lambda^{\mathrm{t}}\mathcal{H}h = 0$. Replacing h by $h(Tx)$ for a suitable $T \in \mathrm{GO}_n(\mathbb{C})$, we obtain $\lambda = e_1 + \mathrm{i}e_n$.

   It follows that $\mathcal{H}(h|_{x_1+\mathrm{i}x_n=1})$ is nilpotent. From i) with $n = 3$, we obtain that $\nabla(h|_{x_1+\mathrm{i}x_n=1})$ is linearly triangularizable. So we may assume that $\nabla(h|_{x_1+\mathrm{i}x_n=1})$ is of the form of (5.21), with x replaced by $x_2, x_3, \dots, x_{n-1}$. By homogenization, we can get h back out of $\nabla(h|_{x_1+\mathrm{i}x_n=1})$, and h is of the form of (5.21).

iii) The proof of this case is similar to that of ii).

iv) Assume $\mathrm{rk}\mathcal{H}h \le 2$. From theorem 5.3.5 and theorem 5.5.3 with $s = 1$, it follows that we may assume that either

$$h = a_1(x_1) + a_2(x_1)x_2 + \cdots + a_n(x_1)x_n \tag{5.8$'$}$$

or

$$h = a_1(x_1 + \mathrm{i}x_2) + a_2(x_1 + \mathrm{i}x_2)x_2 + \cdots + a_n(x_1 + \mathrm{i}x_2)x_n \tag{5.9$'$}$$

We show that h is orthogonally equivalent to

$$g := a_1(x_1 + \mathrm{i}x_2) + a_2x_2 + a_3(x_1 + \mathrm{i}x_2)x_3 + \cdots + a_n(x_1 + \mathrm{i}x_2)x_n$$

with $a_2 \in \mathbb{C}$. Notice that g above is indeed linearly triangularizable. If $\deg h = 2$, then the quadratic part of h is a product of two linear forms. Since $\mathcal{H}h$ is nilpotent, this product is orthogonally equivalent to one of $(x_1+\mathrm{i}x_2)^2$, $(x_1+\mathrm{i}x_2)x_3$, $(x_1+\mathrm{i}x_2)(x_3+\mathrm{i}x_4)$. It follows that h is of the desired form.

So assume $\deg h \ge 3$. Since $\mathrm{tr}\mathcal{H}h = 0$, it follows that h cannot be of the form (5.8$'$). So h is of the form (5.9$'$), but again by the trace condition, $\deg a_2 \le 0$ must be satisfied, as desired.

v) Assume $\mathrm{rk}\mathcal{H}h \le 3$ and h is homogeneous. From theorem 5.3.10, it follows that h can be expressed as a polynomial in three linear forms. So by way of an orthogonal transformation, we can embed h in dimension 6. It follows from iii) that ∇h is linearly triangularizable. One can show that h is orthogonally equivalent to a g of one of the forms below.

$$p(x_1 + \mathrm{i}x_2, x_3 + \mathrm{i}x_4, x_5 + \mathrm{i}x_6)$$

$$p(x_1 + \mathrm{i}x_2, x_3 + \mathrm{i}x_4) + q(x_1 + \mathrm{i}x_2, x_3 + \mathrm{i}x_4)x_5$$

$$p(x_1 + \mathrm{i}x_2, x_3 + \mathrm{i}x_4) + q(x_1 + \mathrm{i}x_2)x_3$$

vi) Assume $n \ge 5$ and $3 \le r \le n-1$. The gradient maps of g_1 and g_2 in example 5.6.8 are not linearly triangularizable, since they do not satisfy DP+. This proves the case $n = 5$. So let $n > 5$. We distinguish two cases:

- $3 \leq r < n-1$.
  By induction, it follows that there exists a $g \in \mathbb{C}[\tilde{x}]$ such that $\mathcal{H}_{\tilde{x}} g$ is nilpotent of rank r, but $\nabla_{\tilde{x}} g$ is not linearly triangularizable, where $\tilde{x} = x_1, x_2, \ldots, x_{n-1}$. From theorem 5.6.6 and the equivalence of linear triangularizability and strong nilpotency of the Jacobian, which is shown in [24, §7.4], we obtain that $\mathcal{H}g$ is nilpotent of rank r, but ∇g is not linearly triangularizable, as desired.

- $r = n-1$.
  If $n = 6$, then one can take
  $$g = g_2 + \Big(2(x_4 + \mathrm{i}x_5)x_1 - \big((x_4+\mathrm{i}x_5)^2 - 1\big)x_2 - \mathrm{i}\big((x_4+\mathrm{i}x_5)^2+1\big)x_6\Big)^2 \tag{5.24}$$
  with g_2 as in example 5.6.8, since $\mathrm{rk}\mathcal{H}g = 5$ and ∇g does not satisfy DP+. If $n > 6$, then the existence of a map g with the desired properties follows from vii) below.

vii) Assume $n \geq 7$ and $4 \leq r \leq n-1$. From corollary 5.6.7 and the equivalence of linear triangularizability and strong nilpotency of the Jacobian, which is shown in [24, §7.4], it follows that homogenizations with $x_{n-1} + \mathrm{i}x_n$ of gradient maps in $\mathbb{C}[\hat{x}]^{n-2}$ that do not satisfy DP+ are not linearly triangularizable, where $\hat{x} = x_1, x_2, \ldots, x_{n-2}$. So the gradient maps of g_3, g_4, g_5 and g_6 in example 5.6.8 are not linearly triangularizable. This gives the case $n = 7$.

The case $4 \leq r < n-1$ is similar to the case $3 \leq r < n-1$ in vi). So let $n > 7$ and $r = n-1$. We distinguish two cases:

- n is odd.
  Notice that the algebraic relations between the components of ∇g_6 in example 5.6.8 are generated by $y_6 + \mathrm{i}y_7$. So assume inductively that there exists a $\hat{g} \in \mathbb{C}[\hat{x}]$ that is homogeneous of degree 6 such that the algebraic relations between the components of $\nabla_{\hat{x}}\hat{g}$ are generated by $y_{n-3} + \mathrm{i}y_{n-2}$.
  Put $g = \hat{g} + x_{n-2}(x_{n-1} + \mathrm{i}x_n)^5$. Then it is clear that $y_{n-1} + \mathrm{i}y_n$ is a relation between the components of ∇g. Now all other such relations are generated by $y_{n-1} + \mathrm{i}y_n$, because x_{n-2} and $x_{n-1} + \mathrm{i}x_n$ are algebraically independent over $\mathbb{C}(\nabla_{\hat{x}}\hat{g}) = \mathbb{C}(\nabla_{x_1,x_2,\ldots,x_{n-3}} g)$, and

$\mathbb{C}(x_{n-2}, x_{n-1} + \mathrm{i}x_n) = \mathbb{C}((\frac{\partial}{\partial x_{n-3}} + \mathrm{i}\frac{\partial}{\partial x_{n-2}})g, \frac{\partial}{\partial x_{n-1}}g)$. By substituting $x_{n-1} + \mathrm{i}x_n = 0$ in $\mathcal{H}g$, we obtain $\mathcal{H}\hat{g}$. From the equivalence of linear triangularizability and strong nilpotency of the Jacobian, which is shown in [24, §7.4], it follow inductively that $\mathcal{H}g$ is not linearly triangularizable, as desired.

- n is even.
  From the proof of the above case, we obtain that we may assume that $n = 8$, provided we give a polynomial g such that the algebraic relations between the components of ∇g are generated by $y_7 + \mathrm{i}y_8$. Now the homogenization with $x_7 + \mathrm{i}x_8$ of $g + x_5$, with g as in (5.24), has the desired properties. This follows from corollary 5.6.7 and a computation of the Hessian rank. $\square$

Chapter 6

Power linear Keller maps and Zhao graphs

6.1 Introduction

Definition 6.1.1. A map $H \in \mathbb{C}[x]^n$ is called *power linear (of degree d)* if H is of the form $(Ax)^{*d}$ (H_i is a power of a linear form for all i). A map $F \in \mathbb{C}[x]^n$ is called *special power linear (of degree d)*, if it is of the form $x + (Ax)^{*d}$ ($F_i - x_i$ is a power of a linear form for all i, or similarly $F - x$ is power linear). A *power linear Keller map* is a special power linear map that satisfies the Keller condition (Keller maps cannot be power linear maps, so the word 'special' is redundant). So if F is a power linear Keller map, then $F - x$ is a power linear map.

The following theorem is proved by several authors: i) is proved by Van den Essen in [23], but the essential idea comes from [18] by Drużkowski. iii) is proved by Cheng in [10, Th. 2].

Theorem 6.1.2 (Cheng, Drużkowski, Van den Essen)**.** *Assume $x + (Ax)^{*d}$ is a power linear Keller map and let $r := \mathrm{rk} A$. If for all homogeneous Keller maps $x + H$ of degree d in dimension r,*

i) $x + H$ is invertible,

ii) $x + H$ is tame,

iii) H is linearly triangularizable, *iv) H satisfies DP(+),*

then $x + (Ax)^{*d}$ or $(Ax)^{*d}$ respectively has this property as well.

Proof. Since $\mathcal{J}(Ax)^{*d}$ is nilpotent, $r < n$ follows. From theorem 6.2.11, it follows that there exists a homogeneous $H \in \mathbb{C}[x_1, x_2, \ldots, x_r]^r$ such that H and $(Ax)^{*d}$ are GZ-paired (a property of a homogeneous map on one hand and a power linear map on the other hand, see definition 6.2.6 for the precise definition). From i) of proposition 6.2.7, we obtain that $x + H$ is a homogeneous Keller map. Now i), ii), and iii) follow from proposition 6.2.7 and iv) follows from proposition 6.3.3. □

We study GZ-pairing in sections 6.2 and 6.3.

Observe that if $H = (Ax)^{*d}$ is power linear, then

$$\mathcal{J}H = \operatorname{diag}\big((Ax)^{*(d-1)}\big) \cdot dA$$

This leads to the following definition.

Definition 6.1.3. We call M a *power linear quasi-Jacobian* if M is of the form

$$M = \operatorname{diag}\big((Ax)^{*(d-1)}\big) \cdot B$$

for some matrices $A \in \operatorname{Mat}_{N,n}(\mathbb{C})$ and $B \in \operatorname{Mat}_N(\mathbb{C})$, where $N \geq n$.

Power linear quasi-Jacobians can be Jacobians of power linear maps, but also the Zhao-matrix in section 6.4, which is associated with homogeneous symmetric Keller maps, is a power linear quasi-Jacobian. We shall derive some results about Zhao matrices in section 6.5. Furthermore, the results of chapter 7 will be formulated for power linear quasi-Jacobians instead of Jacobians where possible, and results about Zhao matrices will be derived.

But first, we formulate a proposition to obtain a cubic linear map with nilpotent Jacobian that does not satisfy DP:

Proposition 6.1.4. *Let $H = (H_1, H_2, \ldots, H_n)$ be homogeneous of degree d such that $\mathcal{J}H$ is nilpotent and the components of H are linearly independent over $\mathbb{C}$. Assume further that there are N d-th powers $L_1^d, L_2^d, \ldots, L_N^d$ of linear forms L_i, such that each component H_i of H can be written as a linear combination of these linear powers. If the N linear powers are linearly independent over $\mathbb{C}$, then there exists a homogeneous power linear map $G = (G_1, G_2, \ldots, G_N)$ of degree d such that $\mathcal{J}G$ is nilpotent, the components of G are linearly independent and* $\operatorname{rk}\mathcal{J}G = \operatorname{rk}\mathcal{J}L \leq n$.

Proof. Since $H_i \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ for all $i \le n$, we may assume that $L_i \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ for all $i \le N$ as well (substitute $x_{n+1} = x_{n+2} = \cdots = 0$). The components of H are linearly independent, so we can extend them to a basis of $\mathbb{C}L_1^d \oplus \mathbb{C}L_2^d \oplus \cdots \oplus \mathbb{C}L_N^d$ by adding linear powers L_i^d. So after renumbering the L_i's, we have

$$\mathbb{C}L_1^d \oplus \mathbb{C}L_2^d \oplus \cdots \oplus \mathbb{C}L_N^d = \mathbb{C}H_1 \oplus \cdots \oplus \mathbb{C}H_n \oplus \mathbb{C}L_{n+1}^d \oplus \cdots \oplus \mathbb{C}L_N^d \quad (6.1)$$

Now let $F := (H_1, H_2, \ldots, H_n, L_{n+1}^d, L_{n+2}^d, \ldots L_N^d)$ and $X := x_1, x_2, \ldots, x_N$. Since $L_i \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ for all i, $\mathcal{J}_X F$ is nilpotent as well. Write

$$F = T \cdot (L_1^d, L_2^d, \ldots, L_N^d)$$

From (6.1), it follows that $T \in \mathrm{GL}_N(\mathbb{C})$. Put

$$G := T^{-1}F(TX) = (L_1^d, L_2^d, \ldots, L_N^d)|_{X=TX}$$

Then G is a power linear map and $\mathcal{J}_X G$ is nilpotent. Furthermore, $\mathrm{rk}\mathcal{J}_X G = \mathrm{rk}\mathcal{J}_X(L_1^d, L_2^d, \ldots, L_N^d) = \mathrm{rk}\mathcal{J}_X(L_1, L_2, \ldots, L_N)$, for $\mathcal{J}_X(L_1^d, L_2^d, \ldots, L_N^d) = d \cdot \mathrm{diag}(L_1^{d-1}, L_2^{d-1}, \ldots, L_N^{d-1}) \cdot \mathcal{J}_X(L_1, L_2, \ldots, L_N)$. □

The above proposition can be used to construct a cubic linear counterexample to the Linear dependence problem in dimension 53. The number 53 appears on the VW beetle Herbie.

Example 6.1.5. The following map is a slight variation on the map of corollary 4.2.5 with $n = 10$:

$$H = \begin{pmatrix} 6x_{10}(x_9x_1 - x_{10}x_2) \\ 6x_9(x_9x_1 - x_{10}x_2) \\ 6x_{10}(x_9x_3 - x_{10}x_4) \\ 6x_9(x_9x_3 - x_{10}x_4) \\ 6x_{10}(x_9x_5 - x_{10}x_6) \\ 6x_9(x_9x_5 - x_{10}x_6) \\ 6x_9(x_1x_4 - x_2x_3) \\ 6x_9(x_3x_6 - x_4x_5) \\ 6(x_8(x_1x_4 - x_2x_3) - x_7(x_3x_6 - x_4x_5)) \\ x_9^3 \end{pmatrix}$$

and by the proof of corollary 4.2.5, we see that H is a cubic homogeneous counterexample to the Linear dependence problem as well ($\mathcal{J}H$ is nilpotent).

Furthermore, there are 53 cubic linear powers such that each component of H can be written as a $\mathbb{Z}$-linear combination of these powers:

$$\begin{aligned}
H_1 &= (x_1+x_9+x_{10})^3-(x_1+x_9)^3-(x_1+x_{10})^3+x_1^3- \\
&\quad (x_9+x_{10})^3+x_9^3+x_{10}^3-(x_2+x_{10})^3-(x_2-x_{10})^3+2x_2^3 \\
H_2 &= -(x_2+x_9+x_{10})^3+(x_2+x_9)^3+(x_2+x_{10})^3-x_2^3+ \\
&\quad (x_9+x_{10})^3-x_9^3-x_{10}^3+(x_1+x_9)^3+(x_1-x_9)^3-2x_1^3 \\
H_3 &= (x_3+x_9+x_{10})^3-(x_3+x_9)^3-(x_3+x_{10})^3+x_3^3- \\
&\quad (x_9+x_{10})^3+x_9^3+x_{10}^3-(x_4+x_{10})^3-(x_4-x_{10})^3+2x_4^3 \\
H_4 &= -(x_4+x_9+x_{10})^3+(x_4+x_9)^3+(x_4+x_{10})^3-x_4^3+ \\
&\quad (x_9+x_{10})^3-x_9^3-x_{10}^3+(x_3+x_9)^3+(x_3-x_9)^3-2x_3^3 \\
H_5 &= (x_5+x_9+x_{10})^3-(x_5+x_9)^3-(x_5+x_{10})^3+x_5^3- \\
&\quad (x_9+x_{10})^3+x_9^3+x_{10}^3-(x_6+x_{10})^3-(x_6-x_{10})^3+2x_6^3 \\
H_6 &= -(x_6+x_9+x_{10})^3+(x_6+x_9)^3+(x_6+x_{10})^3-x_6^3+ \\
&\quad (x_9+x_{10})^3-x_9^3-x_{10}^3+(x_5+x_9)^3+(x_5-x_9)^3-2x_5^3 \\
H_7 &= (x_1+x_4+x_9)^3-(x_1+x_4)^3-(x_1+x_9)^3-(x_4+x_9)^3+ \\
&\quad x_1^3+x_4^3-(x_2+x_3+x_9)^3+(x_2+x_3)^3+(x_2+x_9)^3+ \\
&\quad (x_3+x_9)^3-x_2^3-x_3^3 \\
H_8 &= (x_3+x_6+x_9)^3-(x_3+x_6)^3-(x_3+x_9)^3-(x_6+x_9)^3+ \\
&\quad x_3^3+x_6^3-(x_4+x_5+x_9)^3+(x_4+x_5)^3+(x_4+x_9)^3+ \\
&\quad (x_5+x_9)^3-x_4^3-x_5^3 \\
H_9 &= (x_1+x_4+x_8)^3-(x_1+x_4)^3-(x_8+x_1)^3-(x_8+x_4)^3+ \\
&\quad x_1^3+2x_4^3-(x_2+x_3+x_8)^3+(x_2+x_3)^3+(x_8+x_2)^3+ \\
&\quad (x_8+x_3)^3-x_2^3-(x_3+x_6+x_7)^3+(x_3+x_6)^3+ \\
&\quad (x_7+x_3)^3+(x_7+x_6)^3-x_6^3-2x_3^3+(x_4+x_5+x_7)^3- \\
&\quad (x_4+x_5)^3-(x_7+x_4)^3-(x_7+x_5)^3+x_5^3 \\
H_{10} &= x_9^3
\end{aligned}$$

Since these 53 linear powers are linearly independent, it follow from proposition 6.1.4 that there exists a cubic linear counterexample to the Linear dependence problem in dimension 53: the Herbie example.

6.2 GZ-pairing and the linear dependence problem

Let us start with a very useful lemma:

Lemma 6.2.1. *Let $L_1, L_2, \ldots, L_r \in K[x]$ be linear such that $2 \leq r \leq d+1$ and*

$$\lambda_1 L_1^d + \lambda_2 L_2^d + \ldots + \lambda_r L_r^d = 0 \tag{6.2}$$

for some $\lambda \in \mathbb{C}^r \backslash \{0\}$, where K is a field such that either $K \supseteq \mathbb{Q}$ or the characteristic of K is greater than d. Then there are $i \neq j$ and an $s \in \mathbb{C}$ such that $L_i = sL_j$.

Proof. Since $\lambda \neq 0$, we may assume without loss of generality that $\lambda_1 \neq 0$. If $L_1 \neq sL_r$ or $L_r \neq sL_1$ for some $s \in K$, then we are done, so we may assume that L_1 and L_r are independent. So there exists a linear basis $a_1, a_2, \ldots, a_n$ of $\mathbb{C}[x]$ with $a_1 = L_1$ and $a_2 = L_r$.
The case $d = 1$ is easy, so assume $d \geq 2$. Differentiating (6.2) with respect to a_1 gives

$$\mu_1 L_1^{d-1} + \mu_2 L_2^{d-1} + \ldots + \mu_{r-1} L_{r-1}^{d-1} = 0$$

for certain $\mu_i \in K$. In particular, $\mu_1 = d\lambda_1 \neq 0$, so $\mu \in \mathbb{C}^{r-1} \backslash \{0\}$ and the result follows by induction on d. □

The following proposition shows that one can always find powers of linear forms as in example 6.1.5.

Proposition 6.2.2. *Let $f \in K[x]$ be a homogeneous polynomial of degree d over a field K such that either $K \supseteq \mathbb{Q}$ or the characteristic of K is greater than d. Then f can be written as a linear combination of linear powers of degree d.*

Proof. Since each polynomial is a sum of monomials, we may assume that f is a monomial. Assume first that $f = x_1^r x_2^{d-r}$. We show that f can be written as a sum of $x_1^d, (x_1 + x_2)^d, (x_1 + 2x_2)^d, \ldots, (x_1 + dx_2)^d$. So assume that this is not the case. The space of homogeneous polynomials in x_1 and x_2 of degree d is $(d+1)$-dimensional, for it is generated by the $d+1$ polynomials $x_1^d, x_1^{d-1}x_2, \ldots, x_1 x_2^{d-1}, x_2^d$. Since the $d+1$ linear powers $x_1^d, (x_1+x_2)^d, (x_1 + 2x_2)^d, \ldots, (x_1 + dx_2)^d$ do not generate all homogeneous polynomials in x_1 and x_2 of degree d, they are linearly dependent. This contradicts lemma 6.2.1 above, as desired.

Assume next that f is a monomial in $m \geq 3$ indeterminates and that every monomial of degree d in less than m indeterminates can be written as a sum of d-th powers of linear forms. Then $f = gh$, where g is a $(d-e)$-th power of an indeterminate and h has only $m-1$ indeterminates.

By induction on m, we obtain that we can write

$$h = \lambda_1 L_1^e + \lambda_2 L_2^e + \cdots + \lambda_r L_r^e$$

for linear forms L_i and $\lambda_i \in K$. Since $f = \lambda_1 g L_1^e + \lambda_2 g L_2^e + \cdots + \lambda_r g L_r^e$, it suffices to write gL_i^e as a linear combination of linear d-th powers for each i. For that purpose, we first write gt^e as a linear combination of linear d-th powers, which is possible because gt^e is bivariate. Next, we substitute $t = L_i$. □

Corollary 6.2.3. *Assume $H = (H_1, H_2, \ldots, H_n)$ is homogeneous of degree d with a nilpotent Jacobian. If H does not satisfy DP, then there is an $N \geq n$ and a power linear map $G \in \mathbb{C}[X]^N$ of degree d with a nilpotent Jacobian that does not satisfy DP either. Furthermore, $\mathrm{rk}\mathcal{J}G \leq n$.*

The above corollary was simultaneously obtained by Dayan Liu, Xiankun Du and Xiaosong Sun in [13]. Unfortunately, their Herbie has the number 67 on it.

Corollary 6.2.4. *For each $d \geq 3$, the Jacobian conjecture can be reduced to power linear Keller maps of degree d.*

Proof. The proof is left as an exercise to the reader. First, reduce to special homogeneous Keller maps of degree d, using [24, Prop. 6.2.13]. After that, reduce to d-th power linear Keller maps using proposition 6.2.2. □

The following proposition is the power linear variant of proposition 4.6.3. We will use it later.

Proposition 6.2.5. *A power linear map $(Ax)^{*d}$ satisfies DP(+), if and only if there exists a $T \in \mathrm{GL}_n(\mathbb{C})$ such that $T^{-1}(ATx)^{*d}$ is of the form $(Bx)^{*d}$ with $B_1 = 0$ and B_2 dependent of e_1^{t}.*

Proof. The backward implication goes similar as in the proof of proposition 4.6.3. The forward implication goes similar as well. At first, we can choose the rows of T^{-1} of the form e_i^{t} as far as they are not equal to λ (or μ), in order to obtain that $T^{-1}(ATx)^{*d}$ is power linear. So we obtain $(B_1x)^d \in \mathbb{C}$ and $(B_2x)^d \in \mathbb{C}[x_1]$. But that implies that $B_1 = 0$ and that B_2 is dependent of e_1^{t}, as desired. $\square$

Write $X = x_1, x_2, \ldots, x_n, \ldots, x_N$ and let E_i be the i-th standard basis vector of size N. The following definition of *GZ-paired* is essentially that of *paired* in [32], see also [24, §6.4]. It is somewhat more general in the sense that $d \geq 2$ instead of $d = 3$, and that G does not need to be power linear, but the latter is for technical convenience only.

Definition 6.2.6. Assume $n \leq N$ and let $H = (H_1, H_2, \ldots, H_n) \in \mathbb{C}[x]^n$ and $G = (G_1, G_2, \ldots, G_N)$ be homogeneous of degree $d \geq 2$. Then H and G are *weakly GZ-paired* (through B and C) if there exist $B \in \mathrm{Mat}_{n,N}(\mathbb{C})$ and $C \in \mathrm{Mat}_{N,n}(\mathbb{C})$ such that

i) $H = BG(Cx)$,

ii) $BC = I_n$,

iii) $\ker B \subseteq \ker \mathcal{J}G$.

If $n < N$ and

iii′) $\ker B = \ker \mathcal{J}G$.

then H and G are *GZ-paired* (through B and C).

Notice that ii) implies that $\mathrm{rk}B = \mathrm{rk}C = n$. Assume H and $G = (AX)^{*d}$, are *GZ-paired* through B and C. Then by iii′), $\ker A = \ker \mathcal{J}G = \ker B$, whence

$$\mathrm{rk}A = n \tag{6.3}$$

as well. Since $n < N$, $\mathrm{cork}A = N - n > 0$. So $\det A = 0$.

Proposition 6.2.7. *Let $H \in \mathbb{C}[x]^n$ and $G \in \mathbb{C}[X]^N$ be homogeneous of degree $d \geq 2$ and assume H and G are (weakly) GZ-paired through B and C. Then the following holds.*

i) $S^{-1}H(Sx)$ and $T^{-1}G(TX)$ are (weakly) GZ-paired through $S^{-1}BT$ and $T^{-1}CS$ for all $S \in \mathrm{GL}_n(\mathbb{C})$ and $T \in \mathrm{GL}_N(\mathbb{C})$,

ii) $\det \mathcal{J}(x + H) = 1$, if and only if $\det \mathcal{J}_X(X + G) = 1$.

iii) *$X+G$ is invertible, if and only if $x+H$ is.*

iv) *$X+G$ is tame in dimension $M \geq N$, if and only if $x+H$ is.*

v) *G is linearly triangularizable, if and only if H is.*

vi) *$G = G(CBX)$ and $H(BX) = BG$,*

vii) $\det \mathcal{J}(x+H) = \det(\mathcal{J}_X(X+G))|_{X=Cx}$,

viii) $\det \mathcal{J}_X(X+G) = \det(\mathcal{J}(x+H))|_{x=BX}$,

ix) *If $\tilde{C} \in \mathrm{Mat}_{N,n}(\mathbb{C})$ and $B\tilde{C} = I_n$, then H and G are (weakly) GZ-paired through B and $\tilde{C}$ as well.*

Proof.

i) This follows directly from the definitions,

$$\ker \mathcal{J}(T^{-1}G(TX)) = \ker \mathcal{J}(G(TX)) = T^{-1}\ker(\mathcal{J}G)|_{X=TX}$$

and

$$T^{-1}\ker B|_{X=TX} = T^{-1}\ker B = \ker BT = \ker S^{-1}BT$$

ii) This follows from vii).

iii) Since $B \in \mathrm{Mat}_{n,N}(\mathbb{C})$ has full rank, $\dim\ker B = N - n$. Let $M \in \mathrm{Mat}_{N,N-n}(\mathbb{C})$ such that the columns of M are a basis of $\ker B$ and define $T := (C \mid M)$. Since for each $i \leq n$, the i-th column e_i of $BT = (I_n \mid \emptyset)$ is independent of the other columns of BT, it follows that for each $i \leq n$ the i-th column of T is independent of the other columns of T. Together with the fact that the columns of M are independent, we obtain that T is invertible. Moreover, one can easily see that the first n rows of T^{-1} are the rows of B.

Since $\ker \mathcal{J}G \supseteq \ker B$, $(\mathcal{J}G)M = 0$ follows. Consequently, the last $N-n$ columns of $(\mathcal{J}G)T$ are zero, whence $(\mathcal{J}G)T = ((\mathcal{J}G)C \mid \emptyset)$ and $F := T^{-1}G(TX) = T^{-1}G(Cx)$. Furthermore, by the fact that the first n rows of T^{-1} are the rows of B, we obtain $(F_1, F_2, \ldots, F_n) = BG(Cx) = H$. So for $i = 1$, we have

$$\mathcal{J}_X F^i = \begin{pmatrix} \mathcal{J}H^i & \emptyset \\ * & \emptyset \end{pmatrix} \tag{6.4}$$

and hence also for $i = 2, 3, \ldots$. We obtain that $X + F$ is invertible, if and only if $x + H$ is. So $X + G$ is invertible, if and only if $x + H$ is.

iv) This goes similar to iii), using that $X + (H, F_{n+1}, \ldots, F_{m-1}, 0^m, 0^{m+1}, \ldots, 0^N)$ and $X + (H, F_{n+1}, \ldots, F_{m-1}, F_m, 0^{m+1}, \ldots, 0^N)$ differ an elementary map from the right.

v) This goes more or less similar to iii). From (6.4), we obtain that $\mathcal{J}H$ is nilpotent, if and only if $\mathcal{J}_X F$ is, but in a similar manner, one can see that $\mathcal{J}H$ is strongly nilpotent, if and only if $\mathcal{J}_X F$ is.

   By way of the equivalence of linear triangularizability and strong nilpotency of the Jacobian, which is shown in [24, §7.4], we see that that G is linearly triangularizable, if and only if H is.

vi) Since $F = T^{-1}G(TX) = T^{-1}G(Cx)$ and $G(Cx) = G((C \mid \emptyset)X)$, we obtain $TF(T^{-1}X) = G = G((C \mid \emptyset)T^{-1}X)$, and $G = G(CBX)$ follows because the first n rows of T^{-1} are exactly B. Substituting $x = BX$ in $H = BG(Cx)$ gives $H(BX) = BG(CBX) = BG$, as desired.

vii) Notice that by (6.4) with $i = 1$, $\mathcal{J}_X(X+F) = (\mathcal{J}_X(X+F))|_{X=(x,0,\ldots,0)}$. Consequently,

$$\begin{aligned}
\det \mathcal{J}(x+H) &= \det \mathcal{J}_X(X+F)|_{X=(x,0,\ldots,0)} \\
&= \det \Big(T^{-1}\big(\mathcal{J}_X(X+G)\big)\big|_{X=TX}T\Big)\Big|_{X=(x,0,\ldots,0)} \\
&= \det \Big(T^{-1}\big(\mathcal{J}_X(X+G)\big)\big|_{X=Cx}T\Big) \\
&= \det \big(\mathcal{J}_X(X+G)\big)|_{X=Cx}
\end{aligned}$$

viii) Notice that $(\mathcal{J}(x+H))|_{X=T^{-1}X} = (\mathcal{J}(x+H))|_{x=BX}$, whence

$$\begin{aligned}
\det \mathcal{J}_X(X+G) &= \det \mathcal{J}\big(X + TF(T^{-1}X)\big) \\
&= \det \Big(T\big(\mathcal{J}_X(X+F)\big)\big|_{X=T^{-1}X}T^{-1}\Big) \\
&= \det \big(\mathcal{J}(x+H)\big)\big|_{X=T^{-1}X} \\
&= \det \big(\mathcal{J}(x+H)\big)\big|_{x=BX}
\end{aligned}$$

ix) Assume $\tilde{C} \in \mathrm{Mat}_{N,n}(\mathbb{C})$ and $B\tilde{C} = I_n$. By substituting $X = \tilde{C}x$ in $H(BX) = BG$ obtained in vi), we see that

$$H = H(B\tilde{C}x) = BG\big(\tilde{C}x)\big)$$

so H and G are (weakly) GZ-paired through B and $\tilde{C}$ as well. $\square$

The following theorem says that each homogeneous H of degree ≥ 2 that cannot be made power linear by way of linear conjugations can be GZ-paired with a power linear map, but unlike [32] and [24, §6.4], it takes linear relations between the components into account.

Theorem 6.2.8. *Let $H \in \mathbb{C}[x]^n$ be homogeneous of degree $d \geq 2$. Assume that there are exactly m independent linear relations between the components of H and that H is degenerate of order s (m and s may be zero).*

Assume in addition that there exist $M-m-s$ linear forms $L_{m+1}, L_{m+2}, \dots,$ L_{N-s} that are linearly independent over $\mathbb{C}$, such that each component of H can be written as a linear combination of $L_{m+1}^d, L_{m+2}^d, \dots, L_{M-s}^d$.

Then there exists an $N \geq n$ with $M - s \leq N \leq M$ and a power linear map $G \in \mathbb{C}[X]^N$ of degree d such that H and G are weakly GZ-paired. Up to the condition $N > n$, H and G are even GZ-paired. Furthermore, H and $(G_1, G_2, \dots, G_{M-s})$ are weakly GZ-paired, $G_1 = G_2 = \cdots = G_m = 0$ and $G_{m+1}, G_{m+2}, \dots, G_N$ are linearly independent over $\mathbb{C}$.

Proof. Since H is degenerate of order s, we can obtain $H \in \mathbb{C}[x_1, x_2, \dots,$ $x_{n-s}]$, by replacing H by $S^{-1}H(Sx)$ for a suitable $S \in \mathrm{GL}_n(\mathbb{C})$. This is allowed on account of i) of proposition 6.2.7.

Assume L is degenerate of order s'. Then $0 \leq s' \leq s$, so $N := M-s+s' \leq M$. Again by replacing H by $S^{-1}H(Sx)$ for a suitable $S \in \mathrm{GL}_n(\mathbb{C})$, we may assume that $L_i \in \mathbb{C}[x_1, x_2, \dots, x_{n-s'}]$ for all $i \leq M - s$.

i) Since there are exactly m linear relation between the components of H, it follows that the dimension of the vector space

$$\mathbb{C}H_1 + \mathbb{C}H_2 + \cdots + \mathbb{C}H_n$$

over $\mathbb{C}$ is $n - m$. It is a subspace of codimension $(M - s - m) - (n - m) + s' = N - n$ of

$$V := (\mathbb{C}L_{m+1}^d \oplus \mathbb{C}L_{m+2}^d \oplus \cdots \oplus \mathbb{C}L_{M-s}^d) \oplus (\mathbb{C}x_{n-s'+1}^d \oplus \cdots \oplus \mathbb{C}x_n^d)$$

so after renumbering the L_i's and defining $L_{N-s'+i} := x_{n-s'+i}$ for all $i \geq 1$, we have

$$\begin{aligned} V &= (\mathbb{C}H_1 + \cdots + \mathbb{C}H_n) \oplus (L_{n+1}^d \oplus \cdots \oplus \mathbb{C}L_{N-s'}^d) \oplus \\ &\quad (\mathbb{C}x_{n-s'+1}^d \oplus \cdots \oplus \mathbb{C}x_n^d) \\ &= (\mathbb{C}H_1 + \cdots + \mathbb{C}H_n) \oplus (L_{n+1}^d \oplus \cdots \oplus L_{N-1}^d \oplus \mathbb{C}L_N^d) \quad (6.5) \end{aligned}$$

Furthermore, $L_{m+1}^d, L_{m+2}^d, \dots, L_{N-1}^d, L_N$ are linearly independent over $\mathbb{C}$.

ii) Now let
$$F := (H_1, \dots, H_n, L_{n+1}^d, \dots, L_{N-1}^d, L_N^d)$$
Since $(L_{m+1}^d, L_{m+2}^d, \dots, L_{N-s'}^d)$ is degenerate of order $\le s'$ with respect to x, we obtain that $(L_{m+1}^d, L_{m+2}^d, \dots, L_{N-1}^d, L_N^d)$ is not degenerate with respect to x. It follows that $\ker \mathcal{J}(L_{m+1}^d, L_{m+2}^d, \dots, L_{N-1}^d, L_N^d) = \{0\}^n \times \mathbb{C}^{N-n}$, and by (6.5), we obtain $\ker \mathcal{J}F = \{0\}^n \times \mathbb{C}^{N-n}$ as well. Consequently, H and F are weakly GZ-paired through B and C, where
$$B = (I_n \mid \emptyset) \qquad \text{and} \qquad C = \begin{pmatrix} I_n \\ \emptyset \end{pmatrix}$$
and $\ker B = \ker \mathcal{J}F$.

iii) From (6.5), it follows that there exist a $\tilde{T} \in \mathrm{Mat}_{N,N-m}(\mathbb{C})$ of rank $N - m$ such that
$$F = \tilde{T} \cdot (L_{m+1}^d, L_{m+2}^d, \dots, L_{N-1}^d, L_N^d)$$
By adding independent columns on the left of $\tilde{T}$, we see that there exist a $T \in \mathrm{GL}_n(\mathbb{C})$ of the form $T = (* \mid \tilde{T})$ such that
$$F = T \cdot (0^1, \dots, 0^m, L_{m+1}^d, L_{m+2}^d, \dots, L_{N-1}^d, L_N^d)$$
So if we define $L_1 = \cdots = L_m = 0$, then $F = T \cdot L^{*d}$. It follows from i) of proposition 6.2.7 that H and $G := T^{-1}F(TX) = L(TX)^{*d}$ are weakly GZ-paired through BT and $T^{-1}C$. Furthermore, we obtain by $\ker B = \ker \mathcal{J}F$ that
$$\ker(BT) = \ker(BT)|_{X=TX} = \ker(\mathcal{J}F \cdot T)|_{X=TX} = \ker \mathcal{J}G$$
Since $L_{m+1}^d, L_{m+2}^d, \dots, L_{N-1}^d, L_N^d$ are linearly independent over $\mathbb{C}$, it follows that H and G are GZ-paired in the desired way, except that we have $N \ge n$ instead of $N > n$. This gives the desired result.

If we do not add $\mathbb{C}x_{n-s+1}^d \oplus \cdots \oplus \mathbb{C}x_n^d$ to V, then we can prove in a similar manner that H and $(F_1, F_2, \dots, F_{N-s})$ are weakly GZ-paired through $(I_n|\emptyset)$ and $\binom{I_n}{\emptyset}$ and that H and $(G_1, G_2, \dots, G_{N-s})$ are weakly GZ-paired in the desired way. $\square$

Notice that in the above proof, the last $N-n$ rows of T, and hence those of T^{-1} also, are $E_{n+1}^{\mathrm{t}}, E_{n+2}^{\mathrm{t}}, \ldots, E_N^{\mathrm{t}}$, so the last $N-n$ rows of the matrix C in the above proof are zero. For $n=1$ and $H = x_1^d$, there does not exist a G such that H and G are GZ-paired in the way of theorem 6.2.8, so the condition that $N > n$ is necessary.

One can see that the maps H and G in proposition 6.1.4 are weakly GZ-paired.

Example 6.2.9. The map

$$\hat{H} = (x_2x_5, x_1^2 - x_3x_5, 2x_1x_2 - x_4x_5, x_2^2, 0)$$

is quadratic linear and its first four components are linearly independent over $\mathbb{C}$ ($\hat{H}$ is both the map of lemma 4.2.7 with $d=2$, $A=1$ and $B=x_5$, and a symmetric conjugation of (4.6.8)). Each component of $\hat{H}$ can be written as a linear combination of the nine quadratic powers

$$x_5^2, L_2^2, L_3^2, \ldots, L_9^2$$

where

$$\begin{aligned}(L_2, L_3, \ldots, L_9) \quad &:= \quad (x_2 + x_5, x_2 - x_5, x_1 + x_5, x_1 + x_2, 2x_1 + x_3 + x_5,\\ &\qquad 2x_1 + x_3 - x_5^2, x_3 + x_4 + x_5^2, x_3 + x_4 - x_5)\end{aligned}$$

The coefficients of x_5^2 are $0, -1, 2, -1, 0$ in this order, so each component of

$$\begin{aligned} H \quad &:= \quad \hat{H} - x_5^2 \cdot (0, -1, 2, -1, 0) \\ &= \quad (x_2x_5, x_1^2 - x_3x_5 + x_5^2, 2x_1x_2 - x_4x_5 - 2x_5^2, x_2^2 + x_5^2, 0)\end{aligned}$$

can be written as a linear combination of the eight quadratic powers $L_2^2, L_3^2, \ldots, L_9^2$. Furthermore, the L_i's are already in the right order, i.e.

$$\mathbb{C}L_2^2 \oplus \mathbb{C}L_3^2 \oplus \cdots \oplus \mathbb{C}L_9^2 = \mathbb{C}\tilde{H}_1 + \cdots + \mathbb{C}\tilde{H}_5 \oplus \mathbb{C}L_6^2 \oplus \cdots \oplus \mathbb{C}L_9^2$$

Taking

$$T = \begin{pmatrix} 0 & \frac{1}{4} & -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -\frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & -1 & 1 & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

in the proof of theorem 6.2.8 gives that H and $G := (AX)^{*d}$ are GZ-paired, where

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & -\frac{1}{4} & \frac{1}{4} & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & -\frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 1 & \frac{1}{4} & -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & -\frac{1}{4} & 1 & 0 & -\frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 1 & 0 & -1 & -1 & 1 & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ -1 & 0 & -1 & -1 & 1 & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ 1 & 0 & 0 & -1 & 1 & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ -1 & 0 & 0 & -1 & 1 & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

Notice that G satisfies DP. One can easily see that G does not satisfy DP+, however. Substituting $x_1 = 1$ in $(G_2, G_3, \ldots, G_9)$ gives an 8-dimensional map $\tilde{G}$ over $\mathbb{C}[x_2, x_3, \ldots, x_9]$ with nilpotent Jacobian that does not satisfy DP. The components of $\tilde{G}$ are squares of affinely linear polynomials.

The reader may show that the map $(0, 0, x_2^2 + x_2x_4, x_1x_3 - x_2x_5, x_1^2 + x_1x_4)$, a map of the type of theorem 4.6.7, can be GZ-paired with a quadratic linear map in dimension 9 as well. Dimension 9 is the best one can get for both types of maps. This will be shown in the proof of lemma 7.2.3 in the next chapter.

The following theorem shows us that theorem 6.2.8 can be used to find a G of minimal dimension such that H and G are GZ-paired (in the proof of 6.2.8, G has dimension $N = m + l + s' \le m + l + s = M$):

Theorem 6.2.10. *Let H, n, m, s as in theorem 6.2.8 and assume that exactly l linear d-th powers are necessary in order to write each component of H as a linear combination of these powers. Let $G := (AX)^{*d}$. If H and G are weakly GZ-paired, then $N \ge m + l$. If H and G are GZ-paired, then $N \ge m + l + s$.*

Proof. Say that H and G are weakly GZ-paired through matrices B and C. Since $H(x) = BG(Cx)$, each component of H can be written as a linear combination of the components of $G(Cx)$, which are linear powers. So $l \le N$. In particular, $N \ge m + l + s$ is satisfied in case $m = s = 0$.

Just as in the proof of theorem 6.2.8, we may assume that $H_i \in \mathbb{C}[x_1, x_2, \ldots, x_{n-s}]$ for all $i \le n$.

i) From $BC = I_n$, it follows that the rows of B are independent, i.e. $\lambda^{\mathrm{t}} \mapsto \lambda^{\mathrm{t}} B$ is injective. From this and $H(x) = BG(Cx)$, it follows that there are at least as many linear relations between the components of $G(Cx)$ as there are between the components of H. So there are at least m independent linear relations between the components of $G(Cx)$.

   Say that there are $m' \geq m$ independent linear relations between the components of $G(Cx)$. Replacing G by $P^{-1}G(PX)$ (and B by BP and C by $P^{-1}C$) for a suitable permutation P, we may assume that the last $N - m'$ components of $G(Cx)$ are linearly independent and thus each component of G is linearly dependent of $G_{m'+1}, G_{m'+2}, \ldots, G_N$.

   From $H = BG(Cx)$, we obtain that each component of H is linearly dependent of $G_{m'+1}(Cx), G_{m'+2}(Cx), \ldots, G_N(Cx)$, whence $l \leq N - m'$. So $N \geq m' + l \geq m + l$, as desired.

ii) Assume from now on that H and G are GZ-paired. Since each component of G is linearly dependent of $G_{m'+1}, G_{m'+2}, \ldots, G_N$, each row of $\mathcal{J}G = d \cdot \operatorname{diag}((AX)^{*(d-1)}) \cdot A$ is dependent of the last $N - m'$ rows of $\mathcal{J}G$.

   Since $d \cdot \operatorname{diag}((AX)^{*(d-1)})$ is invertible over $\mathbb{C}(x_1, x_2, \ldots, x_N)$, each row of A is dependent of the last $N - m'$ rows of A. This is not affected by multiplication with the matrix $\hat{C}$ consisting of the last s columns of C. So each row of $A\hat{C}$ is dependent of the last $N - m'$ rows of $A\hat{C}$.

iii) From $BC = I_n$ and $\ker A = \ker B$, it follows that $\ker AC = \{0\}$. So the columns of AC, and hence those of $A\hat{C}$ as well, are independent. Since the first m' rows of $A\hat{C}$ are dependent of the last $N - m'$ rows, there exist a permutation P within the coordinates $x_{m'+1}, x_{m'+2}, \ldots, x_N$, such that the lower right minor of size s of $P^{-1}AC = P^{-1}AP \cdot P^{-1}C$ is invertible.

   Replacing G by $P^{-1}G(PX)$ (and changing B and C accordingly), we may assume that the lower right $(s \times s)$-minor of AC is invertible. On account of this fact, we can choose an $(n \times n)$-matrix S of the form

$$S = \begin{pmatrix} I_{n-s} & \emptyset \\ * & \emptyset \end{pmatrix}$$

   such that the last s rows of ACS are zero.

iv) Since $H_i \in K[x_1, x_2, \ldots, x_{n-s}]$ for all i, we have $H(x) = H(Sx)$. It follows from $H(x) = BG(Cx) = B(ACx)^{*d}$ that

$$H(x) = H(Sx) = B \cdot G(CSx) = B \cdot (ACSx)^{*d} \tag{6.6}$$

Notice that the last s components of $G(CSx) = (ACSx)^{*d}$ are zero.

So both the first m' and the last s components of $G(CSx) = (ACSx)^{*d}$ are dependent of the components in between. So by (6.6), at most $N - m' - s$ components of $G(CSx) = (ACSx)^{*d}$ are needed to write each H_i as a linear combination of them, i.e. $N - m' - s \geq l$. It follows that $N \geq m' + l + s \geq m + l + s$, as desired. $\square$

The following theorem shows that GZ-pairing can be done reversed as well: one can start with a power linear map G and find an H such that H and G are GZ-paired. This was useful for the first theorem in this chapter.

Theorem 6.2.11. *Assume $G \in \mathbb{C}[X]^N$ is power linear of degree $d \geq 2$ and $\mathrm{rk}\mathcal{J}G = n < N$. Then there exists a $H \in \mathbb{C}[x]^n$ such that H and G are GZ-paired.*

Proof. Since G is power linear with Jacobian rank n, G is of the form $G = (AX)^{*d}$ such that $\mathrm{rk}A = n$. It follows that there exists a $T \in \mathrm{GL}_N(\mathbb{C})$ such that the rightmost $N-n$ columns of AT^{-1} are zero. Put $F := TG(T^{-1}X) = T(AT^{-1}X)^{*d}$ and let $H := (F_1, F_2, \ldots, F_n)$.

From i) of proposition 6.2.7, it follows that in order to prove that H and G are GZ-paired, it suffices to show that H and F are GZ-paired. For that purpose, define $B := (E_1^{\mathrm{t}}, E_2^{\mathrm{t}}, \ldots, E_n^{\mathrm{t}})$ and $C := (E_1 \mid E_2 \mid \cdots \mid E_n)$. Then $H = BF(Cx)$ and $BC = I_n$. Since $\mathrm{rk}\mathcal{J}F = n$ and $F \in \mathbb{C}[x]^N$, it follows that $\ker \mathcal{J}F = \{0\}^n \times \mathbb{C}^{N-n} = \ker B$, as desired. $\square$

v) of proposition 6.2.7 was simultaneously obtained as [14, Th. 3(2)] by Dayan Liu, Xiankun Du and Xiaosong Sun. Furthermore, the authors observe in [14, Lm. 5] that $A = ACB$ in case H and $(AX)^{*d}$ are GZ-paired through B and C. This is also true if H and $(AX)^{*d}$ are weakly GZ-paired: since $B(CB - I_n) = 0$ and $\ker B \subseteq \ker A$, $A(CB - I_n) = 0$ as well. [14, Th. 3(1)] can be used to prove the following.

Proposition 6.2.12. *Assume G is power linear over $\mathbb{C}$ and $\mathcal{J}G^3 = 0$. Then $x + G$ is invertible.*

Proof. The proof of this is more or less similar to that of [14, Th. 4], which states that G is linearly triangularizable in case $\deg G = 2$. From [14, Th. 3(1)], it follows that there exists a H with $\mathcal{J}H^2 = 0$ such that H and G are GZ-paired. Since $\mathcal{J}H \cdot H = \deg H \cdot \mathcal{J}H^2 = 0$, we obtain by iii) of proposition 3.1.2 that $x + H$ is invertible (see also [18, Th. 4]). Now apply iii) of proposition 6.2.7. □

More results of this type can be obtained with results of [56] and [57].

6.3 DP-fair GZ-pairing

If H and G are GZ-paired, then H does not need to satisfy DP(+) in case G does. But in some occasions, H does satisfy DP(+) in case G does, namely if G is DP(+)-fair.

Definition 6.3.1. Assume $G \in \mathbb{C}[X]^N$. We call G *DP-fair* if

$$\Lambda \in \mathbb{C}^N \text{ and } \Lambda^{\mathrm{t}} G = 0 \implies \Lambda^{\mathrm{t}} \cdot \ker \mathcal{J}_X G = 0$$

We call G *DP+-fair* if in addition

$$\begin{array}{l} \Lambda \in \mathbb{C}^N \text{ and } \Lambda^{\mathrm{t}} G = 0 \text{ and} \\ \mathrm{M} \in \mathbb{C}^N \text{ and } \mathrm{M}^{\mathrm{t}} G \in \mathbb{C}[\Lambda^{\mathrm{t}} X] \end{array} \implies \mathrm{M}^{\mathrm{t}} \cdot \ker \mathcal{J}_X G = 0$$

Assume $H \in \mathbb{C}[x]^n$ such that H and G are GZ-paired through B and C. We call H and G *DP(+)-fairly GZ-paired* through B and C if G is DP(+)-fair.

Proposition 6.3.2. *Assume $G \in \mathbb{C}[X]^N$ is power linear of degree $d \geq 2$ and $\mathrm{rk} \mathcal{J}G = n < N$. If G is DP(+)-fair, then there exists a $H \in \mathbb{C}[x]^n$ such that H and G are DP(+)-fairly GZ-paired.*

Proof. This follows immediately from theorem 6.2.11 and the definition of DP(+)-fair. □

Proposition 6.3.3. *Let $H \in \mathbb{C}[x]^n$ and $G \in \mathbb{C}[X]^N$ be homogeneous of degree $d \geq 2$.*

i) If H and G are GZ-paired and H satisfies DP(+), then G satisfies DP(+) as well.

ii) *If H and G are DP(+)-fairly GZ-paired, then G satisfies DP(+), if and only if H does.*

iii) *If G is DP(+)-fair, then $T^{-1}G(TX)$ is DP(+)-fair as well.*

iv) *If H and G are GZ-paired, then the number of independent linear relations over $\mathbb{C}$ between the components of G is greater than or equal to that of H.*

v) *If H and G are DP-fairly GZ-paired, then the number of independent linear relations over $\mathbb{C}$ between the components of G is equal to that of H.*

vi) *If H and G are GZ-paired and the number of independent linear relations over $\mathbb{C}$ between the components of G is equal to that of H, then H and G are DP-fairly GZ-paired.*

Proof. Assume H and G are GZ-paired through B and C.

i) Assume H satisfies DP, say $\lambda^{\mathrm{t}}H = 0$ for some nonzero $\lambda \in \mathbb{C}^n$. Since $H(BX) = BG$ it follows that $\lambda^{\mathrm{t}}BG = \lambda^{\mathrm{t}}H(BX) = 0$. Since the rows of B are independent, $\lambda^{\mathrm{t}}B \neq 0$ follows, as desired.

   Assume H satisfies DP+, say $\lambda^{\mathrm{t}}H = 0$ and $\mu^{\mathrm{t}}H \in \mathbb{C}[\lambda^{\mathrm{t}}x]$ for some independent $\lambda, \mu \in \mathbb{C}^n$. Then $\lambda^{\mathrm{t}}BG = 0$ and $\mu^{\mathrm{t}}BG = \mu^{\mathrm{t}}H(BX) \in \mathbb{C}[\lambda^{\mathrm{t}}Bx]$. Since the rows of B are independent, $\mu^{\mathrm{t}}B$ is independent of $\lambda^{\mathrm{t}}B$, as desired.

ii) Assume H and G are DP-fairly paired and G satisfies DP, say that $\Lambda^{\mathrm{t}}G = 0$ for some nonzero $\Lambda \in \mathbb{C}^N$. Then

$$\Lambda^{\mathrm{t}} \cdot \ker B = \Lambda^{\mathrm{t}} \cdot \ker \mathcal{J}_X G = 0$$

   so Λ^{t} is a linear combination of the rows of B, i.e. $\Lambda \in \operatorname{im}(B^{\mathrm{t}})$. Say that $\Lambda = B^{\mathrm{t}}\lambda$ for some $\lambda \in \mathbb{C}^n$. Clearly, $\lambda \neq 0$ as well. From $\Lambda^{\mathrm{t}}G(Cx) = 0$, it follows that $\lambda H = \lambda BG(Cx) = \Lambda G(Cx) = 0$, as desired.

   Assume H and G are DP+-fairly GZ-paired and G satisfies DP+, say that $\Lambda^{\mathrm{t}}G = 0$ and $\mathrm{M}^{\mathrm{t}}G \in \mathbb{C}[\Lambda^{\mathrm{t}}X]$ for some independent $\Lambda, \mathrm{M} \in \mathbb{C}^N$. As above, we obtain that $\Lambda = B^{\mathrm{t}}\lambda$ for some $\lambda \in \mathbb{C}^n$, and similarly $\mathrm{M} = B^{\mathrm{t}}\mu$ for some $\mu \in \mathbb{C}^n$. Clearly, λ and μ are independent as well. Now

$$\mathrm{M}^{\mathrm{t}}G(Cx) \in \mathbb{C}[\Lambda^{\mathrm{t}}Cx] = \mathbb{C}[\lambda^{\mathrm{t}}BCx] = \mathbb{C}[\lambda^{\mathrm{t}}x]$$

It follows that $\mu^{\mathrm{t}} H = \mu^{\mathrm{t}} B G(Cx) = \mathrm{M}^{\mathrm{t}} G(Cx) \in \mathbb{C}[\lambda^{\mathrm{t}} x]$, as desired.

iii) Assume $\mathrm{M}^{\mathrm{t}} T^{-1} G(TX) \in \mathbb{C}[\Lambda^{\mathrm{t}} X]$ for some $\Lambda, \mathrm{M} \in \mathbb{C}^N$. Then

$$\big((T^{-1})^{\mathrm{t}}\mathrm{M}\big)^{\mathrm{t}} \cdot G = \mathrm{M}^{\mathrm{t}} T^{-1} \cdot G(TT^{-1}X) \in \mathbb{C}[\Lambda^{\mathrm{t}} T^{-1} X] = \mathbb{C}\big[\big((T^{-1})^{\mathrm{t}}\Lambda\big)^{\mathrm{t}} X\big]$$

Notice that $\ker \mathcal{J}_X\big(T^{-1}G(TX)\big) = \ker(\mathcal{J}_X G \cdot T) = T^{-1} \ker \mathcal{J}_X G$. If either G is DP-fair and $\Lambda = 0$, or G is DP+-fair and $((T^{-1})^{\mathrm{t}}\Lambda)^{\mathrm{t}} G(TX) = 0$, then

$$\mathrm{M}^{\mathrm{t}} \cdot \ker \mathcal{J}_X\big(T^{-1}G(TX)\big) = \mathrm{M}^{\mathrm{t}} T^{-1} \ker \mathcal{J}_X G = \big((T^{-1})^{\mathrm{t}}\mathrm{M}\big)^{\mathrm{t}} \ker \mathcal{J}_X G = 0$$

where we secretly use the homogeneity of G for the first case. This gives the desired result.

iv) The proof is similar to that of i). The independence of the rows of B makes that $\lambda \mapsto B^{\mathrm{t}}\lambda$ is injective.

v) The proof is similar to that of ii). The fact that $y^{\mathrm{t}} \mapsto y^{\mathrm{t}} B$ is a function makes that $B^{\mathrm{t}}\lambda \mapsto \lambda$ is injective.

vi) Assume $\lambda^{\mathrm{t}} H = 0$ for some $\lambda \in \mathbb{C}^n$. From v) or proposition 6.2.7, we obtain that

$$(B^{\mathrm{t}}\lambda)^{\mathrm{t}} G = \lambda^{\mathrm{t}} B G = \lambda^{\mathrm{t}} H(BX) = 0$$

Assume that the number of independent linear relations over $\mathbb{C}$ between the components of G is equal to that of H. Since $\lambda \mapsto B^{\mathrm{t}}\lambda$ is injective, every Λ satisfying $\Lambda^{\mathrm{t}} G = 0$ is of the form $B^{\mathrm{t}}\lambda$, i.e. $\Lambda^{\mathrm{t}} \ker B = 0$. So $\Lambda^{\mathrm{t}} G = 0$ implies $\Lambda^{\mathrm{t}} \ker \mathcal{J}_X G = \Lambda^{\mathrm{t}} \ker B = 0$, as desired. □

Corollary 6.3.4. *Assume $n \geq 2$ and $H \in \mathbb{C}[x]^n$ is homogeneous of degree $d \geq 2$. Then there exists an $N > n$ and a power linear $G \in \mathbb{C}[X]^N$ such that H and G are DP-fairly GZ-paired. Furthermore, G satisfies all properties of theorem 6.2.8.*

Proof. From proposition 6.2.2, it follows that there are linear forms L_{n-m+1}, $L_{n-m+2}, \dots, L_{N-s}$ that satisfies the properties of theorem 6.2.8, where m and s are as in theorem 6.2.8. From theorem 6.2.8, it follows that there exists a power linear $G \in \mathbb{C}[X]^N$, such that both H and G have m independent linear relations between their components, and such that H and G are GZ-paired up to the condition $N > n$. If $N > n$, then we obtain the desired result by vi) of proposition 6.3.3.

So assume from now on that $N = n$. Then B and C in the definition of GZ-paired are just square matrices. From i) of proposition 6.2.7, it follows that we may assume that H itself is power linear. Since $n \geq 2$ and $d \geq 2$, we have $\binom{n-1+d}{d} > n$, so there exists a homogeneous polynomial h of degree d in $\mathbb{C}[x]$ that is linearly independent over $\mathbb{C}$ of the components of H. From proposition 6.2.2, it follows that we can choose a power of a linear form for h. Now H and (H, h) are GZ-paired in the desired manner. $\square$

The following theorem shows that in case a power linear map $G \in \mathbb{C}[X]^N$ is *not* DP-fair, then G is essentially a map in dimension $N - 1$ up to a linear conjugation. Next, one replaces G by this linear conjugation and wonder whether G is DP-fair as a map in dimension $N - 1$. If not, then G is essentially a map in dimension $N - 2$ up to a linear conjugation, etc.

So we can make G DP-fair in some sense. Furthermore, if H and G are GZ-paired, then making G DP-fair as above does not affect that H and G are GZ-paired.

Theorem 6.3.5. *Assume $G \in \mathbb{C}[X]^N$ is* not *DP-fair and there exists a $T \in \mathrm{GL}_N(\mathbb{C})$ such that $T^{-1}G(TX)$ is power linear. Then we can choose T such that in addition,*

$$T^{-1}G(TX) \in \mathbb{C}[x_1, x_2, \ldots, x_{N-1}]^{N-1} \times \{0\}$$

i.e. $T^{-1}G(TX)$ is essentially a map in $\mathbb{C}[x_1, x_2, \ldots, x_{N-1}]^{N-1}$.

If $H \in \mathbb{C}[x]^n$ and $G \in \mathbb{C}[x_1, x_2, \ldots, x_{N-1}]^{N-1} \times \{0\}$ are GZ-paired, then H and G are also GZ-paired as maps in dimensions n and $N - 1$ respectively.

Proof. Assume $G \in \mathbb{C}[X]^N$ is not DP-fair. On account of iii) of proposition 6.3.3, we may assume without loss of generality that G itself is power linear. Then there exists a $\Lambda \in \mathbb{C}^N$ such that $\Lambda^{\mathrm{t}} G = 0$ and $\Lambda^{\mathrm{t}} \cdot \ker \mathcal{J}G \neq 0$. So we can choose $V \in \ker \mathcal{J}G$ such that $\Lambda^{\mathrm{t}} V = 1$.

Now let σ be a permutation of $\{1, 2, \ldots, N\}$ such that the σ_N-th coordinate of Λ is nonzero. Then we can take T of the form

$$T^{-1} = \left(E_{\sigma_1}^{\mathrm{t}} - V_{\sigma_1}\Lambda^{\mathrm{t}}, E_{\sigma_2}^{\mathrm{t}} - V_{\sigma_2}\Lambda^{\mathrm{t}}, \ldots, E_{\sigma_{N-1}}^{\mathrm{t}} - V_{\sigma_{N-1}}\Lambda^{\mathrm{t}}, \Lambda^{\mathrm{t}}\right)$$

and one can verify that $T^{-1}V = E_N$, i.e. $TE_N = V \in \ker \mathcal{J}G$. From this and $\Lambda^{\mathrm{t}} G(TX) = 0$, we obtain

$$T^{-1}G(TX) \in \mathbb{C}[x_1, x_2, \ldots, x_{N-1}]^{N-1} \times \{0\}$$

Furthermore, it follows from $\Lambda^{\mathrm{t}} G(TX) = 0$ that $T^{-1}G(TX)$ is power linear, as desired.

Assume $H \in \mathbb{C}[x]^n$ and $T^{-1}G(TX)$ are GZ-paired through B and C. Since $T^{-1}G(TX) \in \mathbb{C}[x_1, x_2, \ldots, x_{N-1}]^N$, we have $E_N \in \ker \mathcal{J} T^{-1}G(TX)$. It follows that the last column of B is zero. Consequently, the matrix $\tilde{C}$ that we obtain from C by replacing its last row by the zero row satisfies $B\tilde{C} = I_n$ as well.

From viii) of proposition 6.2.7, it follows that H and $T^{-1}G(TX)$ are GZ-paired through B and $\tilde{C}$ as well. This gives that H and $T^{-1}G(TX)$ are also GZ-paired as maps in dimensions n and $N-1$ respectively, as desired. □

Obtaining similar result as in theorems 6.2.8 and 6.3.5 above for DP+ instead of DP is not directly possible. Assume that in theorem 6.3.5 above, G satisfies DP+. One can obtain that $G \in \mathbb{C}[x_1, x_2, \ldots, x_{N-1}]^N$ by a linear conjugation, but $G_n = 0$ might be impossible. In that case, we can get $G_1 = 0$ by a linear conjugation. Now one can show that G is DP+-fair in case we cannot get $G_N \in \mathbb{C}[x_1]$ in addition. So let us assume that $G_N \in \mathbb{C}[x_1]$.

Now we have the following problem: H and $G_1, G_2, \ldots, G_{N-1}$ do not need to be GZ-paired. Take for example the map $\hat{H}$ in example 6.2.9. That map is GZ-paired with (G, x_5^2), where G is as in example 6.2.9, but the pairing is not DP+-fair, because (G, x_5^2) satisfies DP+ and H does not. G itself does not satisfy DP+, just as $\hat{H}$, but $\hat{H}$ and G are *not* GZ-paired. Therefore $\hat{H}$ was modified to a map H such that H and G were GZ-paired.

6.4 Symmetric Keller maps and powers of linear forms

Since the Jacobian conjecture can be reduced to both power linear maps and gradient maps, one can wonder whether the Jacobian conjecture is satisfied for power linear gradient map. The answer to this question is affirmative. More precisely, power linear gradient maps that satisfy the Keller condition are quasi-translations.

Theorem 6.4.1. *Assume $H \in \mathbb{C}[x]$ is power linear of degree $d \geq 2$. Assume in addition that $\mathcal{J}H$ is symmetric. Then $\mathcal{J}H = \mathcal{H}h$ for some $h \in \mathbb{C}[x]$ of degree $d+1$ and h is a sum of powers $(\lambda^{\mathrm{t}} x)^{d+1}$ with pairwise different variables x_i.*

Assume $\mathrm{tr}\mathcal{J}H = 0$. *Then* $\mathcal{J}H^2 = \mathcal{H}h^2 = 0$ *and* $x+H$ *is a quasi-translation. Furthermore,* $\lambda^{\mathrm{t}}\lambda = 0$ *for all powers* $(\lambda^{\mathrm{t}}x)^{d+1}$ *in the above sum.*

Proof. Write $H = (Ax)^{*d}$. Now let G be the graph of n vertices and connect vertex i and j, if and only if H_i and H_j have a variable in common. Assume H_i and H_j have variable x_k in common. Since $d \geq 2$ and $\frac{\partial}{\partial x_i}H_k = \frac{\partial}{\partial x_k}H_i \neq 0$, it follows that A_i and A_k are dependent and that H_k, H_i and H_j have variables x_k and x_i. Similarly, A_j and A_k are dependent and H_k, H_j and H_i have variables x_k and x_j.

Since H_i has variable x_i, we can write $H_i = (d+1)\lambda_i(\lambda^{\mathrm{t}}x)^d$. Then $\frac{\partial}{\partial x_j}H_i = d(d+1)\lambda_j\lambda_i(\lambda^{\mathrm{t}}x)^{d-1}$. Similarly, we can write $H_j = (d+1)\lambda'_j((\lambda')^{\mathrm{t}}x)^d$. Then $\frac{\partial}{\partial x_i}H_j = d(d+1)\lambda'_i\lambda'_j((\lambda')^{\mathrm{t}}x)^{d-1}$. It follows from $\frac{\partial}{\partial x_j}H_i = \frac{\partial}{\partial x_i}H_j$ that λ and λ' are equal up to a $(d+1)$-th root of unity and that $H_j = (d+1)\lambda_j(\lambda^{\mathrm{t}}x)^d$. So every component of more than one vertex of G has a λ associated to it, and one can see that h is the sum of the powers $(\lambda^{\mathrm{t}}x)^{d+1}$, as desired. Since $\mathrm{tr}\mathcal{H}((\lambda^{\mathrm{t}}x)^{d+1}) = d(d+1)\lambda^{\mathrm{t}}\lambda(\lambda^{\mathrm{t}}x)^{d-1}$ and $\mathcal{H}((\lambda^{\mathrm{t}}x)^{d+1}) = d(d+1)\lambda\lambda^{\mathrm{t}}(\lambda^{\mathrm{t}}x)^{d-1}$, it follows that

$$(\mathcal{H}((\lambda^{\mathrm{t}}x)^{d+1}))^2 = \mathrm{tr}\mathcal{H}((\lambda^{\mathrm{t}}x)^{d+1}) \cdot \mathcal{H}((\lambda^{\mathrm{t}}x)^{d+1})$$

Assume $\mathrm{tr}\mathcal{J}H = 0$. Due to the fact that the powers $(\lambda x)^{d+1}$ have no variables in common, we obtain that $\mathrm{tr}\mathcal{H}((\lambda^{\mathrm{t}}x)^{d+1}) = 0$ for all λ in the above sum of powers that makes h and that $(\mathcal{H}h)^2 = 0$.

Since $\mathrm{tr}\mathcal{H}((\lambda^{\mathrm{t}}x)^{d+1}) = d(d+1)\lambda^{\mathrm{t}}\lambda(\lambda^{\mathrm{t}}x)^{d-1}$, it follows that $\lambda^{\mathrm{t}}\lambda = 0$. Since $\mathcal{J}H^2 = (\mathcal{H}h)^2 = 0$ and H is homogeneous, we have $\mathcal{J}H \cdot H = 0$. So $x + H$ is a quasi-translation on account of iii) of proposition 3.1.2. □

The following lemma shows that the operator $f \mapsto \mathrm{tr}\mathcal{H}f$ is surjective. We use that property to prove proposition 6.4.3 below.

Lemma 6.4.2. *If* $f \in \mathbb{C}[x]$ *is homogeneous of degree* d, *then*

$$f = \mathrm{tr}\mathcal{H}\left(\sum_{s=1}^{\infty} \frac{(-1)^{s-1}\left(\sum_{i=1}^n x_i^2\right)^s}{\prod_{r=1}^s 2r\big(n+2(d+1-r)\big)}(\mathrm{tr}\mathcal{H})^{\circ(s-1)}f\right)$$

Proof. Write $\sigma = \sum_{i=1}^n x_i^2$ and put $\bar{h} = \sigma^r\tilde{h}$. Notice that in order to get (5.6), the assumption $\sum_{i=1}^n x_i^2 \nmid \tilde{h}$ is not necessary. Assume f is homogeneous of degree d. Substituting $\tilde{h} = (\mathrm{tr}\mathcal{H})^{\circ(r-1)} f$ in (5.6), we obtain

$$\begin{aligned}
&\mathrm{tr}\mathcal{H}\big(\sigma^r (\mathrm{tr}\mathcal{H})^{\circ(r-1)} f\big) - \sigma^r (\mathrm{tr}\mathcal{H})^{\circ r} f \\
&\quad = \big(2nr + 4r(r-1)\big)\sigma^{r-1}(\mathrm{tr}\mathcal{H})^{\circ(r-1)} f + 4r\sigma^{r-1}E\big((\mathrm{tr}\mathcal{H})^{\circ(r-1)} f\big) \\
&\quad = \Big(2nr + 4r(r-1) + 4r\big(d - 2(r-1)\big)\Big)\sigma^{r-1}(\mathrm{tr}\mathcal{H})^{\circ(r-1)} f \\
&\quad = \Big(2r\big(n + 2(d+1-r)\big)\Big)\sigma^{r-1}(\mathrm{tr}\mathcal{H})^{\circ(r-1)} f
\end{aligned}$$

Substituting $r = s$ and dividing by $\prod_{r=1}^{s} 2r\big(n + 2(d+1-r)\big)$, we get

$$\begin{aligned}
&\sum_{s=1}^{\infty} \frac{\mathrm{tr}\mathcal{H}\big(\sigma^s (\mathrm{tr}\mathcal{H})^{\circ(s-1)} f\big) - \sigma^s (\mathrm{tr}\mathcal{H})^{\circ s} f}{(-1)^{s-1}\prod_{r=1}^{s} 2r\big(n+2(d+1-r)\big)} \\
&\quad = \sum_{s=1}^{\infty} \frac{\sigma^{s-1}(\mathrm{tr}\mathcal{H})^{\circ(s-1)} f}{(-1)^{s-1}\prod_{r=1}^{s-1} 2r\big(n+2(d+1-r)\big)} \\
&\quad = -\sum_{s=0}^{\infty} \frac{\sigma^{s}(\mathrm{tr}\mathcal{H})^{\circ s} f}{(-1)^{s-1}\prod_{r=1}^{s} 2r\big(n+2(d+1-r)\big)}
\end{aligned}$$

so by canceling out terms we get

$$\begin{aligned}
&\sum_{s=1}^{\infty} \frac{\mathrm{tr}\mathcal{H}\big(\sigma^s (\mathrm{tr}\mathcal{H})^{\circ(s-1)} f\big)}{(-1)^{s-1}\prod_{r=1}^{s} 2r\big(n+2(d+1-r)\big)} \\
&\quad = -\sum_{s=0}^{0} \frac{\sigma^{s}(\mathrm{tr}\mathcal{H})^{\circ s} f}{(-1)^{s-1}\prod_{r=1}^{s} 2r\big(n+2(d+1-r)\big)} = f
\end{aligned}$$

This gives the desired result. ∎

Proposition 6.4.3. *Let $f \in \mathbb{C}[x]$ be a homogeneous polynomial of degree d over $\mathbb{C}$. Assume $\mathrm{tr}\mathcal{H} f = 0$. Then f can be written as a sum of linear powers $(\lambda^{\mathrm{t}} x)^d$ such that $\lambda^{\mathrm{t}}\lambda = 0$.*

Proof. Notice that it suffices to show that f is a linear combination of linear powers $(\lambda^{\mathrm{t}} x)^d$ such that $\lambda^{\mathrm{t}}\lambda = 0$. We distinguish two cases.

- $n \le 3$.
  Let V_d be the vector space over $\mathbb{C}$ of homogeneous polynomials of degree d in $\mathbb{C}[x]$. Notice that $\dim V_d = \binom{n+d-1}{d}$, the number of monomials

of degree d in $\mathbb{C}[x]$. From lemma 6.4.2, it follows that $\mathrm{tr}\mathcal{H}V_d = V_{d-2}$. Now by the isomorphism theorem, we obtain

$$\begin{aligned}\dim\{h \in V_d \mid \mathrm{tr}\mathcal{H}h = 0\} &= \dim V_d - \dim V_{d-2} \\ &= \binom{n+d-1}{n-1} - \binom{n+d-3}{n-1}\end{aligned}$$

This is 0 if $n = 1$, 2 if $n = 2$, and

$$\binom{d+2}{2} - \binom{d+1}{2} + \binom{d+1}{2} - \binom{d}{2} = (d+1) + d = 2d + 1$$

if $n = 3$. Assuming $f \neq 0$, $n \geq 2$ follows. If $n = 2$, then $(x_1 + \mathrm{i}x_2)^d$, $(x_1 - \mathrm{i}x_2)^d$ is a basis of $\{h \in V_d \mid \mathrm{tr}\mathcal{H}h = 0\}$. So assume from now on that $n = 3$.

Notice that we can choose $2d+1$ powers $(\lambda_i^{\mathrm{t}}x)^d$ with $\lambda^{\mathrm{t}}\lambda = 0$ that are pairwise linearly independent. Now I claim that they are automagically linearly independent and therefore a basis of $\{h \in V_d \mid \mathrm{tr}\mathcal{H}h = 0\}$. For that purpose, let μ_i be the cross product of λ_i and λ_{i+d}. Then $\mu_{i1}\frac{\partial}{\partial x_1} + \mu_{i2}\frac{\partial}{\partial x_2} + \mu_{i3}\frac{\partial}{\partial x_3}$ kills both $\lambda_i^{\mathrm{t}}x$ and $\lambda_{i+d}^{\mathrm{t}}x$, whence

$$\left(\prod_{i=1}^{d}\left(\mu_{i1}\frac{\partial}{\partial x_1} + \mu_{i2}\frac{\partial}{\partial x_2} + \mu_{i3}\frac{\partial}{\partial x_3}\right)\right)(\lambda_j^{\mathrm{t}}x)^d = 0$$

for all $j \leq 2d$.

On the other hand, $\mu_{i1}\frac{\partial}{\partial x_1} + \mu_{i2}\frac{\partial}{\partial x_2} + \mu_{i3}\frac{\partial}{\partial x_3}$ cannot kill $\lambda_{2d+1}^{\mathrm{t}}x$, since that would imply that λ_{2d+1} is dependent of λ_i and λ_{d+i}, which is impossible because $\lambda_{2d+1}^{\mathrm{t}}\lambda_{2d+1} = 0$ and λ_{2d+1} is not pairwise dependent of one of λ_i and λ_{d+i}. Consequently,

$$\left(\prod_{i=1}^{d}\left(\mu_{i1}\frac{\partial}{\partial x_1} + \mu_{i2}\frac{\partial}{\partial x_2} + \mu_{i3}\frac{\partial}{\partial x_3}\right)\right)(\lambda_{2d+1}^{\mathrm{t}}x)^d \neq 0$$

So $(\lambda_{2d+1}^{\mathrm{t}}x)^d$ is linearly independent of the other $2d$ powers $(\lambda_i^{\mathrm{t}}x)^d$. The linear independence of each of the other powers $(\lambda_i^{\mathrm{t}}x)^d$ goes in a similar manner.

- $n \geq 4$.
  Write f as a polynomial in x_n over $\mathbb{C}[x_1, x_2, \ldots, x_{n-1}]$. Say that f

has degree r as such. Now look at the coefficient of x_n^r of $\mathrm{tr}\mathcal{H}f$. By induction on n, it follows that the leading coefficient of f is a sum of linear powers $(\lambda^{\mathrm{t}}x)^{d-r}$ such that $\lambda_n = 0$ and $\lambda^{\mathrm{t}}\lambda = 0$. Take such a linear power $(\lambda^{\mathrm{t}}x)^{d-r}$ and assume by orthogonal transformation that $\lambda = (1, \mathrm{i}, 0, \dots, 0)$.

Now choose $\mu = (0, \dots, 0, \mathrm{i}, 1)$ and subtract $(\lambda^{\mathrm{t}}x)^{d-r}(\mu^{\mathrm{t}}x)^r$ from f. We may do that because we can write $y_1^{d-r}y_2^r$ as a sum of d-th powers of linear forms in y. The result is that either the leading coefficient of f becomes a sum of less linear powers or r decreases, and the result follows by induction. □

Assume $\mathcal{J}H$ is nilpotent and symmetric. Then H is of the form $H = \nabla h$ and $\mathrm{tr}\mathcal{J}H = 0$. Assume h is homogeneous of degree $d+1$. Since $\mathrm{tr}\mathcal{J}H = 0$, it follows that h can be written as a sum

$$h = \sum_{i=1}^{N} (A_i x)^{d+1}$$

for some $A \in \mathrm{Mat}_{N,n}(\mathbb{C})$ with pairwise independent rows and $A_i A_i^{\mathrm{t}} = 0$ for all i. Since

$$\mathcal{J}h = \mathcal{J}\left((1 \cdots 1) \cdot (Ax)^{*(d+1)}\right) = (1 \cdots 1) \cdot \mathrm{diag}(Ax)^{*d} \cdot (d+1)A$$

it follows that

$$H := \nabla h = (\mathcal{J}h)^{\mathrm{t}} = (d+1)A^{\mathrm{t}}(Ax)^{*d}$$

and

$$\mathcal{H}h = \mathcal{J}\nabla h = \mathcal{J}H = d(d+1)A^{\mathrm{t}}\mathrm{diag}\big((Ax)^{*(d-1)}\big)A \tag{6.7}$$

from which it follows that the Zhao-matrix

$$\Psi_h := \mathrm{diag}\big((Ax)^{*(d-1)}\big)AA^{\mathrm{t}} \tag{6.8}$$

is a nilpotent power linear quasi-Jacobian, see also [61, Prop. 5.3].

Now W. Zhao looks at the graph G with vertices $i : 1 \le i \le N$ and connect vertices i, j by an edge, if and only if $A_i A_j^{\mathrm{t}} \ne 0$. If $AA^{\mathrm{t}} = 0$, then G is totally disconnected. We shall show that $AA^{\mathrm{t}} = 0$ in some cases.

On the other hand, if $H = (Ax)^{*d}$ is power linear, then

$$\mathcal{J}H = \mathrm{diag}\big((Ax)^{*(d-1)}\big) \cdot dA$$

is power linear quasi-Jacobian as well. So both Jacobians of power linear maps and the matrices Ψ_h above are power linear quasi-Jacobians, and the question is how they look when they are nilpotent.

Before we start studying nilpotent power linear quasi-Jacobians, we first derive some preliminaries.

Definition 6.4.4. Let M be a power linear quasi-Jacobian of size N. We call M *reduced* if for the rows $M_1, M_2, \ldots, M_N$ of M

$$\Lambda_1 M_1 + \Lambda_2 M_2 + \cdots + \Lambda_N M_N = 0 \implies \Lambda_i M_i = 0 \text{ for all } i$$

where $\Lambda_i \in \mathbb{C}$ for all i. We call a power linear map $H = (H_1, H_2, \ldots, H_n)$ *reduced* if $\mathcal{J}H$ is reduced, or similarly, if

$$\lambda_1 H_1 + \lambda_2 H_2 + \cdots + \lambda_n H_n = 0 \implies \lambda_i H_i = 0 \text{ for all } i$$

where $\lambda_i \in \mathbb{C}$ for all i.

Proposition 6.4.5. *Assume M is a power linear quasi-Jacobian of size N. Then there exists a $T \in \mathrm{GL}_N(\mathbb{C})$ such that $T^{-1}MT$ is a reduced power linear quasi-Jacobian.*

In particular, if $H = (H_1, H_2, \ldots, H_n)$ is a power linear map, then there exists a $T \in \mathrm{GL}_n(\mathbb{C})$ such that $T^{-1}H(Tx)$ is reduced.

Proof. Assume $\Lambda_1 M_1 + \Lambda_2 M_2 + \cdots + \Lambda_N M_N = 0$ and $\Lambda_r M_r \neq 0$. In particular, the r-th row of M is nonzero. Take

$$T^{-1} = \begin{pmatrix} I_{r-1} & & & \emptyset & \\ \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{N-1} & \Lambda_N \\ & & \emptyset & & I_{N-r} \end{pmatrix}$$

Then $T^{-1}MT$ has one more zero row than M, namely the r-th row. Furthermore, $T^{-1}MT$ is a power linear quasi-Jacobian as well. So the desired result follow by induction.

Assume H is power linear. Then there exists a $T \in \mathrm{GL}_n(\mathbb{C})$ such that $T^{-1}\mathcal{J}H \cdot T$ is reduced. Hence also $\mathcal{J}(T^{-1}H(Tx)) = (T^{-1}\mathcal{J}H \cdot T)|_{x=Tx}$ is reduced. So $T^{-1}H(Tx)$ is reduced. $\square$

Proposition 6.4.6. *Assume A is a matrix over $\mathbb{C}$ with isotropic rows, and every principal minor of size 2×2 at most of AA^{t} has determinant zero. Then $AA^{\mathrm{t}} = 0$.*

Proof. Notice that $B := AA^{\mathrm{t}}$ is symmetric. Furthermore, its diagonal is zero, because the rows of A are isotropic. Now

$$B_{ij}^2 = B_{ji}B_{ij} = B_{ii}B_{jj} = 0$$

for all i, j, so $B = 0$. ∎

Proposition 6.4.7. *Assume $h = \sum_{i=1}^{N}(A_i x)^{d+1}$ and $AA^{\mathrm{t}} = 0$. Then $H = \nabla h$ satisfies (3.4). In particular, $x + \nabla h$ is a linearly triangularizable quasi-translation.*

Proof. From proposition 5.5.1, it follows that we may assume that $A_i x \in \mathbb{C}[x_1 + \mathrm{i}x_n, x_2 + \mathrm{i}x_{n-1}, \ldots, x_{\lfloor n/2 \rfloor} + \mathrm{i}x_{n+1-\lfloor n/2 \rfloor}]$ for each i. So h is of the form of g in (5.21), with $a_\infty(x_1 + \mathrm{i}x_n, x_2 + \mathrm{i}x_{n-1}, \ldots, x_{\lfloor n/2 \rfloor} + \mathrm{i}x_{n+1-\lfloor n/2 \rfloor})$ as the only nonzero term.

From (5.22), it follows that the first $s := \lfloor n/2 \rfloor$ components of $S(\nabla g)(S^{-1}x)$ are zero, where $S = I_n + \mathrm{i}I_n^{\mathrm{r}}$. The possible component in the middle is zero as well by way of (5.23). Furthermore, $\mathcal{J}S(\nabla g)(S^{-1}x)$ is symmetric with respect to the anti-diagonal, because $S^{\mathrm{r}} = 2\mathrm{i}S^{-1} = 2I(S^{-1})^{\mathrm{t}}$. So iii) of proposition 3.4.3 gives the desired result. ∎

6.5 Symmetrically triangularizable quasi-Jacobians

Definition 6.5.1. We call a power linear quasi-Jacobian M of size N *linearly triangularizable* if there exists a $T \in \mathrm{GL}_N(\mathbb{C})$ such that $T^{-1}MT$ is a triangular matrix.

We call a power linear quasi-Jacobian M *symmetrically triangularizable* if there exists a permutation P such that $P^{-1}MP$ is a triangular matrix.

Let

$$M := \mathrm{diag}\big((Ax)^{*(d-1)}\big) \cdot B$$

be a nilpotent power linear quasi-Jacobian such that the rows of A are pairwise independent. The aim of this section is to show that for large d, M is symmetrically triangularizable. At first we show that M is symmetrically triangularizable if $d \gg \mathrm{cork} A$. This has already been conjectured by He Tong at the conference 'Polynomial Automorphisms and Related Topics' on October 2006 in Hanoi, Vietnam (for power linear Jacobians instead of power linear quasi-Jacobians).

After that, we show that for $d > \operatorname{cork} A + 1$, M is symmetrically triangularizable in case M is already linearly triangularizable. Furthermore, we shall show that this latter bound on d is tight and we derive some corollaries for power linear Keller maps and Zhao graphs. But we start with a key lemma for power linear quasi-Jacobians:

Lemma 6.5.2. *Assume* $M = \operatorname{diag}((Ax)^{*(d-1)}) \cdot B$ *is a* nilpotent *power linear quasi-Jacobian of size* N *and some principal minor of size* m *of* B *has a nonzero determinant. Then there exists a nonzero relation* $R \in \mathbb{C}[Y]$ *of degree* m *such that*

$$R\big((A_1x)^{d-1}, (A_2x)^{d-1}, \ldots, (A_Nx)^{d-1}\big) = 0$$

and $\deg_{y_i} R(y) \le 1$ *for all* $i \le N$*, where* $Y = (y_1, y_2, \ldots, y_N)$*.*

Proof. Write

$$\begin{aligned} &\det\left(tI_N + \begin{pmatrix} B_{11}y_1 & B_{12}y_1 & \cdots & B_{1N}y_1 \\ B_{21}y_2 & B_{22}y_2 & \cdots & B_{2N}y_2 \\ \vdots & \vdots & & \vdots \\ B_{N1}y_N & B_{N2}y_N & \cdots & B_{NN}y_N \end{pmatrix} \right) \\ &= \; t^N + R_1(Y)t^{N-1} + R_2(Y)t^{N-2} + \cdots + R_{N-1}(Y)t + R_N(Y) \end{aligned}$$

Since M is nilpotent, $\det(tI_N + M) = t^N$. The coefficient of t^{N-j} of $\det(tI_N + \mathcal{J}(Ax)^{*d})$ equals

$$R_j\big((A_1x)^{d-1}, (A_2x)^{d-1}, \ldots, (A_Nx)^{d-1}\big) = 0$$

for all $j \ge 1$. Furthermore, it follows from the definition of determinant that $\deg_{y_i} R_j \le 1$ for all i, j. Now B has a principal minor of size $m \times m$ that has a determinant $\alpha \ne 0$, say with rows and columns $i_1, i_2, \ldots, i_m$. Then the coefficient of $y_{i_1} y_{i_2} \cdots y_{i_m}$ of R_m equals α, whence $R_m \ne 0$. From the definition of determinant, it follows that R_m has degree m. $\square$

Lemma 6.5.3. *Let* $R \in \mathbb{C}[y]$ *be a nonzero relation with* $\deg_{y_i} R \le 1$ *such that*

$$\begin{aligned} R\big(x_1^d, x_2^d, \ldots, x_r^d, (\lambda_{1,1}x_1 + \lambda_{1,2}x_2 + \cdots + \lambda_{1,r}x_r)^d,& \\ (\lambda_{2,1}x_1 + \lambda_{2,2}x_2 + \cdots + \lambda_{2,r}x_r)^d,& \\ \ldots, (\lambda_{c,1}x_1 + \lambda_{c,2}x_2 + \cdots + \lambda_{c,r}x_r)^d\big) &= 0 \end{aligned} \tag{6.9}$$

where $n = r + c$. If $d \ge 2^{c+1}(2^{c+1} - 2)$, then two of the arguments of R above are linearly dependent. More precisely, if R has degree m and

$$\begin{aligned} d \ \ \ge \ \ & \left(\binom{c+1}{0} + \binom{c+1}{1} + \cdots + \binom{c+1}{m}\right) \cdot \\ & \left(\binom{c+1}{0} + \binom{c+1}{1} + \cdots + \binom{c+1}{m} - 2\right) \end{aligned} \tag{6.10}$$

then two of the arguments of R above are linearly dependent.

Proof. Let m be the degree of R. Without loss of generality, we may assume that $\frac{\partial}{\partial y_1} R \ne 0$. Then $\lambda_{i,1} \ne 0$ for some i as well, so we may additionally assume that $\lambda_{1,1}\lambda_{2,1}\cdots\lambda_{c',1} \ne 0$ and $\lambda_{c'+1,1} = \cdots = \lambda_{c,1} = 0$ for some $c' \ge 1$. Then there exists a nonzero relation $\tilde{R} \in K[y_1, y_2, \ldots, y_{c'+1}]$ with $\deg \tilde{R} \le m$ and $\deg_{y_i} \tilde{R} \le 1$ for all i, such that

$$\tilde{R}\big(x_1^d, (\lambda_{1,1}x_1 + \cdots + \lambda_{1,r}x_r)^d, \ldots, (\lambda_{c',1}x_1 + \cdots + \lambda_{c',r}x_r)^d\big) = 0 \tag{6.11}$$

where $K = \mathbb{C}(x_2, x_3, \ldots, x_r)$, because the arguments of R that are not included as arguments of $\tilde{R}$ are constants in K. Now write $\tilde{R}$ as a sum of its terms: $\tilde{R} = \alpha_1 t_1 + \alpha_2 t_2 + \cdots + \alpha_k t_k$, where $\alpha_i \in \mathbb{C}^*$ and t_i is a monic term for each i, and define

$$g_i := \sqrt[d]{\alpha_i} \cdot t_i(x_1, \lambda_{1,1}x_1 + \cdots + \lambda_{1,r}x_r, \ldots, \lambda_{c',1}x_1 + \cdots + \lambda_{c',r}x_r)$$

for all i. Since $t_i(y)^d = t_i(y^{*d})$ for all i, (6.11) comes down to

$$g_1^d + g_2^d + \cdots + g_k^d = 0$$

Since $\deg \tilde{R} \le \deg R = m$ and $\tilde{R}$ has degree ≤ 1 with respect to each of its $c' + 1$ arguments, it has

$$\begin{aligned} k \ \ \le \ \ & \binom{c'+1}{0} + \binom{c'+1}{1} + \cdots + \binom{c'+1}{m} \\ \le \ \ & \binom{c+1}{0} + \binom{c+1}{1} + \cdots + \binom{c+1}{m} \\ \le \ \ & 2^{c+1} \end{aligned}$$

terms. So (6.10) implies $d \ge k(k-2)$.

So assume that (6.10) is satisfied and hence $d \geq k(k-2)$. From theorem B.3.2, using Lefschetz' principle on the algebraic closure $\bar{K}$ of K, it follows that there are $i \neq j$ such that $g_i \in \bar{K}[x_1]$ and $g_j \in \bar{K}[x_1]$ are linearly dependent over $\bar{K}$ and hence over K as well.

Since t_i and t_j are *not* linearly dependent over K (otherwise they would not have been separated as individual terms of $\tilde{R}$), there exists a linear factor L of both g_i and g_j such that

$$\begin{aligned} L &= l_i(x_1, \lambda_{1,1}x_1 + \cdots + \lambda_{1,r}x_r, \ldots, \lambda_{c',1}x_1 + \cdots + \lambda_{c',r}x_r) \\ &= l_j(x_1, \lambda_{1,1}x_1 + \cdots + \lambda_{1,r}x_r, \ldots, \lambda_{c',1}x_1 + \cdots + \lambda_{c',r}x_r) \end{aligned}$$

for different $l_i, l_j \in \{z_1, z_2, \ldots, z_{c'+1}\}$. So two arguments of $\tilde{R}$ must be linearly dependent over K. Since the leading coefficients with respect to x_1 of both arguments of $\tilde{R}$ are constants in $\mathbb{C}$, these arguments of $\tilde{R}$ are linearly dependent over $\mathbb{C}$. The arguments of $\tilde{R}$ are a subset of those of R, so the desired result follows. $\square$

Theorem 6.5.4. *Assume* $M = \operatorname{diag}\big((Ax)^{*(d-1)}\big) \cdot B$ *is a* nilpotent *power linear quasi-Jacobian of size* N *and the rows* A_i *of* A *are pairwise independent. If some principal minor determinant of* B *is nonzero, then*

$$d < \left(2^{N-\mathrm{rk}A+1} - 1\right)^2$$

More precisely, if some principal $m \times m$ *minor determinant of* B *is nonzero, then*

$$d < \left(\binom{N-\mathrm{rk}A+1}{0} + \binom{N-\mathrm{rk}A+1}{1} + \cdots + \binom{N-\mathrm{rk}A+1}{m} - 1\right)^2$$

Proof. From lemma 6.5.2, it follows that there exists a nonzero relation R of degree m with $\deg_{y_i} R \leq 1$ for all i, such that

$$R\big((A_1x)^{d-1}, (A_2x)^{d-1}, \ldots, (A_Nx)^{d-1}\big) = 0$$

Put $r = \mathrm{rk}A$. Assume without loss of generality that the first r rows of A are independent, then the remaining rows of A are dependent of these r rows, say that

$$A_{r+i} = \lambda_{i,1}A_1 + \lambda_{i,2}A_2 + \cdots + \lambda_{i,r}A_r$$

for all i. Furthermore, the linear forms $A_1x, A_2x, \ldots, A_rx$ are independent and can thus be seen as variables. From lemma 6.5.3, it follows that

$$\begin{aligned} d-1 \quad < \quad & \left(\binom{N-r+1}{0} + \binom{N-r+1}{1} + \cdots + \binom{N-r+1}{m}\right) \cdot \\ & \left(\binom{N-r+1}{0} + \binom{N-r+1}{1} + \cdots + \binom{N-r+1}{m} - 2\right) \end{aligned}$$

so

$$d < \left(\binom{N-r+1}{0} + \binom{N-r+1}{1} + \cdots + \binom{N-r+1}{m} - 1\right)^2$$

as desired. $\square$

Corollary 6.5.5. *Assume $H = (Ax)^{*d}$ is a reduced power linear map and $\mathcal{J}H$ is nilpotent. Let*

$$m := \mathrm{cork}A - \#\{i \mid A_i = 0\} \leq \mathrm{cork}A$$

and assume that $d \geq (2^{m+1} - 1)^2$. Then H is symmetrically triangularizable.

Proof. Let v be the indicator vector defined by

$$v_i := \begin{cases} 0, \text{if } A_i = 0, \\ 1, \text{if } A_i \neq 0, \end{cases}$$

Replacing the zero rows of A by sufficiently independent rows, we get a new matrix $\hat{A}$ such that $\mathrm{cork}\hat{A} = m$. Furthermore,

$$\mathrm{diag}\big((\hat{A}x)^{*(d-1)}\big) \cdot \mathrm{diag}(v) = \mathrm{diag}\big((Ax)^{*(d-1)}\big)$$

Since $\mathrm{diag}(v) \cdot A = A$, it follows that

$$M := \mathcal{J}(Ax)^{*d} = \mathrm{diag}\big((Ax)^{*(d-1)}\big) \cdot A = \mathrm{diag}\big((\hat{A}x)^{*(d-1)}\big) \cdot A$$

is a nilpotent quasi-Jacobian. It follows from theorem 6.5.4 that either two rows of $\hat{A}$ are dependent or A does not have a nonzero principal minor determinant.

- Two rows of $\hat{A}$ are dependent.
  The rows of $\hat{A}$ that have replaced zero rows of A are independent of any other row of $\hat{A}$ (by construction). So two dependent rows of $\hat{A}$ occur already as nonzero rows of A. So say that $A_1 = \lambda A_2 \neq 0$. Then also $(A_1 x)^d = \lambda^d (A_2 x)^d \neq 0$, which contradicts the assumption that $(Ax)^{*d}$ is reduced.

- A does not have a nonzero principal minor determinant.
  From [19, lemma 1.2] (see also [24, prop. 6.3.9]), it follows that there is a permutation matrix P such that $B := P^{-1}AP$ is triangular. Since $M \mapsto MP$ permutes the columns of its argument M in the same way as $M \mapsto P^{-1}M$ permutes the rows of its argument M, $P^{-1}\mathrm{diag}(v)P = \mathrm{diag}(P^{-1}v)$, whence

$$\begin{aligned}
\mathcal{J}(P^{-1}HP) &= \mathcal{J}(P^{-1}(APx)^{*d}) \\
&= P^{-1}\mathcal{J}\big((PBx)^{*d}\big) \\
&= P^{-1}\mathrm{diag}\big((PBx)^{*(d-1)}\big) \cdot dPB \\
&= P^{-1}\big(\mathrm{diag}(PBx)\big)^{d-1} P \cdot dB \\
&= \big(\mathrm{diag}(P^{-1}PBx)\big)^{d-1} \cdot dB \\
&= \mathrm{diag}\big((Bx)^{*(d-1)}\big) \cdot dB \\
&= \mathcal{J}\big((Bx)^{*d}\big)
\end{aligned}$$

  so H is symmetrically triangularizable. □

Corollary 6.5.6. *Assume $h = \sum_{i=1}^{N} (A_i x)^{d+1}$ and the rows A_i of A are pairwise independent and isotropic. Assume in addition that $\mathcal{H}h$ is nilpotent. If*

$$d \geq \binom{N - \mathrm{rk}A + 2}{2}^2$$

then $AA^{\mathrm{t}} = 0$.

Proof. From (6.8), it follows that

$$\mathrm{diag}\big((Ax)^{*(d-1)}\big) AA^{\mathrm{t}}$$

is nilpotent. Now assume

$$\begin{aligned} d &\geq \binom{N - \mathrm{rk}A + 2}{2}^2 \\ &= \left(\binom{N - \mathrm{rk}A + 1}{0} + \binom{N - \mathrm{rk}A + 1}{1} + \binom{N - \mathrm{rk}A + 1}{2} - 1\right)^2 \\ &\geq \left(\binom{N - \mathrm{rk}A + 1}{0} + \binom{N - \mathrm{rk}A + 1}{1} - 1\right)^2 \end{aligned}$$

then every principal minor of size 2 at most of AA^{t} has determinant zero on account of theorem 6.5.4. Now apply proposition 6.4.6. □

We advance on proving that power linear quasi-Jacobians $(AX)^{*(d-1)} \cdot B$ are symmetrically triangularizable in case they are nilpotent and linearly triangularizable and $d > \mathrm{cork}A + 1$.

Lemma 6.5.7. *Let $M = \mathrm{diag}\big((Ax)^{*(d-1)}\big) \cdot B$ be a power linear quasi-Jacobian of size N, such that the rows A_i of A are pairwise linearly independent and $d \geq N - \mathrm{rk}A + 2$. Then every column of M, seen as a power linear map, is reduced.*

Proof. Let $\mathrm{M} = (\mathrm{M}_1, \mathrm{M}_2, \ldots, \mathrm{M}_n)$ be a column of B. We must show that

$$\sum_{i=1}^{N} \Lambda_i (A_i x)^{d-1} \mathrm{M}_i = 0 \implies \Lambda_i (A_i x)^{d-1} \mathrm{M}_i = 0 \text{ for all } i$$

Substituting $\Lambda_i = \Lambda_i \mathrm{M}_i$ below for all i, we see that it suffices to show that

$$\sum_{i=1}^{n} \Lambda_i (A_i x)^{d-1} = 0 \implies \Lambda_i (A_i x)^{d-1} = 0 \text{ for all } i$$

In fact, we will show that the left hand side implies that $\Lambda = 0$.

Take $\Lambda \in \mathbb{C}^N \backslash \{0\}$ arbitrary and assume without loss of generality that $\Lambda_1 (A_1 x)^{d-1} \neq 0$. Let $r = \mathrm{rk}A$ and assume without loss of generality that the first r rows of A are independent. Then $a_1 := A_1 x, a_2 := A_2 x, \ldots, a_r := A_r x$ are independent linear forms, and each $A_j x$ is a linear combination of the a_i's. So we have

$$\sum_{i=1}^{r} \Lambda_i a_i^{d-1} + \sum_{i=r+1}^{N} \Lambda_i (A_i x)^{d-1} = 0$$

Differentiating the above with respect to a_1 gives

$$(d-1)\Lambda_1 a_1^{d-2} + \sum_{i=r+1}^{N} \mathrm{N}_i (A_i x)^{d-2} = 0$$

for certain $\mathrm{N}_i \in \mathbb{C}$. So we have at most $N - r + 1 \leq (d-2) + 1$ linear powers $(A_i x)^{d-2}$ that are linearly dependent. From lemma 6.2.1, it follows that there are $i \neq j$ such that $A_i = sA_j$ for some $s \in \mathbb{C}$. But the rows of A are pairwise independent. Contradiction, so $\Lambda = 0$, as desired. □

Theorem 6.5.8. *Assume* $M = \operatorname{diag}\big((Ax)^{*(d-1)}\big) \cdot B$ *is a* nilpotent *power linear quasi-Jacobian of size* N *and the rows* A_i *of* A *are pairwise linearly independent. If* M *is linearly triangularizable and* $d \geq N - \operatorname{rk}A + 2$*, then* M *is symmetrically triangularizable.*

Proof. Since M is linearly triangularizable, there exists a $T \in \mathrm{GL}_n(\mathbb{C})$ such that $T^{-1}MT$ is lower triangular.

i) Assume that T cannot be taken equal to the product of a permutation matrix and a diagonal matrix. Let r be the index of the first row of T^{-1} that has more than one nonzero coefficient, and choose T such that r is as large as possible, and let λ be the transpose of the r-th row of T^{-1}. Let $C^{(j)}$ be the j-th column of MT. Since $T^{-1}MT$ is lower triangular,

$$\lambda_1 C_1^{(j)} + \lambda_2 C_2^{(j)} + \cdots + \lambda_n C_n^{(j)} = 0$$

for all $j \geq r$. It follows from lemma 6.5.7 with $M = MT$ substituted that $C^{(j)}$ is reduced, whence

$$\lambda_i C_i^{(j)} = 0$$

for all i and all $j \geq r$. Since T^{-1} is invertible and $\lambda^{\mathrm{t}} = (T^{-1})_r$, we can choose i such that $\lambda_i \neq 0$ and E_i^{t} is independent of the first $r-1$ rows of T^{-1}. So $C_i^{(j)} = 0$ for all $j \geq r$. It follows that

$$C_i^{(j)} - \frac{\mathrm{M}_j}{\mathrm{M}_s} C_i^{(s)} = 0 \tag{6.12}$$

for all M with $\mathrm{M}_s \neq 0$ and all $j, s \geq r$.

ii) Now write E_i^{t} as a linear combination of the rows of T^{-1}:

$$E_i^{\mathrm{t}} = \mathrm{M}_1(T^{-1})_1 + \mathrm{M}_2(T^{-1})_2 + \cdots + \mathrm{M}_s(T^{-1})_s$$

such that $\mathrm{M}_s \neq 0$. Then $s \geq r$ follows from the choice of i. Define S by

$$S^{-1} := \begin{pmatrix} I_{s-1} & \begin{matrix} \vdots \\ 0 \end{matrix} & \emptyset \\ \mathrm{M}_1 \ \cdots & \mathrm{M}_s & 0 \ \cdots \\ \emptyset & \begin{matrix} 0 \\ \vdots \end{matrix} & I_{N-s} \end{pmatrix}$$

the the s-th row of $S^{-1}T^{-1}$ equals E_i^{t}.

Since S is lower triangular, it follows that replacing T by TS (and $T^{-1}MT$ by $S^{-1}T^{-1}MTS$) does not affect the property that $T^{-1}MT$ is lower triangular. Since $s \geq r$, it follows from (6.12) that the property that $C_i^{(j)} = 0$ for all $j \geq r$ is maintained as well. So we may assume that the s-th row of T^{-1} equals E_i^{t}.

iii) It follows from the maximality of r that $s > r$. Now take for P a permutation such that

$$P^{-1} = \mathcal{J}_X(x_1, \ldots, x_{r-1}, x_s, x_r, \ldots, x_{s-1}, x_{s+1}, \ldots, x_N)$$

where $X = (x_1, x_2, \ldots, x_N)$. Then the first r rows of $P^{-1}T^{-1}$ have only one nonzero coefficient each. It follows from the maximality of r that $P^{-1}T^{-1}MTP$ cannot be lower triangular. But as a matter of fact, $P^{-1}T^{-1}MTP$ *is* lower triangular: since P cycles the columns $r, \ldots, s$ of $T^{-1}MT$ to the right and P^{-1} cycles the rows $r, \ldots, s$ of $T^{-1}MT$ downward, the only concern is the r-th row of $P^{-1}T^{-1}MTP$. Since $C_i^{(j)} = 0$ for all $j \geq r$, this row is equal to

$$\begin{aligned} E_i^{\mathrm{t}} \cdot MT \cdot P &= (C_i^{(1)}\ C_i^{(2)}\ \cdots\ C_i^{(r-1)}\ C_i^{(r)}\ C_i^{(r+1)}\ \cdots\ C_i^{(N)})P \\ &= (C_i^{(1)}\ C_i^{(2)}\ \cdots\ C_i^{(r-1)}\ 0\ 0\ \cdots\ 0\ )P \\ &= (C_i^{(1)}\ C_i^{(2)}\ \cdots\ C_i^{(r-1)}\ 0\ 0\ \cdots\ 0\ ) \end{aligned}$$

So $P^{-1}T^{-1}MTP$ is lower triangular.

iv) Contradiction, so T can be chosen equal to the product of a permutation matrix P and a diagonal matrix D. Since conjugations with a diagonal matrix do not affect triangularity, $P^{-1}MP = DT^{-1}MTD^{-1}$ is of the desired form. $\square$

Corollary 6.5.9. *Let $H = (Ax)^{*d}$ be a power linear map such that $\mathcal{J}H$ is nilpotent. Assume that H is linearly triangularizable and reduced. If $d \geq \mathrm{cork}A + 1$, then there exists a permutation P such that $P^{-1}H(Px)$ is lower triangular. Furthermore, if we put*

$$m := \mathrm{cork}A - \#\{i \mid A_i = 0\}$$

then a P as above also exists in case $d \geq m + 2$.

Proof. We distinguish two cases:

- $d \geq m + 2$.
  Define $\hat{A}$ in a similar manner as in the proof of theorem 6.5.5. Again, $\mathrm{cork}\hat{A} = m$ and as before,

  $$M := \mathcal{J}(Ax)^{*d} = \mathrm{diag}\big((Ax)^{*(d-1)}\big) \cdot dA = \mathrm{diag}\big((\hat{A}x)^{*(d-1)}\big) \cdot dA$$

  is a nilpotent quasi-Jacobian. It follows from theorem 6.5.8 that either two rows of $\hat{A}$ are dependent, or there exists a permutation P such that $P^{-1}MP$ is a triangular matrix. The rest of the proof of this case is similar to the proof of theorem 6.5.5.

- $d \geq \mathrm{cork}A + 1$.
  Since F is linearly triangularizable, the components of $(Ax)^{*d}$ are linearly dependent, say that

  $$\lambda_1(A_1x)^{*d} + \lambda_2(A_2x)^{*d} + \cdots + \lambda_r(A_rx)^{*d} = 0$$

  with $\lambda_r \neq 0$. Since $(Ax)^{*d}$ is reduced, it follows that $A_r = 0$ and hence

  $$m \leq \mathrm{cork}A - 1 \tag{6.13}$$

  So $d \geq m + 2$ and the previous case gives us the desired result. $\square$

We show that the estimates on d in corollary 6.5.9 are tight and hence by the proof of corollary 6.5.9, the estimates on d in theorem 6.5.8 are tight as well. For that purpose, notice that

$$\begin{aligned}
&\frac{1}{d}\left(\frac{\partial}{\partial x_1} - \frac{1}{d}\frac{\partial}{\partial x_2}\right)\left(\sum_{i=0}^{d}(-1)^i\binom{d}{i}(x_1 + ix_2)^d - (-1)^d d!\, x_2^d\right)\\
&\quad= \sum_{i=0}^{d}(-1)^i\binom{d}{i}\left(1-\frac{i}{d}\right)(x_1+ix_2)^d - \left(-\frac{1}{d}\right)(-1)^d d!\, x_2^{d-1}\\
&\quad= \sum_{i=0}^{d-1}(-1)^i\binom{d-1}{i}(x_1+ix_2)^{d-1} - (-1)^{d-1}(d-1)!\, x_2^{d-1}
\end{aligned}$$

and

$$\begin{aligned}
&-\frac{1}{d^2}\frac{\partial}{\partial x_2}\left(\sum_{i=0}^{d}(-1)^i\binom{d}{i}(x_1 + ix_2)^d - (-1)^d d!\, x_2^d\right)\\
&\quad= \sum_{i=0}^{d}(-1)^i\binom{d}{i}\frac{-i}{d}(x_1+ix_2)^d - \left(-\frac{1}{d}\right)(-1)^d d!\, x_2^{d-1}\\
&\quad= \sum_{i=1}^{d}(-1)^{i-1}\binom{d-1}{i-1}(x_1+ix_2)^{d-1} - (-1)^{d-1}(d-1)!\, x_2^{d-1}\\
&\quad= \sum_{i=0}^{d-1}(-1)^i\binom{d-1}{i}(x_1+ix_2)^{d-1} - (-1)^{d-1}(d-1)!\, x_2^{d-1}
\end{aligned}$$

By induction on d, we can obtain that

$$\sum_{i=0}^{d}(-1)^i\binom{d}{i}(x_1+ix_2)^d = (-1)^d d!\, x_2^d \tag{6.14}$$

for all $d \geq 1$.

Example 6.5.10. Put $n = d + 2$ and

$$H := \begin{pmatrix} 0 \\ x_1^d \\ (x_1+x_2)^d \\ (x_1+2x_2)^d \\ \vdots \\ (x_1+dx_2)^d \end{pmatrix}$$

then by (6.14),

$$\sum_{i=0}^{d}(-1)^i\binom{d}{i}H_{i+2} = (-1)^d d!\,x_2^d$$

Now define $T \in \mathrm{GL}_{d+2}(\mathbb{C})$ by $T_i^{-1} = e_i^{\mathrm{t}}$ for all $i \neq 2$, $(T^{-1})_{21} = 0$, $(T^{-1})_{22} = 1 = (-1)^0\binom{d}{0}$ and $(T^{-1})_{2(i+2)} = (-1)^i\binom{d}{i}$ for all $i \geq 1$. Then $T_i = e_i^{\mathrm{t}}$ for all $i \neq 2$, $T_{21} = 0$, $T_{22} = 1$ and $T_{2(i+2)} = -(-1)^i\binom{d}{i}$ for all $i \geq 1$. Furthermore,

$$G := T^{-1}H(Tx) = \begin{pmatrix} 0 \\ (-1)^d d!\,(T_2x)^d \\ (x_1 + T_2x)^d \\ (x_1 + 2T_2x)^d \\ \vdots \\ (x_1 + dT_2x)^d \end{pmatrix}$$

is power linear, but the diagonal of its Jacobian is nonzero. So G is not symmetrically triangularizable. From lemma 6.2.1, it follows that G is reduced. So the first estimate on d in corollary 6.5.9 is tight. By adding zero components to G, we see that the second estimate on d in corollary 6.5.9 is tight as well.

We shall give another example that is linearly triangularizable, but not symmetrically triangularizable. But this example will meet the estimates on d. So what is the catch? The Jacobian is not nilpotent. For that purpose, notice first that

$$\begin{aligned}
\sum_{i=0}^{d-1}\zeta_d^i(x_1 + \zeta_d^i x_2)^d &= \sum_{i=0}^{d-1}\zeta_d^i\sum_{j=0}^{d}\binom{d}{j}x_1^j(\zeta_d^i x_2)^{d-j} \\
&= \sum_{j=0}^{d}\binom{d}{j}x_1^j x_2^{d-j}\sum_{i=0}^{d-1}\zeta_d^{i(1+d-j)} \\
&= \binom{d}{1}x_1^1 x_2^{d-1}\sum_{i=0}^{d-1}1 \\
&= d^2 x_1 x_2^{d-1} \qquad (6.15)
\end{aligned}$$

Example 6.5.11. Now let $n = d + 1$ and

$$H := \begin{pmatrix} 0 \\ d^2 x_1 x_2^{d-1} \\ (x_1 + \zeta_d x_2)^d \\ \vdots \\ (x_1 + \zeta_d^{d-1} x_2)^d \end{pmatrix}$$

From (6.15), it follows that

$$H_2 - \sum_{i=1}^{d-1} \zeta_d^i H_{i+2} = (x_1 + x_2)^d \tag{6.16}$$

Now define $T \in \mathrm{GL}_{d+2}(\mathbb{C})$ by $T_i^{-1} = e_i^{\mathrm{t}}$ for all $i \neq 2$, $(T^{-1})_{21} = 0$, $(T^{-1})_{22} = 1$ and $(T^{-1})_{2(i+2)} = -\zeta_d^i$ for all $i \geq 1$. Then $T_i = e_i^{\mathrm{t}}$ for all $i \neq 2$, $T_{21} = 0$, $T_{22} = 1$ and $T_{2(i+2)} = \zeta_d^i$ for all $i \geq 1$. Furthermore,

$$G := T^{-1} H(Tx) = \begin{pmatrix} 0 \\ (x_1 + T_2 x)^d \\ (x_1 + \zeta_d T_2 x)^d \\ \vdots \\ (x_1 + \zeta_d^{d-1} T_2 x)^d \end{pmatrix}$$

is power linear, but its Jacobian has a principal minor of size 2 without a zero entry. So G is not symmetrically triangularizable. From lemma 6.2.1, it follows that G is reduced.

One can easily see that by adding components $x_3^d, x_4^d, \ldots$ and zeros, one can increase the Jacobian rank and dimension of each of the above examples.

6.6 (Ditto) linear triangularizability

Notice that a power linear map remains power linear after a conjugation with a permutation. On the other hand, the map G in example 6.5.10 is not symmetrically triangularizable, but admits a linear conjugation that makes its Jacobian lower triangular without affecting the power linearity, because H in example 6.5.10 is power linear with a lower triangular Jacobian. This leads to the following definition.

Definition 6.6.1. Assume $H = (Ax)^{*d}$ is power linear. Then we call H *ditto linearly triangularizable* if there exists a $T \in \mathrm{GL}_n(\mathbb{C})$ such that $T^{-1}H(Tx)$ is power linear as well and $\mathcal{J}T^{-1}H(Tx)$ is lower triangular.

So symmetric triangularizability implies ditto linear triangularizability. So the map G in example 6.5.10 is ditto linearly triangularizable, but we will prove that the map G in example 6.5.11 is *not* ditto linearly triangularizable. But first we show that power linear maps with nilpotent Jacobians of rank 2 are ditto linearly triangularizable.

Theorem 6.6.2. *Assume $(Mx)^{*d}$ and $(AX)^{*d}$ are GZ-paired and $(Mx)^{*d}$ is ditto linearly triangularizable. Then $(AX)^{*d}$ is ditto linearly triangularizable as well.*

Proof. From i) of proposition 6.2.7, it follows that we may assume that $\mathcal{J}(Mx)^{*d}$ is lower triangular. Say that $(Mx)^{*d}$ and $(AX)^{*d}$ are GZ-paired through matrices B and C. Let T^{-1} be a matrix of the form $\binom{B}{*}$, such that the i-th row of T^{-1} is of the form E_j for all $i > n$. Then the i-th component of $T^{-1}(ATX)^{*d}$ is a power of a linear form all $i > n$ and

$$\ker AT = T^{-1}\ker A = T^{-1}\ker B = \ker BT = \ker(I_n \mid \emptyset) = \{0\}^n \times \mathbb{C}^{N-n}$$

so $T^{-1}(ATX)^{*d} \in \mathbb{C}[x]^N$.

We shall show that $T^{-1}(ATX)^{*d}$ is power linear with a lower triangular Jacobian. Notice that we have shown already that the last $N-n$ components of $T^{-1}(ATX)^{*d}$ are of the desired form. Let $\tilde{C}$ be the matrix consisting of the first n columns of T. Since $T^{-1}(ATX)^{*d} \in \mathbb{C}[x]^N$, it follows that

$$T^{-1}(ATX)^{*d} = T^{-1}(ATX)^{*d}|_{x_{n+1}=x_{n+2}=\cdots=x_N=0} = T^{-1}(A\tilde{C}x)^{*d}$$

By ix) of proposition 6.2.7, the first n components of $T^{-1}(ATX) \in \mathbb{C}[x]^N$ are exactly those of $(Mx)^{*d}$. This gives the desired result. $\square$

Corollary 6.6.3. *Assume A is a square matrix. Then $(Ax)^{*d}$ is ditto linearly triangularizable as a power linear map in each of the following cases:*

i) $\mathrm{rk}A \le 1$,

ii) $\mathrm{rk}A \le 2$ *and $\mathcal{J}(Ax)^{*d}$ is nilpotent.*

Proof. Let $r = \operatorname{rk} A$. From theorem 6.2.11, it follows that there exists a homogeneous $H \in \mathbb{C}[x_1, \dots, x_r]^r$ such that H and $(Ax)^{*d}$ are GZ-paired through matrices B and C. Since either $r \le 1$ or $r \le 2$ and $\mathcal{J}_{x_1,\dots,x_r} H$ is nilpotent, it follows that H is power linear and ditto linearly triangularizable. Now apply the above theorem. $\square$

Let
$$M := \mathcal{J}(Ax)^{*d} = \operatorname{diag}\big((Ax)^{*(d-1)}\big) \cdot d\,A$$
be a nilpotent power linear (quasi-)Jacobian that is linearly triangularizable. Just as with symmetric triangularizability, we will give tight estimates on d such that $(Ax)^{*d}$ is ditto linearly triangularizable in case d meets one of these estimates. But we first give the examples. In order to prove that the examples are not ditto linearly triangularizable, we need some preparations.

Proposition 6.6.4. *Assume $(Ax)^{*d}$ is linearly triangularizable and $A_1 = 0$. Then there exists a $T \in \mathrm{GL}_n(\mathbb{C})$ with $T_1 = e_1^{\mathrm{t}}$ and $Te_1 = e_1$ such that the Jacobian of $T^{-1}(ATx)^{*d}$ is lower triangular.*

i) If $(Ax)^{*d}$ is ditto linearly triangularizable, then there exists a T as above such that in addition, $T^{-1}(ATx)^{*d}$ is power linear.

ii) If $A_2 = \lambda e_1^{\mathrm{t}}$ for some $\lambda \in \mathbb{C}$, then there exists a T as above such that in addition, $T_2 = e_2^{\mathrm{t}}$ and $Te_2 = e_2$.

iii) If $(Ax)^{*d}$ is ditto linearly triangularizable and $A_2 = \lambda e_1^{\mathrm{t}}$ for some $\lambda \in \mathbb{C}$, then there exists a T as above such that in addition, $T^{-1}(ATx)^{*d}$ is power linear and $T_2 = e_2^{\mathrm{t}}$ (but not $Te_2 = e_2$).

iv) If $(Ax)^{*d}$ is ditto linearly triangularizable and $A_2 = 0$ (i.e. $\lambda = 0$ in iii)), then there exists a T as above such that in addition, $T^{-1}(ATx)^{*d}$ is power linear, $T_2 = e_2^{\mathrm{t}}$ and $Te_2 = e_2$.

Proof. Since $(Ax)^{*d}$ is linearly triangularizable, there exists a $T \in \mathrm{GL}_n(\mathbb{C})$ such that $\mathcal{J}T^{-1}(ATx)^{*d}$ is lower triangular. We show that we can choose T such that $T_1^{-1} = e_1^{\mathrm{t}}$, since then $T_1 = e_1^{\mathrm{t}}$.

First we show that we can choose T such that $T_i^{-1} = e_1^{\mathrm{t}}$ for some i. For that purpose, write
$$e_1^{\mathrm{t}} = \mu_1 T_1^{-1} + \cdots + \mu_i T_i^{-1}$$

with $\mu_i \neq 0$. Now define

$$L^{-1} = \begin{pmatrix} & & & 0 & & & \\ & I_{i-1} & & \vdots & & \emptyset & \\ & & & 0 & & & \\ \mu_1 & \cdots & \mu_{i-1} & \mu_i & 0 & \cdots & 0 \\ & & & 0 & & & \\ & \emptyset & & \vdots & & I_{n-i} & \\ & & & 0 & & & \end{pmatrix}$$

then L is lower triangular, so $\mathcal{J}L^{-1}T^{-1}(ATLx)^{*d}$ is also lower triangular. Furthermore, the i-th component of $L^{-1}T^{-1}(ATLx)^{*d}$ is zero and therefore, $L^{-1}T^{-1}(ATLx)^{*d}$ is power linear in case $T^{-1}(ATx)^{*d}$ is. Since $(L^{-1}T^{-1})_i = e_1^{\mathrm{t}}$, we can replace T by TL to obtain $T_i^{-1} = e_1^{\mathrm{t}}$.

In order to get $T_1^{-1} = e_1^{\mathrm{t}}$, we replace T by TP and T^{-1} by $P^{-1}T^{-1}$, where P is the cycle $(x_2, \ldots, x_i, x_1, x_{i+1}, \ldots, x_n)$ and P^{-1} is the cycle $(x_i, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. To show that this works, it suffices to show that $\mathcal{J}P^{-1}H(Px)$ is lower triangular in case $\mathcal{J}H$ is lower triangular and $H_i = 0$. This follows from $\mathcal{J}P^{-1}H(Px) = P^{\mathrm{t}}\mathcal{J}H|_{x=Px}P$ and the fact that $P^{\mathrm{t}}MP$ is lower triangular in case M is lower triangular and $M_i = 0$ (see also the proof of theorem 6.5.8).

So we can choose T such that $T_1 = e_1^{\mathrm{t}}$. In order to get $Te_1 = e_1$ in addition, define

$$\tilde{L} = \left(\begin{array}{c|ccc} & 0 & \cdots & 0 \\ T^{-1}e_1 & & & \\ & & I_{n-1} & \\ & & & \end{array}\right)$$

then $T\tilde{L}e_1 = e_1$, so replacing T by $T\tilde{L}$ and T^{-1} by $\tilde{L}^{-1}T^{-1}$ results in $Te_1 = e_1$. Furthermore, the lower triangularity is preserved because $\tilde{L}$ is lower triangular. $T_1 = e_1^{\mathrm{t}}$ is preserved as well, as desired.

i) Assume that $T^{-1}(ATx)^{*d}$ is power linear with a lower triangular Jacobian. As mentioned above, $L^{-1}T^{-1}(ATLx)^{*d}$ is power linear with a lower triangular Jacobian as well. So $P^{-1}L^{-1}T^{-1}(ATLPx)^{*d}$ is power linear with a lower triangular Jacobian in addition, and by replacing T by TLP just as above, we may assume that $T_1 = e_1^{\mathrm{t}}$.

   Since $T_1^{-1}(Ax)^{*d} = (Ax)_1^{*d} = (A_1x)^{*d} = 0$ and $\tilde{L}^{-1} = (\tilde{L}^{-1}e_1 \mid e_2 \mid \cdots \mid e_n)$, it follows that $\tilde{L}^{-1}T^{-1}(AT\tilde{L}x) = T^{-1}(AT\tilde{L}x)$ is power linear as well. So the above construction for obtaining $T_1 = e_1^{\mathrm{t}}$ and $Te_1 = e_1$ does not affect that $T^{-1}(ATx)^{*d}$ is power linear.

ii) Write
$$e_2^{\mathrm{t}} = \nu_1 T_1^{-1} + \cdots + \nu_j T_j^{-1}$$
with $\nu_j \neq 0$. In a similar manner as above, we obtain $T_j^{-1} = e_2^{\mathrm{t}}$ for some j. Notice that $T_j^{-1}(ATx)^{*d} = (A_2 Tx)^d = (\lambda e_1 Tx)^d = (\lambda T_1 x)^d = (\lambda x_1)^d$.

In order to get $T_2^{-1} = e_2^{\mathrm{t}}$, we replace T by TQ and T^{-1} by $Q^{-1}T^{-1}$, where Q is the cycle $(x_1, x_3, \ldots, x_j, x_2, x_{j+1}, \ldots, x_n)$ and Q^{-1} is the cycle $(x_1, x_j, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$. To show that this works, we must show that $\mathcal{J}Q^{-1}H(Qx)$ is lower triangular in case $\mathcal{J}H$ is lower triangular and $H_j \in \mathbb{C}[x_1]$. This follows from $\mathcal{J}Q^{-1}H(Qx) = Q^{\mathrm{t}} \cdot \mathcal{J}H|_{x=Qx}Q$ and the fact that $Q^{\mathrm{t}}MQ$ is lower triangular in case M is lower triangular and M_j is a multiple of e_1^{t} (see also the proof of theorem 6.5.8).

So we can choose T such that $T_1 = e_1^{\mathrm{t}}$ and $T_2 = e_2^{\mathrm{t}}$. We can get $Te_2 = e_2$ in a similar manner as we obtained $Te_1 = e_1$. Both obtaining $Te_1 = e_1$ and $Te_2 = e_2$ preserve the lower triangularity and $T_1 = e_1^{\mathrm{t}}$ and $T_2 = e_2^{\mathrm{t}}$ are preserved as well, since $(T^{-1})_{21}(T^{-1})_{12} = 0$, as desired.

iii) This follows in a similar way as i) and ii). Since $A_2 \neq 0$, it follows that getting $T_2 = e_2^{\mathrm{t}}$ may affect the power linearity of the triangularization, so that property is not included.

iv) See iii). Now we do have $A_2 = 0$, so getting $T_2 = e_2^{\mathrm{t}}$ does not affect the power linearity of the triangularization, as desired. □

Notice that by (6.16), the components of H in example 6.5.11 generate
$$\mathbb{C}(x_1 + x_2)^d + \mathbb{C}(x_1 + 2x_2)^d + \cdots + \mathbb{C}(x_1 + dx_2)^d$$
Hence, the components of $\mathcal{J}_{x_2}H$ generate
$$\mathbb{C}(x_1 + x_2)^{d-1} + \mathbb{C}(x_1 + 2x_2)^{d-1} + \cdots + \mathbb{C}(x_1 + dx_2)^{d-1}$$
From lemma 6.2.1 with $r = (d-1)+1$, it follows that the dimension of this space is d, so the last d components
$$\frac{\partial}{\partial x_2}H_2, \frac{\partial}{\partial x_2}H_3, \ldots, \frac{\partial}{\partial x_2}H_n$$

of $\mathcal{J}_{x_2} H$ are linearly independent over $\mathbb{C}$. Applying the below lemma with $j = 2$ on the map G in example 6.5.11, we see that this map is not ditto linearly triangularizable.

Lemma 6.6.5. *Assume that H is homogeneous of degree d and $j \leq 3$ such that $H_1 = \cdots = H_{j-1} = 0$ and $H_i \in \mathbb{C}[x_1, x_2, \ldots, x_j]$ for all $i \geq j$. Assume in addition that H_j is not a power of a linear form and*

$$\frac{\partial}{\partial x_j} H_{j+1}, \frac{\partial}{\partial x_j} H_{j+2}, \ldots, \frac{\partial}{\partial x_j} H_n$$

is linearly independent over $\mathbb{C}$ of $\frac{\partial}{\partial x_j} H_j$.

If there exists an $S \in \mathrm{GL}_n(\mathbb{C})$ such that $S_i = e_i^{\mathrm{t}}$ for all $i < j$ and $S^{-1}H(Sx)$ is power linear, then there does not *exist a $T \in \mathrm{GL}_n(\mathbb{C})$ such that $T^{-1}H(Tx)$ is power linear with a lower triangular Jacobian.*

Proof. Assume there does exist a $T \in \mathrm{GL}_n(\mathbb{C})$ such that $G := T^{-1}H(Tx)$ is power linear with a lower triangular Jacobian. Since G is a linear triangularization of $S^{-1}H(Sx)$ and $S_i = e_i^{\mathrm{t}}$ for all $i < j$, it follows from proposition 6.6.4 that we may assume that $T_i = e_i^{\mathrm{t}}$ for all $i < j$ as well.

Since $G_i = 0$ for all $i < j$ and G_j is a power of a linear form, but $H_j = T_j G(T^{-1}x)$ is not, it follows that there exists an $k > j$ such that $T_{jk} \neq 0$. Since $T_i = e_i^{\mathrm{t}}$ for all $i < j$ and $H \in \mathbb{C}[x_1, x_2, \ldots, x_j]^n$, it follows that $H(Tx) = H|_{x_j = T_j x}$ and

$$\frac{\partial}{\partial x_k} H_i(Tx) = T_{jk} \left(\frac{\partial}{\partial x_j} H_i \right) \Big|_{x_j = T_j x}$$

So

$$0 = \frac{\partial}{\partial x_k} G_j = \frac{\partial}{\partial x_k} T_j^{-1} H(Tx) = T_{jk} \left(T_j^{-1} \mathcal{J}_{x_j} H \right) |_{x_j = T_j x}$$

i.e. $T_j^{-1} \mathcal{J}_{x_j} H = 0$. From $H_i = 0$ for all $i < j$ and the fact that

$$\frac{\partial}{\partial x_j} H_{j+1}, \frac{\partial}{\partial x_j} H_{j+2}, \ldots, \frac{\partial}{\partial x_j} H_n$$

is linearly independent over $\mathbb{C}$ of $\frac{\partial}{\partial x_j} H_j$, it follows that $T(^{-1})_{ji} = 0$ for all $i > j$ and that $G_j = (T^{-1})_{jj} H_j(Tx)$.

Since $(T^{-1})_{ji} = 0$ for all $i > j$ and T_j^{-1} is independent of $T_1 = e_1^{\mathrm{t}}, T_2 = e_2^{\mathrm{t}}, \dots, T_{j-1} = e_{j-1}^{\mathrm{t}}$, we obtain $(T^{-1})_{jj} \neq 0$. Since H_j is not a power of a linear form by assumption, this contradicts the fact that $G_j = (T^{-1})_{jj} H_j(Tx)$ is a power of a linear form, as desired. □

Example 6.6.6. Put

$$H := \begin{pmatrix} 0 \\ 0 \\ x_1^d - x_2^d \\ (x_1 + 2x_3)^d \\ \vdots \\ (x_1 + dx_3)^d \\ (x_2 + x_3)^d \\ (x_2 + 2x_3)^d \\ \vdots \\ (x_2 + dx_3)^d \end{pmatrix}$$

Substituting $(x_1, x_2) = (x_1, x_3)$ and $(x_1, x_2) = (x_2, x_3)$ in (6.14), we obtain

$$\sum_{i=0}^{d} (-1)^i \binom{d}{i} (x_1 + ix_3)^d = \sum_{i=0}^{d} (-1)^i \binom{d}{i} (x_2 + ix_3)^d \tag{6.17}$$

so

$$d(x_1+x_3)^d = (x_1^d - x_2^d) + \sum_{i=2}^{d} (-1)^i \binom{d}{i} (x_1 + ix_3)^d - \sum_{i=1}^{d} (-1)^i \binom{d}{i} (x_2 + ix_3)^d$$

It follows that there exists a $T \in \mathrm{GL}_{2d+2}(\mathbb{C})$ such that $T_i = e_i^{\mathrm{t}}$ for all $i \neq 3$ and $TH(T^{-1}x)$ is power linear.

In order to show that $TH(T^{-1}x)$ is not ditto linearly triangularizable, it suffices to show that $\frac{\partial}{\partial x_3} H_4, \frac{\partial}{\partial x_3} H_5, \dots, \frac{\partial}{\partial x_3} H_{2d+2}$ are linearly independent over $\mathbb{C}$, on account of $\frac{\partial}{\partial x_3} H_3 = 0$ and lemma 6.6.5 with $j = 3$. So assume

$$\frac{\partial}{\partial x_3}(\lambda_4 H_4 + \lambda_5 H_5 + \cdots + \lambda_n H_n) = 0$$

Applying $\frac{\partial}{\partial x_1}$ to the left hand side, it follows from lemma 6.2.1 with $r = (d-2)+1$ that $\lambda_4 = \cdots = \lambda_{d+2} = 0$. Using lemma 6.2.1 again with $r = (d-1)+1$ (without applying $\frac{\partial}{\partial x_1}$), we obtain $\lambda_{d+3} = \cdots = \lambda_{2d+2} = 0$ as well. So $TH(T^{-1}x)$ is not ditto linearly triangularizable.

Example 6.6.7. Now let

$$H := \begin{pmatrix} 0 \\ 0 \\ 2d^2 x_1 x_2^{d-1} \\ (x_1 + \zeta_d x_2 + x_3)^d \\ \vdots \\ (x_1 + \zeta_d^{d-1} x_2 + x_3)^d \\ (x_1 + x_2 - x_3)^d \\ (x_1 + \zeta_d x_2 - x_3)^d \\ \vdots \\ (x_1 + \zeta_d^{d-1} x_2 - x_3)^d \end{pmatrix}$$

By substituting $x_1 = x_1 \pm x_3$ in (6.15), we obtain

$$\begin{aligned} &\sum_{i=0}^{d-1} \zeta_d^i (x_1 + \zeta_d^i x_2 + x_3)^d + \sum_{i=0}^{d-1} \zeta_d^i (x_1 + \zeta_d^i x_2 - x_3)^d \\ &= \quad d^2 (x_1 + x_3) x_2^{d-1} + d^2 (x_1 - x_3) x_2^{d-1} \\ &= \quad 2d^2 x_1 x_2^{d-1} \end{aligned} \tag{6.18}$$

whence

$$H_3 - \zeta_d H_4 - \zeta_d^2 H_5 - \cdots - \zeta_d^{2d-1} H_{2d+2} = (x_1 + x_2 + x_3)^d$$

It follows that there exists a $T \in \mathrm{GL}_{2d+2}(\mathbb{C})$ such that $T_i = e_i^{\mathrm{t}}$ for all $i \neq 3$ and $TH(T^{-1}x)$ is power linear.
Showing that $TH(T^{-1}x)$ is not ditto linearly triangularizable goes in a similar manner as with the above example.

By adding components $x_4^d, x_5^d, \ldots$ and zeros, one can increase the Jacobian rank and dimension of each of the above examples. The proof of this and a similar result for example 6.5.11 is left as an exercise to the reader. The following lemma is crucial for the desired estimates on d.

Lemma 6.6.8. *Let $A \in \mathrm{Mat}_{k,n}(\mathbb{C})$ with pairwise independent rows and assume $d \geq 2$ and*

$$\frac{\partial}{\partial x_n} \sum_{i=1}^{k} (A_i x)^d = 0 \tag{6.19}$$

Then the following holds.

i) If there exists an $i \le k$ such that $A_{in} \ne 0$, then $k \ge d-1+\mathrm{rk}A \ge d+1$.

ii) If $A_{in} \ne 0$ for all $i \le k$ and $d \ge 3$, then $k \ge d-3+2\mathrm{rk}A$.

iii) If $A_{in} \ne 0$ for all $i \le k$, $d \ge 3$ and $\sum_{i=1}^{k}(A_i x)^d$ is not a power of a linear form, then $k \ge 2d-3+\mathrm{rk}A \ge 2d$.

Proof. Since the rows of A are pairwise independent, $r := \mathrm{rk}A \ge 2$.

i) Assume that there exists an $i \le k$ such that $A_{in} \ne 0$. By removing the rows A_i with $A_{in} = 0$, we obtain that $A_{in} \ne 0$ for all $i \le k$. Furthermore, the decrement of $\mathrm{rk}A$ does not exceed that of k, so we may assume that $A_{in} \ne 0$ for all $i \le k$. In case $d \ge 3$, ii) gives

$$k \ge d-3+2\mathrm{rk}A \ge d-1+\mathrm{rk}A \ge d+1$$

so assume $d = 2$. Then the terms of (6.19) are linear and $k \ge \mathrm{rk}A+1 \ge 3$. This gives the desired result.

ii) Assume that $A_{in} \ne 0$ for all $i \le k$ and $d \ge 3$. Assume without loss of generality that $A_1, A_2, \ldots, A_r$ are independent. Then $a_1 := A_1x, a_2 := A_2x, \ldots, a_r := A_rx$ are independent linear forms. Since $A_{in} \ne 0$ for all $i \le k$, it follows that

$$\sum_{i=1}^{r} A_{in}(A_i x)^{d-1}$$

has Hessian rank r. In order to have (6.19),

$$\sum_{i=r+1}^{k} A_{in}(A_i x)^{d-1}$$

must have Hessian rank r as well. So $k \ge 2r$. This gives the case $d = 3$.

In case $d \ge 4$, let D be a generic linear combination of $\frac{\partial}{\partial a_2}, \frac{\partial}{\partial a_3}, \ldots, \frac{\partial}{\partial a_n}$. Then

$$0 = D\frac{\partial}{\partial x_n}\sum_{i=1}^{k}(A_i x)^d = \frac{\partial}{\partial x_n}\sum_{i=2}^{k} d(DA_i x)(A_i x)^{d-1}$$

and the result follows by induction on d.

iii) Assume $A_{in} \neq 0$ for all $i \le k$, $d \ge 3$ and $S := \sum_{i=1}^{k}(A_i x)^d$ is not a power of a linear form. By (6.19), we obtain that S is degenerate, so if $r = 2$, then S would be a polynomial in one linear form. Since S is homogeneous and not a power of a linear form, $r \ge 3$ follows. If $d = 3$ then by ii) and $r \ge d$, we obtain the desired result.

So assume $d \ge 4$. Just as in the proof of ii), assume that $a_1 := A_1 x, a_2 := A_2 x, \ldots, a_r := A_r x$ are independent linear forms. We distinguish two cases:

- There is a $j \le r$ such that $\frac{\partial}{\partial a_j} S$ is not a $(d-1)$-th power of a linear form.
  Assume without loss of generality that $j = r$. Since $\frac{\partial}{\partial a_r}(A_i x)^d = 0$ for all $i < r$, it follows that
  $$\sum_{i=r}^{k} \frac{\partial}{\partial a_r}(A_i x)^d = \frac{\partial}{\partial a_r} \sum_{i=1}^{k}(A_i x)^d$$
  is not a $(d-1)$-th power of a linear form. So by induction on d, we obtain $1 + k - r \ge 2(d-1)$, whence $k \ge 2d - 3 + r \ge 2d$, as desired.
- $\frac{\partial}{\partial a_j} S$ is a $(d-1)$-th power of a linear form for all j.
  Notice that if $\frac{\partial}{\partial a_j} S$ and $\frac{\partial}{\partial a_{j'}} S$ would be linearly dependent over $\mathbb{C}$ for all j, j', $S \in \mathbb{C}[a_1, a_2, \ldots, a_r]$ would be degenerate of order $r - 1$. But this is impossible because S is homogeneous and not a power of a linear form. So we may assume that $\frac{\partial}{\partial a_1} S$ and $\frac{\partial}{\partial a_2} S$ are not linearly dependent over $\mathbb{C}$.
  Now each $A_i x$ can be expressed as a linear combination over $\mathbb{C}$ of the a_j's, and either there exists an i such that the coefficients of a_1 and a_2 in $A_i x$ are both nonzero, or such an i does not exist.
  In the first case, say that $A_{r+1} x = \mu_1 a_1 + \mu_2 a_2 + \cdots + \mu_r a_r$ with $\mu_1 \mu_2 \neq 0$. Notice that
  $$\left(\frac{\mu_2}{\mu_1}\frac{\partial}{\partial a_1} - \frac{\partial}{\partial a_2}\right) S \tag{6.20}$$
  is not a power of a linear form. But since
  $$\left(\frac{\mu_2}{\mu_1}\frac{\partial}{\partial a_1} - \frac{\partial}{\partial a_2}\right) A_{r+1} x = 0$$

(6.20) becomes the new $\frac{\partial}{\partial a_2} S$ if we interchange A_1 and A_{r+1}. So $k \geq 2d - 3 + r \geq 2d$ follows from the case above.

In the second case, we have that one of $\frac{\partial}{\partial a_1}$ and $\frac{\partial}{\partial a_2}$ kills at least half of $A_{r+1}x, A_{r+2}x, \ldots, A_n x$. Say that $\frac{\partial}{\partial a_1}$ kills at least half of $A_{r+1}x, A_{r+2}x, \ldots, A_n x$ and let $S' = \frac{\partial}{\partial a_1} S$. Then S' has at most $1 + (k - r)/2$ nonzero terms, and by induction on d, we obtain from i) that $1 + (k - r)/2 \geq (d - 1) + 1$. So $k \geq 2(d - 1) + r > 2d - 3 + r \geq 2d$, as desired. $\square$

Theorem 6.6.9. *Assume $(Ax)^{*d}$ has a nilpotent Jacobian and $(Ax)^{*d}$ is linearly triangularizable. If $\operatorname{cork} A \leq 2d - 2$, then $(Ax)^{*d}$ is ditto linearly triangularizable. Furthermore, if*

$$m := \operatorname{cork} A - \#\{i \mid A_i = 0\}$$

satisfies $m \leq 2d - 4$, then $(Ax)^{*d}$ is ditto linearly triangularizable.

Proof. Since $(Ax)^{*d}$ is linearly triangularizable, there exists a $T \in \mathrm{GL}_n(\mathbb{C})$ such that $H := T^{-1}(ATx)^{*d}$ is lower triangular. In particular, $(Ax)^{*d}$ satisfies $DP(+)$. From proposition 6.2.5, it follows that we may assume that $A_1 = 0$ and A_2 is dependent of e_1^{t}. From proposition 6.6.4, it follows that we may assume that $T_1 = e_1^{\mathrm{t}}$ and $T_2 = e_2^{\mathrm{t}}$.

Assume that $(Ax)^{*d}$ is *not* ditto linearly triangularizable. Then H is not power linear. Let H_s be the first component of H that is not a d-th power of a linear form. Notice that $s \geq 3$ and $H_s = T_s^{-1}(ATx)^{*d}$.

Choose T such that s is as large as possible (without affecting $T_1 = e_1^{\mathrm{t}}$ and $T_2 = e_2^{\mathrm{t}}$). Next, choose T such that the number of nonzero entries of T^{-1} is minimal. We distinguish two cases:

- There exists an i such that $(T^{-1})_{si}(A_i Tx)^d \in \mathbb{C}[x_1, x_2, \ldots, x_{s-1}] \backslash \{0\}$. Since T^{-1} is invertible, we can write

  $$e_i^{\mathrm{t}} = \lambda_1 T_1^{-1} + \lambda_2 T_2^{-1} + \ldots + \lambda_t T_t^{-1}$$

  where $\lambda_t \neq 0$. If $t < s$, then we can clean $(T^{-1})_{si}$ by row operations on T^{-1}, and since these row operations correspond to a multiplication by a lower triangular matrix L^{-1} from the left, we can replace T by TL and T^{-1} by $L^{-1}T^{-1}$ to obtain $(T^{-1})_{si} = 0$, without affecting the lower triangularity of $\mathcal{J}H$. This contradicts the minimality of the number of nonzero entries of T^{-1}.

So $t \geq s$. Again by replacing T by TL and T^{-1} by $L^{-1}T^{-1}$ for a suitable lower triangular L, we can obtain that the t-th row of T^{-1} becomes e_i^{t}. So from the minimality of the number of nonzero entries of T^{-1}, we obtain that this is already the case up to a scalar factor, i.e. $\lambda_1 = \lambda_2 = \cdots = \lambda_{t-1} = 0$ and hence $T_t^{-1} = \lambda_t^{-1} e_i^{\mathrm{t}}$. It follows that

$$H_t = T_t^{-1}(ATx)^{*d} = \lambda_t^{-1}(A_iTx)^d \in \mathbb{C}[x_1, x_2, \ldots, x_{s-1}]$$

So if

$$P := (x_1, x_2, \ldots, x_{s-1}, x_t, x_s, \ldots, x_{t-1}, x_{t+1}, \ldots, x_n)$$

then $(PH(P^{-1}x))_s$ is a power of a linear form and one can easily verify that the Jacobian of $PH(P^{-1}x)$ is lower triangular (see also the proof of theorem 6.5.8). This contradicts the maximality of s.

- $(T^{-1})_{si}(A_iTx)^d \notin \mathbb{C}[x_1, x_2, \ldots, x_{s-1}] \backslash \{0\}$ for all i.
  Let $k := \#\{i \mid T_{si}^{-1}(A_iTx)^d \neq 0\}$ and take D of the form

$$D = \mu_s \frac{\partial}{\partial x_s} + \mu_{s+1} \frac{\partial}{\partial x_{s+1}} + \cdots + \mu_n \frac{\partial}{\partial x_n}$$

such that $D(T_{si}^{-1}(A_iTx)^d) \neq 0$ for all i such that $T_{si}^{-1}(A_iTx)^d \neq 0$. Since H_s is not a power of a linear form and $DH_s = 0$, it follows from iii) of lemma 6.6.8 that

$$k \geq 2d - 3 + \mathrm{rk}\Big(\mathrm{diag}\big((T^{-1})_s^{\mathrm{t}}\big) \cdot AT\Big) \tag{6.21}$$

Since the matrix on the right hand side has $n - k - \#\{i \mid A_i = 0\}$ more zero rows than A, its rank is at least $\mathrm{rk}A - (n - k - \#\{i \mid A_i = 0\})$. It follows that $k + (n - k - \#\{i \mid A_i = 0\}) \geq 2d - 3 + \mathrm{rk}A$ and

$$m = n - \mathrm{rk}A - \#\{i \mid A_i = 0\} \geq 2d - 3$$

so the assumption $m \leq 2d - 4$ cannot be satisfied. So it suffices to show that the assumption $\mathrm{cork}A \leq 2d - 2$ cannot be satisfied either For that purpose, we distinguish two cases:

– A_2 is dependent of $A_3, A_4, \dots, A_n$.
  Since $T_1 = e_1^{\mathrm{t}}$, $T_2 = e_2^{\mathrm{t}}$ and A_2 is dependent of e_1^{t}, it follows that $(T^{-1})_{s2}(A_2Tx)^d \in \mathbb{C}[x_1]$. By assumption, $(T^{-1})_{si}(A_iTx)^d \notin \mathbb{C}[x_1, x_2, \dots, x_{s-1}] \backslash \{0\}$ for all i, so we obtain $(T^{-1})_{s2}(A_2Tx)^d = 0$. Furthermore, $A_1 = 0$ and the matrix on the right hand side of (6.21) has rank $\mathrm{rk}(A_3, A_4, \dots, A_n) - ((n-2)-k) = \mathrm{rk}A - (n-k-2)$ at least. It follows from (6.21) that $k-(n-k-2) \geq 2d-3+\mathrm{rk}A$, so $\mathrm{cork}A \geq 2d-1$. So the assumption $\mathrm{cork}A \leq 2d-2$ cannot be satisfied, as desired.

– A_2 is independent of $A_3, A_4, \dots, A_n$.
  Since A_2 is dependent of e_1^{t}, it follows that both removing the first column and removing the second row of A decreases its rank. Consequently, the first column of $(A_3, A_4, \dots, A_n)$ is dependent of the other columns of this matrix, So by column operations on A, we can clean the first column of $(A_3, A_4, \dots, A_n)$. Let $S \in \mathrm{GL}_n(\mathbb{C})$ be the matrix that corresponds to these column operations. Then $(AS)_{i1} = 0$ for all $i \geq 3$.

  Since $A_1 = 0$ and for each i, S_i is dependent of e_1^{t} and e_i^{t} only, it follows that $S^{-1}(ASx)^{*d}$ is power linear. Replacing $(Ax)^{*d}$ by $S^{-1}(ASx)^{*d}$, we obtain that $A_{i1} = 0$ for all $i \geq 3$. Now let $\tilde{A}$ be the matrix one obtains from A by removing both the first row and first column, and $\tilde{x} = (x_1, x_2, \dots, x_{n-1})$. Then $(\tilde{A}\tilde{x})^{*d}$ is a power linear map and $\mathrm{cork}\tilde{A} = \mathrm{cork}A$. Furthermore, it follows by way of the equivalence of strong nilpotency and linear triangularizability that $(\tilde{A}\tilde{x})^{*d}$ is linearly triangularizable.

  So by induction on n, we obtain that either $\mathrm{cork}A \geq 2d-1$ or $(\tilde{A}\tilde{x})^{*d}$ is ditto linearly triangularizable. So assume $\tilde{T}^{-1}(\tilde{A}\tilde{T}\tilde{x})^{*d}$ is power linear with a lower triangular Jacobian. Then by proposition 6.6.4, we may assume that $\tilde{T}\tilde{e}_1 = \tilde{e}_1$, the first standard unit vector in dimension $n-1$. Now $T^{-1}(ATx)^{*d}$ has a lower triangular Jacobian as well, where

$$T := \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \tilde{T} & \\ 0 & & & \end{pmatrix}$$

  Since $T_2^{-1}(ATx)^{*d} \in \mathbb{C}[x_1]$ and $T^{-1}e_2 = e_2$, it follows that the map $T^{-1}(ATx)^{*d}$ is power linear. This contradicts the assumption that $(Ax)^{*d}$ is not ditto linearly triangularizable, as desired. □

Corollary 6.6.10. *Assume $(Ax)^{*d}$ is linearly triangularizable and has a nilpotent Jacobian. If $n \le 2d+1$, then $(Ax)^{*d}$ is ditto linearly triangularizable.*

Proof. The case $\mathrm{rk} A \le 2$ follows from corollary 6.6.3. So assume $\mathrm{rk} A \ge 3$. Then $\mathrm{cork} A \le n-3 \le 2d-2$, and theorem 6.6.9 gives the desired result. □

In [9], it is show that power linear maps H of degree $d \ge 2$ for which $\mathcal{J}H^2 = 0$ are ditto linearly triangularizable. This result can be generalized. Notice that $\mathcal{J}H^2 = 0$ implies that $x + H$ is a quasi-translation by way of iii) of proposition 3.1.2, for $\mathcal{J}H \cdot H = d\mathcal{J}H^2$ in case H is homogeneous of degree d.

Proposition 6.6.11. *Assume H is power linear over $\mathbb{C}$, such that $x+H$ is a quasi-translation. Then H satisfies (3.4). Furthermore, H is symmetrically triangularizable in case H is reduced.*

Proof. Assume without loss of generality that H is reduced. Write $H = (Ax)^{*d}$. From ii) of proposition 3.1.2, it follows that $\nu(H_i) = 0$ for all i, where $\nu(f) := \deg_t f(x + tH)$ is the exponent with respect to $x + H$. Since $\mathbb{C}$ is a domain, it follows that $\nu(A_i x) = 0$ as well. So the coefficient of t in $A_i(x + tH)$ is zero, i.e. $A_i H = 0$. Since H is reduced, it follows that $H_j = 0$ for all j such that $A_{ij} \ne 0$. So if the variables appearing in H are $x_1, x_2, \ldots, x_s$, then $H_1 = H_2 = \cdots = H_s = 0$. Now apply iii) of proposition 3.4.3 to obtain the desired result. □

Chapter 7

Power linear Keller maps of corank three

7.1 Introduction

The main theorem of this chapter is the following.

Theorem 7.1.1. *Assume* $(Ax)^{*d}$ *is power linear with a nilpotent Jacobian. Put*

$$m := \mathrm{cork}A - \#\{i \mid A_i = 0\}$$

If $m \leq 1 \leq d$ *or* $m \leq 3 \leq d$, *then* $x + (Ax)^{*d}$ *is tame. Furthermore,* $(Ax)^{*d}$ *is (ditto) linearly triangularizable in any of the following cases:*

i) $m \leq d - 2$ *and* $m \leq 3$,

ii) $m \leq d - 1 \leq 3$ *and* $\mathrm{cork}A \leq d$,

iii) $\mathrm{cork}A \leq 3 \leq d$.

The main theorem improves results of [19] (corank ≤ 2 and degree 3 are tame), [7] (corank ≤ 2 and degree 3 are linearly triangularizable) and [11] (corank ≤ 2 and degree ≥ 3 are (ditto) linearly triangularizable). Before worrying about the proof of this theorem, let us first gather some results about power linear Keller maps from the previous chapter and add some new results as well.

Theorem 7.1.2. *Assume $(Ax)^{*d}$ is power linear with a nilpotent Jacobian. If $\mathrm{rk}A \le 3$ or $\mathrm{rk}A + d \le 7$, then $x + (Ax)^{*d}$ is tame. Furthermore, $(Ax)^{*d}$ is linearly triangularizable in any of the following cases:*

i) $\mathrm{rk}A \le 3$,

ii) $\mathrm{rk}A + d \le 6$,

iii) $\mathrm{rk}A + d \le 7$ *and at most six rows of A are nonzero,*

iv) $\mathrm{rk}A + d \le 7$ *and $n \le 8$.*

We will prove theorem 7.1.2 in the next section. After that, we will prove theorem 7.1.1 in a sequence of several sections. The next result is an immediate consequence of both theorems.

Corollary 7.1.3. *Assume $(Ax)^{*d}$ is power linear with a nilpotent Jacobian. If $d \ge 1$ and at most seven rows of A are nonzero, then $x + (Ax)^{*d}$ is tame. If $d = 3$ and eight rows of A are nonzero, then $x + (Ax)^{*d}$ is tame as well. Furthermore, $(Ax)^{*d}$ is linearly triangularizable in any of the following cases:*

i) At most six rows of A are nonzero,

ii) $n \le 8$ and at most seven rows of A are nonzero,

iii) $n \le 8$ and $d = 3$,

iv) $d \ge 5$ and at most seven rows of A are nonzero.

Proof. We first prove the tameness results. Put

$$m := \mathrm{cork}A - \#\{i \mid A_i = 0\}$$

Notice that

$$m = (n - \mathrm{rk}A) - (n - \#\{i \mid A_i \ne 0\}) = \#\{i \mid A_i \ne 0\} - \mathrm{rk}A \tag{7.1}$$

The case $m \le 3 \le d$ follows from theorem 7.1.1, so assume that either $m \ge 4$ or $1 \le d \le 2$.

Assume first that $1 \le d \le 2$. The case $\mathrm{rk}A \le 5$ follows from theorem 7.1.2, so assume $\mathrm{rk}A \ge 6$. Since $d \le 2 \ne 3$, the assumptions tell us that we may assume that at most seven rows of A are nonzero. Then $m \le \#\{i \mid A_i \ne$

$0\} - \mathrm{rk}A \le 7-6 = 1$ on account of (7.1), and theorem 7.1.1 gives the desired result.

So assume $m \ge 4$. The case $\mathrm{rk}A \le 3$. follows from theorem 7.1.2, so assume $\mathrm{rk}A \ge 4$. From (7.1) it follows that

$$\#\{i \mid A_i \ne 0\} = \mathrm{rk}A + m \ge 4 + 4 = 8$$

So by the assumptions on the number of nonzero rows of A, we may assume that $d = 3$ and $\#\{i \mid A_i \ne 0\} = 8$. This is only possible if $\mathrm{rk}A = 4 = m$, so $\mathrm{rk}A + d = 4 + 3 = 7$, and theorem 7.1.2 gives the desired result.

Next, we prove the linear triangularizability results.

i) Assume that at most six rows of A are nonzero. The case $\mathrm{rk}A \le 3$ follows from i) of theorem 7.1.2, so assume $\mathrm{rk}A \ge 4$. From (7.1), it follows that

$$m = \#\{i \mid A_i \ne 0\} - \mathrm{rk}A \le 6 - \mathrm{rk}A \le 2$$

The case $m \le d-2$ follows from i) of theorem 7.1.1, so assume $m \ge d-1$. Since $d-1 \le m \le 6 - \mathrm{rk}A$, we obtain $\mathrm{rk}A + d \le 7$. Now apply iii) of theorem 7.1.2.

ii) Assume that $n \le 8$ and at most seven rows of A are nonzero. The case $\mathrm{rk}A + d \le 7$ follows from iv) of theorem 7.1.2. So assume $\mathrm{rk}A + d \ge 8$. Then

$$\mathrm{cork}A \le n - \mathrm{rk}A \le 8 - \mathrm{rk}A \le d$$

and on account of (7.1),

$$m = \#\{i \mid A_i \ne 0\} - \mathrm{rk}A \le 7 - \mathrm{rk}A \le n - 1 - \mathrm{rk}A = \mathrm{cork}A - 1 \le d - 1$$

So by ii) of theorem 7.1.1, we obtain the case $d - 1 \le 3$. So assume $d - 1 \ge 4$. Then $d \ge 5$, and iv) gives the desired result.

iii) Assume that $n \le 8$ and $d = 3$. The case $\mathrm{rk}A \le 4$ follows from iv) of theorem 7.1.2. In case $\mathrm{rk}A \ge 5$, we have $\mathrm{cork}A \le 3$ on account of $n \le 8$, and iii) of theorem 7.1.1 gives the desired result.

iv) Assume $d \ge 5$ and at most seven rows of A are nonzero. The case $\mathrm{rk}A \le 3$ follows from i) of theorem 7.1.2. So assume $\mathrm{rk}A \ge 4$. Then (7.1) and the assumption $d \ge 5$ give

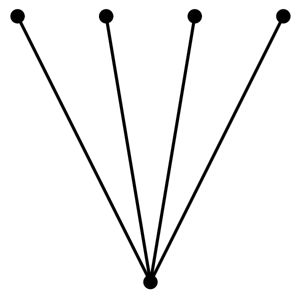
$$m = \#\{i \mid A_i \ne 0\} - \mathrm{rk}A \le 7 - 4 = 3 \le d - 2$$

and i) of theorem 7.1.1 gives the desired result. $\square$

As we have seen in chapter 6, Zhao uses graphs to view homogeneous gradient maps with nilpotent Jacobian. One of this graphs is the one that totally disconnected, but that one corresponds to a quasi-translation that satisfies (3.4) on account of proposition 6.4.7, i.e. a map that satisfies the Jacobian conjecture. The graph corresponding to

$$\begin{aligned} h &:= (x_1 + \mathrm{i}x_2)^4 + (x_1 - \mathrm{i}x_2)(x_3 + \mathrm{i}x_4)^3 \\ &= (x_1 + \mathrm{i}x_2)^4 + \\ &\quad \frac{1}{16}(x_1 - \mathrm{i}x_2 + x_3 + \mathrm{i}x_4)^4 + \frac{1}{16}\mathrm{i}(x_1 - \mathrm{i}x_2 + \mathrm{i}x_3 - x_4)^4 - \\ &\quad \frac{1}{16}(x_1 - \mathrm{i}x_2 - x_3 - \mathrm{i}x_4)^4 - \frac{1}{16}\mathrm{i}(x_1 - \mathrm{i}x_2 - \mathrm{i}x_3 + x_4)^4 \end{aligned}$$

is a so-called shrub.



This graph becomes totally disconnected when one removes the vertex below: the root of the shrub. By proposition 6.4.7, we obtain that a totally disconnected graph corresponds to a quasi-translation that satisfies (3.4). ii) of the following theorem shows that the root may be removed in some sense, and that shrubs correspond to compositions of two quasi-translations that satisfy (3.4).

Theorem 7.1.4. *Assume that* $h = \sum_{i=1}^{N}(A_i x)^{d+1}$ *and that the rows* A_i *of* A *are pairwise independent. Assume in addition that* $\mathcal{H}h$ *is nilpotent and define* $\tilde{h} := \sum_{i=1}^{N-1}(A_i x)^{d+1}$.

i) If either A_N *is independent of* $A_1, A_2, \ldots, A_{N-1}$*, or* $A_N A^{\mathrm{t}} = 0$*, then* $\mathcal{H}\tilde{h}$ *is nilpotent as well.*

ii) If $x + \nabla\tilde{h}$ *is invertible, then there exists a* $T \in \mathrm{GL}_n(\mathbb{C})$ *and an elementary polynomial map* E *such that*

$$x + \nabla h = T^{-1}E\big(T(x + \nabla\tilde{h})\big)$$

The above theorem can be used to prove the Jacobian conjecture for a map $x+\nabla h$ as in this theorem by induction on N. The induction hypothesis that $x+\nabla\tilde{h}$ satisfies the Jacobian conjecture connects the conclusion of i) that $\mathcal{H}\tilde{h}$ is nilpotent and the assumption of ii) that $x+\nabla\tilde{h}$ is invertible. We will prove theorem 7.1.4 in section 7.3.

7.2 Proof of theorem 7.1.2

We first show the tameness result. The case $\mathrm{rk} A \le 3$ follows from theorem 4.1.4 and the case $d \le 1$ is easy, so assume that So assume $4 \le \mathrm{rk} A + d \le 7$ and $d \ge 2$. Put $r := \mathrm{rk} A$. From theorem 6.2.11, it follows that there exists a homogeneous $H \in \mathbb{C}[x_1, x_2, \ldots, x_r]^r$ such that H and $(Ax)^{*d}$ are GZ-paired. From iv) of proposition 6.2.7, it follows that it suffices to show that $x + (H, 0^{r+1}, \ldots, 0^n)$ is tame. In case $r \ge 5$, we have $d \le 7-r \le 2 \le d$, so $d = 2$ and $r = 5$ follow, and theorem 4.6.8 gives the desired result. So assume $r = 4$. Then $d \le 7 - r \le 3$, and theorem 4.6.5 and lemma 7.2.1 below together give the desired result.

Lemma 7.2.1. *Assume H is cubic homogeneous in dimension $n = 4$ with a nilpotent Jacobian. Then $(x, x_5) + (H, 0)$ is tame in dimension 5.*

Proof. If H is linearly triangularizable, then $x + H$ is tame. So assume H is not linearly triangularizable. From theorem 4.6.5, it follows that we may assume that H is of the form

$$H = \begin{pmatrix} 0 \\ c(x_1) \\ x_2(x_1x_3 - x_2x_4) + p(x_1, x_2) \\ x_1(x_1x_3 - x_2x_4) + q(x_1, x_2) \end{pmatrix}$$

In case $c = p = q = 0$, we have that $H = (x_1x_3 - x_2x_4)(0, 0, x_2, x_1)$ and $x + H$ is a quasi-translation of the form $x + gH$ in theorem 3.4.4. This gives the case $c = p = q = 0$. The general case can be reduced to this case by way of 3 elementary maps from the left, namely the maps corresponding to the substitutions $y_2 = y_2 - c(y_1)$, $y_3 = y_3 - p(y_1, y_2)$ and $y_4 = y_4 - q(y_1, y_2)$. $\square$

We advance to the proofs of the linear triangularizability results of theorem 7.1.2. Again, set $r := \mathrm{rk} A$. From theorem 6.2.11, it follows that there exists a homogeneous $H \in \mathbb{C}[x_1, x_2, \ldots, x_r]^r$ such that H and $(Ax)^{*d}$ are GZ-paired.

From v) of proposition 6.2.7, it follows that it suffices to show that H is linearly triangularizable.

i) Assume $r = \mathrm{rk} A \leq 3$. From theorem 4.1.4, it follows that H is linearly triangularizable, which gives the desired result.

ii) Assume $r + d = \mathrm{rk} A + d \leq 6$. The case $r \leq 3$ follows from i), so assume $r \geq 4$. Then $d \leq 2$. The case $d \leq 1$ is easy, so assume $d = 2$. Then $4 \leq r \leq 6 - d = 4$, so $r = 4$. Consequently, H is linearly triangularizable on account of theorem 4.6.5. This gives the desired result.

iii) Assume $r + d = \mathrm{rk} A + d \leq 7$ and at most six rows of A are nonzero. From ii) it follows that we may assume that $r + d = \mathrm{rk} A + d = 7$. The case $d \leq 1$ is easy and the case $r \leq 3$ follows from i), so assume $r \geq 4$ and $d \geq 2$. Since $r + d = 7$, two cases remain

- $r = 4$ and $d = 3$.
  From lemma 7.2.4 below, it follows that H is linearly triangularizable.
- $r = 5$ and $d = 2$.
  In this case, we obtain that H is linearly triangularizable by way of lemma 7.2.3 below.

iv) The proof of this case is similar to that of iii).

In order to obtain the required lemmas 7.2.4 and 7.2.3 below, the following lemma is used.

Lemma 7.2.2. *Assume $n = 5$ and $f \in \mathbb{C}[x]$.*

i) Besides x_1^2, $(x_1+x_2)^2$ and x_2^2, one requires at least two squares of linear forms with x_4 to obtain

$$f := x_1 x_4 + p(x_1, x_2)$$

as a linear combination of them, where p is quadratic homogeneous. If one of the linear forms contains both x_2 and x_4, then at least three squares of linear forms are necessary besides x_1^2, $(x_1 + x_2)^2$ and x_2^2.

ii) *Besides x_1^2, $(x_1 + x_2)^2$ and x_2^2, one requires at least four squares of linear forms, of which four are independent, to add up to*

$$f := x_1x_3 - x_2x_4 + q(x_1, x_2)$$

where q is quadratic homogeneous. But if one of the linear forms must contain x_5, then at least six squares of linear forms are needed besides x_1^2, $(x_1 + x_2)^2$ and x_2^2.

Proof. We first show that restrictions with 'besides' are no real restrictions. For that purpose, assume a, b and c are the coefficients of x_1^2, $(x_1 + x_2)^2$ and x_2^2, in this order, when we write f as a linear combination of squares of linear form with as few linear forms as possible besides x_1^2, $(x_1 + x_2)^2$ and x_2^2. Now define $\tilde{f} = f - ax_1^2 - b(x_1 + x_2)^2 - cx_2^2$. Then $\tilde{f}$ has the same form as f, but x_1^2, $(x_1 + x_2)^2$ and x_2^2 are no longer necessary to write $\tilde{f}$ as a linear combination of the above-mentioned squares of linear forms from which f is a linear combination.

Notice that in order to write f as a linear combination of squares of linear forms, at least $\mathrm{rk}\mathcal{H}f$ such linear forms are necessary.

i) Since the Hessian of f has rank ≥ 2, at least two linear forms are necessary to write f as a linear combination of their squares. Two of these linear forms must contain x_4, because f has x_4 but not x_4^2. Assume that one of the linear forms, say L, has both x_2 and x_4. Now make $\tilde{f}$ in an arbitrary manner by subtracting a nonzero scalar multiple of L^2 from f. Then $\tilde{f}$ has Hessian rank ≥ 2 as well, so two linear forms are necessary to write $\tilde{f}$ as a linear combination of their squares. With similar arguments as with the previous $\tilde{f}$, we obtain the desired result.

ii) Since the Hessian of f has rank 4, at least four linear forms are necessary to write f as a linear combination of their squares. Furthermore, four of these linear forms must be independent. Now assume that one of these linear forms has x_5. By substituting a suitable linear form into x_5, we obtain that that linear form becomes x_5. We may do this substitution, because f has no x_5 and is thus not affected by the substitution. Now make $\tilde{f}$ in an arbitrary manner by subtracting a nonzero scalar multiple of x_5^2 from f. Then $\tilde{f}$ has Hessian rank 5. So five linear forms are necessary to write $\tilde{f}$ as a linear combination of their squares. With similar arguments as with the previous $\tilde{f}$, we obtain the desired result. □

Lemma 7.2.3. *Quadratic linear maps over $\mathbb{C}$ with nilpotent Jacobians of rank 5 are linearly triangularizable in case the dimension is at most 8 or at most 6 components are nonzero. Furthermore, they satisfy DP+ in case at most 7 components are nonzero.*

Proof. From v) of proposition 6.2.7 and theorems 6.2.11, 4.6.7 and 4.6.8, it follows that it suffices to show that the quadratic homogeneous maps of theorems 4.6.7 and 4.6.8 cannot be GZ-paired with quadratic linear maps in dimension ≤ 8 and neither with quadratic linear maps with at most 6 nonzero components.

Assume first that H is the quadratic homogeneous map of theorem 4.6.8, i.e.

$$H \equiv (0, x_1x_3, x_2^2 - x_1x_4, 2x_2x_3 - x_1x_5, x_3^2) \pmod{x_1^2}$$

Claim: besides x_1^2, $(x_1+x_3)^2$ and x_3^2, at least six squares of linear forms are necessary to make H_3 and H_4 as linear combinations of those squares.

Let us first prove the claim. From ii) of lemma 7.2.2, it follows that besides $x_1^2, (x_1+x_3)^2$ and x_3^2, at least four squares of linear forms are necessary to make $H_4 \equiv 2x_2x_3 - x_1x_5 \pmod{x_1^2}$ as a linear combination of their squares. If less than six squares of linear forms are necessary, then these forms do not contain x_4, and another two linear forms are necessary for H_3 on account of i) of lemma 7.2.2. This proves the claim.

Now assume that H and G are GZ-paired through $B \in \mathrm{Mat}_{n,N}(\mathbb{C})$ and $C \in \mathrm{Mat}_{N,n}(\mathbb{C})$. By adding a component $(B_1X)^2$ to G, we can obtain $G_N(Cx) = (B_1Cx)^2 = x_1^2$, but the object becomes showing that $N \ge 10$ and that G requires seven nonzero components. From $H = BG(Cx)$ and $H_5 \equiv x_3^2 \pmod{G_N(Cx)}$, it follows that x_3^2 is a linear combination of the components of $G(Cx)$. From $H = BG(Cx)$ and $H_2 \equiv \frac{1}{2}(x_1+x_3)^2 \pmod{x_1^2, x_3^2}$ it follows that $(x_1+x_3)^2$ is a linear combination of the components of $G(Cx)$

Since x_1^2, $(x_1+x_3)^2$ and x_3^2 are linear combinations of the components of $G(Cx)$ and $H = BG(Cx)$, it follows from the claim that nine squares of linear forms are necessary to make the components of $G(Cx)$ as linear combinations. Since $H_1 = 0$, we obtain by i) of proposition 6.3.3 that the components of $G(Cx)$ are linearly dependent, and $N \ge 10$ follows. So $N \ge 9$ originally, as desired. Furthermore, G has at least nine nonzero components now, so at least eight nonzero components originally, as desired.

Assume next that H is the quadratic homogeneous map of theorem 4.6.7, i.e.

$$H = \big(0, \lambda x_1^2, x_2x_4 + p(x_1, x_2), x_1x_3 - x_2x_5 + q(x_1, x_2), x_1x_4 + r(x_1, x_2)\big)$$

Claim: besides x_1^2, at least seven squares of linear forms are necessary to make the components of H as linear combinations.

Let us first assume that the claim is satisfied. Assume that H and G are GZ-paired through $B \in \mathrm{Mat}_{n,N}(\mathbb{C})$ and $C \in \mathrm{Mat}_{N,n}(\mathbb{C})$. Notice that G has at least seven nonzero components, because at least seven squares of linear forms are necessary to make the components of H as linear combinations. Furthermore, it follows from i) of proposition 6.3.3 that G satisfies DP+ because H does. So it remains to show that $N \geq 9$.

Assume first that two components of H are zero. Then there are two independent relations between the components of G on account of iv) proposition 6.3.3. It follows that the components of $G(Cx)$ are generated by at most $N-2$ squares of linear forms. From the claim and $H = BG(Cx)$, we obtain $N \geq 9$. So assume next that only one component of H is zero. Then the components of $G(Cx)$ are generated by at most $N-1$ squares of linear forms, but since $H = BG(Cx)$ and $H_2 = \lambda x_1^2 \neq 0$, we may assume that one of the linear forms is x_1. Again by the claim and $H = BG(Cx)$, we obtain $N \geq 9$.

So it remains to prove the claim. Assume first that besides x_1^2, less than six squares of linear forms are necessary to make $H_4 = x_1x_3 - x_2x_5 + q(x_1, x_2)$. Then these linear forms do not contain x_4 on account of ii) of lemma 7.2.2. Furthermore, at least four linear forms without x_4 are needed to make $H_4 = x_1x_3 - x_2x_5 + q(x_1, x_2)$ as a linear combination of them. For $H_3 = x_2x_4 + p(x_1, x_2)$, another linear form is needed: one with both x_2 and x_4. It follows from i) of lemma 7.2.2 that three additional linear forms are required for $H_3 = x_2x_4 + p(x_1, x_2)$ and $H_5 = x_1x_4 + r(x_1, x_2)$ together, which proves the claim for this case.

So assume that besides x_1^2, six squares of linear forms are necessary to make $H_4 = x_1x_3 - x_2x_5 + q(x_1, x_2)$. Assume that the claim is not satisfied, i.e. those six linear forms are all linear forms we need for components of H. Then for some $\lambda \in \mathbb{C}$, $H_4 - \lambda H_5$ is a linear combination of five squares of linear forms besides x_1^2, because one square can be eliminated by choosing λ

appropriate. Now set

$$S := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

then $\tilde{H} := SH|_{x_3=x_3+\lambda x_4}$ is of the same form as H, where $n = 5$. Furthermore, $\tilde{H}_4 = S_4 H|_{x_3=x_3+\lambda x_4} = (H_4 - \lambda H_5)|_{x_3=x_3+\lambda x_4}$ is a linear combination of five squares of linear forms besides x_1^2, because x_1^2 is not affected by the substitution $x_3 = x_3 + \lambda x_4$. It follows that the claim is satisfied for $\tilde{H}$ instead of H and hence for H itself as well, because x_1^2 is not affected by the inverse substitution $x_3 = x_3 - \lambda x_4$ either. □

Lemma 7.2.4. *Cubic linear maps over $\mathbb{C}$ with nilpotent Jacobians of rank 4 are (ditto) linearly triangularizable in case the dimension is at most 8 or at most 6 components of it are nonzero.*

Proof. From theorem 6.6.9 (with $m = 2$), it follows that it suffice to show linear triangularizability only. From v) of proposition 6.2.7 and theorems 6.2.11 and 4.6.5, it follows that it suffices to show that the cubic homogeneous map of theorem 4.6.5 cannot be GZ-paired with cubic linear maps in dimension ≤ 8 and neither with quadratic linear maps with at most 6 nonzero components.

So assume that H is the cubic homogeneous map of theorem 4.6.5, i.e.

$$H := \big(0, \lambda x_1^3, x_2(x_1x_3 - x_2x_4) + p(x_1, x_2), x_1(x_1x_3 - x_2x_4) + q(x_1, x_2)\big)$$

Claim: besides x_1^3, at least seven cubes of linear forms are necessary to make H_4 as a linear combination of them.

If the claim is satisfied, then we obtain the desired result by reasoning as in the proof of the above lemma (the second map H there has a similar claim). So it remains to prove the claim. Notice that

$$\frac{\partial}{\partial x_1} H_4 = (2x_1x_3 - x_2x_4) + \tilde{q}(x_1, x_2)$$

for some homogeneous $\tilde{q} \in \mathbb{C}[x_1, x_2]$ of degree 2. So by ii) of lemma 7.2.2, it follows that besides x_1^2, four squares of independent linear forms are needed to write $\frac{\partial}{\partial x_1} H_4$ as a linear combination of them, and maybe some other squares of linear forms as well.

By applying $\frac{\partial}{\partial x_1}$ to the cubes of linear forms that make H_3 as a linear combination, we obtain that besides x_1^3, four cubes of independent linear forms are needed to write H_4 as a linear combination of them and maybe some other cubes of linear forms as well. Let a_1, a_2, a_3, a_4 be such four independent linear forms.

Since a_1, a_2, a_3 and a_4 are independent, we may assume without loss of generality that x_1 is independent of a_1, a_2, a_3. Furthermore, it makes sense to talk about $\frac{\partial}{\partial a_4}$, a linear combination of $\frac{\partial}{\partial x_1}$, $\frac{\partial}{\partial x_2}$, $\frac{\partial}{\partial x_3}$, $\frac{\partial}{\partial x_4}$ such that $\frac{\partial}{\partial a_4} a_1 = \frac{\partial}{\partial a_4} a_2 = \frac{\partial}{\partial a_4} a_3 = 0$ and $\frac{\partial}{\partial a_4} a_4 = 1$.
Since x_1 is independent of a_1, a_2, a_3 and a_1, a_2, a_3 are independent, it follows that $\lambda := \frac{\partial}{\partial a_4} x_1 \neq 0$. Consequently,

$$\begin{aligned}
\frac{\partial}{\partial a_4} H_3 &= \lambda(2x_1x_3 - x_2x_4) - \mu x_1 x_4 + \hat{q}(x_1, x_2 + \lambda^{-1}\mu x_1) \\
&= \big(\lambda(2x_1x_3 - x_2x_4) + \hat{q}(x_1, x_2)\big)\big|_{x_2 = x_2 + \lambda^{-1}\mu x_1}
\end{aligned}$$

for some homogeneous $\hat{q} \in \mathbb{C}[x_1, x_2]$ of degree 2, where $\mu := \frac{\partial}{\partial a_4} x_2$. From ii) of lemma 7.2.2, it follows that besides x_1^2, at least four squares of linear forms are needed to write $\lambda(2x_1x_3 - x_2x_4) + \hat{q}(x_1, x_2)$ as a linear combination of them. Since the substitution $x_2 = x_2 + \lambda^{-1}\mu x_1$ does not affect x_1^2, a similar statement holds for $\frac{\partial}{\partial a_4} H_3$.

By applying $\frac{\partial}{\partial a_4}$ to the cubes of linear forms that make H_4 as a linear combination, we obtain that besides x_1^3, a_1^3, a_2^3 and a_3^3, at least four cubes of linear forms are needed to write H_4 as a linear combination of them. With a_1^3, a_2^3 and a_3^3, this adds up to seven cubes of linear forms besides x_1^3, as desired. □

Notice that the bound of seven cubes of linear forms in the above proof can be reached: $x_1^2 x_3$ can be done with three cubes, $-x_1x_2x_4$ can be done with four cubes, and $q(x_1, x_2)$ may only require x_1^3. The bounds in the proof of lemma 7.2.3 can be reached as well. But that is not really surprising, because in example 6.2.9 and the remark after it in the previous section , we have seen that the bounds of lemma 7.2.3 itself are tight as well.

7.3 Independent powers of linear forms

The following proposition was the starting point of finding the results of theorem 7.1.1.

Proposition 7.3.1. *Let F be an invertible polynomial map over $\mathbb{C}$ and*

$$G := (F_1, \dots, F_{n-1}, G_n)$$

Assume that $\det \mathcal{J}G = \det \mathcal{J}F$. Then there exists an elementary invertible polynomial map E, such that $G = E(F)$. In particular, G is invertible.

Proof. Since F is invertible, we can write $G_n - F_n$ as a polynomial in F, say $G_n - F_n = p(F)$ with $p \in \mathbb{C}[y]$. Since $\det \mathcal{J}G = \det \mathcal{J}F$, it follows that $\det \mathcal{J}H = 0$, where

$$H := (F_1, \dots, F_{n-1}, G_n - F_n)$$

From proposition 1.2.9, it follows that the components of H are algebraically dependent over $\mathbb{C}$ and that F_n is algebraically independent over $\mathbb{C}$ of the components of H, because $H_i = F_i$ for all $i \le n-1$ and $\operatorname{trdeg}_{\mathbb{C}} \mathbb{C}(F) = n$. Consequently, $G_n - F_n = p(F_1, \dots, F_{n-1}, y_n)$. Comparing degrees with respect to y_n, we obtain that $G_n - F_n = p(F_1, \dots, F_{n-1}, 0)$. It follows that $E = x + (0^1, \dots, 0^{n-1}, p(x))$ will do the job. $\square$

If one is only interested in invertibility, then there is a stronger result then the above proposition, due to E. Formanek. Formanek only assumes that $\mathbb{C}[F, G_n] = \mathbb{C}[x]$ and $\det \mathcal{J}G \in \mathbb{C}^*$ to obtain that G is invertible. See [29] or [24, §1.1, Exc. 9].

Formanek states that his result follows from a theorem of S. Wang in [52], although this is not immediately clear at first glance. In Formanek's own proof of his result that does not use Wang's theorem, he essentially remarks that the ideal of algebraic relations over $\mathbb{C}$ between $F_1, F_2, \dots, F_{n+1}$ is principal. So say that this ideal is generated by the relation R. Now by Wang's theorem, one can prove that $\frac{\partial}{\partial y_{n+1}} R \in \mathbb{C}^*$, which is only possible if R is of the form $R = \lambda y_{n+1} + p(y)$ for some $\lambda \in \mathbb{C}^*$ and a $p \in \mathbb{C}[y]$. So $F_{n+1} = -\lambda^{-1} p(F)$, which gives Formanek's result.

Corollary 7.3.2. *Assume $F \in \mathbb{C}[x]^n$ such that the nonlinear part of F_n is contained in $\mathbb{C}[a_n]$ for some linear form a_n, and the nonlinear parts of*

$F_1, \ldots, F_{n-1}$ are contained in $\mathbb{C}[a_1, \ldots, a_{n-1}]$ for some linear forms $a_1, \ldots,$ a_{n-1} such that $\mathbb{C}[a_1, a_2, \ldots, a_n] = \mathbb{C}[x]$.

Assume in addition that $\det \mathcal{J}F \in \mathbb{C}[a_1, \ldots, a_{n-1}]$. Then $\det \mathcal{J}F = \det \mathcal{J}G$, where $G := (F_1, F_2, \ldots, F_{n-1}, 0)$. Furthermore, if either F or G is invertible, then there exists an elementary invertible polynomial map E such that $G = E(F)$. In particular, both F and G are invertible in case one of them is.

Proof. Write $a = a_1, a_2, \ldots, a_n$. Since a_n is algebraically independent of $a_1, \ldots, a_{n-1}$ and the coefficients of $\mathcal{J}F$ are contained in $\mathbb{C}[a]$, the substitution $a_n = 0$ in $\mathcal{J}F$ makes sense. Furthermore, it makes sense to talk about $\frac{\partial}{\partial a_i}$ for all i and we have $\mathcal{J}_a a = I_n$. From the chain rule and the fact that $\mathcal{J}a$ is a matrix over $\mathbb{C}$, it follows that

$$(\mathcal{J}F)|_{a_n=0} = (\mathcal{J}_a F \cdot \mathcal{J}a)_{a_n=0} = (\mathcal{J}_a F)_{a_n=0} \cdot \mathcal{J}a$$

From the conditions on the nonlinear parts of the components of F, it follows that the rightmost hand side is equal to

$$\mathcal{J}_a G \cdot \mathcal{J}a = \mathcal{J}G$$

So $(\mathcal{J}F)|_{a_n=0} = \mathcal{J}G$. From $\det \mathcal{J}F \in \mathbb{C}[a_1, \ldots, a_{n-1}]$, it follows that

$$\det \mathcal{J}F = (\det \mathcal{J}F)|_{a_n=0} = \det \big((\mathcal{J}F)|_{a_n=0}\big) = \det \mathcal{J}G$$

and proposition 7.3.1 gives the desired result. $\square$

Proof of theorem 7.1.4. Let *hatA* be the matrix one obtains from A by replacing the last row by the zero row.

i) Assume first that A_N is independent of $A_1, A_2, \ldots, A_{N-1}$. Then there exists independent linear forms $a_1, a_2, \ldots, a_r$ such that $A_n x = a_r$ and such that $A_i x$ is dependent of $a_1, a_2, \ldots, a_{r-1}$ for each $i \le N-1$. So by (6.7),

$$\begin{aligned} \mathcal{H}\tilde{h} &= d(d+1)A^{\mathrm{t}}\mathrm{diag}\big((Ax)^{*(d-1)}\big)|_{a_r=0}A \\ &= d(d+1)\hat{A}^{\mathrm{t}}\mathrm{diag}\big((\hat{A}x)^{*(d-1)}\big)|_{a_r=0}\hat{A} \\ &= \mathcal{H}h|_{a_r=0} \end{aligned}$$

as desired.

Assume next that $A_N A^{\mathrm{t}} = 0$. Then

$$\operatorname{diag}((Ax)^{*(d-1)})AA^{\mathrm{t}} = \operatorname{diag}((\hat{A}x)^{*(d-1)})\hat{A}\hat{A}^{\mathrm{t}}$$

From (6.8), it follows that the left hand side is nilpotent. By (6.7), we obtain that $\mathcal{H}(\tilde{h} + 0^{d+1})$ is nilpotent, as desired.

ii) Assume that $x + \nabla\tilde{h}$ is invertible. Since $A_N \neq 0$, there exists a $T \in \mathrm{GL}_n(\mathbb{C})$ such that $A_N^{\mathrm{t}} = T^{-1}e_n$. Consequently,

$$\begin{aligned} T(x+\nabla h) - T(x+\nabla\tilde{h}) &= T\nabla(A_N x)^{d+1} \\ &= (d+1)TA_N^{\mathrm{t}}(A_N x)^d \\ &= (d+1)e_n(A_N x)^d \end{aligned}$$

So $T(x+\nabla h)$ and $T(x+\nabla\tilde{h})$ are only different on the last component. Since the former is of Keller type and the latter is invertible,

$$\det \mathcal{J}\big(T(x+\nabla h)\big) = \det T = \det \mathcal{J}\big(T(x+\nabla\tilde{h})\big)$$

It follows that $T(x+\nabla h)$ and $T(x+\nabla\tilde{h})$ satisfy the conditions of G and F in corollary 7.3.2 respectively. This gives the desired result. □

The above techniques can be used to prove some of the tameness results of the main theorem 7.1.1 of this section as well. For that purpose, we first make a definition.

Definition 7.3.3. Let A be a matrix. Define the *dependence number* $DN(A)$ as the number of rows of A that is dependent of the other rows of A.

Proposition 7.3.4. *Assume $(Ax)^{*d}$ is power linear with a nilpotent Jacobian. If $DN(A) \leq \operatorname{cork}A + \max\{3, 7-d\}$, then $x + (Ax)^{*d}$ is tame.*

Proof. Notice that variants of corollary 7.3.2, where F and G are different on another component then the lowest one, hold as well. By applying corollary 7.3.2 and its variants $n - DN(A)$ times, we can reduced to the case $DN(A) = n$, because $\operatorname{cork}A$ increases along with $DN(A)$, whence $DN(A) \leq \operatorname{cork}A + \max\{3, 7-d\}$ is not affected. So we may assume that $n \leq \operatorname{cork}A + \max\{3, 7-d\}$, i.e. $\operatorname{rk}A \leq \max\{3, 7-d\}$. Now apply theorem 7.1.2. □

But we will not use corollary 7.3.2 directly for reducing to the case $DN(A) = n$, because it does not work as desired for some other properties that $(Ax)^{*d}$ can have. For instance, applying corollary 7.3.2 to make the last row of A zero, introduces a new linear dependence relation, namely e_n^{t}. What happens with linear triangularizability is not clear to us, except that in case $(Ax)^{*d}$ is linearly triangularizable, then it remains linearly triangularizable after replacing an independent row of A by the zero row by way of corollary 7.3.2.

In the next section, we will modify the above method of getting rid of rows of A that are independent of the other rows of A. The new method does not introduce new linear dependence relations and 'conducts' symmetrical triangularizability and reducedness in both directions.

7.4 The crop matrix

Assume $A \in \mathrm{Mat}_n(\mathbb{C})$ such that A_n is independent of the other rows of A. In order to avoid some complications in the definition of crop matrix below, it is convenient to have $A_{nn} = 0$. The following proposition shows that $A_{nn} = 0$ in case $\mathcal{J}(Ax)^{*d}$ is nilpotent. So we may assume that $A_{nn} = 0$.

Proposition 7.4.1. *Let A be a matrix of size n such that A_n is independent of $A_1, A_2, \ldots, A_{n-1}$. If $A_{nn} \neq 0$, then $\mathrm{tr}\mathcal{J}(Ax)^{*d} \neq 0$.*

Proof. Assume $A_{nn} \neq 0$. Since

$$\mathcal{J}(Ax)^{*d} = d\,\mathrm{diag}\big((Ax)^{*(d-1)}\big) \cdot A$$

it follows that

$$\mathrm{tr}\mathcal{J}(Ax)^{*d} = d\sum_{i=1}^{n} A_{ii}(A_i x)^{d-1}$$

Now differentiate $d-2$ times with respect to x_n to obtain

$$\frac{\partial^{d-2}}{\partial x_n^{d-2}}\mathrm{tr}\mathcal{J}(Ax)^{*d} = d!\sum_{i=1}^{n} A_{ii}A_{in}^{d-2}A_i x$$

Since the coefficient of $A_n x$ on the right hand side is $d!A_{nn}^{d-1} \neq 0$ and A_n is independent of the other rows of A, it follows that the right hand side is nonzero. So the left hand side is nonzero as well, which gives the desired result. □

Definition 7.4.2. Let A be a matrix of size n such that $A_{nn} = 0$ and A_n is independent of $A_1, A_2, \ldots, A_{n-1}$. Define

$$B := A - uAe_nA_n$$

where u is a new variable (and Ae_n and A_n are the last column and last row of A). We call the upper left principal minor of size $n-1$ of B the *crop matrix of* A.

Theorem 7.4.3. *Let A be a matrix of size n such that $A_{nn} = 0$ and A_n is independent of $A_1, A_2, \ldots, A_{n-1}$ and let $\tilde{A}$ be the crop matrix of A. Write $\tilde{x} = x_1, x_2, \ldots, x_{n-1}$ and assume $d \geq 2$ is an integer. Then*

i) $x + (Ax)^{*d}$ *is an invertible (tame) polynomial map over* $\mathbb{C}[u]$, *if and only if* $x + ((\tilde{A}\tilde{x})^{*d}, 0)$ *is,*

ii) $\mathcal{J}(Ax)^{*d}$ *is nilpotent, if and only if* $\mathcal{J}(\tilde{A}\tilde{x})^{*d}$ *is,*

iii) $\operatorname{cork} A = \operatorname{cork} \tilde{A}$.

Proof. Let $\hat{A}$ be the matrix that consists of the first $n-1$ rows of A and define

$$H := ((\hat{A}x)^{*d}, u(A_nx))$$

Let $\tilde{B}$ be the matrix that consists of the first $n-1$ rows of $B := A - uAe_nA_n$. We first show that

$$\det \mathcal{J}(x + (Ax)^{*d}) = 1 \iff \det \mathcal{J}(x + H) = 1 \tag{7.2}$$

Since

$$\mathcal{J}(Ax)^{*d} = (\mathcal{J}H)|_{u=d\,(A_nx)^{d-1}}$$

the forward implication follows. Notice that each entry of $(\mathcal{J}Ax)^{*d}$ is contained in $\mathbb{C}[A_1x, A_2x, \ldots, A_nx]$. Since A_nx is algebraically independent of $A_1x, A_2x, \ldots, A_{n-1}x$, the substitution $A_nx = \sqrt[d-1]{u/d}$ in $\mathcal{J}(Ax)^{*d}$ makes sense, and the backward implication follows as well.

i) We show that the invertibility (tameness) of $x + (Ax)^{*d}$ is equivalent to that of $x + H$, $x + ((\tilde{B}x)^{*d}, 0)$ and $x + ((\tilde{A}\tilde{x})^{*d}, 0)$.

- Assume that either $x+(Ax)^{*d}$ or $x+H$ is invertible (tame). From (7.2), it follows that

$$\det \mathcal{J}(x+(Ax)^{*d}) = \det \mathcal{J}(x+H) = 1$$

Since $x+(Ax)^{*d}$ and $x+H$ only differ at the last coordinate, it follows from proposition 7.3.1 that both $x+(Ax)^{*d}$ and $x+H$ are invertible (tame). So $x+(Ax)^{*d}$ is invertible (tame), if and only if $x+H$ is.

- Since $B = A(I_n - ue_nA_n)$, it follows that $Bx = Ax|_{x=x-ue_nA_nx}$. Consequently,

$$Bx = Ax|_{x_n=x_n-uA_nx} \tag{7.3}$$

Since $A_{nn} = 0$, it follows that uA_nx does not contain x_n and we obtain

$$Ax = Bx|_{x_n=x_n+uA_nx} \tag{7.4}$$

From (7.3), it follows that

$$(x+H)|_{x_n=x_n-uA_nx} = \big(\tilde{x} + (\tilde{B}x)^{*d}, x_n\big) = x + \big((\tilde{B}x)^{*d}, 0\big)$$

So $x+H$ and $x+((\tilde{B}x)^{*d},0)$ can be obtained from each other by way of an elementary map from the right.

- Notice that $\tilde{A}$ consists of the first $n-1$ columns of $\tilde{B}$. So

$$\tilde{A}\tilde{x} = \tilde{B}x\Big|_{x_n=0} \tag{7.5}$$

Since A_n is independent of $A_1, A_2, \ldots, A_{n-1}$, there exists a $v \in \mathbb{C}^n$ such that $Av = e_n$. Set $\tilde{v} := (v_1, v_2, \ldots, v_{n-1})$. Since $A_{nn} = 0$, it follows that $\hat{A}e_n = \tilde{B}e_n$. From this and $\hat{A}v = 0$, we obtain

$$\begin{aligned} \tilde{A}\tilde{v} &= \tilde{B}v - \tilde{B}e_nv_n \\ &= \big(\hat{A} - u\hat{A}e_nA_n\big)v - \tilde{B}e_nv_n \\ &= -\tilde{B}e_n(uA_nv + v_n) \\ &= -(u+v_n)\tilde{B}e_n \end{aligned}$$

So $(-(u+v_n)\tilde{A} \mid \tilde{A}\tilde{v})x = (-(u+v_n)\tilde{A} \mid -(u+v_n)\tilde{B}e_n)x = -(u+v_n)\tilde{B}x$. Consequently,

$$\tilde{B}x = \tilde{A}\tilde{x}\Big|_{\tilde{x}=\tilde{x}-\frac{1}{u+v_n}\tilde{v}x_n} \tag{7.6}$$

and

$$\tilde{A}\tilde{x} = \tilde{B}x\Big|_{\tilde{x}=\tilde{x}+\frac{1}{u+v_n}\tilde{v}x_n} \tag{7.7}$$

Now let

$$T := \left(\begin{array}{ccc|c} & I_{n-1} & & \frac{1}{u+v_n}\tilde{v} \\ 0 & \cdots & 0 & 1 \end{array}\right)$$

then by (7.7), $\tilde{A}\tilde{x} = \tilde{B}Tx$ and

$$\begin{aligned} x + ((\tilde{A}\tilde{x})^{*d}, 0) &= T^{-1}Tx + \big((\tilde{B}Tx)^{*d}, 0\big) \\ &= T^{-1}\Big(Tx + \big((\tilde{B}Tx)^{*d}, 0\big)\Big) \end{aligned}$$

So $x + ((\tilde{A}\tilde{x})^{*d}, 0)$ and $x + ((\tilde{B}x)^{*d}, 0)$ are linearly conjugate.

So the invertibility (tameness) of $x + (Ax)^{*d}$ is equivalent to that of $x + H$, $x + ((\tilde{B}x)^{*d}, 0)$ and $x + ((\tilde{A}\tilde{x})^{*d}, 0)$, as desired.

ii) Since $\mathcal{J}(Ax)^{*d}$ is nilpotent, if and only if $\det \mathcal{J}(x + (Ax)^{*d}) = 1$, it follows from (7.2) that $\mathcal{J}(Ax)^{*d}$ is nilpotent, if and only if $\det \mathcal{J}(x + H) = 1$. From the proof of i), we obtain that $\det \mathcal{J}(x+H) = 1$, if and only if $\det \mathcal{J}(x + (\tilde{A}\tilde{x})^{*d}, 0)) = 1$, which in turn is equivalent to the nilpotency of $\mathcal{J}_{\tilde{x}}(\tilde{A}\tilde{x})^{*d}$, as desired.

iii) From (7.3), (7.5) and $A_{nn} = 0$, it follows that

$$(\tilde{A}\tilde{x}, A_n x) = Ax\big|_{x_n=-uA_n x} \tag{7.8}$$

From (7.6) and (7.4), it follows that

$$(A_1 x, A_2 x, \dots, A_{n-1}x) = \tilde{A}\tilde{x}\Big|_{\tilde{x}=\tilde{x}-\frac{1}{u+v_n}\tilde{v}(x_n+uA_n x)} \tag{7.9}$$

It follows that $\tilde{A}$ and $\hat{A}$ have the same rank. Since A_n is independent of $A_1, A_2, \dots, A_{n-1}$, the desired result follows. $\square$

As we announced in the previous section already, the crop matrix behaves more desirably with respect to DP than just zeroing independent rows. This is made precise in the following theorem.

Theorem 7.4.4. *Let $A \in \mathrm{Mat}_n(\mathbb{C})$ such that $A_{nn} = 0$ and A_n is independent of $A_1, A_2, \ldots, A_n$, and let $\tilde{A}$ be the crop matrix of A. Write $\tilde{x} = x_1, x_2, \ldots, x_{n-1}$. Then the following holds.*

i) $(Ax)^{*d}$ satisfies DP, if and only if $(\tilde{A}\tilde{x})^{*d}$ satisfies DP over $\mathbb{C}(u)$. More precisely, the linear dependences over $\mathbb{C}[u]$ between the first $n-1$ components of $(Ax)^{*d}$ and those between the components of $(\tilde{A}\tilde{x})^{*d}$ are similar.

ii) $(Ax)^{*d}$ is reduced, if and only if $(\tilde{A}\tilde{x})^{*d}$ is reduced over $\mathbb{C}(u)$.

iii) $DN(A) = DN(\tilde{A})$. More precisely, for each $i \leq n-1$, the i-th row A_i of A is independent of the other rows of A, if and only the i-th row $\tilde{A}_i$ of $\tilde{A}$ is independent of the other rows of $\tilde{A}$.

Proof.

i) Assume $(Ax)^{*d}$ satisfies DP. Say that

$$\lambda_1(\tilde{A}_1\tilde{x})^d + \lambda_2(\tilde{A}_2\tilde{x})^d + \cdots + \lambda_{n-1}(\tilde{A}_{n-1}\tilde{x})^d = 0$$

for certain $\lambda_i \in \mathbb{C}[u]$. From (7.9), it follows that

$$\lambda_1(A_1x)^d + \lambda_2(A_2x)^d + \cdots + \lambda_{n-1}(A_{n-1}x)^d = 0$$

Next, assume $(\tilde{A}\tilde{x})^{*d}$ satisfies DP over $\mathbb{C}(u)$. Say that

$$\lambda_1(A_1\tilde{x})^d + \lambda_2(A_2\tilde{x})^d + \cdots + \lambda_{n-1}(A_{n-1}\tilde{x})^d + \lambda_n(A_n\tilde{x})^d = 0$$

for certain $\lambda_i \in \mathbb{C}[u]$. Since A_n is independent of the other rows of A, we obtain $\lambda_n = 0$. From (7.8), it follows that

$$\lambda_1(\tilde{A}_1\tilde{x})^d + \lambda_2(\tilde{A}_2\tilde{x})^d + \cdots + \lambda_{n-1}(\tilde{A}_{n-1}\tilde{x})^d = 0$$

This gives the desired result.

ii) Assume that $(\tilde{A}\tilde{x})^{*d}$ is reduced and

$$\lambda_1(A_1x)^d + \lambda_2(A_2x)^d + \cdots + \lambda_{n-1}(A_{n-1}x)^d + \lambda_n(A_nx)^d = 0$$

for certain $\lambda_i \in \mathbb{C}[u]$. Since A_n is independent of the other rows of A, we obtain $\lambda_n = 0$. From i), it follows that

$$\lambda_1(\tilde{A}_1\tilde{x})^d + \lambda_2(\tilde{A}_2\tilde{x})^d + \cdots + \lambda_{n-1}(\tilde{A}_{n-1}\tilde{x})^d = 0$$

Since $(\tilde{A}\tilde{x})^{*d}$ is reduced, it follows that $\lambda_i \tilde{A}_i \tilde{x} = 0$ for all $i \leq n-1$. By (7.9), we obtain that $\lambda_i A_i x = 0$ for all $i \leq n-1$. So $(Ax)^{*d}$ is reduced.

The converse is similar, except that (7.9) is replaced by (7.8), to obtain that $(A_i x)|_{x_n = -uA_n x} = \tilde{A}_i \tilde{x}$ for all $i \leq n-1$.

iii) Replacing d by 1 in the proof of i), we obtain that for all $\lambda \in \mathbb{C}[u]^n$,

$$\lambda_1 A_1 x + \lambda_2 A_2 x + \cdots + \lambda_{n-1} A_{n-1} x + \lambda_n A_n x = 0$$

if and only if $\lambda_n = 0$ and

$$\lambda_1 \tilde{A}_1 \tilde{x} + \lambda_2 \tilde{A}_2 \tilde{x} + \cdots + \lambda_{n-1} \tilde{A}_{n-1} \tilde{x} = 0$$

This gives the desired result. □

Corollary 7.4.5. *Assume $A \in \mathrm{Mat}_n(\mathbb{C})$ such that $\mathcal{J}(Ax)^{*d}$ is nilpotent and $d \geq 2$. If*

$$DN(A) - \mathrm{cork} A \leq \max\{3, 7-d\}$$

then $(Ax)^{*d}$ satisfies DP.

Proof. Assume $DN(A) - \mathrm{cork} A \leq \max\{3, 7-d\}$. From iv) of theorem 7.4.4 and iii) of theorem 7.4.3, it follows that $DN(\tilde{A}) - \mathrm{cork} \tilde{A} \leq \max\{3, 7-d\}$ as well. From i) of theorem 7.4.4 and induction on n, using Lefschetz' principle, we may assume that A_n is dependent of the other rows of A. By conjugation with permutations, we may even assume that for each i, A_i is dependent of the other rows of A, i.e. $DN(A) = n$. So $\mathrm{rk} A = DN(A) - \mathrm{cork} A \leq \max\{3, 7-d\}$.

From iv) of theorem 6.1.2, it follows that it suffices to show that homogeneous H in dimension $n \leq \max\{3, 7-d\}$ with $\mathcal{J}H$ nilpotent satisfy DP. The case $n \leq 3$ follows from proposition 4.1.1 and corollary 4.1.3. In case $n = 4$, we have $d \leq 7 - n = 3$, and theorem 4.6.5 gives the desired result. If $n = 5$, we have $d \leq 7 - n = 2$, and H satisfies DP on account of theorem 4.6.8, as desired. □

We will investigate now how the crop matrix acts with respect to DP+. The behavior of the crop matrix with respect to DP+ is not so nice as that with respect to DP. Furthermore, a lot of this behavior is still unknown.

Theorem 7.4.6. *Let $A \in \mathrm{Mat}_n(\mathbb{C})$ such that $A_{nn} = 0$ and A_n is independent of $A_1, A_2, \ldots, A_{n-1}$, and let $\tilde{A}$ be the crop matrix of A. Assume $(Ax)^d$ satisfies DP, say that $\lambda^{\mathrm{t}}(Ax)^d = 0$ for some nonzero $\lambda \in \mathbb{C}^n$. Write $\tilde{x} = x_1, x_2, \ldots, x_{n-1}$. Then the following holds.*

i) If A_n is dependent of λ^{t}, then $(\tilde{A}\tilde{x})^{*d}$ satisfies DP+ over $\mathbb{C}(u)$, if and only if there exists a $\mu \in \mathbb{C}^n$ that is independent of λ and e_n such that $\mu^{\mathrm{t}}(Ax)^d \in \mathbb{C}[\lambda^{\mathrm{t}}x, x_n]$. Furthermore, $\mu^{\mathrm{t}}(Ax)^d$ is a power of a linear form with x_n in case $\mu^{\mathrm{t}}(Ax)^d \in \mathbb{C}[\lambda^{\mathrm{t}}x, x_n]\backslash\{0\}$.

ii) If A_n is not dependent of λ^{t}, then $(\tilde{A}\tilde{x})^{*d}$ satisfies DP+ over $\mathbb{C}(u)$ in case $(Ax)^d$ satisfies DP+.

iii) If $(\tilde{A}\tilde{x})^{*d}$ satisfies DP+ and there exists a $v \in \mathbb{C}^n$ with $\lambda^{\mathrm{t}}v = 0$, such that $Av = e_n$, then $(Ax)^d$ satisfies DP+ as well.

iv) If $(\tilde{A}\tilde{x})^{*d}$ satisfies DP+ and there exists a $w \in \mathbb{C}^n$ with $\lambda^{\mathrm{t}}w \neq 0$, such that $Aw = e_{n-1}$, then $(Ax)^d$ satisfies DP+ as well.

Proof. From i) of theorem 7.4.4, it follows that $\lambda^{\mathrm{t}}((\tilde{A}\tilde{x})^{*d}, 0) = 0$. Since A_n is independent of the other rows of A, it follows that $\lambda_n = 0$. Take $v \in \mathbb{C}^n$ such that $Av = e_n$ and let $\tilde{v} = (v_1, v_2, \ldots, v_{n-1})$.

i) Assume A_n is dependent of λ. Assume first that there exists a $\mu \in \mathbb{C}^n$ that is independent of λ and e_n, such that $\mu^{\mathrm{t}}(Ax)^d \in \mathbb{C}[\lambda^{\mathrm{t}}x, x_n]$. From (7.8), it follows that $\mu^{\mathrm{t}}((\tilde{A}\tilde{x})^{*d}, (A_nx)^d) \in \mathbb{C}[\lambda^{\mathrm{t}}x, -uA_nx]$. Since λ^{t} is dependent of A_n, we obtain $\mu^{\mathrm{t}}((\tilde{A}\tilde{x})^{*d}, 0) \in \mathbb{C}(u)[\lambda^{\mathrm{t}}x]$. Since μ is independent of λ and e_n, it follows that $(\tilde{A}\tilde{x})^d$ satisfies DP+.

Assume next that $(\tilde{A}\tilde{x})^d$ satisfies DP+. From proposition 4.6.2, it follows that there exists a $\mu \in \mathbb{C}(u)^n$ with $\mu_n = 0$ that is independent of λ, such that $\mu^{\mathrm{t}}((\tilde{A}\tilde{x})^{*d}, 0)$ is dependent of $(\lambda^{\mathrm{t}}x)^d$. From (7.9), it follows that $\mu^{\mathrm{t}}(Ax)^{*d}$ is dependent of

$$(A_nx)^d\Big|_{\tilde{x}=\tilde{x}-\frac{1}{u+v_n}\tilde{v}(x_n+uA_nx)} = \left(A_nx + \frac{A_n\binom{\tilde{v}}{0}}{u+v_n}(x_n + uA_nx)\right)^d$$

Since λ^{t} is dependent of A_n, we obtain that $\mu^{\mathrm{t}}(Ax)^{*d} \in \mathbb{C}[\lambda^{\mathrm{t}}x, x_n]$.

Furthermore, μ is independent of λ and e_n, because μ is independent of λ and $\lambda_n = \mu_n = 0$. From the definition of v, it follows that $A_n\binom{\tilde{v}}{0} \neq 0$, whence $\mu^{\mathrm{t}}(Ax)^d$ is a power of a linear form with x_n in case $\mu^{\mathrm{t}}(Ax)^d \ne 0$. This gives the desired result.

ii) Assume that A_n is not dependent of λ^{t} and that $(Ax)^{*d}$ satisfied DP+. From proposition 4.6.2, it follows that there exists a $\mu \in \mathbb{C}^n$ that is independent of λ, such that $\mu^{\mathrm{t}}(Ax)^{*d}$ is dependent of $(\lambda^{\mathrm{t}}x)^d$. As we observed above, $\lambda_n = 0$. If $\mu_n = 0$ as well, then we obtain by (7.8) that $\mu^{\mathrm{t}}(Ax)^{*d}|_{x_n=-uA_nx} = \mu^{\mathrm{t}}((\tilde{A}\tilde{x})^{*d}, 0)$ is dependent $(\lambda^{\mathrm{t}}x)^d|_{x_n=-uA_nx} = (\lambda^{\mathrm{t}}x)^d$, as desired.

So assume $\mu_n \ne 0$. Since $Av = e_n$, it follows that

$$\mathcal{J}(\mu^{\mathrm{t}}(Ax)^{*d}) \cdot v = d\mu^{\mathrm{t}} \operatorname{diag}\big((Ax)^{*(d-1)}\big) \cdot Av = d\mu_n(A_nx)^{(d-1)} \ne 0$$

Since the left hand side is dependent of $\mathcal{J}(\lambda^{\mathrm{t}}x)^d \cdot v$, which in turn is dependent of $(\lambda^{\mathrm{t}}x)^{d-1}$, the above can only be satisfied if A_n is dependent of λ^{t}. This gives the desired result.

iii) Assume that $(\tilde{A}\tilde{x})^{*d}$ satisfied DP+ over $\mathbb{C}(u)$ and that $\lambda^{\mathrm{t}}v = 0$. From proposition 4.6.2, it follows that there exists a $\mu \in \mathbb{C}(u)^n$ with $\mu_n = 0$ that is independent of λ, such that $\mu^{\mathrm{t}}((\tilde{A}\tilde{x})^{*d}, 0)$ is dependent of $(\lambda^{\mathrm{t}}x)^d$. From (7.9), it follows that $\mu^{\mathrm{t}}(Ax)^{*d}$ is dependent of

$$(\lambda^{\mathrm{t}}x)^d\Big|_{\tilde{x}=\tilde{x}-\frac{1}{u+v_n}\tilde{v}(x_n+uA_nx)} = \left(\lambda^{\mathrm{t}}x - \frac{\lambda^{\mathrm{t}}\binom{\tilde{v}}{0}}{u+v_n}(x_n+uA_nx)\right)^d$$

Since $\lambda_n = 0$, we obtain $\lambda^{\mathrm{t}}\binom{\tilde{v}}{0} = \lambda^{\mathrm{t}}v = 0$. It follows that $\mu^{\mathrm{t}}(Ax)^{*d}$ is dependent of $(\lambda^{\mathrm{t}}x + \lambda^{\mathrm{t}}\binom{\tilde{v}}{0}\cdots)^d = (\lambda^{\mathrm{t}}x)^d$, as desired.

iv) Assume that $(\tilde{A}\tilde{x})^{*d}$ satisfied DP+ over $\mathbb{C}(u)$ and that $Aw = e_{n-1}$ and $\lambda^{\mathrm{t}}w \ne 0$ for some $w \in \mathbb{C}^n$. Just as in iii), we obtain that $\lambda^{\mathrm{t}}\binom{\tilde{v}}{0} = \lambda^{\mathrm{t}}v$ and that $\mu^{\mathrm{t}}(Ax)^{*d}$ is dependent of

$$\begin{aligned}\left(\lambda^{\mathrm{t}}x - \frac{\lambda^{\mathrm{t}}\binom{\tilde{v}}{0}}{u+v_n}(x_n+uA_nx)\right)^d &= \left(\lambda^{\mathrm{t}}x - \frac{\lambda^{\mathrm{t}}v}{u+v_n}(x_n+uA_nx)\right)^d \\ &= \big(\hat{\lambda}^{\mathrm{t}}x\big)^d\end{aligned}$$

where $\hat{\lambda} := \lambda - \frac{v^{\mathrm{t}}\lambda}{u+v_n}(e_n + uA_n^{\mathrm{t}})$. Assume that there exists a $w \in \mathbb{C}^n$ such that $\lambda^{\mathrm{t}}w \ne 0$, such that $Aw = e_{n-1}$. From $A_nw = 0$, it follows that

$$\hat{\lambda}^{\mathrm{t}} w = \lambda^{\mathrm{t}} w + \frac{\lambda^{\mathrm{t}} v}{u + v_n}\big(w_n + uA_n w\big) = \frac{\lambda^{\mathrm{t}} w u + \lambda^{\mathrm{t}}(v_n w + w_n v)}{u + v_n}$$

whence by $\lambda^{\mathrm{t}} w \neq 0$,

$$\deg_u \big((u + v_n)\hat{\lambda}^{\mathrm{t}} w\big) = 1 \tag{7.10}$$

So $\hat{\lambda}^{\mathrm{t}} w \neq 0$. It follows that $\mu^{\mathrm{t}}(Aw)^{*d} \neq 0$ as well in case $\mu^{\mathrm{t}}(Ax)^{*d} \neq 0$. Since we are done in case $\mu^{\mathrm{t}}(Ax)^{*d} = 0$, we may assume that $\mu^{\mathrm{t}}(Aw)^{*d} \neq 0$.

Since $Aw = e_{n-1}$, we obtain that $\mu_{n-1} = \mu^{\mathrm{t}}(Aw)^{*d} \neq 0$. Furthermore,

$$\mathcal{J}(\mu^{\mathrm{t}}(Ax)^{*d}) \cdot w = d\mu^{\mathrm{t}} \operatorname{diag}\big((Ax)^{*(d-1)}\big) \cdot Aw = \mu_{n-1} d(A_{n-1}x)^{d-1} \neq 0$$

Since the left hand side is dependent of $\mathcal{J}(\hat{\lambda}^{\mathrm{t}} x)^d \cdot w$, which in turn is dependent of $(\hat{\lambda}^{\mathrm{t}} x)^{d-1}$, the above can only be satisfied if $\hat{\lambda}^{\mathrm{t}}$ is dependent of A_{n-1}.

If $\lambda^{\mathrm{t}} v = 0$, then $\lambda^{\mathrm{t}} = \hat{\lambda}^{\mathrm{t}}$ is dependent of A_{n-1}, whence $(Ax)^{*d}$ satisfies DP+ with $\mu = e_{n-1}$ in the definition of DP+. So assume $\lambda^{\mathrm{t}} v \neq 0$. Then the last coordinate of $(u + v_n)\hat{\lambda}$ is equal to $\lambda^{\mathrm{t}} v \in \mathbb{C}^*$. Since $\hat{\lambda}$ is dependent of A_{n-1}^{t}, all coordinates of $(u + v_n)\hat{\lambda}$ are contained in $\mathbb{C}$. This contradicts (7.10), so $\lambda^{\mathrm{t}} v = 0$ and $(Ax)^{*d}$ satisfies DP+, as desired. □

Although the behavior of the crop matrix with respect to DP+ is partially unknown, we can derive some results. In the proof of corollary 7.4.7 below, we show that we can reduce to the case $DN(A) \geq n - 1$, i.e. the case that at most one row of A is independent of the other rows of A.

Corollary 7.4.7. *Assume $A \in \mathrm{Mat}_n(\mathbb{C})$ such that $\mathcal{J}(Ax)^{*d}$ is nilpotent and $d \geq 2$. If*

$$DN(A) - \mathrm{cork} A \leq \max\{2, 6 - d\}$$

then $(Ax)^{*d}$ satisfies DP+.

Proof. Write N instead of n and X instead of x. Assume $DN(A) - \mathrm{cork} A \leq \max\{2, 6 - d\}$. From iv) of theorem 7.4.4 and iii) of theorem 7.4.3, it follows that $DN(\tilde{A}) - \mathrm{cork}\tilde{A} \leq \max\{2, 6 - d\}$ as well. We distinguish three cases:

- $DN(A) = N$. Then $\mathrm{rk}A = DN(A) - \mathrm{cork}A \le \max\{2, 6-d\} \le \max\{3, 6-d\}$. From iii) of theorem 6.1.2 and proposition 4.6.3, it follows that it suffices to show that homogeneous H in dimension $n \le \max\{3, 6-d\}$ with $\mathcal{J}H$ nilpotent are linearly triangularizable. The case $n \le 2$ follows from proposition 4.1.1 and the trace condition on $\mathcal{J}H$. The case $n = 3$ follows theorem 4.1.4, so assume $n \ge 4$. Then $n \le 6 - d \le 6 - 2 = 4$, so $n = 4$ and $d = 2$, and theorem 4.6.5 gives the desired result.

- $DN(A) = N - 1$.
  Then $\mathrm{rk}A = DN(A) + 1 - \mathrm{cork}A \le \max\{3, 7-d\}$. The case $\mathrm{rk}A \le \max\{3, 6-d\}$ follows in a similar manner as above, so assume $\mathrm{rk}A = 7-d \ge 4$. Since $\mathcal{J}_X(AX)^{*d}$ is nilpotent, $\mathrm{rk}A < N$ follows. From theorems 6.2.11 and 6.3.3, it follows that it suffices to show that homogeneous H in dimension $n = 7 - d \ge 4$ with $\mathcal{J}H$ nilpotent satisfy DP+ in case they are GZ-paired with $(AX)^{*d}$.

  If $n = 4$, then $d = 7 - n = 3$, and theorem 4.6.5 gives the desired result. So assume $n \ge 5$. Then $n \le 7 - d \le 7 - 2 = 5$, so $n = 5$ and $d = 2$. From theorem 4.6.8, it follows that we may assume that

$$H \equiv (0, x_1x_3, x_2^2 - x_1x_4, 2x_2x_3 - x_1x_5, x_3^2) \pmod{x_1^2}$$

  Assume that H and $(AX)^{*2}$ are GZ-paired through B and C. Since $DN(A) = N - 1$, we may assume without loss of generality that A_N is independent of the other rows of A on account of i) of proposition 6.2.7.

  Assume that the last column of B is zero. Let $\hat{A}$ be the matrix we obtain from A by replacing the last row of A by the zero row. Then H and $(\hat{A}X)^{*2}$ are weakly GZ-paired through B and C as well. Since $\mathrm{rk}\hat{A} = \mathrm{rk}A - 1 = 4$, it follows from ii) of theorem 7.1.2 that $(\hat{A}X)^{*2}$ is linearly triangularizable. Now apply v) of proposition 6.2.7 to obtain a contradiction.

  So it suffices to show that the last column of B is zero. So assume that $BE_N \ne 0$, where E_i is the i-th standard basis unit vector of length N. Since A_N is independent of the other rows of A, there exists a $V \in \mathbb{C}^N$ such that $AV = E_N$. Put $w := BV$. Since $(AX)^{*2} = (ACBX)^{*2}$, it follows that $A_i = \pm A_i C B$ for each i. So $ACBV = \pm E_N$ and

$$\begin{aligned} \mathcal{J}H \cdot w &= \mathcal{J}\big(B(ACx)^{*2}\big) \cdot BV \\ &= 2B \operatorname{diag}(ACx) \cdot ACBV \\ &= \pm 2B \operatorname{diag}(ACx) \cdot E_N \\ &= \pm 2BE_N(A_N Cx) \end{aligned}$$

which has Jacobian rank at most 1. Furthermore,

$$\begin{aligned} \mathcal{J}(\mathcal{J}H \cdot w) \cdot w &= \mathcal{J}(\pm 2BE_N(A_N Cx)) \cdot BV \\ &= \pm 2BE_N A_N CBV \\ &= 2BE_N \end{aligned}$$

which is nonzero. This contradicts lemma 7.4.8 below.

- $DN(A) \le N - 2$.
  Then there are at least two rows of A that are independent of the other rows of A. From proposition 4.6.3, it follows that we may assume without loss of generality that both A_{N-1} and A_N are independent of the other rows of A. Since A_{N-1} is independent of the other rows of A, there exists a $W \in \mathbb{C}^N$ such that $AW = E_{N-1}$. Similarly, there exists a $V \in \mathbb{C}^N$ such that $AV = E_N$.

  From corollary 7.4.5, it follows that $(AX)^{*d}$ satisfies DP. Say that $\Lambda(AX)^{*d} = 0$ for some nonzero $\Lambda \in \mathbb{C}^N$. The cases $\Lambda V = 0$ and $\Lambda W \ne 0$ follow from iii) and iv) of theorem 7.4.6 respectively by induction on N, using Lefschetz' principle. So assume $\Lambda V \ne 0$ and $\Lambda W = 0$.

  Since both A_{N-1} and A_N are independent of the other rows of A, $\Lambda_{N-1} = \Lambda_N = 0$ follows. So $\Lambda PV = 0$ and $\Lambda PW \ne 0$, where P is the permutation matrix that corresponds to the interchange of X_{N-1} and X_N. Furthermore, $(PAP^{-1})(PV) = PE_N = E_{N-1}$ and $(PAP^{-1})(PW) = PE_{N-1} = E_N$. So if we replace A by PAP^{-1} and $(V \mid W)$ by $(PW \mid PV)$, we obtain that $AW = E_{N-1}$, $AV = E_N$, $\Lambda V = 0$ and $\Lambda W \ne 0$. So by either iii) or iv) of theorem 7.4.6, we obtain the desired result by induction on N. □

Lemma 7.4.8. *Assume $n = 1$ and*

$$H \equiv (0, x_1x_3, x_2^2 - x_1x_4, 2x_2x_3 - x_1x_5, x_3^2) \pmod{x_1^2}$$

as in theorem 4.6.8. If $w \in \mathbb{C}^5$ *and*

$$\mathrm{rk}\mathcal{J}(\mathcal{J}H \cdot w) \leq 1$$

then $\mathcal{J}(\mathcal{J}H \cdot w) \cdot w = 0$.

Proof. Assume $w \in \mathbb{C}^5$ and $\mathrm{rk}\mathcal{J}(\mathcal{J}H \cdot w) \leq 1$. Notice that $\mathcal{J}(\mathcal{J}H \cdot w)$ is a matrix over $\mathbb{C}$. Since $\mathrm{rk}\mathcal{J}(\mathcal{J}H \cdot w) \leq 1$, it follows that the components of $\mathcal{J}H \cdot w$ are pairwise dependent linear forms.

If $w_1 \neq 0$, then the coefficient of x_4 of $\mathcal{J}H_3 \cdot w$ is nonzero and the coefficient of x_4 of $\mathcal{J}H_2 \cdot w$ is zero. This contradicts that (the coefficient of x_3 of) $\mathcal{J}H_2 \cdot w$ is nonzero. So $w_1 = 0$.

If $w_3 \neq 0$, then the coefficient of x_2 of $\mathcal{J}H_4 \cdot w$ is nonzero and the coefficient of x_2 of $\mathcal{J}H_2 \cdot w$ is zero. This contradicts that (the coefficient of x_1 of) $\mathcal{J}H_2 \cdot w$ is nonzero. So $w_3 = 0$.

If $w_2 \neq 0$, then the coefficient of x_3 of $\mathcal{J}H_4 \cdot w$ is nonzero and the coefficient of x_3 of $\mathcal{J}H_3 \cdot w$ is zero. This contradicts that (the coefficient of x_2 of) $\mathcal{J}H_3 \cdot w$ is nonzero. So $w_2 = 0$.

Since $w_1 = w_2 = w_3 = 0$, we obtain that the components of $\mathcal{J}H \cdot w$ are dependent of x_1, and $w_1 = 0$ gives the desired result. $\square$

7.5 Singular principal minors

We will investigate how the crop matrix behaves with respect to linear triangularization. One thing we will prove is that in case $\mathcal{J}(Ax)^{*d}$ is nilpotent, $(\tilde{A}\tilde{x})^{*d}$ is linearly triangularizable over $\mathbb{C}$, if and only if $(Ax)^{*d}$ is linearly triangularizable over $\mathbb{C}$. This is however not sufficient for induction purposes: since we use Lefschetz' principle on $\mathbb{C}(u)$, we can only obtain the induction assumption that $(\tilde{A}\tilde{x})^{*d}$ is linearly triangularizable over $\mathbb{C}(u)$ instead of $\mathbb{C}$.

But we were unable to prove that $(Ax)^{*d}$ is linearly triangularizable over $\mathbb{C}$ in case $(\tilde{A}\tilde{x})^{*d}$ is linearly triangularizable over $\mathbb{C}(u)$. In case of symmetric triangularizability, the conjugation matrix has only ones and zeros, whence it does not matter whether the base ring is $\mathbb{C}$ or $\mathbb{C}(u)$. We shall show that in case $(Ax)^{*d}$ is a power linear map over $\mathbb{C}$ with nilpotent Jacobian, $(Ax)^{*d}$ is symmetrically triangularizable, if and only if $(\tilde{A}\tilde{x})^{*d}$ is.

Definition 7.5.1. We say that a square matrix has property E_r if the determinant of each principal minor of size r at most vanishes.

Proposition 7.5.2. *A square matrix M has property E_n, if and only if there exists a permutation matrix P such that $P^{-1}MP$ is triangular with zeros on the diagonal.*

For a polynomial map $H \in \mathbb{C}[x]^n$ with nilpotent Jacobian, $\mathcal{J}H$ has property E_n, if and only if there exists a permutation matrix P such that $\mathcal{J}(P^{-1}HPx)$ is triangular.

Proof. The second statement follows from the first statement ands

$$\mathcal{J}\big(P^{-1}H(Px)\big) = P^{-1} \cdot \mathcal{J}H|_{x=Px} \cdot P = (P^{-1} \cdot \mathcal{J}H \cdot P)|_{x=Px}$$

The forward implication of the first statement follows from [19, lemma 1.2] (see also [24, prop. 6.3.9]). The backward implication follows from the fact that triangular nilpotent matrices over $\mathbb{C}$ have property E_n and that property E_r is not affected by symmetrical conjugation. □

Although the forward implication of the first statement of proposition 7.5.2 is missing in both [19, lemma 1.2] and [24, prop. 6.3.9], it is the easy part of this statement.

Theorem 7.5.3. *Let $A \in \mathrm{Mat}_n(\mathbb{C})$ such that $A_{nn} = 0$ and A_n is independent of $A_1, A_2, \ldots, A_n$, and let $\tilde{A}$ be the crop matrix of A. Write $\tilde{x} = x_1, x_2, \ldots, x_{n-1}$. Then*

i) If $(Ax)^{*d}$ is linearly triangularizable over $\mathbb{C}$ and $\mathcal{J}(Ax)^{*d}$ is nilpotent, then there exists a $T \in \mathrm{GL}_{n-1}(\mathbb{C})$ of the form $T = \mathrm{diag}(\tilde{T}, 1)$ such that $T^{-1}(ATx)^{*d}$ has property E_n.

ii) If $T^{-1}(ATx)^{*d}$ has property E_{r+1} for some r and some $T \in \mathrm{GL}_n(\mathbb{C})$ of the form $T = \mathrm{diag}(\tilde{T}, 1)$, then $\tilde{T}^{-1}(\tilde{A}\tilde{T}\tilde{x})^{*d}$ has property E_r.

iii) If there exists a $\tilde{T} \in \mathrm{GL}_{n-1}(\mathbb{C})$ such that $\mathcal{J}(\tilde{T}(\tilde{A}\tilde{T}\tilde{x})^{*d})$ has property E_r, then $\mathcal{J}(T^{-1}(ATx)^{*d})$ has property E_r as well, where $T = \mathrm{diag}(\tilde{T}, 1)$.

iv) If there exist a $T \in \mathrm{GL}_n(\mathbb{C})$ such that $T^{-1}(ATx)^{*d}$ has property E_{n-1} and $\det \mathcal{J}(Ax)^{*d} = 0$, then $\mathcal{J}(Ax)^{*d}$ is nilpotent and $(Ax)^{*d}$ is linearly triangularizable.

v) If $\mathcal{J}(Ax)^{*d}$ is nilpotent, then $(Ax)^{*d}$ is symmetrically triangularizable, if and only if $(\tilde{A}\tilde{x})^{*d}$ is.

Proof.

i) Assume $(Ax)^{*d}$ is linearly triangularizable over $\mathbb{C}$, say $\mathcal{J}(T^{-1}(ATx)^{*d})$ is lower triangular. We first show that we may assume that $T_s^{-1} = e_n^{\mathrm{t}}$ for some s. For that purpose, write

$$e_n^{\mathrm{t}} = \lambda_1 T_1^{-1} + \cdots + \lambda_s T_s^{-1}$$

with $\lambda_s \neq 0$. Now define

$$L^{-1} = \begin{pmatrix} & & & 0 & & & \\ & I_{s-1} & & \vdots & & \emptyset & \\ & & & 0 & & & \\ \lambda_1 & \cdots & \lambda_{s-1} & \lambda_s & 0 & \cdots & 0 \\ & & & 0 & & & \\ & \emptyset & & \vdots & & I_{n-s} & \\ & & & 0 & & & \end{pmatrix}$$

then L is lower triangular, so $\mathcal{J}(L^{-1}T^{-1}(ATLx)^{*d})$ is also lower triangular. Furthermore, $(L^{-1}T^{-1})_s = e_n^{\mathrm{t}}$, so replacing T by TL gives $T_s^{-1} = e_n^{\mathrm{t}}$ for some s, as desired.

Take s' as small as possible, such that the s'-th coordinate of the last column of T^{-1} is nonzero. Since $T_s^{-1} = e_n^{\mathrm{t}}$, the s-th coordinate of the last column of T^{-1} equals 1, so $s' \leq s$. We distinguish two cases:

- $s' = s$.
  Assume first that $s = n$. Then $T_n^{-1} = e_n^{\mathrm{t}}$ and $T^{-1}e_n = e_n$, whence $T_n = e_n^{\mathrm{t}}$ and $Te_n = e_n$ as well. This gives the desired result because lower triangular nilpotent matrices over $\mathbb{C}$ have property E_n.
  So assume $s < n$ and define

$$\hat{L}^{-1} := \begin{pmatrix} & & & 0 & & & \\ & I_{s-1} & & \vdots & & \emptyset & \\ & & & 0 & & & \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ & & & -(T^{-1})_{(s+1)n} & & & \\ & \emptyset & & \vdots & & I_{n-s} & \\ & & & -(T^{-1})_{nn} & & & \end{pmatrix}$$

Then $\mathcal{J}(\hat{L}^{-1}T^{-1}(AT\hat{L}x)^{*d})$ is lower triangular, because $\hat{L}$ is lower triangular. Since $\hat{L}_s^{-1} = e_s^{\mathrm{t}}$, it follows that

$$(\hat{L}^{-1}T^{-1})_s = \hat{L}_s^{-1}T^{-1} = T_s^{-1} = e_n^{\mathrm{t}}$$

Furthermore, $\hat{L}^{-1}T^{-1}e_n$ has only zeros after the s-th coordinate by definition of $\hat{L}^{-1}$. By definition of s' and $\hat{L}^{-1}$, $\hat{L}^{-1}T^{-1}e_n$ has only zeros before the s'-th coordinate as well. Since $s' = s$ and $(\hat{L}^{-1}T^{-1})_s = e_n^{\mathrm{t}}$, we obtain $\hat{L}^{-1}T^{-1}e_n = e_s$.
By replacing T by $T\hat{L}$ and T^{-1} by $\hat{L}^{-1}T^{-1}$, we obtain $T_s^{-1} = e_n^{\mathrm{t}}$ and $T^{-1}e_n = e_s$. Since $\mathcal{J}(T^{-1}(ATx)^{*d})$ is lower triangular, it has property E_n. Let Q^{-1} be the permutation matrix defined by

$$Q^{-1} = \mathcal{J}(x_1, \dots, x_{s-1}, x_n, x_{s+1}, \dots, x_{n-1}, x_s)$$

then $Q = Q^{-1}$ and $Q^{-1}T^{-1}$ is of the form $Q^{-1}T^{-1} = \operatorname{diag}(\tilde{T}^{-1}, 1)$. Hence TQ is of the form $TQ = \operatorname{diag}(\tilde{T}, 1)$. Since property E_r is not affected by conjugation by a permutation for all r, the result follows.

- $s' < s$.
Let P be the permutation matrix $P := \mathcal{J}(x_1, \dots, x_{s'-1}, x_{s'+1}, \dots, x_s, x_{s'}, x_{s+1}, \dots, x_n)$. Then

$$P^{-1} = \mathcal{J}(x_1, \dots, x_{s'-1}, x_s, x_{s'}, \dots, x_{s-1}, x_{s+1}, \dots, x_n)$$

Replacing T^{-1} by $P^{-1}T^{-1}$ and T by TP, we indeed get $s' = s$, since s changes accordingly. But it is not immediately clear that the lower triangularity of $\mathcal{J}T^{-1}(ATx)^{*d}$ is preserved.
In order to see that the lower triangularity of $\mathcal{J}T^{-1}(ATx)^{*d}$ is preserved, notice first that the map $M \mapsto P^{-1}M = P^{\mathrm{t}}M$ is a cyclic shift to below of row $s', s'+1, \dots, s$, and the map $M \mapsto MP$ is a cyclic shift to the right of column $s', s'+1, \dots, s$. Since $\mathcal{J}P^{-1}T^{-1}(ATPx)^{*d}$ is lower triangular, if and only if the matrix $P^{-1}(\mathcal{J}T^{-1}(ATx)^{*d})P$ is, it follows that the only row of concern of $\mathcal{J}P^{-1}T^{-1}(ATPx)^{*d}$ is the s'-th row.
So we must prove that

$$P_{s'}^{-1}T^{-1}(ATPx)^{*d} \in \mathbb{C}[x_1, \dots, x_{s'-1}] \tag{7.11}$$

Since $P_{s'}^{-1}T^{-1} = e_s^{\mathrm{t}}T^{-1} = T_s^{-1} = e_n^{\mathrm{t}}$, we obtain

$$(A_nTx)^d = e_n^{\mathrm{t}}(ATx)^{*d} = P_{s'}^{-1}T^{-1}(ATx)^{*d}$$

Notice that $\mathbb{C}[P_1^{-1}x, \ldots, P_{s-1}^{-1}x] = \mathbb{C}[x_1, \ldots, x_{s'-1}]$. Substituting $x = P^{-1}x$ in (7.11), we see that (7.11) is equivalent to

$$(A_nTx)^d \in \mathbb{C}[x_1, \ldots, x_{s'-1}]$$

Since A_n is independent of the other rows of A, there exists a $v \in \mathbb{C}^n$ such that $Av = e_n$. Since $T_{s'}^{-1}(ATx)^{*d} \in \mathbb{C}[x_1, \ldots, x_{s'-1}]$, $\mathcal{J}(T_{s'}^{-1}(ATx)^{*d}) \cdot T^{-1}v \in \mathbb{C}[x_1, \ldots, x_{s'-1}]$ as well. But

$$\begin{aligned}
&\mathcal{J}\big(T_{s'}^{-1}(ATx)^{*d}\big) \cdot T^{-1}v \\
&\quad= d\,T_{s'}^{-1}\mathrm{diag}\big((ATx)^{*(d-1)}\big)ATT^{-1}v \\
&\quad= d\,T_{s'}^{-1}\mathrm{diag}\big((ATx)^{*(d-1)}\big)e_n \\
&\quad= d\,T_{s'n}^{-1}(A_nTx)^d
\end{aligned}$$

Since $T_{s'n}^{-1} \neq 0$ by definition of s', (7.11) follows, as desired.

ii) Assume $T \in \mathrm{GL}_n(\mathbb{C})$ is of the form $T = \mathrm{diag}(\tilde{T}, 1)$. Put $H := T^{-1}(ATx)^{*d}$ and assume that for some $s \le r$, the lower right principal minor of size $s+1$ of $\mathrm{diag}(t\,e_n) + \mathcal{J}H$ has determinant zero, where t is a new indeterminate. Notice that the determinant of this minor is the sum of t times the determinant of a principal minor of size s of $\mathcal{J}H$ and the determinant of a principal minor of size $s+1$ of $\mathcal{J}H$. Then

$$\begin{aligned}
0 &= \det\big(\mathrm{diag}(t\,e_{s+1}) + \mathcal{J}_{x_{n-s},\ldots,x_n}(H_{n-s}, \ldots, H_{n-1}, H_n)\big) \\
&= \det\big(\mathcal{J}_{x_{n-s},\ldots,x_n}(H_{n-s}, \ldots, H_{n-1}, tx_n + (A_nTx)^d)\big) \quad (7.12)
\end{aligned}$$

Since the last row of $ATT^{-1} = A$ is independent of the other rows of $ATT^{-1} = A$, the last row A_nT of AT is independent of the other rows of AT. Since for each $i \le n-1$, H_i is a polynomial in the first $n-1$ components of ATx, we can substitute $A_nTx = \sqrt[d-1]{tu/d}$ in the matrix on the right hand side of (7.12). This way, the last row of this matrix changes from $\mathcal{J}(t\,x_n + (A_nTx)^d)$ to $\mathcal{J}(t\,x_n + tuA_nTx)$.

By the chain rule, we obtain

$$\begin{aligned} 0 &= \det \mathcal{J}_{x_{n-s},\ldots,x_{n-1},x_n}\big(H_{n-s},\ldots,H_{n-1},tx_n + tuA_nTx\big) \\ &= \det \mathcal{J}_{x_{n-s},\ldots,x_{n-1},x_n}\big((H_{n-s},\ldots,H_{n-1})|_{x_n = x_n - uA_nTx}, tx_n\big) \\ &= t\det \mathcal{J}_{x_{n-s},\ldots,x_{n-1}}\big((H_{n-s},\ldots,H_{n-1})|_{x_n = x_n - uA_nTx}\big) \qquad (7.13) \end{aligned}$$

Since $T_n = e_n^{\mathrm{t}}$, it follows that

$$\begin{aligned} H_i\big|_{x_n = x_n - uA_nTx} &= T_i^{-1}(Ax)^{*d}\big|_{x=Tx}\Big|_{x_n = x_n - uA_nTx} \\ &= T_i^{-1}(Ax)^{*d}\big|_{x_n = x_n - uA_nx}\Big|_{x=Tx} \end{aligned}$$

By (7.3) and $T^{-1}e_n = e_n$, we obtain

$$H_i\big|_{x_n = x_n - uA_nTx} = T_i^{-1}(BTx)^{*d} = \tilde{T}_i^{-1}(\tilde{B}Tx)^{*d}$$

for all $i \le n-1$, where $\tilde{B}$ is the matrix that consists of the first $n-1$ rows of $B := A - uAe_nA_n$.

Assume $Av = e_n$ and write $v = (\tilde{v}, v_n)$. From (7.7), it follows that

$$\begin{aligned} \tilde{T}^{-1}(\tilde{A}\tilde{T}\tilde{x})^{*d} &= \tilde{T}^{-1}(\tilde{B}x)^{*d}\big|_{\tilde{x} = \tilde{x} + \frac{1}{u+v_n}\tilde{v}x_n}\Big|_{x=Tx} \\ &= \tilde{T}^{-1}(\tilde{B}x)^{*d}\big|_{x=Tx}\Big|_{\tilde{x} = \tilde{x} + \frac{1}{u+v_n}\tilde{T}^{-1}\tilde{v}x_n} \end{aligned}$$

So by (7.13), we obtain

$$\begin{aligned} 0 &= \Big(\det \mathcal{J}_{x_{n-s},\ldots,x_{n-1}}\big((\tilde{T}_{n-s}^{-1},\ldots,\tilde{T}_{n-1}^{-1})\cdot(\tilde{B}Tx)^{*d}\big)\Big)\Big|_{\tilde{x} = \tilde{x} - \frac{1}{u+v_n}\tilde{T}^{-1}\tilde{v}x_n} \\ &= \det\Big(\mathcal{J}_{x_{n-s},\ldots,x_{n-1}}\big((\tilde{T}_{n-s}^{-1},\ldots,\tilde{T}_{n-1}^{-1})\cdot(\tilde{B}Tx)^{*d}\big)\Big)\Big|_{\tilde{x} = \tilde{x} - \frac{1}{u+v_n}\tilde{T}^{-1}\tilde{v}x_n} \\ &= \det \mathcal{J}_{x_{n-s},\ldots,x_{n-1}}\big((\tilde{T}_{n-s}^{-1},\ldots,\tilde{T}_{n-1}^{-1})\circ(\tilde{A}\tilde{T}\tilde{x})^{*d}\big) \end{aligned}$$

i.e. the lower right principal minor of size s of $\mathcal{J}(\tilde{T}^{-1}\tilde{A}\tilde{T}\tilde{x})^{*d}$ has determinant zero.

By replacing T by TP for suitable permutation matrices P of the form $P = \mathrm{diag}(\tilde{P}, 1)$, we can prove that if all principal minors of size s and $s+1$ of $\mathcal{J}(T^{-1}ATx)^{*d}$ have determinant zero, then all principal minors of size s of $\mathcal{J}(\tilde{T}^{-1}\tilde{A}\tilde{T}\tilde{x})^{*d}$ have determinant zero as well. This gives the desired result.

iii) The proof is in fact the converse of the proof of ii). Just read that proof from below to above. The three substitutions have converses: (7.6) is the converse of (7.7), (7.4) is the converse of (7.3), and $u = d/t \cdot (A_n Tx)^{d-1}$ is the converse of $A_n Tx = \sqrt[d-1]{tu/d}$. By looking at coefficients with respect to t, we can distinguish two minor determinants of $\mathcal{J}(T^{-1}(ATx)^{*d})$.

iv) Assume there exist a $T \in \mathrm{GL}_n(\mathbb{C})$ such that $T^{-1}(ATx)^{*d}$ has property E_{n-1} and $\det \mathcal{J}(Ax)^{*d} = 0$. Since $\det \mathcal{J}(T^{-1}(ATx)^{*d}) = 0$ as well, it follows that $\mathcal{J}(T^{-1}(ATx)^{*d})$ has property E_n. Hence it follows from proposition 7.5.2 that $\mathcal{J}(P^{-1}T^{-1}(ATPx)^{*d})$ is triangular and nilpotent for some permutation matrix P. So $\mathcal{J}(Ax)^{*d}$ is nilpotent as well.

v) Assume $\mathcal{J}(Ax)^{*d}$ is nilpotent. Assume first that $(Ax)^{*d}$ is symmetrically triangularizable. Then $\mathcal{J}(Ax)^{*d}$ has property E_n on account of proposition 7.5.2. From ii) with $r = n-1$, it follows that $\mathcal{J}(\tilde{A}\tilde{x})^{*d}$ has property E_{n-1}. Using proposition 7.5.2 again, we obtain that $(\tilde{A}\tilde{x})^{*d}$ is symmetrically triangularizable.

Assume next that $(\tilde{A}\tilde{x})^{*d}$ is symmetrically triangularizable. Then $\mathcal{J}(\tilde{A}\tilde{x})^{*d}$ has property E_{n-1} on account of proposition 7.5.2. From iii) with $r = n-1$, it follows that $\mathcal{J}(Ax)^{*d}$ has property E_{n-1}. By iv) and proposition 7.5.2, we obtain that $(Ax)^{*d}$ is symmetrically triangularizable, as desired. □

Corollary 7.5.4. *Assume $A \in \mathrm{Mat}_n(\mathbb{C})$ such that $\mathcal{J}(Ax)^{*d}$ is nilpotent and $d \geq 2$. Assume in addition that $(Ax)^{*d}$ is reduced and*

$$d \geq \min\{\mathrm{cork} A + 1, m + 2\}$$

where $m := \mathrm{cork} A - \#\{i \mid A_i = 0\}$. Then $(Ax)^{*d}$ is symmetrically triangularizable in the following cases:

i) $DN(A) - \mathrm{cork} A \leq 3$,

ii) $DN(A) - \mathrm{cork} A \leq 6 - d$,

iii) $DN(A) - \mathrm{cork} A = 7 - d \leq 6 - m$,

iv) $DN(A) - \mathrm{cork} A = 7 - d \leq 8 - \mathrm{cork} A$.

Proof. From v) of theorem 7.5.3, i), ii) and iii) of theorem 7.4.4, and iii) of theorem 7.4.3, it follows by induction on n, using Lefschetz' principle, that we may assume that $DN(A) = n$. So $DN(A) - \mathrm{cork}A = \mathrm{rk}A$. Now one can show for each of the above cases that $(Ax)^{*d}$ is linearly triangularizable on account of the corresponding case of theorem 7.1.2. For instance, in case iii) is satisfied, we have

$$\begin{aligned}\mathrm{rk}A &= 7 - d \le 6 - m = 6 - n + \mathrm{rk}A + \mathrm{cork}A - m \\ &= 6 - n + \mathrm{rk}A + \#\{i \mid A_i = 0\} = \mathrm{rk}A + 6 - \#\{i \mid A_i \ne 0\}\end{aligned}$$

and hence at most six rows of A are nonzero. Since $d \ge \min\{\mathrm{cork}A + 1, m + 2\}$, we obtain by corollary 6.5.9 that $(Ax)^{*d}$ is symmetrically triangularizable, as desired. □

The bound $d \ge \mathrm{cork}A + 1$ in the above corollary can be improved to $d \ge \mathrm{cork}A$, provided we only prove (ditto) linear triangularizability, and either $DN(A) - \mathrm{cork}A \le 2$ or $DN(A) - \mathrm{cork}A \le 6 - d$. This is shown in the corollary below.

Corollary 7.5.5. *Assume* $A \in \mathrm{Mat}_n(\mathbb{C})$ *such that* $\mathcal{J}(Ax)^{*d}$ *is nilpotent and* $d = \mathrm{cork}A \ge 2$. *If*

$$DN(A) \le \max\{2 + d, 6\}$$

then $(Ax)^{*d}$ *is (ditto) linearly triangularizable.*

Proof. From corollary 7.4.7 and proposition 6.2.5, it follows that we may assume that $A_1 = 0$ and A_2 is dependent of e_1^{t}. We distinguish two cases:

- A_2 is dependent of the other rows of A.
  Let $\hat{A}$ be the matrix one obtains from A by replacing the second row by the zero row. Since the second row of A is dependent of e_1^{t}, it follows that the minors of $\mathcal{J}(Ax)^{*d}$ without the first column are the same as the corresponding minors of $\mathcal{J}(\hat{A}x)^{*d}$. Since the first row of both A and $\hat{A}$ are zero, it follows that the minors of $\mathcal{J}(Ax)^{*d}$ with the first row have the same determinant as the corresponding minors of $\mathcal{J}(\hat{A}x)^{*d}$.

  It follows that the determinants of the principal minors of $\mathcal{J}(Ax)^{*d}$ correspond to those of $\mathcal{J}(\hat{A}x)^{*d}$. Since A_2 is dependent of the other rows of A, it follows that $DN(A) = DN(\hat{A})$ and $\mathrm{cork}A = \mathrm{cork}\hat{A}$. From corollary 7.5.4 with

  $$m = \mathrm{cork}\hat{A} - \#\{i \mid \hat{A}_i = 0\} \le \mathrm{cork}\hat{A} - \#\{i \le 2 \mid \hat{A}_i = 0\} = \mathrm{cork}\hat{A} - 2$$

it follows that $(\hat{A}x)^{*d}$ is symmetrically triangularizable. So $\mathcal{J}(\hat{A}x)^{*d}$ has property E_n on account of proposition 7.5.2. It follows that $\mathcal{J}(Ax)^{*d}$ has property E_n as well. Again by proposition 7.5.2, we obtain that $(Ax)^{*d}$ is symmetrically triangularizable. This gives the desired result.

- A_2 is independent of the other rows of A.
  The proof of this case is similar to the (sub)case that A_2 is independent of $A_3, A_4, \dots, A_n$ in the proof of theorem 6.6.9. In that case in the proof of theorem 6.6.9, a matrix $\tilde{A}$ of dimension $n-1$ is constructed and by induction on n, $(\tilde{A}\tilde{x})^{*d}$ is ditto linearly triangularizable.

  In order to verify that this part of the proof of theorem 6.6.9 works here as well, we must verify that $DN(A) = DN(\tilde{A})$ and $\operatorname{cork} A = \operatorname{cork} \tilde{A}$. The last assertion is already mentioned in the part of the proof of theorem 6.6.9, and the assertion $DN(A) = DN(\tilde{A})$ is easy to prove as well, as desired. □

7.6 Proof of the case $m \le 1$ of theorem 7.1.1

We first show the tameness result. The case $DN(A) \le \operatorname{cork} A + \max\{3, 7-d\}$ follows from proposition 7.3.4. In case $DN(A) \ge \operatorname{cork} A + \max\{4, 8-d\}$, we have $DN(A) \ne \operatorname{cork} A + 2$. We shall show that in that case and in the cases $m = 0$ and $d \ne 2$, we even have symmetrical linear triangularizability for reduced power linear maps $(Ax)^{*d}$ with nilpotent Jacobians. See corollary 7.6.2 below. The case $\operatorname{cork} A \le 1$ of corollary 7.6.2 has already been proved by C. Cheng in [8].

The map

$$\big(0, (x_1 - x_2 - x_3 - x_4)^2, (x_1 + x_2 + x_3 + x_4)^2, -2(x_2 + x_3 + x_4)^2, x_4^2, \dots, x_{n-1}^2\big)$$

shows that in the remaining case $DN(A) \ne \operatorname{cork} A + 2$ and $d = 2$, symmetrical linear triangularizability for reduced power linear maps $(Ax)^{*d}$ with nilpotent Jacobians cannot be guaranteed.

The argument to prove all other cases with $m \le 1$ also works for power linear quasi-Jacobians in general. Since power linear quasi-Jacobians can be used to obtain results about so-called Zhao-graphs as well, we formulate a theorem for power linear quasi-Jacobians now.

Theorem 7.6.1. *Assume*

$$M := \operatorname{diag}\big((Ax)^{*(d-1)}\big) \cdot B$$

is a power linear quasi-Jacobian of size N over $\mathbb{C}$ and $d \ge 2$. Assume that the rows of A are pairwise independent and that $DN(A) = r$. Say that for each $i \ge r+1$, row A_i is independent of the other rows of A. Then the following holds.

i) *Every principal minor of M with rows and columns of M in the range $r+1, r+2, \dots, N$ only, has determinant zero.*

   More generally, if $\tilde{M}$ is the principal minor of M consisting exactly of the rows and columns of M for which the indices are in I, then $\det \tilde{M} = 0$ in case for each $i \in I$, row A_i is independent of the rows A_j of A for which $j \notin I$.

ii) *Let $\tilde{B}$ be the leading principal minor of size r of B. Then*

$$\operatorname{diag}\big((A_1x)^{d-1}, A_2x)^{d-1}, \dots, (A_rx)^{d-1},\big) \cdot \tilde{B}$$

   is a nilpotent power linear quasi-Jacobian of size r over $\mathbb{C}$.

iii) *If $\operatorname{rk} A \le N-1$ and some principal minor $\tilde{M}$ of M has a nonzero determinant, then $\operatorname{rk} A = N-1$ and $d = 2$. Furthermore, any such minor $\tilde{M}$ contains exactly one row and column of M in the range $1, 2, \dots, r$.*

Proof. Notice that the last $N-r$ rows of A are independent. Assume $\operatorname{rk} A = N-c$. Then we may assume without loss of generality that the last $N-c$ rows of A are independent. So for each $i \le c$, we can express A_ix as a linear form in $A_{c+1}x, A_{c+2}x, \dots, A_rx$. It follows that there are linear forms $L_i \in \mathbb{C}[y_{c+1}, y_{c+2}, \dots, y_r]$ such that M and

$$\hat{M} := \operatorname{diag}\begin{pmatrix} L_1(z_{c+1}, z_{c+2}, \dots, z_r)^{d-1} \\ L_2(z_{c+1}, z_{c+2}, \dots, z_r)^{d-1} \\ \vdots \\ L_c(z_{c+1}, z_{c+2}, \dots, z_r)^{d-1} \\ z_{c+1}^{d-1} \\ z_{c+2}^{d-1} \\ \vdots \\ z_N^{d-1} \end{pmatrix} \cdot B$$

can be obtained from each other by substitutions. Assume that $\tilde{M}$ is a principal minor of size s of $\hat{M}$.

i) Assume that $\tilde{M}$ consists of rows and columns of $\hat{M}$ within the range $r+1, r+2, \ldots, N$ only. Then we may assume without loss of generality that $\tilde{M}$ consists of the rows and columns of $\hat{M}$ in the range $N-s+1, N-s+2, \ldots, N$. By substituting $z_{c+1} = z_{c+2} = \cdots = z_{N-s} = 0$ in the sum of determinants of principal minors of size s of $\hat{M}$, we obtain that $\det \tilde{M} = 0$ as desired.

   Assume next that $\tilde{M}$ is the principal minor of M consisting exactly of the rows and columns of M for which the indices are in I, and that for each $i \in I$, row A_i is independent of the rows A_j of A for which $j \notin I$. Assume that the total rank of the rows A_j with $j \notin I$ is equal to t. Then we may assume without loss of generality that $\#(\{c+1, c+2, \ldots, N\} \backslash I) = t$.

   Now substitute $z_j = 0$ in the sum of determinants of principal minors of size s of $\hat{M}$, for all $j \notin I$. Then for each $j \le c$ with $j \notin I$, $L_j(z_{c+1}, z_{c+2}, \ldots, z_r)$ becomes zero after this substitution, but for each $i \in I$, $L_i(z_{c+1}, z_{c+2}, \ldots, z_r)$ remains nonzero, because row A_i is independent of the rows A_j with $j \notin I$. It follows that $\det \tilde{B} = 0$, where $\tilde{B}$ is the principal minor of B consisting exactly of the rows and columns of B for which the indices are in I. This gives the desired result.

ii) Assume that $\tilde{M}$ is the leading principal minor of size r of $\hat{M}$. By substituting $z_{r+1} = z_{r+2} = \cdots = z_N = 0$ in $\hat{M}$, we obtain a matrix that is nilpotent, if and only if $\tilde{M}$ is nilpotent. Furthermore,

$$\begin{aligned}
\tilde{M} &= \Big(\mathrm{diag}\big((A_1x)^{d-1}, A_2x)^{d-1}, \ldots, (A_rx)^{d-1}\big) \,\Big|\, \emptyset\Big) \cdot \begin{pmatrix} \tilde{B} \\ * \end{pmatrix} \\
&= \mathrm{diag}\big((A_1x)^{d-1}, A_2x)^{d-1}, \ldots, (A_rx)^{d-1}\big) \cdot \tilde{B}
\end{aligned}$$

   This gives the desired result.

iii) Assume $\mathrm{rk} A \le N-1$ and some principal minor $\tilde{M}$ of M has a nonzero determinant. If $\mathrm{rk} A = N$, then $r = 0$ and i) gives a contradiction. So $\mathrm{rk} A = N-1$, whence there is essentially only one linear relation between the rows of A. Since $DN(A) = r$, that relation is

$$A_1 = \lambda_2 A_2 + \lambda_3 A_3 + \cdots + \lambda_r A_r \tag{7.14}$$

for unique $\lambda_i \in \mathbb{C}^*$. Furthermore, $c = 1$ and $\lambda_i = \frac{\partial}{\partial y_i} L_1$ for each i.

Assume that $\det \tilde{M} \ne 0$. Since $\lambda_2 \ne 0$, it follows that A_1 is independent of $A_3, A_4, \ldots, A_N$. If $\tilde{M}$ contains the first two rows of M, then A_1 is independent of the rows A_j of A for which M_j is not included in $\tilde{M}$. More generally, every row A_i for which M_i is included in $\tilde{M}$ is independent of the rows A_j of A for which M_j is not included in $\tilde{M}$, in case $\tilde{M}$ contains two rows of M in the range $1, 2, \ldots, r$. From i), it follows that $\tilde{M}$ has exactly one row in the range $1, 2, \ldots, r$.

So it remains to show that $d = 2$. Assume without loss of generality that $\tilde{M}$ consist of the rows and columns of M in the range $1, r+1, r+2, \ldots, r+s-1$. Now let $\hat{B}^{(i)}$ be the principal minor of size s of B with rows and columns $i, r+1, \ldots, r+s-1$ of B. Substituting $z_{r+s} = z_{r+s+1} = \cdots = z_N = 0$ in the sum of determinants of principal minors of size s of M, and omitting terms for which we have already seen that they are zero, we obtain

$$\begin{aligned} q(z)(\lambda_2 z_2 + \cdots + \lambda_r z_r)^{d-1} \det \hat{B}^{(1)} \;+& \\ q(z) z_2^{d-1} \det \hat{B}^{(2)} + \cdots + q(z) z_r^{d-1} \det \hat{B}^{(r)} \;=&\; 0 \end{aligned} \tag{7.15}$$

where $q(z) = (z_{r+1} z_{r+2} \cdots z_{r+s-1})^{d-1}$. Since the rows of A are pairwise independent, it follows that $r \ge 3$ and that $(\lambda_2 z_2 + \cdots + \lambda_r z_r)^{d-1}$ has terms with two of the variables z_i in case $d \ge 3$. Consequently, $0 = \det \hat{B}^{(1)} \mid \det \tilde{M}$ in case $d \ge 3$. So $d = 2$, as desired. □

Corollary 7.6.2. *Assume $(Ax)^{*d}$ is reduced with a nilpotent Jacobian and $d \ge 2$. Put*

$$m := \operatorname{cork} A - \#\{i \mid A_i = 0\}$$

If $m = 0$, then $(Ax)^{*d}$ is symmetrically triangularizable. If $m = 1$, and either $DN(A) \ne \operatorname{cork} A + 2$ or $d \ne 2$, then $(Ax)^{*d}$ is symmetrically triangularizable as well.

Proof. From v) of theorem 7.5.3, i), ii) and iii) of theorem 7.4.4, and iii) of theorem 7.4.3, it follows that we may assume that $DN(A) = n$. Let $\hat{A}$ be a matrix one obtains from A by replacing zero rows by independent rows and $B := dA$. Then $\mathcal{J}(Ax)^{*d} = (\hat{A}x)^{*(d-1)} \cdot B$ and $\operatorname{rk} \hat{A} = n - m$.

The case $m = 0$ follows from i) of theorem 7.6.1 and proposition 7.5.2, so assume $m = 1$. By conjugation with a permutation, we can obtain (7.14) with $\hat{A}$ instead of A and $N = n$, so the case $d \neq 2$ follows from iii) of theorem 7.6.1.

So assume $d = 2$. It follows from (7.14) with $\hat{A}$ instead of A and $N = n$ that the last $n - r$ rows of $B = dA$ are zero. So minors of $\mathcal{J}(Ax)^{*d}$ with these rows have determinant zero. It follows from iii) of theorem 7.6.1 that all principal minors of size ≥ 2 of $\mathcal{J}(Ax)^{*d}$ have determinant zero.

So assume the diagonal of $\mathcal{J}(Ax)^{*d}$ is nonzero. By the trace condition on $\mathcal{J}(Ax)^{*d}$, we obtain by (7.14) with $N = n$ that the diagonal of B is dependent of

$$(-1, \lambda_{c+1}, \lambda_{c+2}, \ldots, \lambda_r, 0^{r+1}, 0^{r+2}, \ldots, 0^n)$$

so the first r entries of the diagonal of B are nonzero.

Assume $n = DN(A) \neq \operatorname{cork} A + 2$. Then $\operatorname{rk} A \neq 2$. Since the diagonal of A is nonzero and $(Ax)^{*d}$ is reduced, it follows that $\operatorname{rk} A \geq 3$. Since the relations between the rows of A are generated by $A_{r+1} = A_{r+2} = \cdots = A_n = 0$ and (7.14), it follows that $r = \operatorname{rk} A + 1 \geq 4$, so A has at least 4 nonzero rows.

Furthermore, every triple of nonzero rows of A is independent. So every triple of rows of $\hat{A}$ is independent. Since every principal minor of size 2 and size 3 of $\mathcal{J}(Ax)^{*d}$ has determinant zero, it follows that every principal minor of size 2 and size 3 of $B = dA$ has already determinant zero.

We will derive a contradiction by showing that $\operatorname{rk} A = \operatorname{rk} B = 1$. In order to do so, it suffices to show that all minors of size 2 of B that contain exactly one diagonal entry of A have determinant zero. If such a minor contains a zero row of B, then we are done, so assume the opposite. Then such a minor is contained in a principal minor of size 3, which looks like

$$\begin{pmatrix} 1 & a & b^{-1} \\ a^{-1} & 1 & c \\ b & c^{-1} & 1 \end{pmatrix} \cdot \operatorname{diag} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix}$$

where $\mu_1\mu_2\mu_3 \neq 0$, and has determinant zero on account of the above properties of A. Computing its determinant gives

$$\mu_1\mu_2\mu_3 \frac{(abc-1)^2}{abc} = 0$$

so $abc = 1$. It follows that the above principal minor of size 3 has rank 1, as desired. $\square$

Besides the tameness results for the case $m \leq 1$, we have proved i) and iii) of theorem 7.1.1 as well for the case $m \leq 1$, because in case $m = 1$, then both i) and iii) of theorem 7.1.1 imply $d \geq 3$. The cases $d \geq 3$ and $DN(A) \neq \operatorname{cork} A + 2$ of ii) of theorem 7.1.1 follow from corollary 7.6.2 as well in case $m \leq 1$.

Since the case $d \leq 1$ is easy, the case $d = 2 = DN(A) - \operatorname{cork} A$ of ii) of theorem 7.1.1 remains. By the condition $\operatorname{cork} A \leq d$ of ii), we may assume that $\operatorname{cork} A \leq 2$. The case $\operatorname{cork} A \leq 1$ follows from corollary 7.5.4 and the case $\operatorname{cork} A = 2$ follows from corollary 7.5.5, as desired.

By way of the techniques in the proof of corollary 7.6.2, one can prove the following.

Proposition 7.6.3. *Assume $(Ax)^{*d}$ is a power linear map over $\mathbb{C}$ such that $\operatorname{tr} \mathcal{J}(Ax)^{*d} = 0$ and every principal minor of size ≥ 2 of $\mathcal{J}(Ax)^{*d}$ has determinant zero. Then $x + (Ax)^{*d}$ is tame.*

The proof of this proposition is left as an exercise to the reader. Besides the results of corollary 7.6.2, theorem 7.6.1 can be used to obtain results about Zhao graphs: corollary 7.6.4 below.

Corollary 7.6.4. *Assume $h = \sum_{i=1}^{N} (A_i x)^{d+1}$, where $d \geq 2$ and the rows of A are pairwise independent and isotropic. Assume in addition that $\mathcal{H}h$ is nilpotent.*

i) Assume $d \geq 2$ and $\operatorname{rk} A \geq N - 1$. Then $x + \nabla h$ is not only tame, but also linearly triangularizable. If $AA^{\mathrm{t}} \neq 0$, then $d = 2$, $\operatorname{rk} A = N - 1$, and AA^{t} is of the form

$$AA^{\mathrm{t}} = P^{-1} \begin{pmatrix} \emptyset & C \\ C^{\mathrm{t}} & \emptyset \end{pmatrix} P \tag{7.16}$$

for some permutation matrix P. Furthermore, $\operatorname{rk} C = 1$ and C has $DN(A) \geq 3$ rows that are all nonzero.

ii) Assume $d \geq 3$ and $\operatorname{rk} A \geq N - d + 1$. Assume in addition that either h is linearly triangularizable or one row of A is independent of the other rows of A and the corresponding row of AA^{t} is nonzero.

Then $x + \nabla h$ is not only tame, but also linearly triangularizable. If $AA^{\mathrm{t}} \neq 0$, then $\operatorname{rk} A = N - d + 1$ and AA^{t} is of the form of (7.16) for some permutation matrix P. Furthermore, $\operatorname{rk} C = 1$ and C has $DN(A) = d + 1$ rows that are all nonzero.

Proof. From (6.8), it follows that

$$M := \operatorname{diag}\bigl((Ax)^{*(d-1)}\bigr)AA^{\mathrm{t}}$$

is nilpotent. Put $B := AA^{\mathrm{t}}$. If $B = 0$, then $x + \nabla h$ is tame and linearly triangularizable on account of proposition 6.4.7. Assume first that $\mathrm{rk} A = N$. Then every principal minor of size ≤ 2 of M, and hence of B as well, has determinant zero. Since B is symmetric, $B = 0$ follows. So assume that $\mathrm{rk} A \le N - 1$.

i) By assumption, $\mathrm{rk} A = N - 1$. Assume without loss of generality that M is as in (7.14) in theorem 7.6.1, with $B = AA^{\mathrm{t}}$. Since the rows of A are pairwise independent, it follows that $r \ge 3$. Since by iii) of theorem 7.6.1, every principal minor of size 2 with rows and columns in the range $1, 2, \ldots, r$ of M is zero and the rows of A are isotropic and nonzero, we obtain that the leading principal minor of size r of $B = AA^{\mathrm{t}}$ is the zero matrix.

   Assume that $B = AA^{\mathrm{t}} \ne 0$. Then $r \le N - 1$. In a similar manner as for the leading principal minor of size r, we obtain by i) of theorem 7.6.1 that the lower right principal minor of size $N - r$ of $B = AA^{\mathrm{t}}$ is the zero matrix, and (7.16) with $P = I_N$ follows, where C has height r. Since $B \ne 0$, $r \le N - 1$ follows, so A_N is independent of the other rows of A. From ii) it follows that $\mathrm{rk} A = N - d + 1 \le N - 2$ in case $d \ge 3$. So $d = 2$.

   If $r = N - 1$, then C consists of only one column and $\mathrm{rk} C = 1$ is trivial. If $r \le N - 2$, then by looking at principal minors of M with two rows and columns in the range $1, 2, \ldots, r$ and another two in the range $r + 1, r + 2, \ldots, N$, we obtain that minors of size 2 of C are singular, and $\mathrm{rk} C = 1$ follows.

   Since AA^{t} is of the above form with $\mathrm{rk} C = 1$, it follows that all principal minors of size $\ne 2$ of M have determinant zero. By looking at the terms with z_i in the sum of determinants of principal minor of size 2 of

$$\hat{M} := \operatorname{diag}\bigl(\lambda_2 z_2 + \lambda_3 z_3 + \cdots + \lambda_r z_r, z_2, z_3, \ldots, z_N\bigr) \cdot B$$

   we see that for the i-th column $\hat{C}^{(i)}$ of C, $(\hat{C}^{(i)})^{*2}$ is dependent of $(-1, \lambda_2, \lambda_3, \ldots, \lambda_r)$. So all rows of C are nonzero.

Since the last row of A is independent of the other rows of A, we can remove it by way of theorem 7.1.4. By induction, it follows that $x+\nabla h$ is tame. Furthermore, if we define $\tilde{h} = \sum_{i=1}^{r+1}(A_i x)^{d+1}$, then $x + \nabla \tilde{h}$ is tame as well.

We shall show that ∇h is even linearly triangularizable. Assume without loss of generality that $n \ge 2N$. From proposition 5.5.1, it follows that by way of an orthogonal transformation, we obtain $A_i x = x_i + \mathrm{i}x_{n+1-i}$ for all $i \ge r+1$. Furthermore, we can obtain

$$\begin{aligned} A_i x \quad = \quad & (x_i + \mathrm{i}x_{n+1-i}) + \frac{1}{2}\big(C_{i1}(x_{r+1} - \mathrm{i}x_{n-r}) \; + \\ & C_{i2}(x_{r+2} - \mathrm{i}x_{n-r-1}) + \cdots + C_{i(N-r)}(x_N - \mathrm{i}x_{n+1-N})\big) \end{aligned}$$

for all $i \le r-1$. Now $A_r x$ is determined, because it is dependent of $A_1 x, A_2 x, \ldots, A_{r-1} x$. Since $\mathrm{rk} C = 1$, it follows that $A_i x \in \mathbb{C}[A_1 x, x_1 + \mathrm{i}x_n, x_2 + \mathrm{i}x_{n-1}, \ldots, x_{r-1} + \mathrm{i}x_{n+2-r}]$ for all $i \le r$.

Let $\tilde{h}$ as above. Since $x + \nabla \tilde{h}$ is tame, we obtain that $\mathcal{H}\tilde{h}$ is nilpotent. Notice that $\tilde{h}$ is degenerate of such a large order that by theorem 5.6.2, the nilpotency of its Hessian comes down to that of $\mathcal{H}_{x_{r+1},x_{n-r}}\tilde{h}$ only. View $\tilde{h}$ as a polynomial over $\mathbb{C}$ in $x_{r+1} + \mathrm{i}x_{n-r}, A_1 x, x_1 + \mathrm{i}x_n, x_2 + \mathrm{i}x_{n-1}, \ldots, x_{r-1} + \mathrm{i}x_{n+2-r}$. Since $\tilde{h}$ has degree $d+1 \ge 2$ with respect to $x_{r+1} + \mathrm{i}x_{n-r}$, the degree of $\tilde{h}$ with respect to $A_1 x$ is at most 1, for $A_1 x = \frac{1}{2}C_{11}(x_{r+1} - \mathrm{i}x_{n-r}) + \cdots$. So $\tilde{h}$ is of the form of g in (5.21). This gives the desired result.

ii) Assume first that some row of A is independent of the other rows of A and the corresponding row of AA^{t} is nonzero. Assume without loss of generality that the last row A_N of A is such a row. Assume $\mathrm{rk} A = N - c$. Then we may assume without loss of generality that the last $N-c$ rows of A are independent. So for each $i \le c$, we can express $A_i x$ as a linear form in $A_{c+1}x, A_{c+2}x, \ldots, A_N x$. But the coefficients of $A_N x$ are zero, for A_N is independent of the other rows of A.

It follows that there are linear forms $L_i \in \mathbb{C}[y_{c+1}, y_{c+2}, \ldots, y_{N-1}]$ such that M and

$$\hat{M} := \operatorname{diag}\begin{pmatrix} L_1(z_{c+1}, z_{c+2}, \dots, z_{N-1})^{d-1} \\ L_2(z_{c+1}, z_{c+2}, \dots, z_{N-1})^{d-1} \\ \vdots \\ L_c(z_{c+1}, z_{c+2}, \dots, z_{N-1})^{d-1} \\ z_{c+1}^{d-1} \\ z_{c+2}^{d-1} \\ \vdots \\ z_N^{d-1} \end{pmatrix} \cdot B$$

can be obtained from each other by substitutions. Now differentiate the sum of principal minors of size 2 of $\hat{M}$ with respect to z_N, to obtain

$$-(d-1)z_N^{d-2}B_N \cdot \left(\begin{pmatrix} L_1(z_{c+1}, z_{c+2}, \dots, z_{N-1})^{d-1} \\ L_2(z_{c+1}, z_{c+2}, \dots, z_{N-1})^{d-1} \\ \vdots \\ L_c(z_{c+1}, z_{c+2}, \dots, z_{N-1})^{d-1} \\ z_{c+1}^{d-1} \\ z_{c+2}^{d-1} \\ \vdots \\ z_N^{d-1} \end{pmatrix} * Be_N\right) = 0$$

Since B_N is nonzero by assumption and B is symmetric, the coefficients of the $(d-1)$-th powers in the above sum, which are contained in $\mathbb{C}z_N^{d-2}$, are not all zero. It follows that the components of $(Ax)^{*(d-1)}$ are linearly dependent.

Let I be the set of indices i for which $(A_ix)^{d-1}$ has a nonzero coefficient in the above linear relation and $\tilde{A}$ be the matrix consisting of the rows A_i of A for which $i \in I$. From ii) of lemma 6.6.8, with a generic linear combination of the $\frac{\partial}{\partial z_i}$'s instead of $\frac{\partial}{\partial x_n}$, it follows that $\tilde{N} := \#I \geq d - 3 + 2\text{rk}\tilde{A}$, so $\tilde{N} - \text{rk}\tilde{A} \geq d - 3 + \text{rk}\tilde{A}$. By $\text{rk}A \geq N - d + 1$, we obtain

$$d - 1 \geq N - \text{rk}A \geq \tilde{N} - \text{rk}\tilde{A} \geq d - 3 + \text{rk}\tilde{A}$$

so $\text{rk}\tilde{A} \leq 2$. But the rows of A are pairwise independent, so $\text{rk}\tilde{A} = 2$. Furthermore, $N - \text{rk}A = \tilde{N} - \text{rk}\tilde{A} = d - 1$, so all dependences between the rows of A correspond to dependences between the rows of $\tilde{A}$. Consequently, $DN(A) = DN(\tilde{A})$ and

$$r := DN(A) = \#I = \mathrm{rk}\tilde{A} + (\tilde{N} - \mathrm{rk}\tilde{A}) = d + 1$$

Assume without loss of generality that for each $i \ge r+1$, row A_i of A is independent of the other rows of A.

Notice that the leading principal minor $\tilde{B}$ of size r of B is equal to $\tilde{A}\tilde{A}^{\mathrm{t}}$. Since $\mathrm{rk}\tilde{A} = 2$, we obtain by proposition 7.6.5 below and ii) of theorem 7.6.1 that $\tilde{B} = 0$. In a similar manner as in i), we obtain that the lower right principal minor of size $N - r$ of $B = AA^{\mathrm{t}}$ is the zero matrix, and (7.16) with $P = I_N$ follows, where C has height r.

Notice that for each column $\hat{C}$ of C, $\hat{C}^{*2}$ is dependent of a fixed vector without any zero, namely the vector of coefficients of the linear relation between the components of $(\tilde{A}x)^{d-1}$. So every row of C is nonzero.

Since the last row of A is independent of the other rows of A, we can remove it by way of theorem 7.1.4. If we do not obtain $B = 0$ after removing A_N, then there is another row of A with the same properties as A_N. By induction, it follows that $x + \nabla h$ is tame.

We shall show that $\mathrm{rk}C = 1$. For that purpose, assume that $\mathrm{rk}C \ge 2$. Assume without loss of generality that the leading minor of size 2 of C has rank 2. Just as in the proof of i), we may assume that $n \ge 2N$, $A_i x = x_i + \mathrm{i}x_{n+1-i}$ for all $i \ge r+1$ and

$$\begin{aligned} A_i x \;\; &= \;\; (x_i + \mathrm{i}x_{n+1-i}) + \frac{1}{2}\big(C_{i1}(x_{r+1} - \mathrm{i}x_{n-r}) \; + \\ & \qquad C_{i2}(x_{r+2} - \mathrm{i}x_{n-r-1}) + \cdots + C_{i(N-r)}(x_N - \mathrm{i}x_{n+1-N})\big) \end{aligned}$$

for $i = 1$ and $i = 2$. Since the leading minor of size 2 of C has rank 2, it follows that there exists linear combinations L_1 and L_2 of $A_1 x$ and $A_2 x$ such that $A_i x = \frac{1}{2}(C_{i1}L_1 + C_{i2}L_2)$ and

$$\begin{aligned} L_i - (x_{r+i} - \mathrm{i}x_{n+1-r-i}) \in \mathbb{C}[&x_1 + \mathrm{i}x_n, x_2 + \mathrm{i}x_{n-1}, x_{r+3} - \mathrm{i}x_{n-2-r}, \\ & x_{r+4} - \mathrm{i}x_{n-3-r}, \ldots, x_N - \mathrm{i}x_{n+1-N}] \end{aligned}$$

for $i = 1$ and $i = 2$.

Just like in the proof of i), define $\tilde{h} = \sum_{i=1}^{r+1}(A_i x)^{d+1}$. Again $x + \nabla\tilde{h}$ is tame and $\tilde{h}$ is degenerate of such a large order that by theorem 5.6.2, the nilpotency of its Hessian comes down to that of $\mathcal{H}_{x_{r+1},x_{n-r}}\tilde{h}$ only. View $\tilde{h}$ as a polynomial over $\mathbb{C}$ in $x_{r+1}+\mathrm{i}x_{n-r}, L_1, L_2$. Since $\tilde{h}$ has degree $d+1 \geq 2$ with respect to $x_{r+1}+\mathrm{i}x_{n-r}$, the degree of $\tilde{h}$ with respect to L_1 is only 1, for $L_1 = x_{r+1} - \mathrm{i}x_{n-r} + \cdots$.

It follows that the degree with respect to L_1 of $\sum_{i=1}^{r}(A_i x)^{d+1}$, seen as polynomial in L_1 and L_2, is equal to 1 as well. With $\sum_{i=1}^{r}(A_i x)^{d+1} + (A_{r+2}x)^{d+1}$ instead of $\tilde{h}$, we can derive that the degree with respect to L_2 of $\sum_{i=1}^{r}(A_i x)^{d+1}$, seen as polynomial in L_1 and L_2, is equal to 1, too. So $d+1 \leq 2$. Contradiction, so $\mathrm{rk} C = 1$, as desired.

In a similar manner as in i), it follows that ∇h is linearly triangularizable. We assume now that h is indeed linearly triangularizable, however not because we derived it, but in order to prove the second case of ii).

Say that h is as in (5.21). If for each i, $A_i x \in \mathbb{C}[x_1 + \mathrm{i}x_n, x_2 + \mathrm{i}x_{n-1}, \ldots, x_{\lfloor n/2 \rfloor} + \mathrm{i}x_{n+1-\lfloor n/2 \rfloor}]$, then $AA^{\mathrm{t}} = 0$. So assume that this is not the case.

Let L_3 be a generic linear combination of $x_1 + \mathrm{i}x_n, x_2 + \mathrm{i}x_{n-1}, \ldots, x_{\lfloor n/2 \rfloor} + \mathrm{i}x_{n+1-\lfloor n/2 \rfloor}$ and $x_{n+1} + \mathrm{i}x_{n+2}$. Then one can easily show that the Hessian of $h + L_3^d$ with respect to $x_1, x_2, \ldots, x_{n+2}$ is nilpotent. Furthermore, $h + L_3^d$ is a sum of $N+1$ powers of linear forms in a trivial manner, say that $h + L_3^d = \sum_{i=1}^{N+1}(\hat{A}_i(x_1, x_2, \ldots, x_{n+2}))^d$.

Then the row of $\hat{A}$ that corresponds to L_3 is independent of the other rows of $\hat{A}$ and the corresponding row of $\hat{A}\hat{A}^{\mathrm{t}}$ is nonzero. So $\hat{A}$ satisfies the desired properties of A and one can easily show that A itself satisfies the desired properties as well. □

The result of the following proposition is a part of [54, Th. 1.2 (i)].

Proposition 7.6.5 (Willems). *Assume $h = \sum_{i=1}^{N}(A_i x)^{d+1}$, where $d \geq 1$ and the rows A_i of A are pairwise independent and isotropic. If $\mathcal{H}h$ is nilpotent and* $\mathrm{rk} A \leq 2$, *then $AA^{\mathrm{t}} = 0$.*

Proof. Assume that the diagonal of AA^{t} is zero, but $AA^{\mathrm{t}} = 0$. Then we may assume without loss of generality that $\lambda := A_1 A_2^{\mathrm{t}} = (AA^{\mathrm{t}})_{12} \neq 0$. Now all subsequent rows are both dependent of A_1 and A_2 and isotropic, so for each $i \geq 3$, there exists a $\mu \in \mathbb{C}$ such that $A_i = \mu A_1$ or $A_i = \mu A_2$.

Since the rows of A are pairwise independent, it follows that A has only two rows. Since $AA^{\mathrm{t}} = \binom{0\,\lambda}{\lambda\,0}$, it follows that the Zhao matrix $\operatorname{diag}(Ax)^{*(d-1)} \cdot AA^{\mathrm{t}}$ has full rank. This contradicts that the Hessian of h is not nilpotent by way of (6.8). So $AA^{\mathrm{t}} = 0$, as desired. □

Notice that the Zhao graphs of the maps in corollary 7.6.4 are complete bipartite graphs that contain a shrub with three branches as a subgraph, if we do not count isolated vertices. For instance,

$$A = \begin{pmatrix} 1 & \mathrm{i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \mathrm{i} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \mathrm{i} & 0 & 0 & 0 \\ -1 & -\mathrm{i} & -1 & -\mathrm{i} & -1 & -\mathrm{i} & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 2\mathrm{i} \end{pmatrix}$$

gives a shrub with four branches. For a shrub with c branches, find $\lambda_i \in \mathbb{C}^*$ such that $\sum_{i=1}^{c} \lambda_i^3 = 0$. Next, find A_i such that $\sum_{i=1}^{c} \lambda_i^2 A_i = 0$. After that, find A_{c+1} such that $A_i A_{c+1}^{\mathrm{t}} = \lambda_i$ for all $i \le c-1$, then by

$$\lambda_c^2 A_c A_{c+1}^{\mathrm{t}} - \lambda_c^3 = \left(\sum_{i=1}^{c} \lambda_i^2 A_i \right) A_{c+1}^{\mathrm{t}} - \sum_{i=1}^{c} \lambda_i^3 = 0$$

it follows that automatically $A_c A_{c+1}^{\mathrm{t}} = \lambda_c$ as well. The reader may provide the details in the construction of A and verify that A gives a shrub with c branches. Furthermore, the reader may show that every complete bipartite graph that contains a shrub with three branches as a subgraph can be obtained.

7.7 Proof of the case $m \le d-1$ of theorem 7.1.1

Just as in the previous section, the case $DN(A) \le \operatorname{cork} A + \max\{3, 7-d\}$ of the tameness result follows from proposition 7.3.4. In case $DN(A) \ge \operatorname{cork} A + \max\{4, 8-d\}$, we have $DN(A) \ne \operatorname{cork} A + 2$. We shall show that in that case and in the case $m \le d-2$, we even have symmetrical linear triangularizability for reduced power linear maps $(Ax)^{*d}$ with nilpotent Jacobians. See corollary 7.7.4 below.

The results of this section are based on lemma 7.7.1 below, which has a technical proof in which the principle of inclusion and exclusion for functions is used:

$$\mathcal{F}_S = \sum_{T \subseteq S} (-1)^{\#S - \#T} \sum_{U \subseteq T} \mathcal{F}_U \tag{7.17}$$

(7.17) can be used to obtain as with Möbius inversion formula for square-free numbers as well (the full version requires a variant for multisets instead of sets).

Lemma 7.7.1. *Let $d \geq 2$ and $R \in \mathbb{C}[y_1, y_2, \dots, y_{r+c}]$ be a nonzero relation with $\deg_{y_i} R \leq 1$ for all i such that*

$$R(z_1^d, z_2^d, \dots, z_r^d, (C_1(z_1, \dots, z_r))^d, \dots, (C_c(z_1, \dots, z_r))^d) = 0 \tag{7.18}$$

Assume furthermore that $C_{11} C_{21} \cdots C_{c1} \neq 0$ and $d \geq c$. Then the rows $e_1^{\mathrm{t}}, C_1, \dots, C_c$ are dependent.

Proof. Without loss of generality, we may assume that $C_{11} = C_{21} = \cdots = C_{c1} = 1$. Let $k = \deg_{y_1, y_{r+1}, \dots, y_{r+c}} R$ and $\bar{R}$ be the largest degree part of R with respect to the variables $y_1, y_{r+1}, \dots, y_{r+c}$. Then $\bar{R}$ is homogeneous as a polynomial over $\mathbb{C}[y_2, \dots, y_r]$ in the variables $y_1, y_{r+1}, \dots, y_{r+c}$. If we view $\bar{R}$ as such, we see that $\bar{R}$ can be written as follows:

$$\bar{R} = \sum_{S \subseteq \{1,2,\dots,c\}} P_S(y_2, \dots, y_r) y_1^{k - \#S} \prod_{s \in S} y_{r+s}$$

Put $\underline{c} := \{1, 2, \dots, c\}$. Using (7.17) with $\mathcal{F}_S = P_{\underline{c} \setminus S}$, we obtain

$$P_S(y_2, \dots, y_r) = \sum_{S \subseteq T \subseteq \underline{c}} (-1)^{\#T - \#S} Q_T(y_2, \dots, y_r)$$

where

$$Q_T(y_2, \dots, y_r) := \sum_{T \subseteq U \subseteq \underline{c}} P_U(y_2, \dots, y_r)$$

So we get

$$\begin{aligned} \bar{R} &= \sum_{S \subseteq \underline{c}} P_S(y_2, \dots, y_r) y_1^{k - \#S} \prod_{s \in S} y_{r+s} \\ &= \sum_{S \subseteq \underline{c}} \sum_{S \subseteq T \subseteq \underline{c}} Q_T(y_2, \dots, y_r) (-1)^{\#T - \#S} y_1^{k - \#S} \prod_{s \in S} y_{r+s} \\ &= \sum_{T \subseteq \underline{c}} Q_T(y_2, \dots, y_r) \sum_{S \subseteq T} (-1)^{\#T - \#S} y_1^{k - \#S} \prod_{s \in S} y_{r+s} \end{aligned}$$

Assume that the rows $e_1^{\mathrm{t}}, C_1, \dots, C_c$ are independent and write $\tilde{z} := (z_1, \dots, z_r)$. Assume first that $Q_T(y_2, \dots, y_r) = 0$ for all $T \subsetneq \underline{c}$. Then $Q_{\underline{c}}(y_2, \dots, y_r) \mid \bar{R} \ne 0$. Furthermore, the term $S = \emptyset$, $T = \underline{c}$ of $\bar{R}$ gives $k \le \deg_{y_1} \bar{R} \le \deg_{y_1} R \le 1$ and the term $S = T = \underline{c}$ gives $c \le k$, so $c \le 1$.

Since $z_1^d, \dots, z_r^d$ are algebraically independent, it follows that $c = 1$ and $\deg_{y_{r+1}} R = 1$. But since $d \ge 2$, we see that $(C_1\tilde{z})^d$ is the only polynomial of $z_1^d, \dots, z_r^d, (C_1\tilde{z})^d = (C_c\tilde{z})^d$ that has terms that are not d-th powers in $z_1, z_2, \dots, z_r$. This is impossible, because the left hand side of (7.18) would have terms that are not d-th powers. Contradiction, so $Q_U(y_2, \dots, y_r) \ne 0$ for some $U \subsetneq \{1, 2, \dots, c\}$.

Say that $y_{i_1} y_{i_2} \cdots y_{i_m}$ has a nonzero coefficient in $Q_U(y_2, \dots, y_r)$. Let λ_T be the coefficient of $y_{i_1} y_{i_2} \cdots y_{i_m}$ in $Q_T(y_2, \dots, y_r)$, for all $T \subseteq \{1, 2, \dots, c\}$. Then $\lambda_U \ne 0$.

Since $\deg_{y_1, y_{r+1}, \dots, y_{r+c}}(\bar{R} - R) \le k - 1$, it follows that

$$\deg_{z_1}(\bar{R} - R)(z_1^d, z_2^d, \dots, z_r^d, (C_1\tilde{z})^d, \dots, (C_c\tilde{z})^d) \le (k-1)d$$

But $R(z_1^d, z_2^d, \dots, z_r^d, (C_1\tilde{z})^d, \dots, (C_c\tilde{z})^d) = 0$, so

$$\frac{\bar{R}(z_1^d, z_2^d, \dots, z_r^d, (C_1\tilde{z})^d, \dots, (C_c\tilde{z})^d)}{z_1^{(k-1)d}} \in \mathbb{C}[z_1^{-1}, z_2, \dots, z_r]$$

Writing out this quotient, noticing that $1/z_1^{(k-1)d} = z_1^d/(z_1^d)^k$, we obtain

$$\sum_{T \subseteq \underline{c}} Q_T(z_2^d, \dots, z_r^d) \sum_{S \subseteq T} (-1)^{\#T - \#S} z_1^{d - d\#S} \prod_{s \in S} (C_s\tilde{z})^d \in \mathbb{C}[z_1^{-1}, z_2, \dots, z_r]$$

Now take a term $t = t_1 t_2$ on the left hand side, such that t_1 is a term of $Q_T(z_2^d, \dots, z_r^d)$ for some $T \subseteq \underline{c}$ and t_2 is a term of

$$(-1)^{\#T - \#S} z_1^{d - d\#S} \prod_{s \in S} (C_s\tilde{z})^d$$

for some $S \subseteq T$, which is divisible by $z_{i_1}^d z_{i_2}^d \cdots z_{i_m}^d$.

Assume first that $\deg_{z_1} t > 0$. Since $\deg_{z_1} t_1 = 0$ and $\deg_{\tilde{z}} t_2 = d$, it follows that $\deg_{z_2, \dots, z_r} t_2 < d$. By $z_{i_1}^d z_{i_2}^d \cdots z_{i_m}^d \mid t$, we obtain that t_1 must be divisible by $z_{i_1} z_{i_2} \cdots z_{i_m}$. But t_1 is a d-th power, so t_1 is already divisible by $z_{i_1}^d z_{i_2}^d \cdots z_{i_m}^d$, whence $t_1 = \lambda_T z_{i_1}^d z_{i_2}^d \cdots z_{i_m}^d$ in case $\deg_{z_2, \dots, z_r} t < (m+1)d$.

Assume next that $\deg_{z_1} t \le 0$. Then $t \in \mathbb{C}[z_1^{-1}, z_2, \dots, z_r]$, regardless of t_1 is divisible by $z_{i_1}^d z_{i_2}^d \cdots z_{i_m}^d$ or not.
So if we select the terms $t = t_1 t_2$ as above with $\deg_{z_2,\dots,z_r} t < (m+1)d$ and $z_{i_1}^d z_{i_2}^d \cdots z_{i_m}^d \mid t$, omitting the terms t with both $\deg_{z_1} t \le 0$ and $z_{i_1}^d z_{i_2}^d \cdots z_{i_m}^d \nmid t_1$, then we obtain

$$z_{i_1}^d z_{i_2}^d \cdots z_{i_m}^d \sum_{T \subseteq \underline{c}} \lambda_T \sum_{S \subseteq T} (-1)^{\#T - \#S} z_1^{d - d\#S} \prod_{s \in S} (C_s \tilde{z})^d \in \mathbb{C}[z_1^{-1}, z_2, \dots, z_r]$$

Since the factor $z_{i_1}^d z_{i_2}^d \cdots z_{i_m}^d$ does not influence the degree with respect to z_1 of terms on the left hand side, we may omit it. Since the rows $e_1^{\mathrm{t}}, C_1, \dots, C_c$ are independent and $C_{11} = C_{21} = \cdots = C_{c1} = 1$, the above is equivalent to

$$\sum_{T \subseteq \underline{c}} \lambda_T \sum_{S \subseteq T} (-1)^{\#T - \#S} z_1^{d - d\#S} \prod_{s \in S} (z_1 + w_s)^d \in \mathbb{C}[z_1^{-1}, w_1, \dots, w_c] \quad (7.19)$$

where $w_1, \dots, w_c$ are new indeterminates.
From (7.17) with $\mathcal{F}_T = z_1^{d - \#T} \prod_{s \in T} (d w_s)$ instead of $\mathcal{F}_S$, we obtain that

$$\begin{aligned} z_1^{d-\#T} \prod_{s \in T} (d w_s) &= \sum_{S \subseteq T} (-1)^{\#T - \#S} \sum_{S' \subseteq S} z_1^{d - \#S'} \prod_{s \in S'} (d w_s) \\ &= \sum_{S \subseteq T} (-1)^{\#T - \#S} z_1^{d - \#S} \sum_{S' \subseteq S} z_1^{\#S - \#S'} \prod_{s \in S'} (d w_s) \\ &= \sum_{S \subseteq T} (-1)^{\#T - \#S} z_1^{d - \#S} \prod_{s \in S} (z_1 + d w_s) \end{aligned}$$

whence

$$\sum_{T \subseteq \underline{c}} \lambda_T z_1^{d - \#T} \prod_{s \in T} (d w_s) = \sum_{T \subseteq \underline{c}} \lambda_T \sum_{S \subseteq T} (-1)^{\#T - \#S} z_1^{d - \#S} \prod_{s \in S} (z_1 + d w_s)$$

But the right hand side are exactly those terms of the left hand side of (7.19) that satisfy $\deg w_i \le 1$ for all i. Consequently,

$$\sum_{T \subseteq \underline{c}} \lambda_T (z_1^{d - \#T}) \prod_{s \in T} (d w_s) \in \mathbb{C}[z_1^{-1}, w_1, \dots, w_c]$$

Since $d \ge c > \#U$, the coefficient $d^{\#U} \lambda_U$ of $(z_1^{d - \#U}) \prod_{s \in U} w_s$ equals zero. Contradiction, so the rows $e_1^{\mathrm{t}}, C_1, \dots, C_c$ are dependent, as desired. $\square$

Theorem 7.7.2. *Assume*

$$M := \operatorname{diag}\big((Ax)^{*(d-1)}\big) \cdot B$$

is a nilpotent power linear quasi-Jacobian of size N over $\mathbb{C}$ and $d \ge 3$. Assume in addition that the rows of A are pairwise linearly dependent over $\mathbb{C}$ and that M is not *symmetrically triangularizable.*

i) *If* $\mathrm{rk} A \ge N-2$ *and* $d \ge 3$*, then* $\mathrm{rk} A = N-2$*,* $d = 3$ *and* $DN(A) = 4$*.*

ii) *If* $\mathrm{rk} A \ge N-3$ *and* $d \ge 4$*, then* $\mathrm{rk} A = N-3$ *and* $DN(A) \le 6$*.*

iii) *If* $\mathrm{rk} A \ge N-3$*,* $d \ge 3$*, and A has two rows that are together dependent of the other $N-2$ rows of A, but individually* not *dependent of the other $N-2$ rows of A, then* $\mathrm{rk} A = N-3$ *and* $d = 3$*. Furthermore, A contains four nonzero rows such that the rank of the matrix of these rows is equal to 2 in this case.*

Proof. Let $r = \mathrm{rk} A$ and $c = N - r$. Without loss of generality, we assume that the rows $A_1, A_2, \ldots, A_r$ of A are independent and that

$$\begin{pmatrix} A_{r+1} \\ \vdots \\ A_n \end{pmatrix} = C \cdot \begin{pmatrix} A_1 \\ \vdots \\ A_r \end{pmatrix}$$

for some $C \in \mathrm{Mat}_{c,r}(\mathbb{C})$. Since M is not symmetrically triangularizable, it follows from proposition 7.5.2 and lemma 6.5.2 that

$$R\big((A_1x)^{d-1}, (A_2x)^{d-1}, \ldots, (A_rx)^{d-1}, (A_{r+1}x)^{d-1}, \ldots, (A_Nx)^{d-1}\big) = 0$$

for some nonzero $R \in \mathbb{C}[y_1, y_2, \ldots, y_N]$. By definition of C, this is equivalent to

$$R(z_1^{d-1}, z_2^{d-1}, \ldots, z_r^{d-1}, (C_1(z_1, \ldots, z_r))^{d-1}, \ldots, (C_c(z_1, \ldots, z_r))^{d-1}) = 0 \tag{7.20}$$

i.e. (7.18) with $d-1$ instead of d. Furthermore, $c \ge 1$ follows.

i) Assume $r \ge N-2$ and $d \ge 3$. Then $c \le 2$. Assume without loss of generality that $C_{11} \ne 0$. If $c = 1$, then e_1^{t} and C_1 are dependent on account of lemma 7.7.1, which contradicts that A_1 and A_{r+1} are independent. So $c = 2$ and $\mathrm{rk} A = r = N - 2$. Furthermore, $r \geq 2$ and we may even assume that $C_{11}C_{12} \neq 0$.

If $C_{21} = 0$, then A_1 and A_{r+1} are individually independent of the other $N - 2$ rows of A, and we get a contradiction by iii), so $C_{21} \neq 0$. A similar argument with A_2 and A_{r+1} gives $C_{21}C_{22} \neq 0$. From lemma 7.7.1, it follows that e_1^{t} is dependent of C_1 and C_2. A slight variation of lemma 7.7.1 tells us that e_2^{t} is dependent of C_1 and C_2 as well. It follows that $C_{1i} = C_{2i} = 0$ for all $i \geq 3$.

So $DN(A) = 4$. In order to show that $d = 3$, we assume that $d \geq 4$ and derive a contradiction.

Since $z_3^{d-1}, z_4^{d-1}, \dots, z_r^{d-1}$ are algebraically independent over $\mathbb{C}$ of z_1^{d-1}, $z_2^{d-1}, (C_{11}z_1 + C_{12}z_2)^{d-1}, (C_{21}z_1 + C_{22}z_2)^{d-1}$, it follows from (7.20) that every coefficient $\hat{C}$ of R, where R is viewed as a polynomial in $y_3, y_4, \dots, y_r$, with coefficients in $\mathbb{C}[y_1, y_2, y_{r+1}, y_{r+2}]$, satisfies

$$\hat{C}\big(z_1^{d-1}, z_2^{d-1}, (C_{11}z_1 + C_{12}z_2)^{d-1}, (C_{21}z_1 + C_{22}z_2)^{d-1}\big) = 0$$

Since $R \neq 0$, we can choose $\hat{C} \neq 0$.

It follow that we can choose $\hat{R} \neq 0$ homogeneous with $\deg_{y_i} \hat{R} \leq 1$, such that

$$\hat{R}\big(z_1^{d-1}, z_2^{d-1}, (z_1 + \lambda z_2)^{d-1}, (z_1 + \mu z_2)^{d-1}\big) = 0 \tag{7.21}$$

where $\lambda := C_{12}/C_{11}$ and $\mu := C_{22}/C_{21}$. Take $\hat{R}$ in this way, such that $\deg \hat{R}$ is minimal.

If $\deg \hat{R} \geq 3$, then by looking at the leading coefficient with respect to z_1 in (7.21), we obtain $y_2 \mid \hat{R}$, and it follows that the degree of $\hat{R}$ is not minimal. If $\deg \hat{R} = 1$, then it follows from lemma 6.2.1 with $r = 4$ that $\lambda = \mu$, and that C_1 and C_2 are dependent, which contradicts that A_{r+1} and A_{r+2} are dependent. So $\deg \hat{R} = 2$.

Write

$$\hat{R} = a_1 y_{r+1} y_{r+2} + a_2 y_1 y_{r+2} + a_3 y_1 y_{r+1} + \tilde{R}$$

with $\tilde{R} \in \mathbb{C}[y_1, y_2, y_{r+1}, y_{r+2}]$ such that $\deg_{y_1, y_{r+1}, y_{r+2}} \tilde{R} \leq 1$, and $a_i \in \mathbb{C}$ for all i. Looking at the coefficient of $z_1^{2(d-1)-1} z_2$ in (7.21) gives

$$(a_1 + a_3)\lambda = -(a_1 + a_2)\mu \tag{7.22}$$

after dividing by $d-1$. Looking at the coefficient of $z_1^{2(d-1)}$ in (7.21), gives $a_1+a_2+a_3=0$. By subtracting $(a_1+a_2+a_3)\lambda = -(a_1+a_2+a_3)\mu$ from (7.22), we get $-a_2\lambda = a_3\mu$.

At last, the coefficient of $z_1^{2(d-1)-2}z_2^2$ in (7.21) is equal to

$$\begin{aligned}
0 &= (d-1)^2 a_1\lambda\mu + \binom{d-1}{2}(a_1+a_3)\lambda^2 + \binom{d-1}{2}(a_1+a_2)\mu^2 \\
&= (d-1)\left(\frac{d}{2}+\frac{d-2}{2}\right)a_1\lambda\mu - \binom{d-1}{2}a_2\lambda^2 - \binom{d-1}{2}a_3\mu^2 \\
&= \left(\binom{d}{2}+\binom{d-1}{2}\right)a_1\lambda\mu + \binom{d-1}{2}a_3\lambda\mu + \binom{d-1}{2}a_2\lambda\mu \\
&= \binom{d}{2}a_1\lambda\mu
\end{aligned}$$

So $a_1=0$. By $a_1+a_2+a_3=0$, we obtain $-a_2=a_3$. From $-a_2\lambda = a_3\mu$, it follows that either $a_2=a_3=0$ or $\lambda=\mu$. But $\lambda=\mu$ was impossible, so $a_2=a_3=0$. Consequently, $y_2 \mid \hat{R}$, contradicting the minimality of $\deg \hat{R}$, as desired.

ii) Assume $\mathrm{rk}A \ge N-3$ and $d \ge 4$. From i), it follows that $\mathrm{rk}A = N-3$. If column j of C has exactly one nonzero entry C_{ij}, then row A_j and row A_{r+i} are together dependent of the other $N-2$ rows of A, but individually independent of the other $N-2$ rows of A. This is however impossible on account of iii), so columns of C do not have exactly one nonzero entry.

So assume without loss of generality that $C_{11}C_{21} \ne 0$. We first show that we may assume that $C_{31} \ne 0$. So assume that $C_{31}=0$ and $C_{32} \ne 0$. Since the second column of C has at least two nonzero entries, we may assume that $C_{22} \ne 0$. If $C_{12} \ne 0$, then we can interchange the first two columns of C by interchanging A_1 and A_2, and conjugating B with the corresponding flip. So assume $C_{12}=0$. By decomposing $M_i = (A_i x)^{d-1} \cdot B_i$ differently for $i=r+1$ and $i=r+2$, we can obtain $C_{11}=C_{21}=1$. Next, we can adapt the decomposition of M_2 to obtain $C_{22}=1$. So we have only ones and zeros in the first two columns of C

except C_{32}. Now

$$\begin{aligned} z_1 &= \big(z_{r+2} - C_2(0, z_2, z_3)\big) \\ z_{r+1} &= C_1(0, z_2, z_3) + z_1 \\ &= C_1(0, z_2, z_3) + \big(z_{r+2} - C_2(0, z_2, \dots, z_r)\big) \\ z_{r+3} &= C_3(0, z_2, z_3) + C_{31} z_1 \\ &= C_3(0, z_2, z_3) + C_{31}(z_{r+2} - C_2(0, z_2, \dots, z_r)) \end{aligned} \tag{7.23}$$

and the coefficients

$$-C_{22} = -1 \qquad C_{12} - 1 = -1 \qquad C_{32} - C_{31} \tag{7.24}$$

of z_2 are nonzero, because $C_{31} = 0 \neq C_{32}$. Since $C_{21} \neq 0$, we can interchange A_1 and A_{r+2}, and conjugate B with the corresponding flip. Adapting C with respect to the latter maneuver, we get $C_{12}C_{22}C_{32} \neq 0$, and the former maneuver gives $C_{11}C_{21}C_{31} \neq 0$, as desired.

We show that we may assume that $C_{12}C_{22}C_{32} \neq 0$ as well. For that purpose, assume without loss of generality that $C_{12} = 0$. Then $C_{22}C_{23} \neq 0$ and again, we may assume that $C_{11} = C_{21} = C_{22} = 1$. Assume first that $C_{32} \neq C_{31}$. Then we obtain $C_{12}C_{22}C_{32} \neq 0$ after applying the maneuver of (7.23), because the coefficients of z_2 are again as in (7.24). In order to see that $C_{11}C_{21}C_{31} \neq 0$ is preserved, we look at the coefficients of z_{r+2} in (7.23):

$$1 \qquad 1 \qquad C_{31} \tag{7.25}$$

Not only $C_{12}C_{22}C_{32} \neq 0$ is preserved, but the C_{1i} even keep their values.

So assume $C_{32} = C_{31}$. By adapting the decomposition of M_{r+3}, we get $C_{31} = C_{32} = 1$. Assume now that $DN(A) \geq 7$. Then C has at least four nonzero columns. Assume without loss of generality that the first four columns of C are nonzero. If the third column is just as cooperative as the second one, then it has one zero and two equal nonzero entries. If the zero is on the same spot as in the second column, i.e. $C_{13} = 0$, then A_2 and A_3 are together dependent of the other $N-2$ rows of A, but individually independent of the other $N-2$ rows of A. This is impossible on account of iii), so assume $C_{22} = 0$ and $C_{13} = C_{33}$.

If the fourth column is thwarting again, then $C_{34} = 0$ and $C_{14} = C_{24}$. But the first column is entirely nonzero, and we can use lemma 7.7.1 and

$$\det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & C_{13} & C_{14} \\ 1 & 1 & 0 & C_{14} \\ 1 & 1 & C_{13} & 0 \end{pmatrix} = C_{13}C_{14} \det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} = 2 \neq 0$$

to get a contradiction. So we may assume that $C_{12}C_{22}C_{32} \neq 0$ as well, as desired.

Assume first that C_1, C_2 and C_3 are independent. Then by lemma 7.7.1, e_1^{t} and e_2^{t} are dependent of C_1, C_2 and C_3. That gives two independent linear relations between the five rows $A_1, A_2, A_{r+1}, A_{r+2}, A_{r+3}$ already. Since $\mathrm{rk} A = N-3$, there is one remaining relation. If at most one row besides these five is involved in this remaining relation, then $DN(A) = 6$, as desired. But if there are two or more such rows, then any two such rows are together dependent of the other $N-2$ rows of A, but individually independent of the other $N-2$ rows of A. This is however impossible on account of iii).

Assume next that C_1, C_2 and C_3 are dependent. Since the third column of C is nonzero, it follows that not both e_1^{t} and e_2^{t} are dependent of C_1, C_2 and C_3. So assume without loss of generality that e_1^{t} is independent of C_1, C_2 and C_3, and $C_{11} = C_{21} = C_{31} = 1$. If $C_{12} = C_{22} = C_{32}$, then the first two rows of A are together dependent of the other $N-2$ rows of A, but individually independent of the other $N-2$ rows of A. So we may assume that $C_{12} \neq C_{22} \neq C_{32}$ (possibly $C_{12} = C_{32}$). Assume again that $C_{22} = 1$.

Notice that C_1 and C_3 are independent, for otherwise the rows of A would not be pairwise independent. So C_1, e_1^{t} and C_3 are independent. Furthermore, the coefficients of z_2 on the right hand sides of (7.23) are again as in (7.24), except that just $C_{12}-1 \neq 0$ instead of $C_{12}-1 = -1$. The coefficients of z_{r+2} on the right hand sides of (7.23) are all as in (7.25). Since $C_{21} \neq 0$, we can apply the maneuver of (7.23). This way, we obtain the case above that C_1, C_2 and C_3 are independent after recomputing C, for C_1, e_1^{t} and C_3 were independent, as desired.

iii) Assume that $d \ge 3$, and that A has two rows that are together dependent of the other $N-2$ rows of A, but individually not dependent of the other $N-2$ rows of A. Assume without loss of generality that one of both rows is A_1. Since the other of both rows is dependent of A_1 and the other $N-2$ rows of A, we may assume that the other of both rows is A_{r+1}.

Since A_1 and A_{r+1} are individually independent of the other $N-2$ rows of A, it follows that A_{r+1} is independent of $A_2, A_3, \ldots, A_{r+i}$ for each $i \geq 2$, whence $C_{11} \neq 0$ and $C_{i1} = 0$ for each $i \geq 2$. Write

$$\begin{aligned} R \;=\; & \tilde{R}(y_1, y_2, \ldots, y_r, y_{r+2}, \ldots, y_N) \;+ \\ & y_{r+1}\hat{R}(y_1, y_2, \ldots, y_r, y_{r+2}, \ldots, y_N) \end{aligned}$$

Looking at terms of (7.20) for which the degree with respect to z_1 is $d-2$ or $2d-3$, we obtain that

$$\begin{aligned} &(d-1)(C_{11}z_1)^{d-2}C_1(0, z_2, \ldots, z_r) \cdot \\ &\hat{R}(z_1^{d-1}, \ldots, z_r^{d-1}, (C_2(z_1, \ldots, z_r))^{d-1}, \ldots, (C_c(z_1, \ldots, z_r))^{d-1}) = 0 \end{aligned} \tag{7.26}$$

and hence

$$\tilde{R}(z_1^{d-1}, \ldots, z_r^{d-1}, (C_2(z_1, \ldots, z_r))^{d-1}, \ldots, (C_c(z_1, \ldots, z_r))^{d-1}) = 0 \tag{7.27}$$

as well.

Let $\hat{A}$ be the matrix one obtains from A by adding a column e_{r+1} to the right, and $\hat{x} = (x_1, x_2, \ldots, x_{n+1})$. Then

$$\hat{M} := \mathrm{diag}\big((\hat{A}\hat{x})^{*(d-1)}\big) \cdot B$$

is a power linear quasi-Jacobian, and $\hat{c} := N - \mathrm{rk}\hat{A}$ satisfies $\hat{c} \leq c-1$.

We show that $\hat{M}$ is nilpotent. For that purpose, we show that for every $m \leq N$, the sum of determinants of principal minors of size m is zero. So let $m \leq N$. If every principal minor of size m of B has determinant zero, then every principal minor of size m of $\hat{M}$ has determinant zero as well.

So assume that B has a principal minor of size m that does not have determinant zero. By lemma 6.5.2, we obtain that

$$R\big((A_1x)^{d-1}, \ldots, (A_rx)^{d-1}, (A_{r+1}x)^{d-1}, \ldots, (A_Nx)^{d-1}\big) = 0$$

for some nonzero $R \in \mathbb{C}[y_1, y_2, \ldots, y_N]$. In fact, this is the same R as above, and (7.26) and (7.27) are satisfied for this R as well.

Furthermore, by construction of $\hat{A}$ and the proof of lemma 6.5.2, in which R depends on B only, one can see that the sum of determinants of principal minors of size m of $\hat{M}$ is equal to

$$R\big((\hat{A}_1\hat{x})^{d-1}, \ldots, (\hat{A}_r\hat{x})^{d-1}, (\hat{A}_{r+1}\hat{x})^{d-1}, \ldots, (\hat{A}_N\hat{x})^{d-1}\big)$$

which refines to

$$\tilde{R}\big((A_1x)^{d-1}, \ldots, (A_rx)^{d-1}, (A_{r+2}x)^{d-1}, \ldots, (A_Nx)^{d-1}\big) + \\ (\hat{A}_{r+1}\hat{x})^{d-1}\hat{R}\big((A_1x)^{d-1}, \ldots, (A_rx)^{d-1}, (A_{r+2}x)^{d-1}, \ldots, (A_Nx)^{d-1}\big)$$

The latter formula is equal to zero on account of (7.27) and (7.26), so $\hat{M}$ is indeed nilpotent.

Assume $\mathrm{rk}A \ge N-3$. Then $\hat{c} = c-1 \le 2$. By induction on c, it follows from i) that either $\hat{M}$ is symmetrically triangularizable or $d = 3$ and $\mathrm{rk}\hat{A} = N-2$. If $\hat{M}$ is symmetrically triangularizable, the $\hat{M}$ and hence B and M as well have property E_n on account of proposition 7.5.2, whence M is symmetrically triangularizable by the same proposition.

This contradicts the assumptions, so the case $d = 3$ and $\mathrm{rk}\hat{A} = N-2$ applies. Furthermore, we obtain $DN(\hat{A}) = 4$ by i), whence the claim that A contains four nonzero rows such that the rank of the matrix of these rows is equal to 2 follows. This gives the desired result.

Notice that i) and iii) refer to each other, but iii) commits induction on c before using i). So there is no cyclic reasoning. □

The claim that $d = 3$ in i) of theorem 7.7.2 was obtained by H. Tong. It is in fact [11, Lm. 4.5]. This lemma was used by R. Willems as [54, Lm. 3.6] to obtain $d = 4$ in [54, Th. 1.2 (ii)] (his d is one bigger than ours). Proposition 7.6.5 and corollary 7.7.3 together are a generalization of [54, Th. 1.2].

Corollary 7.7.3. *Assume $h = \sum_{i=1}^{N}(A_ix)^{d+1}$, where $d \ge 2$ and the rows of A are pairwise independent and isotropic. Assume in addition that $\mathcal{H}h$ is nilpotent.*
Then h and A are like in ii) of corollary 7.6.4 in the following cases:

i) $d \ge 3$ *and* $\mathrm{rk}A \ge N-2$,

ii) $d \ge 4$ *and* $\mathrm{rk}A \ge N-3$.

In particular, the inequalities in i) and ii) respectively are equalities in case $AA^{\mathrm{t}} \ne 0$.

Proof. The case $\mathrm{rk} A \ge N - 1$ follows from i) of corollary 7.6.4, so assume $N-3 \le \mathrm{rk} A \le N-2$. If $AA^{\mathrm{t}} = 0$, then we are done. If A has a row A_i that is independent of the other rows of A, such that $A_i A^{\mathrm{t}} \ne 0$, then we obtain the desired result by way of ii) of corollary 7.6.4.

So it suffices to show that either $AA^{\mathrm{t}} = 0$ or A has a row A_i that is independent of the other rows of A, such that $A_i A^{\mathrm{t}} \ne 0$. If row A_i is independent of the other rows of A, but $A_i A^{\mathrm{t}} = 0$, then we can remove the term $(A_i x)^{d+1}$ by way of proposition 7.1.4 and obtain the desired result by induction on N. So assume that $DN(A) = N$.

i) Assume $d \ge 3$ and $\mathrm{rk} A = N-2$. If $N \le 4$, then $\mathrm{rk} A = N - 2 \le 2$, and $AA^{\mathrm{t}} = 0$ by proposition 6.4.6. So assume $N \ge 5$. Then $DN(A) = 4 < N$ on account of i) of theorem 7.7.2. This contradicts $DN(A) = N$, as desired.

ii) Assume $d \ge 4$ and $N-3 \le \mathrm{rk} A \le N-2$. If $\mathrm{rk} A = N-2$, then i) gives the desired result. So assume $\mathrm{rk} A = N-3$. If $N \ge 7$, then $DN(A) \le 6$ on account of ii) of theorem 7.7.2, which contradicts $DN(A) = N$. So assume $N \le 6$. Then $\mathrm{rk} A = N - 3 \le 3$, whence $\mathrm{rk} \mathcal{H} h \le 3$ as well. By v) of theorem 5.8.6, ∇h is linearly triangularizable, and ii) of corollary 7.6.4 gives by way of $DN(A) = N$ that $AA^{\mathrm{t}} = 0$, as desired. □

Corollary 7.7.4. *Assume* $(Ax)^{*d}$ *is reduced with a nilpotent Jacobian and* $d \ge 2$. *Put*

$$m := \mathrm{cork} A - \#\{i \mid A_i = 0\}$$

Then $(Ax)^{*d}$ *is symmetrically triangularizable in the following cases:*

i) $m \le \min\{d-2, 3\}$,

ii) $m \le \min\{d-1, 3\}$ *and* $DN(A) \ne \mathrm{cork} A + 2$.

Proof. One can easily verify that the case $m \le 1$ follows from corollary 7.7.4. So assume $2 \le m \le 3$. From v) of theorem 7.5.3, i), ii) and iii) of theorem 7.4.4, and iii) of theorem 7.4.3, it follows that we may assume that $DN(A) = n$.

In case $\text{rk}A \le 1$, there is at most one component of $(Ax)^{*d}$ that is nonzero and the result follows easily, so assume that $\text{rk}A \ge 2$. Then

$$DN(A) = n = \text{cork}A + \text{rk}A \ge \text{cork}A + 2$$

Let $\hat{A}$ be a matrix one obtains from A by replacing zero rows by independent rows and $B := dA$. Then $\mathcal{J}(Ax)^{*d} = (\hat{A}x)^{*(d-1)} \cdot B$ and $\text{cork}\hat{A} = m$.

So $\text{rk}\hat{A} = n - m$. In case $m = 2$, the case $DN(A) \ge 5$ follow from i) of theorem 7.7.2. If $m = 2$, then the case $DN(A) \ge 6$ follows from i) of theorem 7.7.2. If $m = 3$, then the case $DN(A) \ge 7$ follows from ii) of theorem 7.7.2. Since $m \in \{2, 3\}$, we may assume that $n = DN(A) \le m + 3$.

Since $\text{cork}A \ge m$, we obtain $DN(A) - \text{cork}A \le 3$. So if $d \le m + 2$, then i) of theorem 7.5.4 gives the desired result. So assume $d \le m + 1$. Then $m \ge d - 1$ and we get a contradiction in i), so we may assume the condition of ii) that $DN(A) \ne \text{cork}A + 2$.

Since $DN(A) \ge \text{cork}A + 2$, it follows that $\text{cork}A + 3 \le DN(A) \le m + 3$. So $m \le \text{cork}A \le m$. It follows that $m = \text{cork}A$ and $n = DN(A) = \text{cork}A + 3$. From corollary 7.4.5, it follows that $(Ax)^{*d}$ satisfies DP. Since $(Ax)^{*d}$ is reduced, we obtain that one row of A is zero, whence $m < \text{cork}A$. Contradiction. □

Besides the tameness results for the case $m \le d - 1$, we have proved i) of theorem 7.1.1 as well in corollary 7.7.4. In order to prove ii) and the case $m \le d - 1$ of iii) of theorem 7.1.1, notice that ii) and iii) of theorem 7.1.1 imply that $m \le 3$ and $\text{cork}A \le d$. So assume that $m \le d - 1$, $m \le 3$ and $\text{cork}A \le d$.

From corollary 7.7.4 the case $DN(A) \ne \text{cork}A + 2$ follows. So the case $DN(A) - \text{cork}A = 2$ remains. The case $d \ge \text{cork}A + 1$ follows from i) of corollary 7.5.4, so assume $d \le \text{cork}A$. Since $\text{cork}A \le d$, we obtain $d = \text{cork}A$ and $DN(A) = \text{cork}A + 2 = d + 2$. Now corollary 7.5.5 gives the case $d \ge 2$ of ii). The case $d = 1$ is easy, as desired.

7.8 Proof of the case $m = 3 = d$ of theorem 7.1.1

Just as in the previous sections, the case $DN(A) \le \text{cork}A + \max\{3, 7 - d\}$ of the tameness result follows from proposition 7.3.4. Assume $m = 3 = d$ and $\mathcal{J}(Ax)^{*d}$ is nilpotent. In case $DN(A) \ge \text{cork}A + \max\{4, 8 - d\}$, we have $DN(A) \ge \text{cork}A + 8 - d = \text{cork}A + 5$. We distinguish two cases.

- $DN(A) \neq \operatorname{cork} A + 2$ and A has four nonzero rows such that the rank of the matrix of these rows is equal to 2.
  Notice that there are two independent relations between these four rows. Since $m = 3$, there exists three independent relations between the nonzero rows of A. So there is one additional independent relation. If there would be a fifth row of A being dependent of the above four rows of A, then $DN(A)$ would be equal to
  $$5 + \#\{i \mid A_i = 0\} = 5 + \operatorname{cork} A - m = \operatorname{cork} A + 2$$
  So the additional independent relation between the rows of A involves at least two new rows of A.

  These two rows of A are individually independent of the other $n - 2$ rows of A, but together dependent of the other $n-2$ rows of A. Assume that one of both rows can be replaced by the zero row without affecting the nilpotency of the Jacobian of $(Ax)^{*d}$. Then by proposition 7.3.1, we can replace this row of A by the zero row by way of an elementary polynomial map from the left, provided the resulting map is invertible. But the resulting map is even tame, because m decreases to $2 = d - 1$ when the above row of A is replaced by the zero row. This was proved in the previous section. It follows that $x + (Ax)^{*d}$ is tame.

  So it suffices to prove the claim that A has two rows, say A_1 and A_{r+1}, that are individually independent of the other $n - 2$ rows of A, but together dependent of the other $n - 2$ rows of A, and that replacing A_{r+1} by the zero row does not affect the nilpotency of the Jacobian of the power linear map at hand. This follows by substituting $x_{n+1} = -A_{r+1}x$ in the power linear quasi-Jacobian $\hat{M}$ in the proof of iii) of theorem 7.7.2, with $B = dA$.

- $DN(A) \geq \operatorname{cork} A + 5$ and A does not have four nonzero rows such that the rank of the matrix of these rows is at most 2.
  In this case, we even have symmetrical linear triangularizability, provided $(Ax)^{*d}$ is reduced. See corollary 7.8.3 below.

Since lemma 7.7.1 in the previous section can only be utilized in case $m \leq d - 1$, we need another lemma with a technical proof.

Lemma 7.8.1. *Let $R \in \mathbb{C}[y_2, y_3, \ldots, y_{r+4}]$ be nonzero with $\deg_{y_i} R \leq 1$ for all $i \geq 2$, such that*
$$R\big(z_2^2, \ldots, z_r^2, (\tilde{C}_1(z_1, \ldots, z_r))^2, \ldots, (\tilde{C}_4(z_1, \ldots, z_r))^2\big) = 0 \qquad (7.28)$$

Assume furthermore that $\tilde{C}_{11}\tilde{C}_{21}\tilde{C}_{31}\tilde{C}_{41} \neq 0$. *Then either the rows* $e_1^{\mathrm{t}}, \tilde{C}_1, \tilde{C}_2, \tilde{C}_3, \tilde{C}_4$ *are dependent or there exists a* $j \geq 2$ *such that at most two columns* $\tilde{C}e_i$ *of* $\tilde{C}$ *are not dependent of* $\tilde{C}e_1$ *and* $\tilde{C}e_j$.

Proof. Without loss of generality, we may assume that $\tilde{C}_{11} = \tilde{C}_{21} = \tilde{C}_{31} = \tilde{C}_{41} = 1$. If the i-th column of $\tilde{C}$ is zero, then the argument z_i^2 of R is algebraically independent of the other arguments of R. It follows that we can argue with either $\frac{\partial}{\partial y_i} R$ or $R|_{y_i=0} = R - y_i \frac{\partial}{\partial y_i} R$ instead of R, since one of $\frac{\partial}{\partial y_i} R$ and $R|_{y_i=0} = R - y_i \frac{\partial}{\partial y_i} R$ is nonzero. So we may assume that all columns of $\tilde{C}$ are nonzero. Furthermore, since the arguments of R are homogeneous of degree 2, we can replace R by any of its nonzero homogeneous components, so we may assume that R is homogeneous.

Let $k = \deg_{y_{r+1}, y_{r+2}, y_{r+3}, y_{r+4}} R$. Since $z_2^2, \ldots, z_r^2$ are algebraically independent, it follows that $k \geq 1$. Put $\tilde{z} = (z_1, z_2, \ldots, z_r)$. Looking at the coefficient of z_1^{2k} of

$$R\big(z_2^2, \ldots, z_r^2, (\tilde{C}_1 \tilde{z})^2, \ldots, (\tilde{C}_4 \tilde{z})^2\big) = 0$$

we get (7.29) below with $z_2^2, z_3^2, \ldots, z_r^2$ instead of $y_2, y_3, \ldots, y_r$, but

$$\sum_{r+1 \leq i_1 < \cdots < i_k \leq r+4} \left(\frac{\partial}{\partial y_{i_1}} \cdots \frac{\partial}{\partial y_{i_k}} R \right) = 0 \tag{7.29}$$

is satisfied as well, because $z_2^2, \ldots, z_r^2$ are algebraically independent. If $4 \mid k$, then the left hand side of (7.29) would consist of exactly one nonzero term. So $1 \leq k \leq 3$.

Assume that $e_1^{\mathrm{t}}, \tilde{C}_1, \tilde{C}_2, \tilde{C}_3, \tilde{C}_4$ are independent. We shall show that

$$\sum_{r+1 < i_2 < \cdots < i_k \leq r+4} \frac{\partial}{\partial y_{r+1}} \frac{\partial}{\partial y_{i_2}} \cdots \frac{\partial}{\partial y_{i_k}} R = 0 \tag{7.30}$$

Take a term $\mu y_{j_1} \cdots y_{j_l}$ on the left hand side of (7.30), such that $\mu \in \mathbb{C}^*$. By definition of k, $2 \leq j_1 < \cdots < j_l \leq r$.

Since $l = \deg R - k$, we see that terms of (7.28) have a degree that exceeds that of $t := z_1^{2k-1} z_{j_1}^2 \cdots z_{j_l}^2$ by one. Due to the $2k-1$ factors z_1 in t, terms of (7.28) that are divisible by t correspond to terms of R that have degree k with respect to $y_{r+1}, y_{r+2}, y_{r+3}, y_{r+4}$. Furthermore, those terms of R are divisible by $y_{j_1} \cdots y_{j_l}$, because at most one factor z_i with $i \neq 1$ comes from the factor in $\mathbb{C}[y_{r+1}, y_{r+2}, y_{r+3}, y_{r+4}]$ of such a term of R.

So if we select the terms that are divisible by t but not by $z_1 t$ in (7.28), we obtain

$$\sum_{r+1\leq i_1<i_2<\cdots<i_k\leq r+4} \sum_{j=1}^{k} 2\lambda_{i_1,i_2,\ldots,i_k}(\tilde{C}_{i_j}\tilde{z} - z_1)t = 0$$

where $\lambda_{i_1,\ldots,i_k}$ is the coefficient of $y_{j_1}\cdots y_{j_l} y_{i_1}\cdots y_{i_k}$ of R. If we divide the above formula by t, then we obtain a linear combination of $z_1, \tilde{C}_1\tilde{z}, \tilde{C}_2\tilde{z}, \tilde{C}_3\tilde{z}, \tilde{C}_4\tilde{z}$. Since these linear forms are independent, the coefficient of $\tilde{C}_1\tilde{z}$ is zero. So

$$\sum_{r+1<i_2<\cdots<i_k\leq r+4} \sum_{j=1}^{1} 2\lambda_{r+1,i_2,\ldots,i_k} = 0$$

But this is exactly the equality $2\mu = 0$. Contradiction, so (7.30) is satisfied.

In order to show that there at most three columns $\tilde{C}e_i$ of $\tilde{C}$ that are not dependent of $\tilde{C}e_1$ and $\tilde{C}_j$ for some $j \neq i$, we first show that $k = 2$. Since $k \neq 0$ and $k \neq 4$, two cases remain:

- $k = 1$.
  Now (7.30) comes down to $\frac{\partial}{\partial y_{r+1}} R = 0$, but with symmetric variants of (7.30), we have
  $$\frac{\partial}{\partial y_{r+1}} R = \frac{\partial}{\partial y_{r+2}} R = \frac{\partial}{\partial y_{r+3}} R = \frac{\partial}{\partial y_{r+4}} R = 0$$
  This contradicts $k = 1$.

- $k = 3$.
  Taking the difference of (7.29) and (7.30), we get $\frac{\partial}{\partial y_{r+2}} \frac{\partial}{\partial y_{r+3}} \frac{\partial}{\partial y_{r+4}} R = 0$. Symmetric variants of this equality can be obtained with symmetric variants of (7.30), and we get a contradiction to $k = 3$ in a similar manner as in the case $k = 1$.

So $k = 2$. The case $k = 2$ is a lot harder than the cases above. Furthermore, a contradiction with the assumption that $e_1^{\mathrm{t}}, \tilde{C}_1, \tilde{C}_2, \tilde{C}_3, \tilde{C}_4$ are independent cannot be obtained in general. So we must show that there exists a $j \geq 2$ such that at most two columns $\tilde{C}e_i$ of $\tilde{C}$ are not dependent of $\tilde{C}e_1$ and $\tilde{C}e_j$.

Let us write down (7.30) and one of its symmetric variants.

$$\begin{aligned}
\frac{\partial}{\partial y_{r+1}}\left(\frac{\partial}{\partial y_{r+2}} + \frac{\partial}{\partial y_{r+3}} + \frac{\partial}{\partial y_{r+4}}\right) R &= 0 \\
\frac{\partial}{\partial y_{r+2}}\left(\frac{\partial}{\partial y_{r+1}} + \frac{\partial}{\partial y_{r+3}} + \frac{\partial}{\partial y_{r+4}}\right) R &= 0
\end{aligned}$$

Adding these equalities and subtracting (7.29), we get

$$S := \frac{\partial}{\partial y_{r+1}} \frac{\partial}{\partial y_{r+2}} R = \frac{\partial}{\partial y_{r+3}} \frac{\partial}{\partial y_{r+4}} R$$

which has symmetric variants

$$T := \frac{\partial}{\partial y_{r+1}} \frac{\partial}{\partial y_{r+3}} R = \frac{\partial}{\partial y_{r+2}} \frac{\partial}{\partial y_{r+4}} R$$

and

$$U := \frac{\partial}{\partial y_{r+1}} \frac{\partial}{\partial y_{r+4}} R = \frac{\partial}{\partial y_{r+2}} \frac{\partial}{\partial y_{r+3}} R$$

From (7.30), it follows that $U = -(S + T)$.

Since $k = 2$, it follows that either $S \neq 0$ or $T \neq 0$. Assume without loss of generality that $S \neq 0$, and let $\sigma y_{j_1} \cdots y_{j_l}$ be a term of S, such that $\sigma \in \mathbb{C}^*$. Let $\tau y_{j_1} \cdots y_{j_l}$ be the corresponding term of T. Since $l = \deg R - k$, we see that terms of (7.28) have a degree that exceeds that of $z_1^{2k-2} z_{j_1}^2 \cdots z_{j_l}^2$ by two. Now let us look at an arbitrary term t of (7.28) that is divisible by $z_1^{2k-2} z_{j_1}^2 \cdots z_{j_l}^2$ and not a square.

Assume that s is the corresponding term of R. In case the degree with respect to $y_{r+1}, y_{r+2}, y_{r+3}, y_{r+4}$ of s is less than k, then the factor z_1^{2k-2} of t corresponds to the product of factors y_{r+i} with $i \geq 1$ of s, whence t/z_1^{2k-2} corresponds to a product $y_{j_1} \cdots y_{j_l} y_{j_{l+1}}$ with $j_{l+1} \leq r$. This contradicts that t is not a square. So the degree with respect to $y_{r+1}, y_{r+2}, y_{r+3}, y_{r+4}$ of s is equal to k. A more or less similar argument tells us that $y_{j_1} \cdots y_{j_l} \mid s$.

It is however possible that s is divisible by y_{r+1} and that the factor z_1^{2k-2} of t corresponds to the product of factors y_{r+i} with $i \geq 2$ of s. But since $y_{j_1} \cdots y_{j_l} \mid s$, it follows from (7.30) that the terms t that correspond to such an s cancel out. More generally, all terms of (7.28) that are divisible by z_1^{2k-2}, for which the factor z_1^{2k-1} corresponds to a product of only $k-1$ factors y_{r+i} with $i \geq 1$, cancel out.

So if we select those terms of (7.28) that are divisible by $z_1^{2k-2} z_{j_1}^2 \cdots z_{j_l}^2$ and not a square, we see that

$$\begin{aligned} 2^2\sigma(\tilde{C}_1\tilde{z} \cdot \tilde{C}_2\tilde{z} + \tilde{C}_3\tilde{z} \cdot \tilde{C}_4\tilde{z}) \ +& \\ 2^2\tau(\tilde{C}_1\tilde{z} \cdot \tilde{C}_3\tilde{z} + \tilde{C}_2\tilde{z} \cdot \tilde{C}_4\tilde{z}) \ -& \\ 2^2(\sigma+\tau)(\tilde{C}_1\tilde{z} \cdot \tilde{C}_4\tilde{z} + \tilde{C}_2\tilde{z} \cdot \tilde{C}_3\tilde{z}) \ &\in \ \mathbb{C}[z_1^2, z_2^2, \ldots, z_r^2] \end{aligned}$$

By substituting $z_1 = 0$ and dividing by 4, we obtain

$$\sigma(a_1 - a_3)(a_2 - a_4) + \tau(a_1 - a_2)(a_3 - a_4) \in \mathbb{C}[z_2^2, \ldots, z_r^2]$$

where $a_i := \tilde{C}_i\tilde{z} - z_1 = \tilde{C}_i\tilde{z}|_{z_1=0}$ for all i. So if we define $b_i := a_i - a_{i+1}$, then we obtain

$$\sigma(b_1 + b_2)(b_2 + b_3) + \tau b_1 b_3 \in \mathbb{C}[z_2^2, \ldots, z_r^2] \tag{7.31}$$

Since the left hand side of (7.31) is a polynomial in three linear forms, it follows that the Hessian of the left hand side of (7.31) has rank ≤ 3. The right hand side tells us that this Hessian is a diagonal matrix, so we may assume without loss of generality that the left hand side of (7.31) is a linear combination of z_2^2, z_3^2 and z_4^2.

If column $\tilde{C}e_i$ of $\tilde{C}$ is dependent of $\tilde{C}e_1$ for all $i \geq 5$, then we are done, because in that case any column $\tilde{C}e_j$ with $j \in \{2, 3, 4\}$ has the desired properties. So assume without loss of generality that $\tilde{C}e_5$ is not dependent of $\tilde{C}e_1$. Now differentiate the left hand side of (7.31) (which is a linear combination of z_2^2, z_3^2 and z_4^2) to z_5, to obtain

$$\begin{aligned} 0 &= \sigma\big((c_1 + c_2)(b_2 + b_3) + (c_2 + c_3)(b_1 + b_2)\big) + \tau\big(c_1 b_3 + c_3 b_1\big) \\ &= \big(\sigma(c_2 + c_3) + \tau c_3\big)b_1 + \sigma\big(c_1 + 2c_2 + c_3\big)b_2 + \big(\sigma(c_1 + c_2) + \tau c_1\big)b_3 \end{aligned} \tag{7.32}$$

where $c_i = \frac{\partial}{\partial z_5} b_i \in \mathbb{C}$ for all i. Since b_1, b_2, b_3, a_4, z_1 span $\mathbb{C}\, z_1 + \mathbb{C}\, \tilde{C}_1\tilde{z} + \mathbb{C}\, \tilde{C}_2\tilde{z} + \mathbb{C}\, \tilde{C}_3\tilde{z} + \mathbb{C}\, \tilde{C}_4\tilde{z}$, it follows that b_1, b_2, b_3 are independent linear forms. So the coefficients of b_1, b_2, b_3 of (7.32) are zero. If we solve these coefficients with respect to c_2, then we obtain

$$c_2 = -\left(1 + \frac{\tau}{\sigma}\right)c_3 = -\frac{1}{2}(c_1 + c_3) = -\left(1 + \frac{\tau}{\sigma}\right)c_1$$

If $c_1 = c_3 = 0$, then $c_2 = 0$ as well. By definition of c_i, b_i and a_i for all i, this is only possible if all entries of $\tilde{C}e_5$ are the same, which contradicts the assumption that $\tilde{C}e_5$ is not dependent of $\tilde{C}e_1$. If $c_1 = c_3 \neq 0$, then $1 + \frac{\tau}{\sigma} = \frac{1}{2} + \frac{1}{2}$, whence $\tau = 0$. If $c_1 \neq c_3$, then $1 + \frac{\tau}{\sigma} = 0$, whence $-(\sigma + \tau) = 0$. By way of symmetry considerations, we may assume without loss of generality that $\tau = 0$.

Now (7.31) comes down to

$$\sigma(b_1 + b_2)(b_2 + b_3) \in \mathbb{C}[z_2^2, \ldots, z_r^2]$$

from which the left hand side is the product of two linear forms. Now a similar argument as immediately after (7.31) tells us that we may assume that the left hand side of (7.31) is a linear combination of z_2^2 and z_3^2. The only possibility is that both

$$\tilde{C}_1\tilde{z} - \tilde{C}_3\tilde{z} = a_1 - a_3 = b_1 + b_2$$

and

$$\tilde{C}_2\tilde{z} - \tilde{C}_4\tilde{z} = a_2 - a_4 = b_2 + b_3$$

are contained in $\mathbb{C}[z_2, z_3]$. It follows that each column $\tilde{C}e_i$ of $\tilde{C}$ with $i \neq 2, 3$ is a linear combination of $(1, 0, 1, 0)$ and $(0, 1, 0, 1)$. This gives the desired result. □

Theorem 7.8.2. *Assume*

$$M := \operatorname{diag}\bigl((Ax)^{*(d-1)}\bigr) \cdot B$$

is a nilpotent power linear quasi-Jacobian of size N over $\mathbb{C}$ and $d \geq 3$. Assume in addition that the rows of A are pairwise linearly independent over $\mathbb{C}$ and that M is not *symmetrically triangularizable.*

If $\operatorname{rk} A \geq N - 3$ then either $DN(A) \leq 7$ or A has four nonzero rows such that the rank of the matrix of these rows is equal to 2.

Proof. The cases $d \geq 4$ and $\operatorname{rk} A \geq N - 2$ follow from theorem 7.7.2, so assume $d = 3$ and $r := \operatorname{rk} A = N - 3$. Without loss of generality, we assume that the rows $A_1, A_2, \dots, A_r$ of A are independent and that

$$\begin{pmatrix} A_{r+1} \\ \vdots \\ A_n \end{pmatrix} = C \cdot \begin{pmatrix} A_1 \\ \vdots \\ A_r \end{pmatrix}$$

for some $C \in \operatorname{Mat}_{3,r}(\mathbb{C})$. Since M is not symmetrically triangularizable, it follows from proposition 7.5.2 and lemma 6.5.2 that

$$R\bigl((A_1x)^2, (A_2x)^2, \dots, (A_rx)^2, (A_{r+1}x)^2, \dots, (A_Nx)^2\bigr) = 0$$

for some nonzero $R \in \mathbb{C}[y_1, y_2, \dots, y_N]$. By definition of C, this is equivalent to

$$R\bigl(z_1^2, z_2^2, \dots, z_r^2, (C_1\tilde{z})^2, (C_2\tilde{z})^2, (C_3\tilde{z})^{d-1}\bigr) = 0$$

where $\tilde{z} = z_1, z_2, \ldots, z_r$.

From iii) of theorem 7.7.2, we obtain the case that A has two rows that are together dependent of the other $N-2$ rows of A, but individually not dependent of the other $N-2$ rows of A. So assume the opposite. From the proof of ii) of theorem 7.7.2, we obtain that C does not have columns with exactly one nonzero entry and that we may assume that $C_{11}C_{21}C_{31} \neq 0$.

Assume that $DN(A) \geq 8$. Then at least 5 columns of C are nonzero. Assume without loss of generality that the first 5 columns of C are nonzero. We distinguish two cases.

- e_1^{t}, C_1, C_2 and C_3 are independent.
  Since $\mathrm{rk}(e_1^{\mathrm{t}}, C) = 4 < 5$, there exists a k with $2 \leq k \leq 5$ such that e_k^{t} is independent of $e_1^{\mathrm{t}}, C_1, C_2, C_3$. Notice that $k \leq r$. Now define $\tilde{C} \in \mathrm{Mat}_{4,r}(\mathbb{C})$ by $\tilde{C}_1 = e_1^{\mathrm{t}} - e_k^{\mathrm{t}}$ and $\tilde{C}_{i+1} = C_i - C_{i1}e_k$. Notice that $\tilde{C}$ can be made out of the matrix (e_1^{t}, C) by elementary column operations. Hence, the rows of $\tilde{C}$ are independent. Since $e_1^{\mathrm{t}} + e_k^{\mathrm{t}}$ is independent of $e_1^{\mathrm{t}}, C_1, C_2, C_3$, it follows that e_1 is independent of the rows of $\tilde{C}$.

  Since $\tilde{C}$ can be made out of (e_1^{t}, C) by elementary column operations, it follows from lemma 6.5.2 that there exists a relation $R \in \mathbb{C}[y_1, y_2, \ldots, y_{r+3}]$ with $\deg_{y_i} R \leq 1$ for all i such that

  $$R\big(z_2^2, \ldots, z_r^2, (\tilde{C}_1\tilde{z})^2, \tilde{C}_2\tilde{z})^2, (\tilde{C}_3\tilde{z})^2, (\tilde{C}_4\tilde{z})^2\big) = 0$$

  where $\tilde{z} = z_1, z_2, \ldots, z_r$. From lemma 7.8.1, it follows that there exists a $j \geq 2$ such that at most two columns $\tilde{C}e_i$ of $\tilde{C}$ are not dependent of $\tilde{C}e_1$ and $\tilde{C}e_j$.

  Assume without loss of generality that $j = 4$ and that $\tilde{C}e_5$ is dependent of $\tilde{C}e_4$ and $\tilde{C}e_1$. This dependence property is not affected by adding the first column of $\tilde{C}$ to the k-th column of $\tilde{C}$. So the fifth column of (e_1^{t}, C) is dependent of the fourth and the first, but since the first column is clearly independent, the fourth and fifth column of (e_1^{t}, C) are dependent.

  It follows that A_4 and A_5 are together dependent of the other $N-2$ rows of A, but individually not dependent of the other $N-2$ rows of A. Contradiction, so $DN(A) \leq 7$.

- e_1^{t}, C_1, C_2, C_3 are dependent.
  From the case above, it follows that we may assume that

$$C_{11}C_{21}C_{31} \neq 0 \implies e_1^{\mathrm{t}}, C_1, C_2, C_3 \text{ are dependent}$$

  remains valid if we conjugate M and A by a permutation such that the first r rows of A stay independent. So we have essentially the same situation as ii) of theorem 7.7.2, because there we obtained the above implication and its symmetric variants by lemma 7.7.1. So by the proof of ii) of theorem 7.7.2, we obtain $DN(A) \leq 6$, as desired. □

Corollary 7.8.3. *Assume $(Ax)^{*d}$ is reduced with a nilpotent Jacobian. Put*

$$m := \mathrm{cork} A - \#\{i \mid A_i = 0\}$$

Assume that A does not have four nonzero rows such that the matrix of these rows has rank 2. If $m \leq 3 \leq d$ and $DN(A) \geq \mathrm{cork} A + 5$, then $(Ax)^{*d}$ is symmetrically triangularizable.

Proof. The case $m \leq d-1$ follows from corollary 7.8.3, so assume $m = 3 = d$. Let $\hat{A}$ be a matrix one obtains from A by replacing zero rows by independent rows and $B := dA$. Then $\mathcal{J}(Ax)^{*d} = (\hat{A}x)^{*(d-1)} \cdot B$ and $\mathrm{cork}\hat{A} = m = 3$.
Assume $DN(A) \geq \mathrm{cork} A + 5$. Then

$$DN(\hat{A}) = DN(A) - \#\{i \mid A_i = 0\} = DN(A) + m - \mathrm{cork} A \geq m + 5 = 8$$

So by theorem 7.8.2, we obtain that $(\hat{A}x)^{*(d-1)} \cdot B$ is symmetrically triangularizable. This gives the desired result. □

We complete the proof of theorem 7.1.1 now. We only need to show the case $m = d$ of iii) of theorem 7.1.1. So assume $m = d$ and $\mathrm{cork} A \leq 3 \leq d$. Since $m \leq \mathrm{cork} A$ by definition of m, it follows that $d = m \leq \mathrm{cork} A \leq 3 \leq d$, so $d = m = \mathrm{cork} A = 3$. Assume without loss of generality that $(Ax)^{*d}$ is reduced. We will derive a contradiction.
Since $m = \mathrm{cork} A$, it follows that all rows of A are nonzero. So $(Ax)^{*d}$ does not satisfy DP, for $(Ax)^{*d}$ is reduced. By corollary 7.4.5, we obtain that $DN(A) - 3 = DN(A) - \mathrm{cork} A > 7 - d = 7 - 3$, so $DN(A) \geq 8 = \mathrm{cork} A + 5$. From corollary 7.8.3, it follows that A contains four nonzero rows such that the matrix of these rows has rank 2.

Just as in the case that $DN(A) \neq \mathrm{cork}A + 2$ and A has four nonzero rows such that the rank of the matrix of these rows is equal to 2 in the tameness result at the beginning of this section, we obtain that A has two rows, say A_1 and A_{r+1}, that are individually independent of the other $n-2$ rows of A, but together dependent of the other $n-2$ rows of A.

Let $\hat{A}$ and $\hat{x}$ be as in the proof of theorem 7.7.2 and $\hat{M} := \mathrm{diag}((\hat{A}\hat{x})^{*(d-1)}) \cdot B$ be the nilpotent power linear quasi-Jacobian $\hat{M}$ in the proof of iii) of theorem 7.7.2, with $B = dA$. Notice that the nilpotency of $\hat{M}$ is not affected by adding an arbitrary column to the right first and adding a zero row after that. It follows that $\mathcal{J}_{\hat{x}}((\hat{A}\hat{x})^{*d}, 0)$ is nilpotent.

Since $\mathrm{rk}\hat{A} = n - 2$, the relations between the rows of $\hat{A}$ correspond to those between the four nonzero rows of A that have two independent relations. So $DN(\hat{A}) = 4$. Since $DN((\hat{A}, 0)) = 5$ and $d = \mathrm{cork}(\hat{A}, 0) = 3$, it follows from corollary 7.5.5 that $((\hat{A}\hat{x})^{*d}, 0)$ is linearly triangularizable.

By way of equivalence of linear triangularizability and strong nilpotency of the Jacobian, which is shown in [24, §7.4], one can see that $(Ax)^{*d}$ is linearly triangularizable as well. So the components of $(Ax)^{*d}$ are dependent over $\mathbb{C}$. Since $m = \mathrm{cork}A$, this is only possible if $(Ax)^{*d}$ is not reduced. Contradiction, so $(Ax)^{*d}$ is (ditto) linearly triangularizable.

Corollary 7.8.4. *Assume* $h = \sum_{i=1}^{N} (A_i x)^{d+1}$*, where* $d \geq 3$ *and the rows of* A *are pairwise independent and isotropic. Assume in addition that* $\mathcal{H}h$ *is nilpotent.*

Assume $\mathrm{rk}A \geq N - 3$*. Then* $x + \nabla h$ *is not only tame, but also linearly triangularizable. If* $AA^{\mathrm{t}} \neq 0$*, then* AA^{t} *is of the form of (7.16) for some permutation matrix* P*. Furthermore,* $\mathrm{rk}C = 1$*,* $DN(A) \geq 4$ *and* C *has* $r \in \{4, 5, 6\}$ *rows for some* $r \in \{4, DN(A)\}$*, all of which are nonzero.*

Proof. The cases $d \geq 4$ and $\mathrm{rk}A \geq N - 2$ follows from corollary 7.7.3, so assume that $d = 3$ and $\mathrm{rk}A = N - 3$. Assume that $AA^{\mathrm{t}} \neq 0$. We first show that $x + \nabla h$ is tame and that AA^{t} is of the desired form. We distinguish four cases.

- A has two rows that are together dependent, but individually independent of the other $N - 2$ rows of A.
  Assume without loss of generality that those rows are A_{N-1} and A_N. Let $\hat{A}$ be the matrix one obtains from A by adding a column e_N and

another column $\mathrm{i}e_N$ to the right and define $\hat{x} = (x_1, x_2, \ldots, x_{n+2})$. Then $\mathrm{rk}\hat{A} = N - 2$ and $\hat{A}\hat{A}^{\mathrm{t}} = AA^{\mathrm{t}}$.

From the proof of iii) of theorem 7.7.2, we can extract that $(\hat{A}\hat{x})^{*(d-1)} \cdot AA^{\mathrm{t}}$ is nilpotent. By (6.7) and $\hat{A}\hat{A}^{\mathrm{t}} = AA^{\mathrm{t}}$, we obtain that the Hessian of $\hat{h} := \sum_{i=1}^{N} (\hat{A}_i \hat{x})^d$ is nilpotent. From i) of corollary 7.7.3, it follows that $\hat{h}$ and $\hat{A}$ are like h and A in ii) of corollary 7.6.4. Since $\hat{A}\hat{A}^{\mathrm{t}} = AA^{\mathrm{t}}$, we obtain that AA^{t} is like in ii) of corollary 7.6.4.

By substituting $x_{n+1} = -A_N x - \mathrm{i}x_{n+2}$ in the Hessian of $\hat{h}$, we get the Hessian of $\tilde{h} := \sum_{i=1}^{N-1} (A_i x)^d$, so $\mathcal{H}\tilde{h}$ is nilpotent. Since $\mathrm{rk}(A_1, A_2, \ldots, A_{N-1}) = N - 3 = (N-1) - 2$, it follows from ii) of theorem 7.1.4 and induction on N that $x + \nabla h$ is tame.

- A has a row A_i that is independent of the other rows of A, such that $A_i A^{\mathrm{t}} = 0$.
  Assume without loss of generality that $i = N$. Then we can remove the term $(A_N x)^{d+1}$ by way of proposition 7.1.4 and we obtain by induction on N that $x + \nabla h$ is tame and that AA^{t} is of the desired form.

- A has a row A_i that is independent of the other rows of A, such that $A_i A^{\mathrm{t}} \neq 0$.
  Assume again without loss of generality that $i = N$. Since $A_N A^{\mathrm{t}} \neq 0$, we obtain in a similar manner as in the proof of ii) of corollary 7.6.4 that the components of $(Ax)^{*2} = (Ax)^{*(d-1)}$ are linearly dependent over $\mathbb{C}$.

  Let I be the set of indices i for which $(A_i x)^2$ has a nonzero coefficient in the above linear relation and $\tilde{A}$ be the matrix consisting of the rows A_i of A for which $i \in I$. From ii) of lemma 6.6.8, with a generic linear combination of the $\frac{\partial}{\partial z_i}$'s instead of $\frac{\partial}{\partial x_n}$, it follows that $\tilde{N} := \#I \geq d - 3 + 2\mathrm{rk}\tilde{A} = 2\mathrm{rk}\tilde{A}$, so $\tilde{N} - \mathrm{rk}\tilde{A} \geq \mathrm{rk}\tilde{A}$. Since $\mathrm{rk}A = N - 3$, it follows that $\tilde{N} - \mathrm{rk}\tilde{A} \leq N - \mathrm{rk}A = 3$, so

$$\mathrm{rk}\tilde{A} \leq \tilde{N} - \mathrm{rk}\tilde{A} \leq 3 \tag{7.33}$$

  Due to the assumption that the rows of $\tilde{A}$ are pairwise independent, we obtain

$$(N - \mathrm{rk}A) - (\tilde{N} - \mathrm{rk}\tilde{A}) \leq 3 - 2 = 1 \tag{7.34}$$

  We shall show that we may assume that for each row A_j of A that is dependent of the other rows of A, A_j is contained in the row space of

$\tilde{A}$. So assume that A_j is dependent of the other rows of A, but A_j is not contained in the row space of $\tilde{A}$. Then A_j is dependent of a set of other rows of A, which contains a row A'_j with $j' \notin I$. From (7.34), it follows that A_j and $A_{j'}$ are together dependent of the other $N-2$ rows of A but individually independent of the other rows of A, and we obtain from the first case that $x + \nabla h$ is tame and AA^{t} has the desired form.

So we may assume that for each row A_j of A that is dependent of the other rows of A, A_j is contained in the row space of $\tilde{A}$. Let J be the set of indices j such that A_j is dependent of the rows of $\tilde{A}$. From ii) of theorem 7.6.1, it follows that the Hessian of $\sum_{j \in J}(A_j x)^d$ is nilpotent.

We shall show that $\tilde{A}\tilde{A}^{\mathrm{t}} = 0$. The case $\mathrm{rk}\tilde{A} \le 2$ follows from proposition 7.6.5 and the nilpotency of $\mathcal{H}\sum_{j \in J}(A_j x)^d$. The case $\mathrm{rk}\tilde{A} = 3$ follows from lemma 7.8.5 below, because by (7.33), $\#I = \tilde{N} = 6$, as desired.

It follows from i) of theorem 7.6.1 that AA^{t} is of the form of (7.16) for some permutation matrix P. Assume without loss of generality that $P = I_N$. Choosing r appropriate, we can obtain that AA^{t} is of the form of (7.16) for some permutation matrix P, such that all rows of C are nonzero. Furthermore, $x + \nabla h$ is tame.

We show that $r \in \{4, DN(A)\}$. In case $\mathrm{rk}\tilde{A} = 2$ and $r \ge 5$, we have $r = 5 = DN(A)$, because for each row A_j of A that is dependent of the other rows of A, A_j is contained in the row space of $\tilde{A}$. In case $\mathrm{rk}\tilde{A} = 3$ and $r \ge 5$, we have $r = 6 = DN(A)$, because $\tilde{A}$ has 6 rows on account of $\dim\ker\tilde{A} \ge \mathrm{rk}\tilde{A}$ In case $r \le 4$, we have $r = 4$ for a similar reason. So $r \in \{4, DN(A)\}$.

We show that $\mathrm{rk}C = 1$. The case $\mathrm{rk}\tilde{A} = 2$ follows in a similar manner as $\mathrm{rk}C = 1$ in the proof of ii) of corollary 7.6.4. The case $\mathrm{rk}\tilde{A} = 3$ follows from lemma 7.8.5 below, because by (7.33), $\#I = \tilde{N} = 6$, as desired.

- None of the above.
  Then $DN(A) = N$. From theorem 7.8.2, it follows that either $N = DN(A) \le 7$ or A has four nonzero rows such that the rank of the matrix of these rows is equal to 2. In the latter case, we even have $N = DN(A) \le 5$, because A does not have two rows that are together

dependent, but individually independent of the other $N-2$ rows of A. So $N \leq 7$.

It follows that $\mathrm{rk}A \leq 4$. We shall show that ∇h is linearly triangularizable in case $\mathrm{rk}A \leq 4$. From proposition 5.5.1, it follows that we may assume that for each $i \leq N$, $A_i x$ is dependent of either

$$x_1 + \mathrm{i}x_5, x_2 + \mathrm{i}x_6, x_3 + \mathrm{i}x_7, x_4 + \mathrm{i}x_8$$

or

$$x_1, x_2 + \mathrm{i}x_5, x_3 + \mathrm{i}x_6, x_4 + \mathrm{i}x_7$$

or

$$x_1, x_2, x_3, x_4, x_5, x_6$$

In the first case, ∇h is linearly triangularizable on account of proposition 6.4.7. In the second case, we get the same conclusion, because the rows of A must be isotropic and therefore $Ae_1 = 0$. In the last case, $h \in \mathbb{C}[x_1, x_2, x_3, x_4, x_5, x_6]$ is degenerate, so the columns of $\mathcal{H}h$ and hence the rows as well are dependent over $\mathbb{C}$. So by iii) of theorem 5.8.6, ∇h is linearly triangularizable.

Since ∇h is linearly triangularizable, we can use the same trick as at the end of the proof of ii) of corollary 7.6.4, to reduce to the previous case that A has a row A_i that is independent of the other rows of A, such that $A_i A^{\mathrm{t}} \neq 0$.

In a similar manner as in the proof of i) of corollary 7.6.4, it follows that ∇h is linearly triangularizable, as desired. □

Lemma 7.8.5. *Assume $h = \sum_{i=1}^{N}(A_i x)^4$, and the rows of A are pairwise independent and isotropic. Assume in addition that $\mathcal{H}h$ is nilpotent.*
Let $\tilde{A}$ be the matrix consisting of the first 6 rows of A and assume that $\mathrm{rk}A = N - 3$, $\mathrm{rk}\tilde{A} = 3$, and

$$\sum_{i=1}^{6} \lambda_i (A_i x)^2 = 0 \tag{7.35}$$

for certain $\lambda_i \in \mathbb{C}^*$. Then $\tilde{A}\tilde{A}^{\mathrm{t}} = 0$ and AA^{t} is of the form of (7.16) for $P = I_N$. Furthermore, $\mathrm{rk}C = 1$.

Proof. Since $\mathrm{rk} A = N - 3$ and $\mathrm{rk} \tilde{A} = 3 = \tilde{N} - 3$, where $\tilde{N} := 6$, it follows that $DN(A) = DN(\tilde{A})$. By applying theorem 7.1.4 $N-6$ times, we see that $\tilde{h} := \sum_{i=1}^{6} (A_i x)^4$ has a nilpotent Hessian.

We first show that $\tilde{A}\tilde{A}^{\mathrm{t}} = 0$. Since $\mathrm{rk}\tilde{A} = 3$, it follows from proposition 5.5.1, that we may assume that for each $i \le N$, $A_i x$ is dependent of either

$$x_1 + \mathrm{i}x_4, x_2 + \mathrm{i}x_5, x_3 + \mathrm{i}x_6$$

or

$$x_1, x_2 + \mathrm{i}x_4, x_3 + \mathrm{i}x_5$$

or

$$x_1, x_2, x_3 + \mathrm{i}x_4$$

or

$$x_1, x_2, x_3$$

So $\tilde{A}\tilde{A}^{\mathrm{t}} = 0$ in the first case. In the second case, we get the same conclusion, because the rows of $\tilde{A}$ must be isotropic and therefore $\tilde{A}e_1 = 0$. Assume next that $\tilde{h} \in \mathbb{C}[x_1, x_2, x_3]$. Then $\tilde{h}$ is degenerate and $(\mu_1 \frac{\partial}{\partial x_1} + \mu_2 \frac{\partial}{\partial x_2} + \mu_3 \frac{\partial}{\partial x_3})\tilde{h} = 0$ for a certain nonzero isotropic $\mu \in \mathbb{C}^3$. From proposition 5.5.1, it follows that we may assume that $\mu = (1, \mathrm{i}, 0)$.

Since the rows of $\tilde{A}$ are isotropic and pairwise independent, there can only be one $i \le 6$ such that $(\frac{\partial}{\partial x_1} + \mathrm{i}\frac{\partial}{\partial x_2})\tilde{A}_i x = 0$, for A_i must be dependent of $(1, \mathrm{i}, 0)^{\mathrm{t}}$. Since $DN(\tilde{A}) = 6$ on account of (7.35), it follows that removing the i-th row of $\tilde{A}$ does not decrease its rank. Now by ii) of lemma 6.6.8, we obtain that $5 \ge 4 - 3 + 2 \cdot 3 = 7$. Contradiction.

So assume that $\tilde{h} \in \mathbb{C}[x_1, x_2, x_3 + \mathrm{i}x_4]$. Then for each $i \le 6$, either $A_i x$ is a linear combination of $x_1 + \mathrm{i}x_2$ and $x_3 + \mathrm{i}x_4$, or $A_i x$ is a linear combination of $x_1 - \mathrm{i}x_2$ and $x_3 + \mathrm{i}x_4$. Applying $\frac{\partial}{\partial x_1} \pm \mathrm{i}\frac{\partial}{\partial x_2}$ on (7.35), we see that there are at least three $A_i x$ of each of both types. If $A_i x$ is dependent of $x_3 + \mathrm{i}x_4$, then $A_i x$ belongs to both types, but since there is at most one i such that $A_i x$ is dependent of $x_3 + \mathrm{i}x_4$, we may assume without loss of generality that $A_i x$ is a linear combination of $x_1 + \mathrm{i}x_2$ and $x_3 + \mathrm{i}x_4$ for all $i \le 3$ and $A_i x$ is a linear combination of $x_1 - \mathrm{i}x_2$ and $x_3 + \mathrm{i}x_4$ for all $i \ge 4$

If $\tilde{h} \in \mathbb{C}[x_1 + \mathrm{i}x_2, x_1 - \mathrm{i}x_2, x_3 + \mathrm{i}x_4]$ has degree ≤ 1 in $x_1 - \mathrm{i}x_2$, then $(\frac{\partial}{\partial x_1} + \mathrm{i}\frac{\partial}{\partial x_2})^2 h = 0$, and by i) of lemma 6.6.8, we obtain a contradiction to that $A_i x$ is a linear combination of $x_1 + \mathrm{i}x_2$ and $x_3 + \mathrm{i}x_4$ for all $i \le 3$. So $\tilde{h} \in \mathbb{C}[x_1 + \mathrm{i}x_2, x_1 - \mathrm{i}x_2, x_3 + \mathrm{i}x_4]$ has degree ≥ 2 in $x_1 - \mathrm{i}x_2$, and similarly,

$\tilde{h} \in \mathbb{C}[x_1 + \mathrm{i}x_2, x_1 - \mathrm{i}x_2, x_3 + \mathrm{i}x_4]$ has degree ≥ 2 in $x_1 + \mathrm{i}x_2$. This contradicts that $\mathcal{H}\tilde{h}$ is nilpotent.

So $\tilde{A}\tilde{A}^{\mathrm{t}} = 0$. It follows from i) of theorem 7.6.1 that AA^{t} is of the form of (7.16) with $P = I_N$. We shall show that $\mathrm{rk} C = 1$. Since $\mathrm{rk}\tilde{A} = 3$, we may assume without loss of generality that A_3 is independent of A_1 and A_2.

Just as in the proof of i) of corollary 7.6.4, we may assume that $n \geq 2N$, $A_i x = x_i + \mathrm{i}x_{n+1-i}$ for all $i \geq 7$ and

$$\begin{aligned} A_i x \quad = \quad & (x_i + \mathrm{i}x_{n+1-i}) + \frac{1}{2}\big(C_{i1}(x_{r+1} - \mathrm{i}x_{n-r}) \; + \\ & C_{i2}(x_{r+2} - \mathrm{i}x_{n-r-1}) + \cdots + C_{i(N-r)}(x_N - \mathrm{i}x_{n+1-N})\big) \end{aligned}$$

for $i = 1$, $i = 2$ and $i = 3$, where $r = 6$. Define L_1 and L_2 in a similar manner as in the proof of ii) of corollary 7.6.4 with $r = 6$ and put

$$L_3 := A_3 x - \frac{1}{2}(C_{31}L_1 + C_{32}L_2)$$

Then we can write $\sum_{i=1}^{6}(A_i x)^4$ as a polynomial in $\mathbb{C}[L_1, L_2, L_3]$. Furthermore, L_3 is defined in such a way that

$$\begin{aligned} L_3 - (x_3 + \mathrm{i}x_{n-2}) \in \mathbb{C}[x_1 + \mathrm{i}x_n, x_2 + \mathrm{i}x_{n-1}, x_{r+3} - \mathrm{i}x_{n-2-r},& \\ x_{r+4} - \mathrm{i}x_{n-3-r}, \ldots, x_N - \mathrm{i}x_{n+1-N}]& \end{aligned}$$

and just as in the proof of ii) of corollary 7.6.4, we can derive that the degrees with respect to L_1 and L_2 of $\sum_{i=1}^{r}(A_i x)^4$ are at most 1.

We shall show that $L_3^3 \mid \tilde{h}$. For that purpose, define $\hat{h} := \sum_{i=1}^{8}(A_i x)^4$. Then $x + \hat{h}$ is tame and $\hat{h}$ is degenerate of such a large order that the nilpotency of its Hessians comes down to that of $\mathcal{H}_{x_7, x_8, x_{N-7}, x_{N-6}}\hat{h}$ only. Now a computation reveals that $\hat{h}$, seen as polynomial over $\mathbb{C}$ in $x_7 + \mathrm{i}x_{N-6}$, $x_8 + \mathrm{i}x_{N-7}$, L_1, L_2, L_3, does not have a term that is divisible by $L_1 L_2$. Since we already showed the absence of terms divisible by L_1^2 or L_2^2, $L_3^3 \mid \tilde{h}$ follows.

So $\sum_{i=1}^{6}(A_i x)^4 = L_3^3 L_4$ for some linear form L_4 such that $L_4 = \lambda_1 L_1 + \lambda_2 L_2 + \lambda_3 L_3$. Take $\mu = (\mu_1, \mu_2) \neq 0$ such that $\mu_1\lambda_1 + \mu_2\lambda_2 = 0$. Then $(\mu_1 \frac{\partial}{\partial x_7} + \mu_2 \frac{\partial}{\partial x_8}) L_3^3 L_4 = 0$, but $(\mu_1 \frac{\partial}{\partial x_7} + \mu_2 \frac{\partial}{\partial x_8}) L_i = \mu_i$ for $i = 1$ and $i = 2$.

Let $\hat{A}$ be the matrix containing of the rows $\tilde{A}_i$ of $\tilde{A}$ for which $(\mu_1 \frac{\partial}{\partial x_7} + \mu_2 \frac{\partial}{\partial x_8})\tilde{A}_i x \neq 0$ From ii) of lemma 6.6.8, with $\mu_1 \frac{\partial}{\partial x_7} + \mu_2 \frac{\partial}{\partial x_8}$ instead of $\frac{\partial}{\partial x_n}$, it follows that the height of $\hat{A}$ is at least $4 - 3 + 2\mathrm{rk}\hat{A} = 1 + 2\mathrm{rk}\hat{A}$, so $\mathrm{rk}\hat{A} = 2$ and the height of $\hat{A}$ is five. This contradicts (7.35), so $\mathrm{rk} C = 1$. □

The reader may verify that in case $\mathrm{rk}\tilde{A} = 2$, there exist maps with the properties of corollary 7.8.4. Such maps do exist for $\mathrm{rk}\tilde{A} = 3$ as well. We give two examples where $\sum_{i=1}^{r}(\tilde{A}_i x)^4$ cannot be expressed as a polynomial in less than three linear forms.

By antidifferentiating (6.17) with respect to x_3, we obtain

$$\begin{aligned} x_1^d\,(x_3 - \lambda_d x_1) + \sum_{i=1}^{d} \frac{(-1)^i}{(d+1)i}\binom{d}{i}(x_1 + ix_3)^{d+1} \\ = \quad x_2^d\,(x_3 - \lambda_d x_2) + \sum_{i=1}^{d} \frac{(-1)^i}{(d+1)i}\binom{d}{i}(x_2 + ix_3)^{d+1} \end{aligned}$$

where

$$\lambda_d := \sum_{i=1}^{d} \frac{(-1)^i}{(d+1)i}\binom{d}{i}$$

So $(x_1^d - x_2^d)x_3 - \lambda_d(x_1^{d+1} - x_2^{d+1})$ can be written as a sum of $2d$ linear forms. Consequently,

$$\begin{aligned} \big((x_1 + \mathrm{i}x_n)^d - (x_2 + \mathrm{i}x_{n-1})^d\big)(x_3 - \mathrm{i}x_{n-2}) \; - \\ \lambda_d\big((x_1 + \mathrm{i}x_n)^{d+1} - (x_2 + \mathrm{i}x_{n-1})^{d+1}\big) \quad = \quad \sum_{i=1}^{r}(\tilde{A}_i x)^d \end{aligned}$$

for a suitable matrix $\tilde{A}$ and $r = 2d$.

By antidifferentiating (6.18) with respect to x_3, we obtain

$$\begin{aligned} \sum_{i=0}^{d-1} \frac{\zeta_d^i}{d+1}(x_1 + \zeta_d^i x_2 + x_3)^{d+1} \; - \\ \sum_{i=0}^{d-1} \frac{\zeta_d^i}{d+1}(x_1 + \zeta_d^i x_2 - x_3)^{d+1} \quad = \quad 2d^2 x_1 x_2^{d-1} x_3 \end{aligned}$$

and in a similar manner as above, it follows that

$$(x_1 + \mathrm{i}x_n)(x_2 + \mathrm{i}x_{n-1})^{d-1}(x_3 - \mathrm{i}x_{n-2}) = \sum_{i=1}^{r}(\tilde{A}_i x)^d$$

for a suitable matrix $\tilde{A}$ and $r = 2d$.

Chapter 8

Conclusions and suggestions for further research

8.1 Conclusions

At the end of our journey for the Jacobian conjecture, we must conclude that the Jacobian conjecture is still unsolved. But we met many nice results on our journey. The most spectacular things were probably the gradient reduction of the Jacobian conjecture in chapter 1 and the cubic homogeneous counterexample to the dependence problem in chapter 4. The homogeneous dependence problem has been open for more than ten years, and the Vodka made solving it even more special. The gradient reduction is one of the many reductions of the Jacobian conjecture, but no new reductions had been added since 1983, except the '$A^2 = 0$'-reduction for power linear Keller maps by Drużkowski in 2000, which is actually somewhat similar to the gradient reduction.

Another nice result is the linear triangularizability of homogeneous Keller maps in dimension 3. The homogeneity reduction of the Jacobian conjecture is classical, so others could have found the same result. Since they did not, it makes me feel that this result is a deep result, although others focus more on degree 3 only than we do. (The case of degree 3 was already solved in 1992 by Wright.) For people that are not interested in the Jacobian conjecture, the classification of homogeneous maps over $\mathbb{C}$ with Jacobians of rank 2 might still be interesting. The recent result that the dependence problem has an affirmative answer for homogeneous maps with Jacobian rank 2 in dimension

4, is interesting as well, since dimension 4 is the only dimension for which the homogeneous dependence problem is unsolved.

As for the symmetric Keller maps, we were pioneers, so finding new results in small dimensions was not really a challenge. Nevertheless, I think we did a good job here, because the proofs are quite involved, especially for the unipotent case in dimension 4. Furthermore, we found many results for symmetric Jacobians of small rank. One of them is that homogeneous polynomials for which the Hessian rank r is at most three can be expressed as a polynomial in r linear forms. These results might be interesting for people that do not feel any affection for reductions of the Jacobian conjecture as well.

In chapter 3, we have seen that quasi-translations are a very interesting kind of automorphisms. This is stressed by the construction of homogeneous counterexamples to the dependence problem in in sections 3.7 and 4.2, and the results about singular Hessians. Quasi-translations can also be seen as so-called locally nilpotent derivations. See [30] for more about locally nilpotent derivations. The quasi-translations of section 3.7 can also be found in [30, §3.10.3]. In [30], quasi-translation are called nice derivations, but that definition does not match that in [24], which is used here. Homogeneous quasi-translations are quasi-linear, because $\frac{1}{d}\mathcal{J}H$ suffices as the matrix M in the definition of quasi-linear in [30, §3.2.1], where the quasi-translation $x + H$ has degree d. So the results of chapter 3 are useful for many areas in affine geometry. Furthermore, it is of interest for the history of mathematics as well, since the article [34] from 1876 by Gordan and Nöther is studied.

Chapters 6 and 7 are the least interesting for people that do not feel affection for the Jacobian conjecture. It contains many new results for power linear Keller maps and some for so called Zhao-graphs, but one can ask for what benefit? One can view finding new results as a type of sports as well. Furthermore, the reduction of the Jacobian conjecture to power linear maps is classical, so finding new results is surely a challenge, especially for cubic linear maps. And new results for cubic linear maps can be found in chapters 6 and 7, for instance the counterexample in dimension 53 to the dependence problem and the main result of chapter 7, that power linear maps $(Ax)^{*d}$ with $d \geq 3$ and $\operatorname{cork} A = 3$ are linearly triangularizable. The results of chapter 7 improve those of [I.11]. The results are not very interesting for people that do not like power linear Keller maps. But the results are new and the power linear reduction of the Jacobian conjecture is old.

The result of reducing the Jacobian conjecture to gradient maps has already been used by Wenhua Zhao in [61]. He derives formulas for the inverse and formulates the so-called 'vanishing conjecture'. This is a conjecture for differential operators, and the vanishing conjecture for the Laplace operator is equivalent to the unipotent/homogeneous Jacobian conjecture for gradient maps.

The advisor had many Ph.D. students before the author, but I consider Engelbert Hubbers as the most important predecessor, because his research topic was more or less similar to that of mine. The structure of the final section of his Ph.D. thesis [37] has been copied here. Hubbers' suggestions for future work consists of a list of six items. Half of this list has been completed or extended in this thesis, namely the third, fifth and sixth item.

The third item is the cubic homogeneous counterexample to the dependence problem. The fifth item is about the incompleteness of the classification of quadratic homogeneous Keller maps in dimension 5. But this classification has been completed due to better hardware nowadays. The sixth item of the list has been completed by Charles Ching-An Cheng, and was extended to dimension 6 in section 6. Section 7 extends this result to dimension 7 with reasoning only and dimension 8 with Hubbers' own classification of cubic homogeneous Keller maps in dimension 4 as the only part with computations.

8.2 Suggestions for future work

After obtaining the results of this thesis, several questions arose to the author. Below we list the chapter numbers and the questions belonging to that chapter.

1. In the introduction, I mentioned the question whether quadratic maps for which the Jacobian determinant does not vanish anywhere are automorphisms over commutative rings with $\frac{1}{2}$. As pointed out in the introduction, the answer is affirmative for algebraically closed fields.

2. A question after the symmetry reductions of the Jacobian conjecture is the following: what is the status of $\boxdot(\mathbb{R}, 2n)$? We know that $\boxdot(\mathbb{C}, 2n)$ for all n is equivalent to the Jacobian conjecture in all dimensions, but it might be possible that instances of $\boxdot(\mathbb{R}, 2n)$ satisfy the Jacobian conjecture in a trivial manner.

Another question is about the gradient reduction. The reduction of Meng was stated independent of the other reductions of the Jacobian conjecture. So a natural question is whether the Jacobian conjecture is true for gradient maps in small dimensions n. The case $n = 1$ is trivial, but the case $n = 2$ is solved in the affirmative as well for fields of characteristic zero. See [15, Cor. 1] by Franki Dillen. This is a corollary of the theorem above it in [15], which is equivalent to [24, Prop. 8.2.7].

The case $n = 4$ implies the two-dimensional Jacobian conjecture, and since incorrect proofs of the latter conjecture appear several times a year, I think the case $n = 4$ is too far-reaching. No, let us first start with $n = 3$. Maybe, the methods in the wrong proofs of the two-dimensional Jacobian conjecture can even be helpful.

3. For quasi-translations $x + H$ over $\mathbb{C}$ in dimensions ≤ 3, the components of H are linearly dependent over $\mathbb{C}$. In dimension 4 and up, the rows of $\mathcal{J}H$ might be independent over $\mathbb{C}$. For homogeneous quasi-translations, we have similar results in dimensions ≤ 4 and in dimension 6 and up. So the only missing dimension is dimension 5.

   So the question is whether for homogeneous quasi-translations $x + H$ over $\mathbb{C}$ in dimension 5, the components of H need to be linearly dependent over $\mathbb{C}$? A bottle of Joustra Beerenburg (Frisian spirit) will be offered by the author to the one who first answers this question. My feeling is that the answer is affirmative. Besides that a counterexample should have degree 12 at least, the fact that there are two independent linear relations in dimension 4 and for Jacobian rank 2 even without the trace condition can be seen as a sign in that direction.

4. For the homogeneous dependence problem, the missing dimension is dimension 4. A bottle of Hooghoudt Vodka (Dutch spirit) will be offered by the author to the one who first answers the question whether the homogeneous dependence problem holds in dimension 4. My feeling is that the answer is affirmative for similar reasons as with the above Beerenburg problem, except that there is only one linear relation in dimension 3 and for Jacobian rank 2 in dimension 4, but the trace condition can still be omitted.

   As we have seen, omitting the trace condition gives counterexamples of degree 4 and up, but not of degree 2. The question is what happens with degree 3. Omitting the trace condition out of the calculations of cubic homogeneous Keller maps in dimension 4 made the formulas too large. Maybe, the formulas can be kept under control with new ideas.

Another question that is good for a bottle of Hooghoudt Vodka is the dependence problem for quadratic homogeneous Keller maps. This question was originally stated by Kamil Rusek. As we have shown, it only needs to be proved for quadratic linear maps or gradient maps, and it is trivially satisfied for quadratic linear gradient maps.

For the homogeneous dependence problem, all known counterexamples have Jacobian rank 4 at least. My guess is that the homogeneous dependence problem is true for Jacobian rank 2. Recall that we came as far as dimension 4 without using the trace condition. So the question is what happens with Jacobian rank three and Jacobian rank 2. For Jacobian rank 2, another question is whether the homogeneous dependence problem is even true without the trace condition.

The trace condition is necessary for dimension 4 and hence for Jacobian rank 3. If one solves the homogeneous dependence problem for Jacobian rank 3 in the affirmative, then one receives two bottles: one bottle of Beerenburg for the homogeneous quasi-translations in dimension 5 and one bottle of Vodka for the homogeneous dependence problem in dimension 4.

5. For singular Hessians over $\mathbb{C}$, we do not know yet whether for corresponding quasi-translation $x + Q$, the components of Q need to be linearly dependent over $\mathbb{C}$. I offer a bottle of Joustra Beerenburg as well for the one who first solves the question whether for quasi-translations $x + Q$ from singular Hessians in dimension 4 over $\mathbb{C}$, the components of Q need to be linearly dependent over $\mathbb{C}$.

   Another question is whether for homogeneous Hessians $\mathcal{H}h$ of rank 4, we have a similar subdivision as for dimension 5. That is, $h = g(Tx)$ for some $T \in \mathrm{GL}_n(\mathbb{C})$ with either

$$g \in \mathbb{C}[x_1, x_2, x_3, x_4]$$

   or

$$g \in \mathbb{C}[x_1, x_2, a_3(p,q)x_3 + a_4(p,q)x_4 + \cdots + a_n(p,q)x_n]$$

   where $p, q \in \mathbb{C}[x_1, x_2]$.

A third question is whether the dependence problem is satisfied for homogeneous gradient maps in dimension 6. An affirmative answer would imply linear triangularizability.

6. For a power linear Keller map $x + (Ax)^{*d}$ over $\mathbb{C}$, the components of $(Ax)^{*d}$ do not need to be linearly dependent over $\mathbb{C}$, as the Herbie example makes clear. But if the diagonal of A is nonzero, then the trace condition on $\mathcal{J}(Ax)^{*d}$ tells us that the components of $(Ax)^{*(d-1)}$ are linearly dependent over $\mathbb{C}$. So the question is whether the components of $(Ax)^{*(d-1)}$ are linearly dependent over $\mathbb{C}$ if the diagonal of A is zero.

   Another question is trying to find an algorithm that, given a homogeneous map H, computes a power linear map G in dimension N such that H and G are GZ-paired and N is minimal. Only the quadratic case in dimension $n = 1$ is solved, as far as I know. See example 6.2.9 and section 7.2 for some results for particular examples H.

   The case $n = 1$ is also interesting in connection with Zhao graphs. The problem is of course to find powers of linear forms that are killed by the Laplace operator and add up to a given homogeneous $h \in \mathbb{C}[x]$ with $\mathcal{H}h$ nilpotent. I do not know if this problem is already solved for degree 2.

7. In section 7.4, we introduced the concept of crop matrix. Many connections between a matrix A and its crop matrix $\tilde{A}$ are known, but some are not. One of those things is whether $(Ax)^{*d}$ satisfies DP+ in case its Jacobian is nilpotent and $(\tilde{A}\tilde{x})^{*d}$ satisfies DP+, where $\tilde{x} = x_1, x_2, \ldots, x_n$. Another thing is whether $(Ax)^{*d}$ is (ditto) linearly triangularizable in case its Jacobian is nilpotent and $(\tilde{A}\tilde{x})^{*d}$ is (ditto) linearly triangularizable.

   Chapter 7 contains many results about Zhao graphs, but more results can be obtained. It is only that I do not have time any more to work out the details. I hope to get additional results in the next year.

Appendix A

Computations of nilpotent Jacobians

A.1 Normal forms for nilpotent Jacobians

For nilpotent matrices, the conjugation classes are given by Jordan normal forms. Now it would be useful to have a similar reduction by linear conjugations for non-linear maps with nilpotent Jacobians. Notice that for maps of degree d, the Jacobian has degree $d-1$, and linear conjugation do not change this. So it is impossible to get a Jordan normal form by linear conjugations of maps of degree 2 at least.

But one can substitute some constant vector in x in the Jacobian and hope that the Jacobian will be a Jordan normal form after this substitution. We will show that this is indeed possible after a suitable linear conjugation. Furthermore, we can obtain that the substitution vector is the sum of at most $\sqrt{n}$ distinct unit vectors.

Definition A.1.1. Let $v \in \mathbb{C}^n$ be nonzero and $M \in \mathrm{Mat}_n(\mathbb{C})$ be nilpotent. Define the *image exponent* of v with respect to M as

$$\mathrm{IE}(M, v) := \max\{i \in \mathbb{N} \mid M^i v \neq 0\}$$

and the *preimage exponent* of v with respect to M as

$$\mathrm{PE}(M, v) := \max\{i \in \mathbb{N} \mid M^i w = v \text{ for some } w \in \mathbb{C}^n\}$$

Notice that if $N \in \mathrm{Mat}_n(\mathbb{C})$ has ones on the subdiagonal and zeros elsewhere, then $\mathrm{IE}(N,v) + \mathrm{PE}(N,v) = n-1$ for each nonzero $v \in \mathbb{C}^n$. If M is nilpotent and $\mathrm{cork}M = 1$, then $N = T^{-1}MT$ for some $T \in \mathrm{GL}_n(\mathbb{C}(x))$. So if M is nilpotent and $\mathrm{cork}M = 1$, then $\mathrm{IE}(M,v) + \mathrm{PE}(M,v) = n-1$ for each nonzero $v \in \mathbb{C}^n$ as well, because N is the Jordan normal form of M, $\mathrm{IE}(T^{-1}MT, T^{-1}v) = \mathrm{IE}(M,v)$ and $\mathrm{PE}(T^{-1}MT, T^{-1}v) = \mathrm{PE}(M,v)$.

Proposition A.1.2. *Assume $M \in \mathrm{Mat}_n(\mathbb{C})$ is nilpotent and $v \in \mathbb{C}^n$ is nonzero. Then there exists a $T \in \mathrm{GL}_n(\mathbb{C})$ such that $N := T^{-1}MT$ is the Jordan normal form of M and*

$$w := T^{-1}v = e_{i_1} + e_{i_2} + \cdots + e_{i_m}$$

where

$$\mathrm{IE}(N, e_{i_1}) < \mathrm{IE}(N, e_{i_2}) < \cdots < \mathrm{IE}(N, e_{i_m}) = \mathrm{IE}(N, w) = \mathrm{IE}(M, v)$$

and

$$\mathrm{PE}(M, v) = \mathrm{PE}(N, w) = \mathrm{PE}(N, e_{i_1}) < \mathrm{PE}(N, e_{i_2}) < \cdots < \mathrm{PE}(N, e_{i_m})$$

Proof. We distinguish three cases:

- $\mathrm{cork}M = 1$.
  Let N be the matrix with ones on the subdiagonal and zeros elsewhere. Then N is the Jordan normal form of M, say $N = T^{-1}MT$. Let $w := T^{-1}v$ and i be the index of the first nonzero coordinate of w. Notice that $n - i = \mathrm{IE}(N,w) = \mathrm{IE}(M,v)$ and $i - 1 = \mathrm{PE}(N,w) = \mathrm{PE}(M,v)$.

  The operator $x \mapsto Nx$ shifts the coordinates of its argument one step downward, inserting a zero above. The operator $x \mapsto N^{\mathrm{t}}x$ shifts the coordinates of its argument one step upward, inserting a zero below. Now define the matrix $S \in \mathrm{GL}_N(\mathbb{C})$ by $Se_i = w$, $Se_{i+j} = N^j w$ and $Se_{i-j} = (N^{\mathrm{t}})^j w$, for all $j \geq 1$. Then $(TS)^{-1}v = S^{-1}w = e_i$, so it suffices to show that $(TS)^{-1}MTS = S^{-1}NS$ is the Jordan normal form of M. Indeed $S^{-1}NSe_j = Ne_j$ for all j, because by definition of i, S is constructed in such a way that $NSe_j = Se_{j+1}$ for all $j < n$ and $NSe_n = 0$.

- $\mathrm{cork}M = 2$.
  Again, let $N = T^{-1}MT$ be the subdiagonal Jordan normal form of

M and $w = T^{-1}v$. Notice that N has two Jordan blocks, say $N_1 \in \mathrm{Mat}_r(\mathbb{C})$ and $N_2 \in \mathrm{Mat}_{n-r}(\mathbb{C})$. Since $\mathrm{cork} N_1 = \mathrm{cork} N_2 = 1$, it follows from the case $\mathrm{cork} M = 1$ that we may assume that w is the sum of at most two unit vectors e_i and e_j, such that $1 \le i \le r < j \le n$. If $w = e_i$ or $w = e_j$, then we are done, so assume $w = e_i + e_j$.

Assume without loss of generality that $\mathrm{PE}(N, i) \le \mathrm{PE}(N, j)$ and, in case $\mathrm{PE}(N, i) = \mathrm{PE}(N, j)$, that $\mathrm{IE}(N, i) \ge \mathrm{IE}(N, j)$. Since we are done in case both $\mathrm{IE}(N, e_i) < \mathrm{IE}(N, e_j)$ and $\mathrm{PE}(N, i) < \mathrm{PE}(N, j)$, we may assume that $\mathrm{IE}(N, i) \ge \mathrm{IE}(N, j)$ in any case.

Since $\mathrm{IE}(N, e_j) \le \mathrm{IE}(N, e_i) = r - i$ it follows that $\mathrm{IE}(N, w) = r - i$. Since $\mathrm{PE}(N, e_j) \ge \mathrm{PE}(N, e_i) = i - 1$ it follows that $\mathrm{PE}(N, w) = i - 1$. In fact, $N^{i-1}(e_1 + e_{j-i+1}) = w$.

Now define the matrix $S \in \mathrm{GL}_N(\mathbb{C})$ by $Se_k = e_k + e_{j-i+k}$ if $j-i+k \le n$ and $Se_k = e_k$ if $j - i + k > n$. Then $Se_i = e_i + e_j = w$, so $S^{-1}w = e_i$. Since $NSe_k = Se_{k+1}$ for all $k \notin \{r, n\}$ and $NSe_r = 0 = NSe_n$, it follows that $S^{-1}NS = N$. This gives the desired result.

- $\mathrm{cork} M \ge 3$.
  Again, let $N = T^{-1}MT$ be the subdiagonal Jordan Normal Form of M and $w = T^{-1}v$. From the case $\mathrm{cork} M = 1$, we obtain that we may assume that w is the sum of at most one unit vector e_i for each Jordan block. From the case $\mathrm{cork} M = 2$, we obtain that we may assume two summands e_i and e_j of w satisfy $\mathrm{IE}(N, e_i) < \mathrm{IE}(N, e_j)$ and $\mathrm{PE}(N, e_i) < \mathrm{PE}(N, e_j)$. That gives the desired result. □

Notice that m in proposition A.1.2 is at most $\sqrt{n}$. This is because the size of the Jordan block with coordinate i_{k+1} must be at least 2 larger than that with i_k (in order to have both $\mathrm{IE}(N, e_{i_k}) < \mathrm{IE}(N, e_{i_{k+1}})$ and $\mathrm{PE}(N, e_{i_k}) < \mathrm{PE}(N, e_{i_{k+1}})$) so the sizes are at least $1, 3, 5, \ldots, 2m - 1$, and the series of the odd numbers are the squares.

Theorem A.1.3. *Assume $\mathcal{J}H$ is nilpotent. Then there exists a $T \in \mathrm{GL}_n(\mathbb{C})$ such that*

$$\left(\mathcal{J}T^{-1}H(Tx)\right)\Big|_{x=w} = N$$

where N is the Jordan Normal Form of $\mathcal{J}H$ and

$$w = e_{i_1} + e_{i_2} + \cdots + e_{i_m}$$

such that

$$\mathrm{IE}(N, e_{i_1}) < \mathrm{IE}(N, e_{i_2}) < \cdots < \mathrm{IE}(N, e_{i_m}) = \mathrm{IE}(\mathcal{J}H, x)$$

and

$$\mathrm{PE}(\mathcal{J}H, x) = \mathrm{PE}(N, e_{i_1}) < \mathrm{PE}(N, e_{i_2}) < \cdots < \mathrm{PE}(N, e_{i_m})$$

Proof. Take $v \in \mathbb{C}^n$ generic and put $M := (\mathcal{J}H)|_{x=v}$. Then M has the same Jordan Normal Form as $\mathcal{J}H$, $\mathrm{IE}(M, v) = \mathrm{IE}(\mathcal{J}H, x)$ and $\mathrm{PE}(M, v) = \mathrm{PE}(\mathcal{J}H, x)$. So there exists a $T \in \mathrm{GL}_n(\mathbb{C})$ such that $N := T^{-1}MT$ and $w := T^{-1}v$ satisfy the properties of proposition A.1.2.

Now

$$\begin{aligned} \Big(\mathcal{J}\big(T^{-1}H(Tx)\big)\Big)\Big|_{x=w} &= T^{-1}(\mathcal{J}H)|_{x=Tx}|_{x=w}T \\ &= T^{-1}(\mathcal{J}H)|_{x=v}T \\ &= T^{-1}MT \\ &= N \end{aligned}$$

as desired. □

By the above theorem, we can do some normalization. In case $n = 4$ and $\mathrm{rk}\mathcal{J}H = 3$, we get four cases, namely $w = e_1, e_2, e_3, e_4$, where

$$(\mathcal{J}H)|_{x=w} = N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Now the case $w = e_n$ does not exists, because that implies $\mathcal{J}H \cdot x = 0$. This is even impossible if H is not homogeneous, for by separating homogeneous parts of H, we see that H is constant. In case H is homogeneous of degree d and $w = e_3$, we have $d\mathcal{J}H \cdot H = \mathcal{J}H^2 \cdot x = 0$, which is impossible since for homogeneous quasi-translations $x + H$, $\mathrm{cork}\mathcal{J}H \geq 2$.

So two cases remain: the case $w = e_1$ and $w = e_2$. Now it seems to be a tradition to substitute $x = e_1$, see e.g. [36] and [55]. For that purpose, one can take

$$P := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

for the case $w = e_2$, and conjugate with P to obtain

$$\begin{aligned}
\Big(\mathcal{J}\big(P^{-1}H(Px)\big)\Big)\Big|_{x=e_1} & \\
&= P^{-1}(\mathcal{J}H)|_{x=Pe_1}P \\
&= P^{-1}(\mathcal{J}H)|_{x=e_2}P \\
&= P^{-1}NP \\
&= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

By way of a conjugation with a power of P, one can always transform a substitution $x = e_i$ to a substitution $x = e_1$. E.g. in the quasi-translation case $w = e_3$, we conjugate with P^2, since $P^2 e_1 = e_3 = w$, to obtain

$$\begin{aligned}
\Big(\mathcal{J}\big(P^{-2}H(P^2x)\big)\Big)\Big|_{x=e_1} &= P^{-2}(\mathcal{J}H)|_{x=P^2e_1}P^2 \\
&= P^{-2}(\mathcal{J}H)|_{x=e_3}P^2 \\
&= P^{-2}NP^2 \\
&= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}
\end{aligned}$$

Now you might say that this latter case does not exist, but where did we assume that H is homogeneous? Take for instance the quasi-translation $x + H$ with

$$H := \big(b(ax_1 - bx_2), a(ax_1 - bx_2), b(ax_3 - bx_4), a(ax_3 - bx_4)\big)\Big|_{a=1, b=x_1x_4 - x_2x_3}$$

to obtain $(\mathcal{J}H)|_{x=e_1} = P^{-2}NP^2$. But this is not a good example, because the substitution $x = e_1$ is *not* a generic substitution here. Since $\mathcal{J}H^2 \cdot x \neq 0$, we have $\mathrm{IE}(e_1, P^{-2}NP^2) < \mathrm{IE}(x, \mathcal{J}H)$. Since $\mathcal{J}H^3 \cdot x = 0$, $\mathrm{IE}(e_1, P^{-1}NP) = \mathrm{IE}(x, \mathcal{J}H)$, and $P^{-1}NP$ is the substitution matrix corresponding to $(\mathcal{J}H)|_{x=v}$ for generic v.

In dimension $n = 4$, w is not always the sum of exactly one unit vector. In case $\mathrm{rk}\mathcal{J}H = 2$ and

$$(\mathcal{J}H)_{x=w} = N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

w might be $e_1 + e_3$. Since it is more desirable to substitute a sole unit vector, we take

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and conjugate with T to obtain

$$\begin{aligned} \Big(\mathcal{J}\big(T^{-1}H(Tx)\big)\Big)\Big|_{x=e_1} & \\ &= T^{-1}(\mathcal{J}H)|_{x=Te_1}T \\ &= T^{-1}(\mathcal{J}H)|_{x=e_1+e_3}T \\ &= T^{-1}NT \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{aligned}$$

Notice that in case H above is homogeneous, then H is a quasi-translation. For homogeneous quasi-translations and other H that satisfy $\mathcal{J}H^2 \cdot x = 0$, we have $\mathrm{IE}(\mathcal{J}H, x) \le 2$ and therefore, w is the sum of at most two unit vectors. On the other hand, we have

$$\big(\mathrm{IE}(\mathcal{J}H, x), \mathrm{PE}(\mathcal{J}H, x)\big) \ne \big(\mathrm{IE}(N, e_i), \mathrm{PE}(N, e_i)\big)$$

for all i for the map H above, so w cannot be a sole unit vector.

For the remaining three cases of $(\mathcal{J}H)|_{x=w}$ in dimension $n = 4$ with $\mathrm{rk}\mathcal{J}H = 2$, w is indeed the sum of only one unit vector and we can cycle the substitution $x = w$ to $x = e_1$.

The computations take less memory than those of Hubbers, because the equations are remembered without substitutions of eliminated variables and the equations are not duplicated for case-selection. But it might take more time to retrieve the equations *with* substitutions of eliminated variables. I think this is only true if you want to find the useful equations, since when you know already what the useful equations are, it is only a waste of time to substitute in the equations you do not need.

And finding the useful equations has already been done by the author. All actual calculations can be found at [62]. Before deriving the other substitution matrices, we make one interesting observation.

Theorem A.1.4. *Assume $H \in \mathbb{C}[x]^n$ such that $\mathcal{J}H^n = 0$ and $\mathrm{PE}(\mathcal{J}H, x) \geq 1$. Then the rows of $\mathcal{J}H$ are linearly independent over $\mathbb{C}$*

Proof. Since x has a preimage under $y \mapsto \mathcal{J}H \cdot y$, every dependence between the rows of $\mathcal{J}H$ is a dependence between the components of x as well. But the components of x are linearly independent over $\mathbb{C}$. □

Corollary A.1.5. *Assume $H \in \mathbb{C}[x]^n$ such that $\mathcal{J}H^{n-1} \cdot x = 0 = \mathcal{J}H^n$ and $\mathrm{rk}\mathcal{J}H = n - 1$. Then the rows of $\mathcal{J}H$ are linearly independent over $\mathbb{C}$.*

Proof. Since $\mathcal{J}H$ is nilpotent of corank 1, $\mathrm{IE}(\mathcal{J}H, x) + \mathrm{PE}(\mathcal{J}H, x) = n - 1$. From $\mathcal{J}H^{n-1} = 0$ we obtain $\mathrm{IE}(\mathcal{J}H, x) < n - 1$, so $\mathrm{PE}(\mathcal{J}H, x) > 0$. Now apply the above theorem. □

For the quasi-translation of dimension 6 in example 3.7.3, one can compute that $\mathrm{PE}(\mathcal{J}H, x) = 1$ and that w is the sum of exactly two unit vectors. The Jordan blocks have sizes 2 and 4 (in case you have not already derived that from the preceding data).

A.2 Quadratic homogeneous Keller maps in dimension 5

We computed all quadratic homogeneous Keller maps $x + H$ in dimension $n = 5$ for which $\mathrm{rk}\mathcal{J}H \geq 3$. Recall that the Keller condition and the homogeneity of H imply that $\mathcal{J}H$ is nilpotent. We started with $\mathrm{rk}\mathcal{J}H = 4$, in which case the normal form N is unique and we have

$$(\mathcal{J}H)|_{x=w} = N = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

After removing the quasi-translation case $w = e_4$ and the other impossible case $w = e_5$, the cases $w = e_1$, $w = e_2$ and $w = e_3$ remained. After cycling the substitutions $x = e_2$ and $x = e_3$ to $x = e_1$ by conjugation with a power of an n-cycle as described in the previous section, we got the following three cases:

$$(\mathcal{J}H)|_{x=e_1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{A.1}$$

$$(\mathcal{J}H)|_{x=e_1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{A.2}$$

$$(\mathcal{J}H)|_{x=e_1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{A.3}$$

Next, we solved the three cases to the condition that the characteristic polynomial $\det(tI_n - \mathcal{J}H)$ equals t^5, a condition that is equivalent to the Keller condition in this context. This process only gave solutions for (A.1), namely a linear conjugation of

$$\begin{pmatrix} 0 \\ x_1x_3 \\ x_2^2 - x_1x_4 \\ 2x_2x_3 - x_1x_5 \\ x_3^2 \end{pmatrix} + x_1^2 \cdot \begin{pmatrix} 0 \\ \frac{h_2k_2+h_3m_2}{4m_2^{3/2}} \\ \frac{h_2}{4m_2^{2/3}} \\ \frac{k_2}{4m_2^{5/6}} \\ \frac{1}{2} \end{pmatrix}$$

i.e. the solution of (4.14) with additional terms x_1^2, and furthermore a linear conjugation of a map of the type of theorem 4.6.7 and a map with a lower triangular Jacobian.

Now you might think that (A.2) and (A.3) simply did not give solutions because quadratic homogeneous maps with nilpotent Jacobians satisfy DP. But we did not use that $\mathrm{IE}(\mathcal{J}H, x) = 3$ and $\mathrm{IE}(\mathcal{J}H, x) = 2$ respectively for these cases.

Although (A.2) does not have solutions of degree 2, the counterexample of degree 6 in corollary 4.2.3 is of this type. For that example, x does have a preimage under $y \mapsto \mathcal{J}H \cdot y$. But there are no quadratic counterexamples to the homogeneous dependence problem in dimension 5, let alone that x has a preimage under $y \mapsto \mathcal{J}H \cdot y$.

For $\mathrm{rk}\mathcal{J}H = 3$, we first did the cases

$$(\mathcal{J}H)|_{x=w} = N = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and $w = e_i$ for some i. After removing the impossible cases $w = e_4$ and $w = e_5$, the cases $w = e_1$, $w = e_2$ and $w = e_3$ remained. Cycling the substitutions $x = e_2$ and $x = e_3$ to $x = e_1$ as described above, we got the following three cases:

$$(\mathcal{J}H)|_{x=e_1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{A.4}$$

$$(\mathcal{J}H)|_{x=e_1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{A.5}$$

$$(\mathcal{J}H)|_{x=e_1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{A.6}$$

Next, we solved the three cases to the condition that the characteristic polynomial $\det(tI_n - \mathcal{J}H)$ equals t^5, and the condition that $\mathcal{J}H^{8-i} \cdot x = 0$ for case (A.i).

As you can see, $\mathrm{PE}(\mathcal{J}H, x) = 0$ for case (A.4) only, and only that case gave solutions: a linear conjugation of a map of the type of theorem 4.6.7 and linearly triangularizable maps.

Next we advanced with the cases

$$(\mathcal{J}H)|_{x=w} = N = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

For these cases, $w = e_i$ for some i. After removing the impossible cases $w = e_3$ and $w = e_5$, the cases $w = e_1$, $w = e_2$ and $w = e_4$ remained. Cycling the substitutions $x = e_2$ and $x = e_4$ to $x = e_1$ as described above, we got the following three cases:

$$(\mathcal{J}H)|_{x=e_1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{A.7}$$

$$(\mathcal{J}H)|_{x=e_1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{A.8}$$

$$(\mathcal{J}H)|_{x=e_1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{A.9}$$

Next, we solved the three cases to the condition that the characteristic polynomial $\det(tI_n - \mathcal{J}H)$ equals t^5, and the condition that $\mathcal{J}H^3 \cdot x = 0$ for the first case and $\mathcal{J}H^2 \cdot x = 0$ for the other two cases. Case (A.7) only gave linearly triangularizable solutions. Case (A.8) did not give any solutions and case (A.9) gave the only quasi-translation of rank 3: a linear conjugation of

$$(0, 0, x_1x_2, x_1^2, x_1x_3 - x_2x_4)$$

and the first map of example 3.5.5. So the other quasi-translation cases (A.6), (A.8), and (A.11) below did not give any solutions.

At last we did the cases

$$(\mathcal{J}H)|_{x=w} = N = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

and $w = e_1 + e_i$ for some $i \in \{3, 4\}$. These i's are the only cases, because $\mathrm{IE}(N, e_5) = 0 = \mathrm{IE}(N, e_1)$ and $\mathrm{PE}(N, e_1) = 0 = \mathrm{PE}(N, e_2)$. By way of a linear conjugation, we got the cases:

$$(\mathcal{J}H)|_{x=e_1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{A.10}$$

$$(\mathcal{J}H)|_{x=e_1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{A.11}$$

Next, we solved both cases to the condition that the characteristic polynomial $\det(tI_n - \mathcal{J}H)$ equals t^5, and the condition that $\mathcal{J}H^{13-i} \cdot x = 0$ for case (A.i). Case (A.10) only gave a linear conjugation of a map of the type of theorem 4.6.7 and case (A.11) did not give any solutions.

A.3 Unipotent Keller maps of degree 4 in dimension 3

We computed all unipotent Keller maps of degree 4 exactly in dimension 3. We did not compute unipotent Keller maps of degree less than 4 in dimension 3, because these have been classified already in section 4.6.

So assume $x + H$ is an unipotent Keller map in dimension 3 such that the leading homogeneous part $\bar{H}$ of H has degree 4. Since $\mathcal{J}H$ is nilpotent, $\mathcal{J}\bar{H}$ is nilpotent as well. Now there are two cases: $\mathcal{J}\bar{H}^2 \cdot x \neq 0$ and $\mathcal{J}\bar{H}^2 \cdot x \neq 0$, with substitution matrices

$$(\mathcal{J}\bar{H})|_{x=e_1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{A.12}$$

$$(\mathcal{J}\bar{H})|_{x=e_1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{A.13}$$

respectively. We computed these cases separately, but not before thinking out parametrizations of $\bar{H}$ in each of both cases.

From theorem 4.1.4, it follows that $\bar{H}$ is linearly triangularizable. Hence both the rows and the columns of $\mathcal{J}\bar{H}$ are linearly dependent over $\mathbb{C}$. This gives $\bar{H}_1 = 0$ and $\bar{H} \in \mathbb{C}[x_1, x_2]^3$ respectively in the case (A.12). Next, the trace condition on $\mathcal{J}H$ gives that $\mathcal{J}H$ is lower triangular in the case (A.12).

In the case (A.13), we have that $x + \bar{H}$ is a quasi-translation. Now it follows from proposition 3.3.2 that $\bar{H}_1 = 0 = \bar{H}_3$ and $\bar{H}_2 \in \mathbb{C}[x_1, x_3]$.

The case (A.12) gave one solution for which the Jacobian is not lower triangular. That solution can be obtained by first substituting $x_1 = x_1 + \lambda$ for a suitable $\lambda \in \mathbb{C}$, in

$$(0, x_1^2 x_2 - x_1 x_3, x_1^3 x_2 - x_1^2 x_3)$$

which is not linearly triangularizable, next adding polynomials in x_1 to the second and third component, and at last applying a linear conjugation.

The computation of the case (A.13) was the largest of all computations presented in this appendix: the biggest formulas and the most calculations. The case (A.13) gave one solution H for which the rows of the Jacobian are linearly independent over $\mathbb{C}$. That solution can be obtained by applying a linear conjugation to

$$\big(x_2 - x_1^2, x_3 + 2x_1(x_2 - x_1^2), -(x_2 - x_1^2)^2\big)$$

As far as they are not linearly triangularizable, the remaining solutions can be obtained by first substituting $x_1 = x_1 + \lambda$ for a suitable $\lambda \in \mathbb{C}$, in either

$$(0, x_1^2 x_2 - x_1 x_3, x_1^3 x_2 - x_1^2 x_3)$$

or

$$(0, x_1 x_2 - x_3, x_1^2 x_2 - x_1 x_3)$$

which are not linearly triangularizable, next adding polynomials in x_1 to the second and third component, and at last applying a linear conjugation.

A.4 Cubic and quadratic homogeneous Keller maps in dimension 4 and the trace condition

For $n = 4$, H homogeneous and $\mathcal{J}H$ nilpotent, we have the following two cases with $\mathrm{rk}\mathcal{J}H = 3$:

$$(\mathcal{J}H)|_{x=e_1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{A.14}$$

$$(\mathcal{J}H)|_{x=e_1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{A.15}$$

For both cases and both degree 2 and 3, we solved $\det(tI_4 - \mathcal{J}H) = t^4$. This process only gave solutions for A.14, and those solutions satisfy $H_1 = 0$ and $H_2 \in \mathbb{C}[x_1]$. So homogeneous maps H in dimension $n = 4$ with $\mathcal{J}H^n = 0$ and $\mathrm{rk}\mathcal{J}H = n - 1$ satisfy DP+.

For quadratic homogeneous maps, we additionally computed the solutions of the nilpotency condition $\det(tI_4 - \mathcal{J}H) = t^4$ without the trace condition, in order to see whether the linear dependence of the components of H would still hold. This appeared to be the case. So we solved the coefficients of t^0, t^1, t^2 of $\det(tI_4 - \mathcal{J}H)$.

We shall derive the substitution matrices for these computations. Since the case $\mathrm{tr}\mathcal{J}H = 0$ is solved above, we may assume that $\mathrm{tr}\mathcal{J}H$ is a nonzero homogeneous polynomial of degree 2. Let $v \in \mathbb{C}^n$ be generic. Then the Jordan Normal Form of $(\mathcal{J}H)|_{x=v}$ is equal to

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \tau \end{pmatrix} \tag{A.16}$$

where $\tau := \mathrm{tr}(\mathcal{J}H)|_{x=v} \neq 0$ because v is generic. Now by replacing H by a linear conjugation of H, we can obtain that $(\mathcal{J}H)|_{x=w}$ is of the form (A.16), where w is of the form $e_i + \lambda e_4$ for some $i \leq 3$ and a $\lambda \in \mathbb{C}$, or just $w = \lambda e_4$ for some $\lambda \in \mathbb{C}$. But the latter case $w = \lambda e_4$ can immediately be discarded. The same holds for the case $i = 3$ and $\lambda = 0$. The case $i = 2$ and $\lambda = 0$ is impossible because homogeneous quasi-translations $x + H$ satisfy $\mathrm{cork}\mathcal{J}H \geq 2$ and $\mathrm{tr}\mathcal{J}H = 0$. So one case with $\lambda = 0$ and three cases with $\lambda \neq 0$ remain.

Assume first that $\lambda \neq 0$. Put

$$T := \begin{pmatrix} \frac{1}{\tau} & 0 & 0 & 0 \\ 0 & \frac{1}{\tau^2} & 0 & 0 \\ 0 & 0 & \frac{1}{\tau^3} & 0 \\ \frac{\lambda}{\tau^i} & \frac{\lambda}{\tau^i} & \frac{\lambda}{\tau^i} & \frac{\lambda}{\tau^i} \end{pmatrix}$$

and $\tilde{H} = \tau^{(d-1)i-1} T^{-1} H(Tx)$. Assuming that $\mathcal{J}H$ is homogeneous of degree $d-1$, we obtain

$$\begin{aligned} \mathcal{J}\tilde{H}|_{x=e_i} &= \tau^{(d-1)i-1} \mathcal{J}\big(T^{-1}H(Tx)\big)\big|_{x=T^{-1}(\tau^{-i}e_i+\tau^{-i}\lambda e_4)} \\ &= \tau^{-1}T^{-1}\big(\mathcal{J}H\big)\big|_{x=e_i+\lambda e_4} T \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{aligned}$$

By cycling $x = e_i$ to $x = e_1$, we get the following three cases:

$$(\mathcal{J}H)|_{x=e_1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \tag{A.17}$$

$$(\mathcal{J}H)|_{x=e_1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{A.18}$$

$$(\mathcal{J}H)|_{x=e_1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{A.19}$$

Now assume that $\lambda = 0$ and $i = 1$. Put

$$T := \begin{pmatrix} \frac{1}{\tau} & 0 & 0 & 0 \\ 0 & \frac{1}{\tau^2} & 0 & 0 \\ 0 & 0 & \frac{1}{\tau^3} & 0 \\ 0 & 0 & 0 & \frac{1}{\tau^i} \end{pmatrix}$$

and $\tilde{H} = \tau^{(d-1)i-1} T^{-1} H(Tx)$. Assuming that $\mathcal{J}H$ is homogeneous of degree $d-1$, we obtain that

$$\begin{aligned} \mathcal{J}\tilde{H}|_{x=e_i} &= \tau^{(d-1)i-1} \mathcal{J}\big(T^{-1}H(Tx)\big)\big|_{x=T^{-1}(\tau^{-i}e_i)} \\ &= \tau^{-1}T^{-1}\big(\mathcal{J}H\big)\big|_{x=e_i} T \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

and the fourth case is

$$(\mathcal{J}H)|_{x=e_1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{A.20}$$

Now we solved all four cases to the condition that the coefficients of t^0, t^1, t^2 of $\det(tI_n - \mathcal{J}H)$ are zero. This way, we obtained solutions for all four cases, but the solutions of (A.18) and (A.19) did not satisfy the assumption $(\mathcal{J}H^3 - (\mathrm{tr}\mathcal{J}H)\mathcal{J}H^2)x = 0$, and those of (A.20) did not satisfy $\mathcal{J}H^4x = 0$. So only the solutions of (A.17) were proper solutions. The first solution of it was interesting: a linear conjugation of

$$\begin{pmatrix} 0 \\ \frac{1}{2}x_3^2 \\ x_1x_2 - x_3x_4 \\ x_1x_3 \end{pmatrix} + x_3^2 \begin{pmatrix} 0 \\ 0 \\ \mu \\ 0 \end{pmatrix} + x_1^2 \begin{pmatrix} 0 \\ \nu_2 \\ 0 \\ \nu_4 \end{pmatrix}$$

It resembles the map of 4.6.7 somewhat. The other solutions were not so interesting because they were linearly triangularizable (with three zeros on the diagonal of the Jacobian of the triangularization).

Appendix B

Another generalization of Mason's ABC-theorem

The well-known ABC-conjecture is generally formulated as follows:

The ABC-conjecture. *Consider the set S of triples $(A, B, C) \in \mathbb{N}^3$ such that $ABC \neq 0$, $\gcd\{A, B, C\} = 1$ and*

$$A + B = C$$

Then for every $\epsilon > 0$, there exists a constant K_ϵ such that

$$C \leq K_\epsilon \cdot R(ABC)^{1+\epsilon}$$

for all triples $(A, B, C) \in S$, where $R(ABC)$ denotes the square-free part of the product ABC.

The ABC-conjecture is studied in many papers, and this article will not be another of them. Instead, we consider an analog of this conjecture for polynomials over $\mathbb{C}$ instead of integers: Mason's ABC-theorem:

Mason's ABC-theorem. *Let f_1, f_2, f_3 be polynomials over $\mathbb{C}$ without a common factor, not all constant, such that*

$$f_1 + f_2 + f_3 = 0$$

Then

$$\max_{1 \le m \le 3} \deg f_m \le r(f_1 f_2 f_3) - 1$$

where $r(g)$ denotes the number of distinct zeros of g.

This theorem was proved at first by Stothers in [E.13]. So Mason did what Stayman did with the bridge convention that has his name: he made the theorem known, even popular.

The bound in Mason's theorem can be reached by examples of arbitrary large degree, namely $f_1 = f^3$, $f_2 = \mathrm{i}g^2$, $f_3 = -(f^3 - g^2)$, where f and g reach H. Davenport's bound:

$$\deg(f^3 - g^2) \ge \frac{1}{2} \deg f + 1$$

All f and g that reach the Davenport bound are determined in [E.17]. The easiest example is

$$(x^2 + 2)^3 - (x^3 + 3x)^2 = 3x^2 + 8$$

So Mason's theorem seems the best you can get. But there is room for generalization. One direction is followed for the ABC-conjecture as well, namely adding more integers/polynomials to (get) the sum that vanishes. Another direction is allowing more indeterminates in the polynomials. We will discuss both generalizations. There has already been done a lot of work in these direction, mainly using so called *Wronskians*, but it seems that no one has combined all ideas to get the best generalized results one can get by means of Wronskians.

A third direction of generalization is to use elements of so-called *function fields* instead of univariate polynomials [E.3,E.5,E.16], or using *meromorphic functions* instead of multivariate polynomials [E.6]. These generalizations will decrease the readability of this expository paper, so we restrict ourselves to polynomials.

B.1 Generalizations of Mason's ABC-theorem

Let p be a (possibly multivariate) polynomial over $\mathbb{C}$. Then we can factorize p:

$$p = p_1^{e_1} p_2^{e_2} \cdots p_s^{e_s}$$

with all p_i irreducible and pairwise relatively prime, and all $e_i \geq 1$. Let

$$\mathfrak{r}(p) := p_1 p_2 \cdots p_s$$

be the square-free part of p and denote by $r(p)$ the degree of $\mathfrak{r}(p)$.

Associating polynomials with principal ideals, we have that $\mathfrak{r}(p)$ is the radical of p; hence the symbol $\mathfrak{r}$ is used.

Mason's ABC-theorem for three polynomials is generally formulated as follows [E.7, E.11–E.13]:

Theorem B.1.1. *Let f_1, f_2, f_3 be pairwise relatively prime univariate polynomials (in the same variable) over $\mathbb{C}$, not all constant, such that*

$$f_1 + f_2 + f_3 = 0$$

Then

$$\max_{1 \leq m \leq 3} \deg f_m \leq r(f_1 f_2 f_3) - 1$$

In [E.10, Theorem 1.2], H.N. Shapiro and G.H. Sparer generalize theorem B.1.1 as follows, see also [E.6]:

Theorem B.1.2. *Let $n \geq 3$ and $f_1, f_2, \ldots, f_n$ be pairwise relatively prime (possibly multivariate) polynomials over $\mathbb{C}$, not all constant, such that*

$$f_1 + f_2 + \cdots + f_n = 0$$

Then

$$\max_{1 \leq m \leq n} \deg f_m \leq (n-2)\Big(r(f_1 f_2 \cdots f_n) - 1\Big)$$

In [E.1, Theorem 5], M. Bayat and H. Teimoori formulate the following improvement of the estimation bound of theorem B.1.2 (so with all f_i's pairwise relatively prime) as follows: they replace $(n-2)(r(f_1 f_2 \cdots f_n) - 1)$ by

$$(n-2)\left(r(f_1 f_2 \cdots f_n) - \frac{n-1}{2}\right)$$

for the case that at most one of the f_i's is constant and by

$$(n-k-1)\left(r(f_1 f_2 \cdots f_n) - \frac{n-k}{2}\right)$$

for the case that exactly $k \geq 1$ of the f_i's are constant. This is indeed an improvement, for if $k < n$ of the f_i's are constant, then $n - k - 1 \leq n - 2$ and

$$r(f_1 f_2 \cdots f_n) \geq n - k \geq \frac{n-k}{2} \geq 1$$

because there cannot be exactly one f_i that is not constant

Unfortunately, the proof of [E.1, Theorem 5] is incorrect: [E.1, Lemma 4] has counterexamples. But we shall see that the theorem itself is correct. In [E.5], the univariate case of theorem B.1.2 is proved, and also the erratic [E.1, Theorem 5] can be viewed as a correct proof for the univariate case.

But let us first discuss the condition that the f_i's are pairwise relatively prime. This condition is quite restrictive, so it is a good idea to try and get rid of it, and replace it by something weaker. The example $n = 3$, $f_1 = f_2 = x^{100}$, $f_3 = -2x^{100}$ shows that we cannot just forget the condition that all f_i's are relatively prime. So let us replace it by the condition that just

$$\gcd\{f_1, f_2, \ldots, f_n\} = 1 \tag{B.1}$$

Now theorem B.1.2 remains valid for $n = 3$, because the conditions $\gcd\{f_1, f_2, f_3\} = 1$ and $f_1 + f_2 + f_3 = 0$ *imply* that f_1, f_2, f_3 are pairwise relatively prime.

This is no longer the case if $n \geq 4$. Reading the proof of theorem B.1.2 above as given in [E.10], it seems that $r(f_1 f_2 \cdots f_n)$ is just a shorthand notation for $r(f_1) + r(f_2) + \cdots + r(f_n)$, but if the f_i's are not relatively prime, then both expressions are different. So we replace $r(f_1 f_2 \cdots f_n)$ by $r(f_1) + r(f_2) + \cdots + r(f_n)$ as well. There are, however, also generalizations with $r(f_1 f_2 \cdots f_n)$, which we will discuss later.

Now the example $n = 4$, $f_1 = -f_2 = x^{100}$, $f_3 = -f_4 = (x + 1)^{100}$ shows us that we are not ready yet to prove something. The problem is that $f_1 + f_2 + \cdots + f_n$ has a proper subsum that vanishes. Actually, such proper subsums can be seen as instances of the original sum with smaller n, and it seems reasonable that (B.1) is satisfied for these subsums as well, i.e.

$$f_{i_1} + f_{i_2} + \cdots + f_{i_s} = 0 \Longrightarrow \gcd\{f_{i_1}, f_{i_2}, \ldots, f_{i_s}\} = 1$$

where $1 \leq i_1 < i_2 < \cdots < i_s \leq n$. This way we get a valid assertion:

Theorem B.1.3. *Let $n \geq 3$ and $f_1, f_2, \ldots, f_n$ be (possibly multivariate) polynomials over $\mathbb{C}$, not all constant, such that*

$$f_1 + f_2 + \cdots + f_n = 0$$

Assume furthermore that for all $1 \le i_1 < i_2 < \cdots < i_s \le n$,

$$f_{i_1} + f_{i_2} + \cdots + f_{i_s} = 0 \Longrightarrow \gcd\{f_{i_1}, f_{i_2}, \ldots, f_{i_s}\} = 1$$

Then

$$\max_{1\le m\le n} \deg f_m \le (n-2)\Big(r(f_1) + r(f_2) + \cdots + r(f_n) - 1\Big) \tag{B.2}$$

If we replace the constant term -1 on the right hand side of (B.2) by $+n$, then the case in which the f_i's are univariate without a vanishing proper subsum of $f_1 + f_2 + \cdots + f_n$ follows from [E.3, Th. B] and the proof of [E.3, Cor. II]. An improvement of the proof of [E.3, Cor. II] as indicated in section B.5 below subsequently replaces the term $+n$ by $+(n-1)/2$.

If one does not wish to replace $r(f_1 f_2 \cdots f_n)$ by $r(f_1) + r(f_2) + \cdots + r(f_n)$ (and neither requires the f_i's to be prime by pairs), then one can use the inequality $r(f_i) \le r(f_1 f_2 \cdots f_n)$ to obtain a coefficient $n(n-2)$, but in [E.14] and [E.3, Cor. I], it is shown that in the univariate case, $(n-1)(n-2)/2$ is enough and that -1 can be maintained within the parentheses. We will prove the multivariate version of this result:

Theorem B.1.4. *Under the conditions of theorem B.1.3,*

$$\max_{1\le m\le n} \deg f_m \le \frac{(n-1)(n-2)}{2}\Big(r(f_1 f_2 \cdots f_n) - 1\Big) \tag{B.3}$$

B.2 Improvements of theorems B.1.3 and B.1.4

But theorems B.1.3 and B.1.4 are not the best one can get. One improvement on B.1.4 is by U. Zannier in [E.16], but his idea also applies to B.1.3. The coefficient $n-2$ in (B.2) should be expressed in the dimension d of the vector space over $\mathbb{C}$ spanned by the f_i's. Since $f_1 + f_2 + \cdots + f_n = 0$, d is at most $n-1$, so the straightforward improvement is replacing $n-2$ by $d-1$. But also the residual term $(n-2) \cdot -1$ can be improved: the natural improvement of the corresponding term $(n-1)(n-2)/2$ in (9) of [E.1, Theorem 5] is $d(d-1)/2$, so we get

$$\max_{1\le m\le n} \deg f_m \le (d-1)\Big(r(f_1) + r(f_2) + \cdots + r(f_n) - \frac{d}{2}\Big)$$

Another improvement is due to P.-C. Hu and C.-C. Yang in [E.5,E.6]. They extend the definition of the $r(g)$ by defining

$$\mathfrak{r}_e(g) = \gcd\{g, \mathfrak{r}(g)^e\}$$

and $r_e(g) = \deg \mathfrak{r}_e(g)$. So $\mathfrak{r}_1(g) = \mathfrak{r}(g)$ is the square-free part of g and $\mathfrak{r}_2(g)$ is the cube-free part of g, etc. Now we have a trivial inequality

$$r_e(g) \leq e\, r(g)$$

and taking $e = n - 2$ indicates precisely how Hu and Yang improve the estimate: they migrate the coefficient $n-2$ to a subscript of r. This migration has the drawback that the residual term $(n-2) \cdot -1$ does not survive several reductions any more (reductions that decrease the dimension of the vector space over $\mathbb{C}$ spanned by the f_i's). This can be overcome by only stating that there is a ρ with $2 \leq \rho \leq n - 1$, such that

$$\max_{1 \leq m \leq n} \deg f_m \leq (\rho - 1)\Big(r(f_1) + r(f_2) + \cdots + r(f_n) - \frac{\rho}{2}\Big)$$

and combining the above idea with that of Zannier, we even assume that $\rho \leq d$ instead of $\rho \leq n - 1$.

Theorem B.2.1. *Let $n \geq 3$ and $f_1, f_2, \ldots, f_n$ be (possibly multivariate) polynomials over $\mathbb{C}$, not all constant, such that*

$$f_1 + f_2 + \cdots + f_n = 0$$

Assume furthermore that for all $1 \leq i_1 < i_2 < \cdots < i_s \leq n$,

$$f_{i_1} + f_{i_2} + \cdots + f_{i_s} = 0 \Longrightarrow \gcd\{f_{i_1}, f_{i_2}, \ldots, f_{i_s}\} = 1$$

Now let d be the dimension of the vector space over $\mathbb{C}$ spanned by the f_i's. Then there exists a ρ with $2 \leq \rho \leq d$, such that

$$\begin{aligned} \max_{1 \leq m \leq n} \deg f_m &\leq r_{\rho-1}(f_1) + r_{\rho-1}(f_2) + \cdots + r_{\rho-1}(f_n) - \frac{\rho(\rho-1)}{2} && \text{(B.4)} \\ &\leq (d'-1)\left(r(f_1) + r(f_2) + \cdots + r(f_n) - \frac{d'}{2}\right) && \text{(B.5)} \end{aligned}$$

for all d' between d and $n-k+1$ inclusive, where k is the number of constant f_i's.

Proof of [E.1, Theorem 5]. Since $f_1 + f_2 + \cdots + f_n = 0$, it follows that $d \leq n-1$. So the first inequality (9) of [E.1, Theorem 5] follows. Assume that exactly k of the f_i's are constant for some k with $1 \leq k \leq n-1$, and assume without loss of generality that f_n is not constant. Since the vector space over $\mathbb{C}$ spanned by the k constant f_i's has dimension 1 at most, the vector space over $\mathbb{C}$ spanned by $f_1, f_2, \ldots, f_{n-1}$ has dimension $(n-1)-(k-1) = n-k$ at most. But since $f_1 + f_2 + \cdots + f_n = 0$, the latter vector space is also the vector space over $\mathbb{C}$ spanned by $f_1, f_2, \ldots, f_n$. So $d \leq n-k$ and the second inequality (10) of [E.1, Theorem 5] follows as well. $\square$

The improvements on theorem B.1.4 are similar to those on theorem B.1.3:

Theorem B.2.2. *Under the conditions of theorem B.2.1, there exists a σ with $1 \leq \sigma \leq d(d-1)/2$ such that*

$$\begin{aligned} \max_{1 \leq m \leq n} \deg f_m &\leq r_\sigma(f_1 f_2 \cdots f_n) - \sigma && \text{(B.6)} \\ &\leq \frac{d'(d'-1)}{2}\Big(r(f_1 f_2 \cdots f_n) - 1\Big) && \text{(B.7)} \end{aligned}$$

for all $d' \geq d$.

We postpone the proofs of theorems B.2.1 and B.2.2 until section B.6, since we first consider some applications.

B.3 Applications to Fermat-Catalan equations

Just like the ABC-conjecture for integers can be used to tackle Fermat's Theorem for integers, versions of Mason's Theorem can be used to tackle polynomial Diophantic equations:

Theorem B.3.1 (Generalized Fermat-Catalan)**.** *Assume*

$$g_1^{e_1} + g_2^{e_2} + \cdots + g_n^{e_n} = 0$$

and $f_1, f_2, \ldots, f_n$ satisfy the conditions of theorem B.2.1, where $f_i = g_i^{e_i}$ for all i. Then

$$\sum_{i=1}^{n} \frac{1}{e_i} > \frac{1}{d-1}$$

where d is the dimension of the vector space over $\mathbb{C}$ spanned by the f_i's

Proof (based on ideas in [E.5]). Assume f_m has the largest degree among the f_i's. From theorem B.2.1, and $r(f_i) \le \deg g_i = e_i^{-1} \deg f_m$, it follows that

$$\deg f_m \le (d-1)\left(\sum_{i=1}^{n} \frac{1}{e_i} \deg f_m - \frac{d}{2}\right)$$

which rewrites to

$$\left(\sum_{i=1}^{n} \frac{1}{e_i} - \frac{1}{d-1}\right) \deg f_m \ge \frac{d}{2} \tag{B.8}$$

which completes the proof. □

In [E.10, Th. 3.1] and [E.1, Th. 8], theorem B.3.1 is proved by way of the following inequality:

$$\left(\sum_{i=1}^{n} \frac{1}{e_i} - \frac{1}{d-1}\right) \sum_{i=1}^{n} \deg g_i \ge \frac{d}{2} \sum_{i=1}^{n} \frac{1}{e_i} \tag{B.9}$$

but the proof of (B.9) will not be copied in a third article today.

In [E.10, (3.3)] and [E.1, Cor. 10], the result of theorem B.3.1 is rewritten into a Fermat-type equation, i.e. with all e_i equal. But it is not observed that in the Fermat case, the condition that the f_i's are relatively prime by pairs can be omitted. Having a version of a generalized Mason's theorem in which the f_i's must be relatively prime by pairs is only partially an excuse for that, since it suffices to use the case that $f_1, f_2, \ldots, f_{n-1}$ are linearly independent of theorem B.1.3, which can be proved with the methods of [E.10] and [E.1], see also [E.5, E.6, Th. 1.3].

We say that polynomials f_1 and f_2 are *similar* if $f_2 = \lambda f_1$ for some $\lambda \in \mathbb{C}^*$.

Theorem B.3.2 (Generalized Fermat)**.** *Assume*

$$g_1^d + g_2^d + \cdots + g_n^d = 0$$

for some polynomials g_i, not all zero, and suppose that

$$d \ge n(n-2)$$

Then the vanishing sum $g_1^d + g_2^d + \cdots + g_n^d$ decomposes into vanishing subsums

$$g_{i_1}^d + g_{i_2}^d + \cdots + g_{i_s}^d = 0$$

with $1 \le i_1 < i_2 < \cdots < i_s \le n$, for which all g_{i_j}'s are pairwise similar.

Proof. Assume without loss of generality that $g_1 \neq 0$, Since $g_1^d + g_2^d + \cdots + g_n^d = 0$, g_1^d is contained in the vector space over $\mathbb{C}$ spanned by $g_2^d, \ldots, g_n^d$. Assume without loss of generality that

$$g_2^d, \ldots, g_l^d$$

is a basis of this vector space and that

$$g_1^d = \lambda_2 g_2^d + \cdots + \lambda_s g_s^d$$

with $s \leq l \leq n$ and $\lambda_2 \cdots \lambda_s \neq 0$. In order to reduce to the case that the g_i's are relatively prime and $d = n - 1$, we define

$$h_i := \frac{\sqrt[d]{\lambda_i} g_i}{\gcd\{g_1, g_2, \ldots, g_s\}}$$

for all $i \leq s$, where $\lambda_1 = -1$, since then we get

$$h_1^d + \cdots + h_s^d = 0$$

Furthermore, $h_2^d, \ldots, h_s^d$ are linearly independent over $\mathbb{C}$, and

$$\gcd\{h_1^d, \ldots, h_s^d\} = \gcd\{h_1, \ldots, h_s\}^d = 1$$

In order to prove this theorem, it suffices to show that all h_i's are constant at this stage. So assume that this is not the case. Then it follows from theorem B.3.1 that

$$\frac{s}{d} > \frac{1}{s-2}$$

i.e. $d < s(s-2) \leq n(n-2)$. Contradiction, so all h_i's are constant. $\square$

B.4 A theorem of Davenport

Now let us look at sums of powers that do *not* vanish:

$$g_1^{e_1} + g_2^{e_2} + \cdots + g_{n-1}^{e_{n-1}} = g_n \neq 0$$

and suppose that no subsum of $g_1^{e_1} + g_2^{e_2} + \cdots + g_{n-1}^{e_{n-1}}$ vanishes. Now the question is how far the degree of g_n can drop. In [E.4], H. Davenport studied the case $n = 3$, $e_1 = 3$, $e_2 = 2$, and showed that

$$\deg(f^3 - g^2) \geq \frac{1}{2} \deg g + 1$$

see also [E.13]. We shall formulate a generalization of this result that improves [E.5, (6)], by weakening the conditions.

But first, we need some preparations. Notice that (B.7) of theorem B.2.2 follows immediately from (B.6), once you realize that not all f_i's are constant. It is somewhat more work to get (B.5) of theorem B.2.1 from (B.4). At first, we remark that we can take all constant f_i's together, resulting in exactly one constant f_i if they do not cancel out and no constant f_i's if they do. This reduction alters k and n. But $n-k$ is not affected and d only might decrease by one, whence the range of d' is at least preserved. Next, it suffices to prove that

$$\Big((d'-1)-(\rho-1)\Big)\Big(r(f_1)+r(f_2)+\cdots+r(f_n)\Big) \geq \frac{d'(d'-1)}{2} - \frac{\rho(\rho-1)}{2}$$

which follows since the right hand side equals $\rho+(\rho+1)+\cdots+(d'-1) \leq ((d'-1)-(\rho-1))(d'-1)$ and

$$r(f_1)+r(f_2)+\cdots+r(f_n) \geq n-k \geq d'-1$$

where $k \leq 1$ now.

If d' is bounded by $n-k$ instead of $n-k+1$ (and such a d' exists for $d \leq n-k$), then one of the f_i's, say f_n, does not need to be estimated in order to boost the residual term to $d'(d'-1)/2$:

$$\max_{1\leq m\leq n} \deg f_m \leq (d'-1)\left(r(f_1)+r(f_2)+\cdots+r(f_{n-1})-\frac{d'}{2}\right)+r_{\rho-1}(f_n)$$

Estimating $r_{\rho-1}(f_n)$ by $\deg f_n$ and realizing that at least two f_i's have maximum degree, we get (B.10) of theorem B.4.1 below under the conditions of theorem B.2.1:

Theorem B.4.1. *Let $f_1, f_2, \ldots, f_n$ be (possibly multivariate) polynomials over $\mathbb{C}$, not all similar, such that*

$$f_1+f_2+\cdots+f_n=0$$

Assume furthermore that for all $1 \leq i_1 < i_2 < \cdots < i_s \leq n$,

$$f_{i_1}+f_{i_2}+\cdots+f_{i_s}=0 \Longrightarrow \deg\gcd\{f_{i_1}, f_{i_2}, \ldots, f_{i_s}\} \leq \deg f_n$$

Let d be the dimension of the vector space over $\mathbb{C}$ spanned by the f_i's. Then

$$\max_{1\le m\le n-1} \deg f_m - \deg f_n \le (d'-1)\left(r(f_1)+r(f_2)+\cdots+r(f_{n-1})-\frac{d'}{2}\right) \tag{B.10}$$

for all d' between d and $n-k$ inclusive, where k is the number of constant f_i's. Furthermore, equality is only possible in (B.10) if $\gcd\{f_1,f_2,\dots,f_n\}=1$.

Proof. Take $m' \le n-1$ such that $\max_{1\le m\le n-1}\deg f_m = \deg f_{m'}$. We reduce to the case that the conditions of theorem B.2.1 are satisfied. If f_n is constant, then the conditions of theorem B.2.1 are satisfied and hence we are done. So assume that f_n is not constant. Then we can remove all constant f_i's and add them to f_n without affecting the estimate, because $\deg f_n$ and $n-k$ will not change due to this maneuver. Furthermore, subsums $f_{i_1}+f_{i_2}+\cdots+f_{i_s}=0$ for which $\gcd\{f_{i_1},f_{i_2},\dots,f_{i_s}\}\ne 1$ are not affected. Now we distinguish two cases.

- There is a minimal vanishing subsum of $f_1+f_2+\cdots+f_n=0$ that contains both $f_{m'}$ and f_n as summands.
  Assume without loss of generality that $f_{m'}+f_{m'+1}+\cdots+f_n=0$ and let $h:=\gcd\{f_{m'},f_{m'+1},\dots,f_n\}$. Then

$$\begin{aligned}
&\deg f_{m'} - \deg f_n\\
&\quad= \deg\frac{f_{m'}}{h} - \deg\frac{f_n}{h}\\
&\quad\le (d'-1)\left(r\left(\frac{f_{m'}}{h}\right)+r\left(\frac{f_{m'+1}}{h}\right)+\cdots+r\left(\frac{f_{n-1}}{h}\right)-\frac{d'}{2}\right)\\
&\quad\le (d'-1)\left(r(f_{m'})+r(f_{m'+1})+\cdots+r(f_{n-1})-\frac{d'}{2}\right)\\
&\quad\le (d'-1)\left(r(f_1)+r(f_2)+\cdots+r(f_{n-1})-\frac{d'+m'-1}{2}\right)
\end{aligned}$$

  where d' is at least the dimension of the vector space spanned by $f_{m'}$, $f_{m'+1},\dots,f_n$ and at most $n-m'+1$, and equality is only possible if h is constant and $m'=1$. This gives the desired result.

- There is no minimal vanishing subsum of $f_1+f_2+\cdots+f_n=0$ that contains both $f_{m'}$ and f_n as summands.
  Assume without loss of generality that $f_1+f_2+\cdots+f_{m'}=0$ and let

$h := \gcd\{f_1, f_2, \ldots, f_{m'}\}$. Then $\deg h \le \deg f_n$. In case $f_1, f_2, \ldots, f_{m'}$ are all similar, then the left hand side of (B.10) is zero and the right hand side is positive, as desired. So assume that that is not the case. By (B.5) in theorem B.2.1,

$$\begin{aligned}
&\deg f_{m'} - \deg f_n \\
&\quad\le\ \deg \frac{f_{m'}}{h} \\
&\quad\le\ (d'-1)\left(r\left(\frac{f_1}{h}\right) + r\left(\frac{f_2}{h}\right) + \cdots + r\left(\frac{f_{m'}}{h}\right) - \frac{d'}{2}\right) \\
&\quad\le\ (d'-1)\left(r(f_1) + r(f_2) + \cdots + r(f_{m'}) - \frac{d'}{2}\right) \\
&\quad\le\ (d'-1)\left(r(f_1) + r(f_2) + \cdots + r(f_{n-1}) - \frac{d'+n-m'-1}{2}\right)
\end{aligned}$$

where d' is at least the dimension of the vector space spanned by $f_1, f_2, \ldots, f_{m'}$ and at most $m'+1$, and equality is not possible because $m' = n-1$ implies $f_n = 0$. This gives the desired result. □

Now substitute $f_i = g_i^{e_i}$ for all $i \le n-1$ and also $f_n = -g_n = \sum_{i=1}^{n} g_i^{e_i}$, in (B.10). Then

$$\left(\sum_{i=1}^{n-1} \frac{1}{e_i} - \frac{1}{d'-1}\right) \max_{1\le m\le n-1} \deg g_m^{e_m} \ge \frac{d'}{2} - \frac{1}{d'-1} \deg \sum_{i=1}^{n-1} g_i^{e_i} \qquad \text{(B.11)}$$

follows from (B.10) in a similar way as (B.8) follows from (B.5) of theorem B.2.1, see also [E.5, (6)].

Indeed, applying (B.11) on the sum $f^3+(\mathrm{i}g)^2$ gives $-\frac{1}{6}\deg(f^3) \ge 1-\deg(f^3-g^2)$ for $d'=2$, which is equivalent to $\deg(f^3-g^2) \ge \frac{1}{2}\deg f + 1$. For $d'=3$, we get $\frac{1}{3}\deg f^3 \ge \frac{3}{2} - \frac{1}{2}\deg(f^3-g^2)$, i.e. $\deg(f^3-g^2) \ge 3 - 2\deg f$, which is useless.

By replacing n by $n+1$ in (B.11), we obtain the following from theorem B.4.1.

Theorem B.4.2. *Assume*

$$g_1^{e_1} + g_2^{e_2} + \cdots + g_n^{e_n} \ne 0$$

and no subsum of $g_1^{e_1} + g_2^{e_2} + \cdots + g_n^{e_n}$ *vanishes. Then*

$$\left(\sum_{i=1}^{n} \frac{1}{e_i} - \frac{1}{d'-1}\right) \max_{1 \le m \le n} \deg g_m^{e_m} \ge \frac{d'}{2} - \frac{1}{d'-1} \deg \sum_{i=1}^{n} g_i^{e_i}$$

for all d' *between* d *and* $n+1$ *inclusive, where* d *is the dimension of the vector space over* $\mathbb{C}$ *spanned by* $g_1^{e_1}, g_2^{e_2}, \ldots, g_n^{e_n}$*. Furthermore, equality cannot be reached in case* $\gcd\{g_1, g_2, \ldots, g_n\} \ne 1$*.*

In [E.17], it is proved that for all even degrees of f, there are univariate polynomials f, g over $\mathbb{C}$ such that $\deg(f^3 - g^2) = \frac{1}{2}\deg f + 1$. Now assume $\deg(f^3 - g^2) = \frac{1}{2}\deg f + 1$. Then $\gcd\{f, g\} = 1$ and the Mason bound on $-f^3 + g^2 + (f^3 - g^2) = 0$ gives us

$$\deg f^3 \le r_1\Big(fg(f^3 - g^2)\Big) - 1 \le \deg\Big(fg(f^3 - g^2)\Big) - 1$$

which is bound to be an equality. Furthermore, $fg(f^3 - g^2)$ is bound to be square-free. But any linear combination $\lambda f^3 + \mu g^2$ with $\lambda\mu \ne 0$ is bound to be square-free, since otherwise the inequality

$$\deg f^3 \le \frac{1}{2}\Big(r_1(f^3) + r_1(g^2) + r_1(f^3 - g^2) + r_1(\lambda f^3 + \mu g^2) - 1\Big)$$

would be violated. The above estimate is an instance of (B.12) in section B.5 below, since there exists a vanishing linear combination without zero coefficients of the arguments of r_1 on the right hand side.

B.5 Some discussion on theorems B.2.1 and B.2.2

We describe now why the condition that all f_i's are relatively prime by pairs is needed in [E.1, E.5, E.6, E.10]. They reduce to the case of maximal dimension $d = n - 1$ as follows. Assume that f_n has the largest degree and say that $f_1, f_2, \ldots, f_d$ is a basis of the vector space over $\mathbb{C}$ spanned by $f_1, f_2, \ldots, f_n$. Then

$$f_n = \lambda_1 f_1 + \lambda_2 f_2 + \cdots + \lambda_d f_d$$

for some $\lambda_i \in \mathbb{C}$. The greatest common divisor of the f_i's in the above sum is still the same as in the original sum, but some f_i's might have a coefficient

λ_i that is zero; say that $\lambda_1\lambda_2\cdots\lambda_\rho \neq 0$ and $\lambda_{\rho+1} = \lambda_{\rho+2} = \cdots = \lambda_d = 0$. Then

$$\lambda_1 f_1 + \lambda_2 f_2 + \cdots + \lambda_\rho f_\rho + (-f_n) = 0$$

is a vanishing sum of maximal dimension ρ. But the problem is that the greatest common divisor of the the f_i's in the last sum might be larger than that of the original sum.

But the above method does work when each set of d f_i's generates the whole vector space over $\mathbb{C}$ spanned by the f_i's, because that implies that $\rho = d$ above. So in this case one can get the estimates of theorems B.2.1 and B.2.2. But one can get even better estimates in this particular case, namely

$$\max_{1\le m\le n} \deg f_m \le \frac{1}{n-d}\left(r_{\rho-1}(f_1) + r_{\rho-1}(f_2) + \cdots + r_{\rho-1}(f_n) - \frac{\rho(\rho-1)}{2}\right) \tag{B.12}$$

and

$$\max_{1\le m\le n} \deg f_m \le \frac{1}{n-d}\big(r_\sigma(f_1 f_2 \cdots f_n) - \sigma\big) \tag{B.13}$$

combining techniques of [E.6] and the proof of [E.16, Th. 2], and also ideas in section B.7 to get $\rho(\rho-1)/2 \le \sigma$. We sketch the proof at the very end of this article.

In [E.2, Th. 2] it is shown that the coefficient $d'(d'-1)/2$ of (B.7) in theorem B.2.2 cannot be replaced by something less than $2n-5$, and the author conjectures that this coefficient can indeed be improved to $2n-5$, i.e.

$$\max_{1\le m\le n} \deg f_m \le (2n-5)\big(r(f_1 f_2 \cdots f_n) - 1\big)$$

I did not find similar considerations on (B.5) in theorem B.2.1 in literature. So let us do something ourselves. The factor $(d'-1)$ in (B.5) cannot be improved, as is shown by the example

$$\begin{aligned} f_i &= \binom{n-2}{i-1}\left(x^{10^{100}}\right)^{i-1} \qquad (1 \le i < n) \\ f_n &= -\left(x^{10^{100}} + 1\right)^{n-2} \end{aligned}$$

The term $d'/2$ in (B.5) cannot be improved to $3d'/4$, as is shown by the example

$$\begin{aligned} f_i &= \lceil n/2\rceil \binom{\lceil n/2\rceil(\lfloor n/2\rfloor+1)-2}{\lceil n/2\rceil i - 1} x^{\lceil n/2\rceil i-1} \qquad (i \le \lfloor n/2\rfloor) \\ f_i &= -\zeta_{\lceil n/2\rceil}^i \left(x + \zeta_{\lceil n/2\rceil}^i\right)^{\lceil n/2\rceil(\lfloor n/2\rfloor+1)-2} \qquad (i > \lfloor n/2\rfloor) \end{aligned}$$

for the case that none of the f_i's is constant, and by the example

$$\begin{aligned} f_i &= \lceil n/2 \rceil \binom{\lceil n/2 \rceil \lfloor n/2 \rfloor - 1}{\lceil n/2 \rceil (i-1)} x^{\lceil n/2 \rceil (i-1)} \quad (i \le \lfloor n/2 \rfloor) \\ f_i &= -\zeta_{\lceil n/2 \rceil}^{-i} \left(x + \zeta_{\lceil n/2 \rceil}^{i} \right)^{\lceil n/2 \rceil \lfloor n/2 \rfloor - 1} \quad (i > \lfloor n/2 \rfloor) \end{aligned}$$

for the case that f_1 is constant, but it might be possible to improve it to $3(d'-1)/4$.

In section B.4, we have reduced (B.5) in theorem B.2.1 to (B.4) and (B.7) in theorem B.2.2 to (B.6). Therefore it remains to prove (B.4) and (B.6). But before we do that, we ask ourselves the question whether (B.4) and (B.6) can be seen as instances of one single, more general estimate. [E.3] has some valuable ideas in that direction. Under the extra assumption that the f_i's are univariate and $d = n - 1$, (B.7) for $d' = d = n$ follows immediately from [E.2, Cor. I], and [E.2, Cor. II] implies

$$\max_{1 \le m \le n} \deg f_m \le (n-2)\big(r(f_1) + r(f_2) + \cdots + r(f_n) + 1\big)$$

but, since the f_i's are linearly independent, the number k of constant f_i's is at most 1. Since the number of empty S_i's in [E.2, Cor. II] equals k as well, one can improve [E.2, Cor. II] to

$$\begin{aligned} H(u_1, u_2, \ldots, u_n) \quad \le \quad & (n-2)\left(|S_1| + |S_2| + \cdots + |S_n| + k - \frac{n+1}{2} \right) - \\ & \frac{(n-1)(n-2)}{2}(2g-2) \end{aligned} \tag{B.14}$$

and (B.5) in theorem B.2.1 for $d' = d = n$ follows.

The proof of (B.14) is left as an exercise to the interested reader. The general result that implies both [E.2, Col. I] and (the improved version (B.14) of) [E.2, Col. II] is [E.2, Theorem A].

The rest of this article is organized as follows. In sections B.6 to B.8, we prove (B.4) of theorem B.2.1 and (B.6) of theorem B.2.2. In section B.6, we reduce to the univariate case. In section B.7, we present the Wronskian, the key element in all generalized versions of Mason's theorem, except [E.14]. Section B.8 consists of the actual proofs of (B.4) and (B.6). At last, in section B.9, we combine (B.4) and (B.6) with ideas of [E.2].

B.6 Some reductions of the main theorem

By replacing the original sum by the minimal vanishing subsum containing $f_{m'}$ as a term, where $\deg f_{m'} = \max_{1\le m\le n} \deg f_m$, we see that in order to prove (B.4) of theorem B.2.1 and (B.6) of theorem B.2.2, we can restrict ourselves to the case that $f_1 + f_2 + \cdots + f_n$ has no proper subsum that vanishes.

We show now that we can restrict ourselves to the case that the f_i's are univariate. More particular, a generic substitution $x_i = p_i y + q_i$ will do the reduction. Assume that no proper subsum of $f_1 + f_2 + \cdots + f_n$ vanishes and say that there are l variables in the f_i's. Let G be the set of nonempty proper subsums

$$f_{i_1} + f_{i_2} + \cdots + f_{i_s}$$

and

$$\bar{G} = \{\bar{g} \mid g \in G\}$$

where $\bar{g}$ is the largest degree homogeneous part of g (i.e. the sum of all terms that have the same degree as g). Now pick a $p \in \mathbb{C}^l$ such that

$$\bar{g}(p) \neq 0$$

for all $\bar{g} \in \bar{G}$ (a p that has coordinates that are transcendental over the field of coefficients of the $\bar{g}$'s will do).

Assume without loss of generality that $p_1 \neq 0$ and define

$$\hat{f}_i := f_i(p_1x_1, x_2 + p_2x_1, \ldots, x_l + p_lx_1)$$

for all i. Since $\gcd\{f_1, f_2, \ldots, f_n\} = 1$, $\gcd\{\hat{f}_1, \hat{f}_2, \ldots, \hat{f}_n\} = 1$ as well. So if we apply the extended gcd-theorem with respect to x_1, we find $a_i \in \mathbb{C}(x_2, \ldots, x_l)[x_1]$ such that

$$1 = a_1\hat{f}_1 + a_2\hat{f}_2 + \cdots + a_n\hat{f}_n$$

For each i, write $a_i = \sum_{j=1}^{\infty} a_{i,j}x_1^j$ with $a_{i,j} \in \mathbb{C}(x_2, \ldots, x_l)$ and only finitely many $a_{i,j}$ nonzero. Now put $q_1 := 0$ and take $(q_2, \ldots, q_l) \in \mathbb{C}^{k-1}$ such that the denominators of the nonzero $a_{i,j}$'s do not vanish on $(q_2, \ldots, q_l)$. Then

$$\begin{aligned} 1 \;=\;& a_1(q_2, \ldots, q_l)[x_1]\hat{f}_1(x_1, q_2, \ldots, q_l) + \\ & a_2(q_2, \ldots, q_l)[x_1]\hat{f}_2(x_1, q_2, \ldots, q_l) + \cdots + \\ & a_n(q_2, \ldots, q_l)[x_1]\hat{f}_n(x_1, q_2, \ldots, q_l) \end{aligned} \tag{B.15}$$

Put

$$\tilde{f}_i := \hat{f}_i(y, q_2, \ldots, q_l) = f_i(q + yp) = f_i(p_1 y + q_1, p_2 y + q_2, \ldots, p_l y + q_l)$$

for all i. From (B.15), it follows that $\gcd\{\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n\} = 1$.

Since $r_{\rho-1}(\tilde{f}_i) \le r_{\rho-1}(f_i)$ for all i and $r_\sigma(\tilde{f}_1 \tilde{f}_2 \cdots \tilde{f}_n) \le r_\sigma(f_1 f_2 \cdots f_n)$, it suffices to show that $\deg \tilde{f}_i = \deg f_i$ for all i and no proper subsum of $\tilde{f}_1 + \tilde{f}_2 + \cdots + \tilde{f}_n = 0$ vanishes. We do so by proving that for all proper subsets I of $\{1, 2, \ldots, n\}$:

$$\deg\left(\sum_{i \in I} \tilde{f}_i\right) = \deg\left(\sum_{i \in I} f_i\right)$$

i.e.

$$\deg g(q + yp) = \deg g$$

for all $g \in G$. This is true, since the coefficient of $y^{\deg g}$ in $g(q + yp)$ is equal to $\bar{g}(p)$, which is nonzero by assumption.

B.7 The Wronskian

Let $f_1, f_2, \ldots, f_n$ be polynomials in one and the same variable, say y. Then the *Wronskian determinant* of $f_1, f_2, \ldots, f_n$ is defined as

$$W(f_1, f_2, \ldots, f_n) := \det \begin{pmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{pmatrix}$$

and the *Wronskian matrix* is the corresponding matrix on the right hand side.

Since differentiating is a linear operator, it follows that $W(f_1, f_2, \ldots, f_n) = 0$ in case

$$\lambda_1 f_1 + \lambda_2 f_2 + \cdots + \lambda_n f_n = 0 \tag{B.16}$$

for some nonzero $\lambda \in \mathbb{C}^n$. Now a classical theorem tells us that the reverse is true as well: if $f_1, f_2, \ldots, f_n$ are linearly independent (i.e. (B.16) implies $\lambda = 0$), then $W(f_1, f_2, \ldots, f_n) \neq 0$. The example $f_1(x) = x^3$, $f_2(x) = |x|^3$ shows us that the f_i's need to be polynomials.

Despite that the oldest known proof of this theorem by Frobenius is elementary, we give another proof, inspired by the proof of [E.15, Lm. 8]. The reason for that will be given below.

So let us assume that $f_1, f_2, \ldots, f_n$ are linearly dependent. If there are two f_i's with the same degree, then we can subtract a multiple of the first from the second to reduce the degree of the second, since this operation does not affect the Wronskian determinant. Progressing in this direction gives us that all f_i's have different degrees. Now order the f_i's by increasing degrees. This might only change the sign of the Wronskian determinant.

The matrix

$$\begin{pmatrix} f_1^{(\deg f_1)} & f_2^{(\deg f_1)} & \cdots & f_n^{(\deg f_1)} \\ f_1^{(\deg f_2)} & f_2^{(\deg f_2)} & \cdots & f_n^{(\deg f_2)} \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(\deg f_n)} & f_2^{(\deg f_n)} & \cdots & f_n^{(\deg f_n)} \end{pmatrix}$$

is upper triangular and does not have zeros on the diagonal. Hence, its determinant does not vanish. Since it is a submatrix of

$$M := \begin{pmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ f_1^{(2)} & f_2^{(2)} & \cdots & f_n^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(\deg f_n)} & f_2^{(\deg f_n)} & \cdots & f_n^{(\deg f_n)} \end{pmatrix}$$

this latter matrix has full rank n. Now we can make a square matrix M' of full rank n out of M by throwing away redundant rows of M, i.e. throwing away rows that are dependent of the rows above it. It suffices to prove that M' is the Wronskian matrix, i.e.

$$M' = \begin{pmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ f_1^{(2)} & f_2^{(2)} & \cdots & f_n^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{pmatrix}$$

Write $f^{(i)}$ for the vector

$$(f_1^{(i)}, f_2^{(i)}, \cdots, f_n^{(i)})$$

and $f = f^{(0)}$ and $f' = f^{(1)}$. Assume that the m-th row of M' is $(f^{(m-1)})^{\mathrm{t}}$, but the $(m+1)$-th row of M' is not $(f^{(m)})^{\mathrm{t}}$, say it is $(f^{(j)})^{\mathrm{t}}$ with $j > m$. Then $(f^{(j-1)})^{\mathrm{t}}$ is in the space generated by the first m rows of M', i.e.

$$f^{(j-1)} = a_0 f + a_1 f' + a_2 f^{(2)} + \cdots + a_{m-1} f^{(m-1)} \tag{B.17}$$

where the a_i are rational functions, i.e. quotients of polynomials, for all i. Differentiating (B.17) gives

$$f^{(j)} = (a_0' f + a_0 f') + (a_1' f' + a_1 f'') + \cdots + (a_{m-1}' f^{(m-1)} + a_{m-1} f^{(m)})$$

Since each of the $2m$ terms on the right hand side is contained in the space generated by the first m rows of M', $f^{(j)}$ is contained in this space as well. Contradiction, so the m-th row of M' is $(f^{(m-1)})^{\mathrm{t}}$ for all m.

In [E.9, Lemma 6, pp. 15-16], a generalization of the Wronskian theorem for more variables is formulated. The operators $\frac{\partial^i}{\partial y^i}$ are in fact replaced by operators Δ_i, each of which is a product of partial derivatives. The number of partial derivatives that Δ_i decomposes into, multiple appearances counted by their frequency, is called the *order* $o(\Delta_i)$ of Δ_i.

The usual Wronskian determinant is replaced by

$$W_\Delta(f_1, f_2, \ldots, f_n) := \det \begin{pmatrix} \Delta_1 f_1 & \Delta_1 f_2 & \cdots & \Delta_1 f_n \\ \Delta_2 f_1 & \Delta_2 f_2 & \cdots & \Delta_2 f_n \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_n f_1 & \Delta_n f_2 & \cdots & \Delta_n f_n \end{pmatrix} \tag{B.18}$$

and the author T. Schneider of [E.9] proves that if $f_1, f_2, \ldots, f_n$ are linearly independent, then $W_\Delta(f_1, f_2, \ldots, f_n) \neq 0$ for certain operators Δ_i of order $i-1$ at most. In particular, Δ_1 is the identity operator, and the first row looks the same as in the case of one variable.

Unlike the above proof of the classical Wronskian theorem, the proof of this theorem by Frobenius cannot be generalized to more indeterminates. The way Schneider proves his multivariate result is by reducing to the univariate Wronskian theorem. But his theorem does not show that there are Δ_i's of all orders $0, 1, 2, \ldots, \rho$, where ρ is the maximum order of the Δ_i's, unlike a straightforward generalization of the above proof of the classical Wronskian theorem to more indeterminates. Neither does his methods give tools to prove that

$$W_\Delta(hf_1, hf_2, \ldots, hf_n) = h^n W_\Delta(f_1, f_2, \ldots, f_n) \tag{B.19}$$

(B.19) can be found in [E.6, Lm. 2.1]. But this lemma is somewhat different to both our methods and [E.9, Lemma 6, pp. 15-16], since the Wronskian determinant might be zero.

Take for instance $f = (1, xy, x^2y^2)$. Notice that

$$W_{1,\frac{\partial}{\partial x},\frac{\partial^2}{\partial x^2}}(1, xy, x^2y^2) = \det \begin{pmatrix} 1 & xy & x^2y^2 \\ 0 & y & 2xy^2 \\ 0 & 0 & 2y^2 \end{pmatrix} = 2y^3$$

and this is also a generalized Wronskian one can get by the multivariate variant of the above method, since $\frac{\partial}{\partial y}x^iy^i = ix^iy^{i-1} = \frac{x\partial}{y\partial y}x^iy^i$. The above Wronskian matrix is however *not* of the form of [E.6, Lm. 2.1] and [E.15, Lm. 8], because $\frac{\partial}{\partial y}f$ is *not* linearly dependent over $\mathbb{C}$ of its rows. The Wronskian matrix of both lemma's must be that of

$$W_{1,\frac{\partial}{\partial x},\frac{\partial}{\partial y}}(1, xy, x^2y^2) = 0$$

instead.

In the proofs of theorems B.2.1 and B.2.2, we shall employ a special generalized Wronskian, one without an identity operator:

Lemma B.7.1. *Let $f_1, f_2, \ldots, f_n$ be polynomials over $\mathbb{C}$ in the variables $y, z_1, z_2, \ldots, z_l$, such that each f_i is of the following form:*

$$f_i = (\lambda_{1,i}z_1 + \lambda_{2,i}z_2 + \cdots + \lambda_{l,i}z_l) \cdot \tilde{f}_i$$

where $\tilde{f}_i$ is a polynomial over $\mathbb{C}$ in the variable y. Assume that $f_1, f_2, \ldots, f_n$ are linearly independent. Then there exists a $\Delta = (\Delta_1, \Delta_2, \ldots, \Delta_n)$ with

$$W_\Delta(f_1, f_2, \ldots, f_n) \neq 0$$

such that for each i, either

$$\Delta_i = \frac{\partial}{\partial z_j}$$

for some j, or (if $i \geq 2$)

$$\Delta_i = \frac{\partial}{\partial y}\Delta_{i-1}$$

Proof. Choose j such that $\lambda_{j,n} \neq 0$. Say that $\lambda_{j,1} = \cdots = \lambda_{j,m} = 0$ and $\lambda_{j,m+1} \cdots \lambda_{j,n} \neq 0$. We distinguish three cases:

- $\frac{\partial}{\partial z_j} f_{m+1}, \ldots, \frac{\partial}{\partial z_j} f_n$ are linearly dependent.
  Say that
  $$\frac{\partial}{\partial z_j} f_{m+1} = \mu_{m+2} \frac{\partial}{\partial z_j} f_{m+2} + \cdots + \mu_n \frac{\partial}{\partial z_j} f_n$$
  Replace f_{m+1} by $f_{m+1} - (\mu_{m+2} f_{m+2} + \cdots + \mu_n f_n)$ and apply induction on $-m$.

- $\frac{\partial}{\partial z_j} f_{m+1}, \ldots, \frac{\partial}{\partial z_j} f_n$ are linearly independent and $m = 0$.
  Then the result follows by applying the Wronskian theorem (in one variable) on $\frac{\partial}{\partial z_j} f_1, \frac{\partial}{\partial z_j} f_2, \ldots, \frac{\partial}{\partial z_j} f_n$. The operators are $\Delta_i = \frac{\partial^i}{\partial y^{i-1} \partial z_j}$.

- $\frac{\partial}{\partial z_j} f_{m+1}, \ldots, \frac{\partial}{\partial z_j} f_n$ are linearly independent and $m \geq 1$.
  From the above case, it follows that $W_D(f_{m+1}, \ldots, f_n) \neq 0$, where $D_i := \frac{\partial^i}{\partial y^{i-1} \partial z_j}$. By induction on n, we have $W_\Delta(f_1, f_2, \ldots, f_m) \neq 0$. Now extend Δ by defining $\Delta_{m+i} = D_i$ for all $i \geq 1$. Since $\frac{\partial}{\partial z_j} f_i = 0$ for all $i \leq m$, it follows that
  $$W_\Delta(f_1, f_2, \ldots, f_n) = W_\Delta(f_1, \ldots, f_m) \cdot W_D(f_{m+1}, \ldots, f_n) \neq 0$$
  and Δ remains of the desired form. □

Notice that the above lemma can be generalized to more variables as well.

B.8 Proof of the main theorem

From the reductions in sections B.4 and B.6, it follows that in order to prove theorems B.2.1 and B.2.2, it suffices to prove the following:

Theorem B.8.1. *Let $\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n$ be nonzero polynomials over $\mathbb{C}$ in the variable y such that $\gcd\{\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n\} = 1$ and*

$$\tilde{f}_1 + \tilde{f}_2 + \cdots + \tilde{f}_n = 0$$

Let d be the dimension of the vector space over $\mathbb{C}$ spanned by the $\tilde{f}_i$'s and assume furthermore that no proper subsum of $\tilde{f}_1 + \tilde{f}_2 + \cdots + \tilde{f}_n$ vanishes. Then

$$\max_{1 \leq m \leq n} \deg \tilde{f}_m \leq r_{\rho-1}(\tilde{f}_1) + r_{\rho-1}(\tilde{f}_2) + \cdots + r_{\rho-1}(\tilde{f}_n) - \frac{\rho(\rho-1)}{2}$$

for some ρ *with* $2 \le \rho \le d$*, and*

$$\max_{1\le m\le n} \deg \tilde{f}_m \le r_\sigma(\tilde{f}_1\tilde{f}_2\cdots\tilde{f}_n) - \sigma$$

for some σ *with* $1 \le \sigma \le d(d-1)/2$*.*

Assume without loss of generality that $\tilde{f}_1, \tilde{f}_2, \dots, \tilde{f}_d$ is a basis of the vector space over $\mathbb{C}$ spanned by the $\tilde{f}_i$'s. For each $j > d$, there exists unique $\lambda_{j,i}$ such that

$$\tilde{f}_j = \sum_{i=1}^{d} \lambda_{j,i}\tilde{f}_i \tag{B.20}$$

In order to get rid of all linear relations between the $\tilde{f}_i$'s except the sum relation, we define

$$f_i := \left(\sum_{j=d+1}^{n} \lambda_{j,i} z_j\right) \cdot \tilde{f}_i$$

for all $i \le d$, and

$$f_i := -z_i \cdot \tilde{f}_i$$

for all $i > d$. It follows from (B.20) that

$$\begin{aligned}\sum_{i=1}^{n} f_i &= \sum_{i=1}^{d}\sum_{j=d+1}^{n} \lambda_{j,i} z_j \tilde{f}_i - \sum_{j=d+1}^{n} z_j \tilde{f}_j \\ &= \sum_{j=d+1}^{n} z_j \left(\sum_{i=1}^{d} \lambda_{j,i}\tilde{f}_i - \tilde{f}_j\right) \\ &= 0\end{aligned}$$

Furthermore, it follows from (B.20) that

$$\sum_{i=1}^{d}\left(1 + \sum_{j=d+1}^{n} \lambda_{j,i}\right)\tilde{f}_i = \sum_{i=1}^{d}\tilde{f}_i + \sum_{j=d+1}^{n}\sum_{i=1}^{d}\lambda_{j,i}\tilde{f}_i = \sum_{i=1}^{n}\tilde{f}_i = 0$$

whence

$$\sum_{j=d+1}^{n} \lambda_{j,i} = -1 \quad (1 \le i \le d) \tag{B.21}$$

for $\tilde{f}_1, \tilde{f}_2, \dots, \tilde{f}_d$ are linearly independent.

Lemma B.8.2. *$\mu_1 f_1 + \mu_2 f_2 + \cdots + \mu_n f_n = 0$ implies $\mu_1 = \mu_2 = \cdots = \mu_n$.*

Proof. Let G be the graph with vertices $\{1, 2, \ldots, n\}$ and connect two vertices j, i by an edge if $\lambda_{j,i} \neq 0$. Notice that G is a bipartite graph between $\{1, 2, \ldots, d\}$ and $\{d+1, \ldots, n\}$. We first show that G is connected. Assume the opposite. Say that G does not have an edge between $\{1, \ldots, d', d+1, \ldots, n'\}$ and $\{d'+1, \ldots, d, n'+1, \ldots, n\}$, where either $d' < d$ or $n' < n$. Then $\lambda_{j,i} = 0$ for all $j > n'$ and $i \leq d'$, whence by (B.21)

$$\sum_{j=d+1}^{n'} \lambda_{j,i} = -1 \tag{B.22}$$

for all $i \leq d'$. On the other hand, $\lambda_{j,i} = 0$ for all $j \leq n'$ and $i > d'$, whence

$$\sum_{j=d+1}^{n'} \lambda_{j,i} = 0 \tag{B.23}$$

for all $i > d'$.

Substituting $z_j = 1$ for all $j \leq n'$ and $z_j = 0$ for all $j > n'$ in $\sum_{i=1}^{n} f_i$, it follows from (B.22) and (B.23) that we obtain

$$\sum_{i=1}^{d} \left(\sum_{j=d+1}^{n'} \lambda_{j,i} \right) \tilde{f}_i - \sum_{j=d+1}^{n'} \tilde{f}_j = -\sum_{i=1}^{d'} \tilde{f}_i - \sum_{j=d+1}^{n'} \tilde{f}_j$$

which is zero, since $\sum_{i=1}^{n} f_i$ is zero. Since no proper subsum of $\sum_{i=1}^{n} \tilde{f}_i$ vanishes, we have $d' = d$ and $n' = n$. Contradiction, so G is connected.

Now assume $\mu_1 f_1 + \mu_2 f_2 + \cdots + \mu_n f_n = 0$. Pick a $j > d$. Substituting $z_j = 1$ and $z_m = 0$ for all $m \neq j$ in $\sum_{i=1}^{n} \mu_i f_i$ gives us

$$\sum_{i=1}^{d} \mu_i \lambda_{j,i} \tilde{f}_i - \mu_j \tilde{f}_j = 0$$

but on account of (B.20), also

$$\sum_{i=1}^{d} \mu_j \lambda_{j,i} \tilde{f}_i - \mu_j \tilde{f}_j = 0$$

so by subtraction

$$\sum_{i=1}^{d}(\mu_i - \mu_j)\lambda_{j,i}\tilde{f}_i = 0$$

Since $\tilde{f}_1, \tilde{f}_2, \cdots, \tilde{f}_d$ are linearly independent over $\mathbb{C}$, $(\mu_i - \mu_j)\lambda_{j,i} = 0$ for all $i \leq d$. So

$$\lambda_{j,i} \neq 0 \implies \mu_i = \mu_j \tag{B.24}$$

Since G is connected, the desired result follows. □

From lemma B.8.2, it follows that $f_1, f_2, \ldots, f_{n-1}$ are linearly independent, whence we can apply lemma B.7.1 to get

$$W_\Delta(f_1, f_2, \ldots, f_{n-1}) \neq 0$$

where $\Delta = (\Delta_1, \Delta_2, \ldots, \Delta_{n-1})$ satisfies the properties of lemma B.7.1. Since $f_1 + f_2 + \cdots + f_n = 0$, we have

$$\begin{aligned} W_\Delta(f_1, f_2, \ldots, f_{n-1}) &= (-1)^{n-i} W_\Delta(f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_n) \\ &= (-1)^{n-1} W_\Delta(f_2, \ldots, f_{n-1}, f_n) \end{aligned} \tag{B.25}$$

Let ρ be the maximum among the orders $o(\Delta_1), o(\Delta_2), \ldots, o(\Delta_{n-1})$, i.e. the maximum number of partial derivatives which any Δ_m may decomposes into. Put

$$\sigma := \sum_{i=1}^{n-1}(o(\Delta_i) - 1)$$

Let $j > d$. Since $\frac{\partial}{\partial z_m} f_j = 0$ for all $j \neq m$, and the left hand side of (B.25) does not vanish, $\frac{\partial}{\partial z_j} \in \{\Delta_1, \Delta_2, \ldots, \Delta_{n-1}\}$. A similar argument on the right hand side of (B.25) gives $\frac{\partial}{\partial z_n} \in \{\Delta_1, \Delta_2, \ldots, \Delta_{n-1}\}$. So $n - d$ of the $n - 1$ Δ_i's have order 1. It follows from lemma B.7.1 that

$$2 \leq \rho \leq d \qquad \text{and} \qquad 1 \leq \frac{\rho(\rho - 1)}{2} \leq \sigma \leq \frac{d(d-1)}{2}$$

Lemma B.8.3.

$$\tilde{f}_1 \tilde{f}_2 \cdots \tilde{f}_n \,\Big|\, \mathfrak{r}_{\rho-1}(\tilde{f}_1)\mathfrak{r}_{\rho-1}(\tilde{f}_2) \cdots \mathfrak{r}_{\rho-1}(\tilde{f}_n) \cdot W_\Delta(f_1, f_2, \ldots, f_{n-1})$$

and

$$\tilde{f}_1 \tilde{f}_2 \cdots \tilde{f}_n \,\Big|\, \mathfrak{r}_\sigma(\tilde{f}_1 \tilde{f}_2 \cdots \tilde{f}_n) \cdot W_\Delta(f_1, f_2, \ldots, f_{n-1})$$

Proof. It suffices to prove that irreducible polynomials g over $\mathbb{C}$ in the variable y divide the right hand side at least as often as the left hand side. So let $g \in \mathbb{C}[y]$ be irreducible. Since $\gcd\{\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n\} = 1$, one of the $\tilde{f}_i$'s is not divisible by g, say that $g \nmid \tilde{f}_1$. It follows from (B.25) that it suffices to show that g divides $\tilde{f}_2 \cdots \tilde{f}_n$ at most as often as

$$\mathfrak{r}_{\rho-1}(\tilde{f}_1)\mathfrak{r}_{\rho-1}(\tilde{f}_2) \cdots \mathfrak{r}_{\rho-1}(\tilde{f}_n) \cdot W_\Delta(f_2, \ldots, f_{n-1}, f_n)$$

and

$$\mathfrak{r}_\sigma(\tilde{f}_1\tilde{f}_2 \cdots \tilde{f}_n) \cdot W_\Delta(f_2, \ldots, f_{n-1}, f_n)$$

Now pick any term of the determinant expression $W_\Delta(f_2, \ldots, f_{n-1}, f_n)$. After permuting $f_2, \ldots, f_n$, the term at hand becomes

$$\Delta_1 f_2 \cdot \Delta_2 f_3 \cdot \cdots \cdot \Delta_{n-1} f_n$$

Now if g divides $\tilde{f}_i$ exactly l times and hence also f_i exactly l times, then g divides $\Delta_{i-1} f_i$ at least $l - \rho$ times, since partial derivatives kill at most one instance of a factor g in their argument. But one of the partial derivatives is a $\frac{\partial}{\partial z_j}$ which does not kill any instance of g, so g divides $\Delta_{i-1} f_i$ at least $l - (\rho - 1)$ times.

The factor $\mathfrak{r}(\tilde{f}_i)^{\rho-1}$ compensates the decrease of $\rho - 1$ factors g, so g divides $\mathfrak{r}(\tilde{f}_i)^{\rho-1}\Delta_{i-1}\tilde{f}_i$ at least as often as it divides $\tilde{f}_i$, and the first inequality of this lemma follows. The second inequality follows from the fact that the Δ_i's together have σ partial derivatives of the form $\frac{\partial}{\partial y}$ that might kill instances of g. ☐

Lemma B.8.4.

$$\deg W_\Delta(f_1, f_2, \ldots, f_{n-1}) \le \deg(\tilde{f}_1\tilde{f}_2 \cdots \tilde{f}_{n-1}) - \sigma$$

Proof. The idea is that a partial derivative decreases the degree by one. Consider a term on the left hand side of the above formula. After reordering the f_i's, this term becomes

$$\Delta_1 f_1 \cdot \Delta_2 f_2 \cdot \cdots \cdot \Delta_{n-1} f_{n-1}$$

Since $o(\Delta_i) \ge 1$ for all i, the degree of this term is at most $\deg(f_1 f_2 \cdots f_{n-1}) - (n-1) = \deg(\tilde{f}_1\tilde{f}_2 \cdots \tilde{f}_{n-1})$. But there are also Δ_i's of orders larger than one, which are responsible for the term σ. ☐

Proof of theorem B.8.1. Assume without loss of generality that $\tilde{f}_n$ has the largest degree among the $\tilde{f}_i$'s. From lemmas B.8.3 and B.8.4, it follows that

$$\sum_{i=1}^{n} \deg \tilde{f}_i \leq r_{\rho-1}(\tilde{f}_1) + r_{\rho-1}(\tilde{f}_2) + \cdots + r_{\rho-1}(\tilde{f}_n) + \deg(\tilde{f}_1 \tilde{f}_2 \cdots \tilde{f}_{n-1}) - \sigma$$

whence

$$\deg \tilde{f}_n \leq r_{\rho-1}(\tilde{f}_1) + r_{\rho-1}(\tilde{f}_2) + \cdots + r_{\rho-1}(\tilde{f}_n) - \frac{\rho(\rho-1)}{2}$$

which is the first inequality of theorem B.8.1. The second inequality follows similarly. □

B.9 Joining theorems B.2.1 and B.2.2

The general result that implies both [E.2, Col. I] and (the improved version (B.14) of) [E.2, Col. II] is [E.2, Theorem A], which we will describe now for the polynomial case. For irreducible polynomials p, let m_p denote the number of f_i's that is *not* divisible by p. Then [E.2, Theorem A] implies

$$\max_{1 \leq m \leq n} \deg f_m \leq -\binom{n-1}{2} + \sum_p \left(\binom{n-1}{2} - \binom{m_p - 1}{2} \right) \tag{B.26}$$

where $\sum_p$ ranges over all irreducible polynomials p. It follows from (B.26) that

$$\max_{1 \leq m \leq n} \deg f_m \leq -\binom{n-1}{2} + \sum_{p \nmid f_1 \cdots f_n} \left(\binom{n-1}{2} - \binom{n-1}{2} \right) + \sum_{p \mid f_1 \cdots f_n} \binom{n-1}{2}$$

which is exactly the case $d' = n-1$ of the univariate case of (B.7) in theorem B.2.2.

In order to get a similar result on (B.26) and (B.5) in theorem B.2.1, we first need some preparations. Assume

$$f_i \nmid f_{i+1} \tag{B.27}$$

The reason for (B.27) is that there exists an irreducible p that divides f_i more times than it divides f_{i+1}, say that p divides f_i $l+j$ times and f_{i+1} l times.

Now replace f_i by $f_i p^j$ and f_{i+1} by $f_{i+1} p^{-j}$. Then (B.27) might still be the case, but the divisibility by p is not the reason any more. Furthermore, for any power q of an irreducible polynomial, q divides as many f_i's as before. If we proceed in this direction, we finally arrive at

Proposition B.9.1. *There exist $h_1, h_2, \ldots, h_n$ such that*

1. $h_1 \mid h_2 \mid \cdots \mid h_n$,
2. *For any power q of an irreducible polynomial, q divides as many h_i's as it divides f_i'.*

Notice that $h_1 = \gcd\{f_1, f_2, \ldots, f_n\} = 1$. More generally, h_i is the greatest common divisor over all subsets $\{j_1, j_2, \ldots, j_i\}$ of $\{1, 2, \ldots, n\}$ of $\operatorname{lcm}\{f_{j_1}, f_{j_2}, \ldots, f_{j_i}\}$.

Since m_p is also the number of h_i's that is not divisible by p,

$$\binom{m_p - 1}{2} = \sum_{i=2}^{m_p} (i-2) = \sum_{\substack{2 \le i \le n \\ p \nmid h_i}} (i-2)$$

whence

$$\binom{n-1}{2} - \binom{m_p - 1}{2} = \sum_{i=m_p+1}^{n} (i-2) = \sum_{\substack{1 \le i \le n \\ p \mid h_i}} (i-2)$$

Summing this over all p, it follows from (B.26) that

$$\max_{1 \le m \le n} \deg f_m \le \sum_{i=1}^{n} (i-2) r(h_i) - \binom{n-1}{2} \tag{B.28}$$

which implies the case $d' = n-1$ of the univariate case of (B.5) in theorem B.2.1, for

$$\sum_{i=1}^{n} r(h_i) = \sum_{i=1}^{n} r(f_i)$$

By $r(h_1) = 0$ and $r(h_i) \le r(h_n)$, the case $d' = n-1$ of the univariate case of (B.7) in theorem B.2.2 follows from (B.28) as well. (B.28) can be improved to

$$\max_{1 \le m \le n} \deg f_m \le \sum_{i=1}^{n} r_{i-2}(h_i) - \binom{n-1}{2}$$

which implies (B.4) in theorem B.2.1 for $\rho = n - 2$ and (B.6) in theorem B.2.2 for $\sigma = (n-1)(n-2)/2$, since $r_i(a)r_j(b) \le r_{i+j}(ab)$. The general multivariate result that includes both theorems B.2.1 and B.2.2 is

$$\max_{1 \le m \le n} \deg f_m \le \sum_{i=2}^{n} r_{(o_{i-1})-1}(h_i) - \sigma$$

where

$$o_1 \le o_2 \le \cdots \le o_{n-1}$$

are the orders of the Δ_i's. The proof is left as an exercise to the reader.

At last we sketch the proof of (B.12) and (B.13). Assume that each set of d f_i's forms a basis of the space generated by all f_i and order the f_i's by increasing degree. As indicated in section B.5, we do not need to multiply the f_i's by linear forms in order to get rid of unwanted linear dependences. Similar to (B.25), one can prove that all sequences of d f_i's have the same Wronskian determinant $W_\Delta(f_1, f_2, \ldots, f_d)$ up to a nonzero constant in $\mathbb{C}$. Since each set of d f_i's generates the whole space, the greatest common divisor of such a set is 1, whence there can only be $d-1$ f_i's at most that are divisible by a given irreducible polynomial p. So $h_1 = h_2 = \cdots = h_{n-d+1} = 1$ and

$$\begin{aligned} f_1 f_2 \cdots f_n \ &| \ h_{n-d+2} h_{n-d+3} \cdots h_n \\ &| \ \mathfrak{r}_1(h_{n-d+2})\mathfrak{r}_2(h_{n-d+3}) \cdots \mathfrak{r}_{d-1}(h_n) W_\Delta(f_1, f_2, \ldots, f_d) \end{aligned} \tag{B.29}$$

because focusing on one irreducible divisor p, one can replace $f_1, f_2, \ldots, f_d$ on the right hand side of (B.29) by the d f_i's of maximum divisibility by p. Next, since each set of d f_i's has a polynomial of maximum degree, the $n-d$ f_i's on the left hand side of (B.29) that are *not* on the right side of (B.29) have maximum degree. That gives the factor $1/(n-d)$ in (B.12) and (B.13).

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Curriculum Vitae

The author was born on May 3, 1973. On primary school already, he revealed distinctive skills and interest in arithmetic. In 1989, he took part in the Dutch mathematical olympiad and missed the second round by two points, but a year later, he won the first round and became fourth after the second and final round. After that, he was exercised and in 1991, he was selected for the international mathematical olympiad in Sweden, where he won a honorable mention. In the same year, the author finished secondary school and started studying at the University of Nijmegen.

In the year 1992-1993, the author participated in the 'Universitaire Wiskunde Competitie', a Dutch mathematical contest for academic undergraduates, where he was the only person that won a prize without having solved three problems during the six weeks of the competition. His contribution was a partial solution of the camel-banana problem, an open problem in those days. It was selected for publication, but in August 1993, he solved the camel-banana problem completely. This last result was published three years later.

In the years 1993-1994, 1996-1997, 1997-1998 and 1998-1999, the author took again part in the 'Universitaire Wiskunde Competitie', winning prizes each year he joined, with a second place in the year 1997-1998. In 2001, he graduated cum laude in both mathematics and 'mathematics and computer science'. In 2003, the author became a Ph.D. student of dr. A. van den Essen at the University of Nijmegen, nowadays the Radboud University Nijmegen. During the years after that, he investigated the Jacobian conjecture, which has led to a significant number of articles, culminating in this thesis.