TWO RESULTS ON HOMOGENEOUS HESSIAN NILPOTENT POLYNOMIALS

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Abstract. Let $z = (z_1, \cdots, z_n)$ and $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial z_i^2}$ the Laplace operator. A formal power series $P(z)$ is said to be *Hessian Nilpotent*(HN) if its Hessian matrix $\operatorname{Hes} P(z) = (\frac{\partial^2 P}{\partial z_i \partial z_j})$ is nilpotent. In recent developments in [BE1], [M] and [Z], the Jacobian conjecture has been reduced to the following so-called *vanishing conjecture*(VC) of HN polynomials: *for any homogeneous HN polynomial $P(z)$ (of degree $d = 4$), we have $\Delta^m P^{m+1}(z) = 0$ for any $m >> 0$.* In this paper, we first show that, the VC holds for any homogeneous HN polynomial $P(z)$ provided that the projective subvarieties $\mathcal{Z}_P$ and $\mathcal{Z}_{\sigma_2}$ of $\mathbb{C}P^{n-1}$ determined by the principal ideals generated by $P(z)$ and $\sigma_2(z) := \sum_{i=1}^n z_i^2$, respectively, intersect only at regular points of $\mathcal{Z}_P$. Consequently, the Jacobian conjecture holds for the symmetric polynomial maps $F = z - \nabla P$ with $P(z)$ HN if F has no non-zero fixed point $w \in \mathbb{C}^n$ with $\sum_{i=1}^n w_i^2 = 0$. Secondly, we show that the VC holds for a HN formal power series $P(z)$ if and only if, for any polynomial $f(z)$, $\Delta^m(f(z)P(z)^m) = 0$ when $m >> 0$.

1. Introduction and Main Results

Let $z = (z_1, z_2, \cdots, z_n)$ be commutative free variables. Recall that the well-known Jacobian conjecture claims that: *any polynomial map $F(z) : \mathbb{C}^n \to \mathbb{C}^n$ with the Jacobian $j(F)(z) \equiv 1$ is an autompophism of $\mathbb{C}^n$ and its inverse map must also be a polynomial map.* Despite intense study from mathematicians in more than sixty years, the conjecture is still open even for the case $n = 2$. In 1998, S. Smale [S] included the Jacobian conjecture in his list of 18 important mathematical problems for 21st century. For more history and known results on the Jacobian conjecture, see [BCW], [E] and references there.

Recently, M. de Bondt and the first author [BE1] and G. Meng [M] independently made the following remarkable breakthrough on the Jacobian conjecture. Namely, they reduced the Jacobian conjecture to

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the so-called *symmetric* polynomial maps, i.e the polynomial maps of the form $F = z - \nabla P$, where $\nabla P := (\frac{\partial P}{\partial z_1}, \frac{\partial P}{\partial z_2}, \cdots, \frac{\partial P}{\partial z_n})$, i.e. $\nabla P(z)$ is the *gradient* of $P(z) \in \mathbb{C}[z]$.

For more recent developments on the Jacobian conjecture for symmetric polynomial maps, see [BE1]–[BE4].

Based on the symmetric reduction above and also the classical homogeneous reduction in [BCW] and [Y], the second author in [Z] further reduced the Jacobian conjecture to the following so-called vanishing conjecture.

Let $\Delta := \sum_{i=1}^{n} \frac{\partial^2}{\partial z_i^2}$ the Laplace operator and call a formal power series $P(z)$ *Hessian nilpotent*(HN) if its Hessian matrix $\operatorname{Hes} P(z) := (\frac{\partial^2 P}{\partial z_i \partial z_j})$ is nilpotent. It has been shown in [Z] that the Jacobian conjecture is equivalent to

Conjecture 1.1. (Vanishing Conjecture of HN Polynomials) *For any homogeneous HN polynomial $P(z)$ (of degree $d = 4$), we have $\Delta^m P^{m+1} = 0$ when $m >> 0$.*

Note that, it has also been shown in [Z] that $P(z)$ is HN if and only if $\Delta^m P^m = 0$ for $m \geq 1$.

In this paper, we will prove the following two results on HN polynomials.

Let $P(z)$ be a homogeneous HN polynomial of degree $d \geq 3$ and $\sigma_2(z) := \sum_{i=1}^{n} z_i^2$. We denote by $\mathfrak{Z}_P$ and $\mathfrak{Z}_{\sigma_2}$ the projective subvarieties of $\mathbb{C}P^{n-1}$ determined by the principal ideals generated by $P(z)$ and $\sigma_2(z)$, respectively. The first main result of this paper is the following theorem.

Theorem 1.2. *Let $P(z)$ be a homogeneous HN polynomial of degree $d \geq 4$. Assume that $\mathfrak{Z}_P$ intersects with $\mathfrak{Z}_{\sigma_2}$ only at regular points of $\mathfrak{Z}_P$, then the vanishing conjecture holds for $P(z)$. In particular, the vanishing conjecture holds if the projective variety $\mathfrak{Z}_P$ is regular.*

Remark 1.3. *Note that, when $\deg P(z) = d = 2$ or 3, the Jacobian conjecture holds for the symmetric polynomial map $F = z - \nabla P$. This is because, when $d = 2$, F is a linear map with $j(F) \equiv 1$. Hence F is an automorphism of $\mathbb{C}^n$; while when $d = 3$, we have $\deg F = 2$. By Wang's theorem* [W]*, the Jacobian conjecture holds for F again. Then, by the equivalence of the vanishing conjecture for the homogeneous HN polynomial $P(z)$ and the Jacobian conjecture for the symmetric map $F = z - \nabla P$ established in* [Z]*, we see that, when $\deg P(z) = d = 2$ or 3, Theorem 1.2 actually also holds even without the condition on the projective variety $\mathfrak{Z}_P$.*

For any non-zero $z \in \mathbb{C}^n$, denote by $[z]$ its image in the projective space $\mathbb{C}P^{n-1}$. Set

$$\widetilde{\mathcal{Z}}_{\sigma_2} := \{z \in \mathbb{C}^n \mid z \neq 0;\ [z] \in \mathcal{Z}_{\sigma_2}\}. \tag{1.1}$$

In other words, $\widetilde{\mathcal{Z}}_{\sigma_2}$ is the set of non-zero $z \in \mathbb{C}^n$ such that $\sum_{i=1}^n z_i^2 = 0$.

Note that, for any homogeneous polynomial $P(z)$ of degree d, it follows from the Euler's formula $dP = \sum_{i=1}^n z_i \frac{dP}{dz_i}$, that any non-zero $w \in \mathbb{C}^n$, $[w] \in \mathbb{C}P^{n-1}$ is a singular point of $\mathcal{Z}_P$ if and only if w is a fixed point of the symmetric map $F = z - \nabla P$. Furthermore, it is also well-known that, $j(F) \equiv 1$ if and only if $P(z)$ is HN.

By the observations above and Theorem 1.2, it is easy to see that we have the following corollary on symmetric polynomial maps.

Corollary 1.4. *Let $F = z - \nabla P$ with P homogeneous and $j(F) \equiv 1$ (or equivalently, P is HN). Assume that F does not fix any $w \in \widetilde{\mathcal{Z}}_{\sigma_2}$. Then the Jacobian holds for $F(z)$. In particular, if F has no non-zero fixed point, the Jacobian conjecture holds for F.*

Our second main result is following theorem which says that the vanishing conjecture is actually equivalent to a formally much stronger statement.

Theorem 1.5. *For any HN polynomial $P(z)$, the vanishing conjecture holds for $P(z)$ if and only if, for any polynomial $f(z) \in \mathbb{C}[z]$, $\Delta^m(f(z)P(z)^m) = 0$ when $m >> 0$.*

2. **Proof of the Main Results**

Let us first fix the following notation. Let $z = (z_1, z_2, \cdots, z_n)$ be free complex variables and $\mathbb{C}[z]$ (resp. $\mathbb{C}[[z]]$) the algebra of polynomials (resp. formal power series) in z. For any $d \geq 0$, we denote by V_d the vector space of homogeneous polynomials in z of degree d.

For any $1 \leq i \leq n$, we set $D_i = \frac{\partial}{\partial z_i}$ and $D = (D_1, D_2, \cdots, D_n)$. We define a $\mathbb{C}$-bilinear map $\{\cdot, \cdot\} : \mathbb{C}[z] \times \mathbb{C}[z] \to \mathbb{C}[z]$ by setting

$$\{f,\, g\} := f(D)g(z)$$

for any $f(z), g(z) \in \mathbb{C}[z]$.

Note that, for any $m \geq 0$, the restriction of $\{\cdot, \cdot\}$ on $V_m \times V_m$ gives a $\mathbb{C}$-bilinear form of the vector subspace V_m, which we will denote by $B_m(\cdot, \cdot)$. It is easy to check that, for any $m \geq 1$, $B_m(\cdot, \cdot)$ is symmetric and non-singular.

The following lemma will play a crucial role in our proof of the first main result.

Lemma 2.1. *For any homogeneous polynomials $g_i(z)$ $(1 \leq i \leq k)$ of degree $d_i \geq 1$, let S be the vector space of polynomial solutions of the following system of PDEs:*

$$\begin{cases} g_1(D)\, u(z) = 0, \\ g_2(D)\, u(z) = 0, \\ \quad \ldots\ldots \\ g_k(D)\, u(z) = 0. \end{cases} \tag{2.2}$$

Then, $\dim S < +\infty$ *if and only if $g_i(z)$ $(1 \leq i \leq k)$ have no non-zero common zeroes.*

Proof: Let I the homogeneous ideal of $\mathbb{C}[z]$ generated by $\{g_i(z) | 1 \leq i \leq k\}$. Since all $g_i(z)$'s are homogeneous, S is a homogeneous vector subspace S of $\mathbb{C}[z]$.

Write

$$S = \bigoplus_{m=0}^{\infty} S_m, \tag{2.3}$$

$$I = \bigoplus_{m=0}^{\infty} I_m. \tag{2.4}$$

where $I_m := I \cap V_m$ and $S_m := I \cap V_m$ for any $m \geq 0$.

<u>*Claim:*</u> *For any $m \geq 1$ and $u(z) \in V_m$, $u(z) \in S_m$ if and only if $\{u, I_m\} = 0$, or in other words, $S_m = I_m^{\perp}$ with respect to the $\mathbb{C}$-bilinear form $B_m(\cdot, \cdot)$ of V_m.*

<u>*Proof of the Claim:*</u> First, by the definitions of I and S, we have $\{I_m, S_m\} = 0$ for any $m \geq 1$, hence $S_m \subseteq I_m^{\perp}$. Therefore, we need only show that, for any $u(z) \in I_m^{\perp} \subset V_m$, $g_i(D)u(z) = 0$ for any $1 \leq i \leq n$.

We first fix any $1 \leq i \leq n$. If $m < d_i$, there is nothing to prove. If $m = d_i$, then $g_i(z) \in I_m$, hence $\{g_i, u\} = g_i(D)u = 0$. Now suppose $m > d_i$. Note that, for any $v(z) \in V_{m-d_i}$, $v(z)g_i(z) \in I_m$. Hence we have

$$\begin{aligned} 0 &= \{v(z)g_i(z), u(z)\} \\ &= v(D)g_i(D)u(z) \\ &= v(D)\left(g_i(D)u\right)(z) \\ &= \{v(z), \left(g_i(D)u\right)(z)\}. \end{aligned}$$

Therefore, we have

$$B_{m-d_i}\left((g_i(D)u)(z),\, V_{m-d_i}\right) = 0.$$

Since $B_{m-d_i}(\cdot,\cdot)$ is a non-singular $\mathbb{C}$-bilinear form of V_{m-d_i}, we have $g_i(D)u = 0$. Hence, the Claim holds. $\square$

By a well-known fact in Algebraic Geometry (see Exercise 2.2 in [H], for example), we know that the homogeneous polynomials $g_i(z)$ $(1 \leq i \leq k)$ have no non-zero common zeroes if and only if $I_m = V_m$ when $m >> 0$. While, by the Claim above, we know that, $I_m = V_m$ when $m >> 0$ if and only if $S_m = 0$ when $m >> 0$, and if and only if the solution space S of the system (2.2) is finite dimensional. Hence, the lemma follows. $\square$

Now we are ready to prove our first main result, Theorem 1.2.

Proof of Theorem 1.2: Let $P(z)$ be a homogeneous HN polynomial of degree $d \geq 4$ and S the vector space of polynomial solutions of the following system of PDEs:

$$\begin{cases} \frac{\partial P}{\partial z_1}(D)\, u(z) = 0, \\ \frac{\partial P}{\partial z_2}(D)\, u(z) = 0, \\ \quad \ldots\ldots \\ \frac{\partial P}{\partial z_n}(D)\, u(z) = 0, \\ \Delta\, u(z) = 0. \end{cases} \tag{2.5}$$

First, note that the projective subvariety $\mathcal{Z}_P$ intersects with $\mathcal{Z}_{\sigma_2}$ only at regular points of $\mathcal{Z}_P$ if and only if $\frac{\partial P}{\partial z_i}(z)$ $(1 \leq i \leq n)$ and $\sigma_2 = \sum_{i=1}^n z_i^2$ have no non-zero common zeros (agian use Euler's formula). Then, by Lemma 2.1, we have $\dim S < +\infty$.

On the other hand, by Theorem 6.3 in [Z], we know that $\Delta^m P^{m+1} \in S$ for any $m \geq 0$. Note that $\deg \Delta^m P^{m+1} = (d-2)m + d$ for any $m \geq 0$. So $\deg \Delta^m P^{m+1} > \deg \Delta^k P^{k+1}$ for any $m > k$. Since $\dim S < +\infty$ (from above), we have $\Delta^m P^{m+1} = 0$ when $m >> 0$, i.e. the vanishing conjecture holds for $P(z)$. $\square$

Next, we give a proof for our second main result, Theorem 1.5.

Proof of Theorem 1.5: The $(\Leftarrow)$ part follows directly by choosing $f(z)$ to be $P(z)$ itself.

To show $(\Rightarrow)$ part, let $d = \deg f(z)$. If $d = 0$, f is a constant. Then, $\Delta^m(f(z)P(z)^m) = f(z)\Delta^m P^m = 0$ for any $m \geq 1$.

So we assume $d \geq 1$. By Theorem 6.2 in [Z], we know that, if the vanishing conjecture holds for $P(z)$, then, for any fixed $a \geq 1$, $\Delta^m P^{m+a} = 0$ when $m >> 0$. Therefore there exists $N > 0$ such that, for any $0 \leq b \leq d$ and any $m > N$, we have $\Delta^m P^{m+b} = 0$.

By Lemma 6.5 in [Z], for any $m \geq 1$, we have

$$\Delta^m(f(z)P(z)^m) = \sum_{\substack{k_1+k_2+k_3=m \\ k_1,k_2,k_3\geq 0}} 2^{k_2} \binom{m}{k_1,k_2,k_3} \sum_{\substack{\mathbf{s}\in\mathbb{N}^n \\ |\mathbf{s}|=k_2}} \binom{k_2}{\mathbf{s}} \frac{\partial^{k_2}\Delta^{k_1} f(z)}{\partial z^{\mathbf{s}}} \frac{\partial^{k_2}\Delta^{k_3} P^m(z)}{\partial z^{\mathbf{s}}}, \tag{2.6}$$

where $\binom{m}{k_1,k_2,k_3}$ and $\binom{k_2}{\mathbf{s}}$ denote the usual binomials.

Note first that, the general term in the sum above is non-zero only if $2k_1 + k_2 \leq d$. But on the other hand, since

$$0 \leq k_1 + k_2 \leq 2k_1 + k_2 \leq d, \tag{2.7}$$

by the choice of $N \geq 1$, we have $\Delta^{k_3} P^m(z) = \Delta^{k_3} P^{k_3+(k_1+k_2)}(z)$ is non-zero only if

$$k_3 \leq N. \tag{2.8}$$

From the observations above and Eqs. (2.6), (2.7), (2.8) it is easy to see that, $\Delta^m(f(z)P(z)^m) \neq 0$ only if $m = k_1 + k_2 + k_3 \leq d + N$. In other words, $\Delta^m(f(z)P(z)^m) = 0$ for any $m > d + N$. Hence Theorem 1.5 holds. $\square$

Note that all results used in the proof above for the ($\Leftarrow$) part of the theorem also hold for all HN formal power series. Therefore we have the following corollary.

Corollary 2.2. *Let $P(z)$ be a HN formal power series such that the vanishing conjecture holds for $P(z)$. Then, for any polynomial $f(z)$, we have $\Delta^m(f(z)P(z)^m) = 0$ when $m >> 0$.*

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