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TWO "SIMPLE" SETS THAT ARE NOT POSITIVELY BOREL

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In memoriam patris Dirk Jan Veldman (1910-2004)

Fructueuse erreur, que je fus bien inspiré de la commettre

H. Lebesgue, in [12], p. vii

Tyger! Tyger! burning bright
In the forests of the night,
What immortal hand or eye
Could frame thy fearful symmetry?

W. Blake, Songs of Experience, 1789-94

0 Introduction

0.1 Descriptive set theory, a subject that started with the work of É. Borel and H. Lebesgue in the early decades of last century, see [12], [15], [14] and [8], is the study of certain classes of subsets of separable metric spaces like $\mathbb{R}$, the set of the real numbers, and *Baire space* $\mathcal{N}$, the set of all infinite sequences of natural numbers. Some examples of the classes of sets one considers are the class of the Borel sets, the class of the analytic sets and the class of the projective sets.
We investigate this subject from an *intuitionistic* point of view.
The founding fathers of descriptive set theory were driven by a certain distrust of set-theoretic arguments like E. Zermelo's proof of the Well-Ordering Theorem and set-theoretic notions like the power set of $\mathcal{N}$ or the first uncountable ordinal, see [2]. L.E.J. Brouwer, who initiated the intuitionistic reform of mathematics at about the same time descriptive set theory found its origin, shared this distrust but, eventually, his critique of classical mathematics went further and deeper. Meditating upon the possible meaning of mathematical statements he suggested a refinement of the language of mathematics and a strong and constructive interpretation of the logical connectives and quantifiers. Reflecting on the way one should handle the concept of the continuum he also proposed several new axioms, some of them outright contradictions if one reads them without taking notice of the changed sense of the logical constants.
We follow Brouwer's suggestion to interpret the logical constants and the set-theoretic

operations constructively. We also use the axioms he advocated, in particular his Continuity Principle. Brouwer's Thesis on bars will play a less prominent role. It will be used for proving some results towards the end of the paper. The main results, as they will be announced in Subsection 0.9, depend on the Continuity Principle alone.
The reader should be aware that Brouwer's axioms are not generally accepted in the wider circle of constructive mathematicians. The famous constructive mathematician E.R. Bishop agreed with Brouwer that mathematicians should use intuitionistic logic, but judged the axioms to be the result of wild and "semi-mystical" speculation.

We restrict our attention to classes of subsets of Baire space $\mathcal{N}$.
We use $m, n, \ldots$ as variables over the set $\mathbb{N}$ of natural numbers, and $\alpha, \beta, \ldots$ as variables over Baire space $\mathcal{N}$.

0.2 Let X be a subset of $\mathcal{N}$.
X is called *basic open* if and only if either X is empty or there exists a finite sequence s of natural numbers such that X consists of all infinite sequences α that have s as an initial part.
X is called *open* if and only if X is a countable union of basic open sets and X is called *closed* if and only if there exists an open set Y such that X is the complement of Y. X is called *positively Borel* if and only if X may be obtained from basic open subsets of $\mathcal{N}$ by the repeated application of the operations of countable union and countable intersection.
We let $\langle \, , \rangle$ be a suitable *pairing function* on $\mathcal{N}$. We may define, for instance, for all α, β in $\mathcal{N}$, for all n in $\mathbb{N}$, $\langle \alpha, \beta \rangle(2n) := \alpha(n)$ and $\langle \alpha, \beta \rangle(2n+1) := \beta(n)$.
Again, let X be a subset of $\mathcal{N}$.
X is called *analytic* if and only if there exists a closed subset Y of $\mathcal{N}$ such that X is the *projection* of Y, that is: for every α, α belongs to X if and only if, for some β, $\langle \alpha, \beta \rangle$ belongs to Y.
X is called *co-analytic* if and only if there exists an open subset Y of $\mathcal{N}$ such that X is the *universal projection* of Y, that is: for every α, α belongs to X if and only if, for all β, $\langle \alpha, \beta \rangle$ belongs to Y.

0.3 The logical operations of disjunction and existential quantification over both $\mathbb{N}$ and $\mathcal{N}$ and also the corresponding set-theoretic operations finite union, countable union, and projection are interpreted constructively. We only say that an element α of $\mathcal{N}$ belongs to some countable union $\bigcup_{n \in \mathbb{N}} X_n$ of subsets of $\mathcal{N}$, if we are able to indicate a natural number n such that α belongs to X_n. We only say that an element α of $\mathcal{N}$ belongs to the projection of a subset X of $\mathcal{N}$ if we are able to indicate an element β of $\mathcal{N}$ such that $\langle \alpha, \beta \rangle$ belongs to X.
For this reason, the class of positively Borel sets does not coincide with, for instance, the class of all subsets of $\mathcal{N}$ that may be obtained from basic open subsets of $\mathcal{N}$ by the repeated application of the operation of countable union and the operation of taking the complement. Also, the complement of an analytic set is a co-analytic set only in very exceptional cases, and the complement of a co-analytic set almost never is an analytic set.

We want to avoid the operation of taking the complement as much as possible. Negative statements and properties seem to be more difficult to understand and in some sense less useful than positive ones.

0.4 The discovery of subsets of $\mathcal{N}$ that are not (positively) Borel but still in some sense "definable", or, to use the term Lebesgue proposed, "mentionable/nameable", "nommable", has been a dramatic moment in the history of classical descriptive set theory. In his memoir [10], Lebesgue had assumed that the projection of a (positively) Borel subset of $\mathcal{N}$ is again (positively) Borel. The Russian mathematicians N. Lusin and A. Souslin observed that he was wrong. Souslin introduced the class of the analytic subsets of $\mathcal{N}$ in [16] and proved that some analytic subsets of $\mathcal{N}$ are not (positively) Borel. One may prove, intuitionistically as well as classically, that a subset X of $\mathcal{N}$ is analytic in the sense of Subsection 0.3, that is, the projection of a closed subset of $\mathcal{N}$, if and only if it is the projection of a positively Borel subset of $\mathcal{N}$.

0.5 We introduce a highly important notion that will enable us to give a succinct description of Souslin's discovery of an analytic set that is not positively Borel.
Let X, Y be subsets of $\mathcal{N}$. We say that X *reduces* to Y if and only if there exists a continuous function f from $\mathcal{N}$ to $\mathcal{N}$ *reducing* X to Y, that is, for every α, α belongs to X if and only if $f(\alpha)$ belongs to Y.
Intuitively, a set X reduces to a set Y if we have a method to translate every question about membership on X, that is every question of the form: "does α belong to X?", into a question about membership on Y, that is, a question of the form: "does β belong to Y?".
We now give an alternative characterization of the classes of the analytic and co-analytic subsets of $\mathcal{N}$.
We first define subsets E^1_1 and A^1_1 of $\mathcal{N}$.
We let $\mathbb{N}^*$ be the set of all finite sequences of natural numbers and assume that some bijective mapping $(n_0, \ldots n_{k-1}) \mapsto \langle n_0, \ldots, n_{k-1}\rangle$ of $\mathbb{N}^*$ onto $\mathbb{N}$ has been defined. We also assume that there are given a function length from $\mathbb{N}$ to $\mathbb{N}$ and a binary operation $(n, i) \mapsto n(i)$ on $\mathbb{N}$ such that, for all n, k, if $k = \text{length}(n)$, then $n = \langle n(0), n(1), \ldots, n(k-1)\rangle$. For every n in $\mathbb{N}$, for every $i \leq \text{length}(n)$, we denote the number $\langle n(0), n(1), \ldots, n(i-1)\rangle$ by $\overline{n}(i)$, or, if confusion is unlikely, simply by $\overline{n}i$. Similarly, for every α in $\mathcal{N}$, for every n in $\mathbb{N}$, we denote $\langle \alpha(0), \ldots \alpha(n-1)\rangle$ by $\overline{\alpha}(n)$, or, if confusion is unlikely, simply by $\overline{\alpha}n$.
For every α, for every s, we say that s *contains* α or that α *passes through* s if and only if there exists n such that $\overline{\alpha}n = s$.
Let α, β belong to $\mathcal{N}$. We say that α *admits* β if and only if, for every n, $\alpha(\overline{\beta}n) = 0$ and we say that α *bars* β if and only if there exists n such that $\alpha(\overline{\beta}n) \neq 0$.
We let E^1_1 be the set of all α in $\mathcal{N}$ that admit at least one β in $\mathcal{N}$, and we let A^1_1 be the set of all α in $\mathcal{N}$ that bar every β in $\mathcal{N}$.
Let α, β belong to $\mathcal{N}$. If α admits β, we also say that β is an *(infinite) path* of α. For this reason, the set E^1_1 is sometimes called **Path**.
Let α belong to $\mathcal{N}$. If α bars every β in $\mathcal{N}$, we also say that α is a *bar in* $\mathcal{N}$. For this

reason, the set A_1^1 is sometimes called **Bar**.
The set E_1^1 is an analytic subset of $\mathcal{N}$. One may prove, intuitionistically as well as classically, that E_1^1 is a *complete* element of the class of the analytic subsets of $\mathcal{N}$, that is, every analytic subset of $\mathcal{N}$ reduces to E_1^1.
The set A_1^1 is a co-analytic subset of $\mathcal{N}$. One may prove, intuitionistically as well as classically, that A_1^1 is a complete element of the class of the co-analytic subsets of $\mathcal{N}$: every co-analytic subset of $\mathcal{N}$ reduces to A_1^1. .

0.6 The following three facts imply that the set E_1^1 is analytic but not positively Borel:

(i) *The Borel Hierarchy Theorem:*
Given any positively Borel subset X of $\mathcal{N}$, one may construct a positively Borel subset Y of $\mathcal{N}$ such that Y does not reduce to X.
(ii) Every positively Borel subset X of $\mathcal{N}$ is analytic.
(iii) Every analytic subset of $\mathcal{N}$ reduces to E_1^1.

In [18] and [24] it is shown that these facts and therefore also the conclusion that the set E_1^1 is not positively Borel hold intuitionistically. The Borel Hierarchy Theorem, as formulated here, is intuitionistically not a trivial result and a consequence of a more strongly formulated theorem proved in [24]. As the complement of a positively Borel set in general is not positively Borel, we can not prove the Borel Hierarchy Theorem in the way Borel and Lebesgue do it, see [10], by an extension of Cantor's diagonal argument. Essential use has to be made of Brouwer's Continuity Principle.

0.7 In classical descriptive set theory, the co-analytic set A_1^1 is the complement of the analytic set E_1^1, and the fact that A_1^1 is not Borel follows from the fact that E_1^1 is not Borel.
It is not so easy to prove intuitionistically that A_1^1 is not positively Borel. A_1^1 is not the complement of E_1^1 and the class of the positively Borel sets is not closed under the operation of taking the complement. An argument similar to the one sketched in Subsection 0.6 is also out of the question as not every positively Borel subset of $\mathcal{N}$ is co-analytic: there exists a union of two closed sets that fails to be so, as we will prove in Theorem 3.3.
There is no proof of the fact that the set A_1^1 is not positively Borel in [18]. The proof is given in [24]. The argument uses the strongly formulated intuitionistic Borel Hierarchy Theorem proven in the same paper.

0.8 The purpose of the present paper is to show that, in intuitionistic descriptive set theory, there exist, *firstly*, an analytic subset of $\mathcal{N}$ much more "simple" than E_1^1 that is not positively Borel, and *secondly*, a co-analytic subset of $\mathcal{N}$ much more "simple" than A_1^1 that is not positively Borel. These are very surprising facts.
In classical descriptive set theory one may prove, using the axiom of **$\mathbf{\Sigma}_1^1$-Determinacy** that every analytic set that is not Borel is a complete element of the class of analytic sets, or, equivalently, that every co-analytic set that is not Borel must be a complete element of the class of co-analytic sets. The argument essentially is due to W. Wadge.

The axiom of $\mathbf{\Sigma^1_1}$**-Determinacy** claims that every set X in the Boolean algebra generated by the analytic sets is determined, that is, in the usual game for two players in Baire space with X as the payoff set, one of the two players must have a winning strategy. For more information, the reader is referred to [8], Sections 26.B and 26.C.
Our results contrast sharply with the classical situation. A classical mathematician would call the "simple" analytic set we are to propose a closed set and the "simple" co-analytic set we are to propose a countable union of closed sets. Intuitionistically, these qualifications are wrong, as both sets fail to be positively Borel. Nevertheless, one may guess and we will prove that the two sets are indeed "simple" in the sense that the first one is not a complete element of the class of the analytic sets and the second one is not a complete element of the class of the co-analytic sets.

0.8.0 Before giving the examples we have to make a remark on the concept of a closed subset of $\mathcal{N}$.
Let X be a subset of $\mathcal{N}$.
X is open, in the sense of the definition given in Subsection 0.2, if and only if there exists γ in $\mathcal{N}$ such that for every α in $\mathcal{N}$, α belongs to X if and only if, for some n, α passes through $\gamma(n) - 1$.
Let γ belong to $\mathcal{N}$. We define β in $\mathcal{N}$ as follows. For each n, $\beta(n) := 1$ if there exist $i, j < \text{length}(n)$ such that $\gamma(i) = \overline{n}(j) + 1$, and $\beta(n) := 0$ if not. Observe that, for every α, there exists n such that α passes through $\gamma(n) - 1$ if and only if there exists n such that $\beta(n) = 1$ and α passes through n.
It follows that X is open if and only if there exists β in $\mathcal{N}$ such that for every α, α belongs to X if and only if for some n, $\beta(\overline{\alpha}n) = 1$. It also follows that X is closed, in the sense of the definition in Subsection 0.2, if and only there exists β in $\mathcal{N}$ such that for every α, α belongs to X if and only if, for every n, $\beta(\overline{\alpha}n) \neq 1$.
Again, let X be a subset of $\mathcal{N}$. We let the *closure* $\overline{X}$ *of* X be the set of all α such that for every n, $\overline{\alpha}n$ contains an element of X. In general X is not a closed subset of $\mathcal{N}$ in the sense of the definition in Subsection 0.2. However, we will only apply the operation of taking the closure to subsets X of $\mathcal{N}$ that are *located* in the following sense: there exists β in $\mathcal{N}$ such that for every n, $\beta(n) = 1$ if and only if n contains an element of X. If X is a located subset of $\mathcal{N}$, then $\overline{X}$ will indeed be a closed set in the sense of the definition in Subsection 2.0. Using a term introduced by Brouwer we call a subset of $\mathcal{N}$ that is the closure of a located subset of $\mathcal{N}$ a *spread*. We have to warn the reader that it is not true that every closed subset of $\mathcal{N}$ is a spread. The reason is that, given s in $\mathbb{N}$ and β in $\mathcal{N}$, one can not always decide if s contains some α admitted by β or not.

0.8.1 Cantor space $\mathcal{C}$ is the set of all α in $\mathcal{N}$ that assume no other values than $0, 1$. Consider the set $\mathbf{Path}_{01}$ consisting of all α in $\mathcal{N}$ that admit a path in Cantor space, that is, there exists β in $\mathcal{C}$ such that, for every n, $\alpha(\overline{\beta}n) = 0$. Observe that $\mathbf{Path}_{01}$ is a located subset of $\mathcal{N}$. The closure $\overline{\mathbf{Path}_{01}}$ of the set $\mathbf{Path}_{01}$ is the set of all α in $\mathcal{N}$ such that for every n there exists β in $\mathcal{C}$ such that for every m, if $m < n$, then $\alpha(\overline{\beta}m) = 0$. In classical descriptive set theory, it is a consequence of König's Lemma that $\mathbf{Path}_{01}$ coincides with its closure $\overline{\mathbf{Path}_{01}}$. In intuitionistic descriptive set theory,

however, the set $\mathbf{Path}_{01}$ turns out to be not closed and not even positively Borel.
Indeed, we shall prove, in Section 2, that even the more simple set $\mathbf{MonPath}_{01}$ consisting of all α in $\mathcal{N}$ that admit a *monotone non-decreasing* path in Cantor space, that is, there exists β in $\mathcal{C}$ such that, for every n, $\beta(n) \leq \beta(n+1)$ and $\alpha(\overline{\beta}n) = 0$, is not positively Borel.
Observe that both $\mathbf{Path}_{01}$ and $\mathbf{MonPath}_{01}$ are analytic subsets of $\mathcal{N}$. The second set reduces to the first one but not conversely, see [18] and Subsection 2.26.

0.8.2 Let $\mathbf{Fin}$ be the set of all α in $\mathcal{C}$ such that for some m, for every $n > m$, $\alpha(n) = 0$. For every α in $\mathcal{C}$, α belongs to $\mathbf{Fin}$ if and only if α is the characteristic function of a finite subset of $\mathbb{N}$.
Let $\mathbf{Almost}^*\mathbf{Fin}$ be the set of all α in $\mathcal{C}$ such that for every one-to-one sequence γ in $\mathcal{N}$ there exists n such that $\alpha(\gamma(n)) = 0$. For every decidable subset A of $\mathbb{N}$, the characteristic function of A will belong to $\mathbf{Almost}^*\mathbf{Fin}$ if and only if we have an effective method to find in every infinite subset of $\mathbb{N}$ an element that does not belong to A. In classical descriptive set theory $\mathbf{Fin}$ and $\mathbf{Almost}^*\mathbf{Fin}$ are one and the same set. In intuitionistic descriptive set theory, however, $\mathbf{Fin}$ is a subset of $\mathbf{Almost}^*\mathbf{Fin}$ but the converse is false. $\mathbf{Fin}$ is a countable set and therefore a countable union of closed sets, but $\mathbf{Almost}^*\mathbf{Fin}$ is not a countable union of closed sets and, as we are to prove in Section 3, not even positively Borel.
Observe that $\mathbf{Almost}^*\mathbf{Fin}$ is a co-analytic subset of $\mathcal{N}$.

0.9 Any subset of $\mathcal{N}$ reducing to a positively Borel subset of $\mathcal{N}$ is positively Borel. The two results announced in Subsection 0.8 therefore imply that the sets E^1_1 and A^1_1 themselves are not positively Borel. We thus find an argument proving this that avoids the Borel Hierarchy Theorem. Moreover, the Borel Hierarchy Theorem, in the weak formulation it has been given in Subsection 0.5, will be seen to follow as a corollary first, from the proof of the fact that the set $\mathbf{MonPath}_{01}$ is not positively Borel, and then also from the proof of the fact that the set $\mathbf{Almost}^*\mathbf{Fin}$ is not positively Borel. This will be clear from the following more detailed description of the two results.

0.9.1 *First result:*
(i) Let X be a positively Borel subset of $\mathcal{N}$. One may construct a positively Borel subset Y of $\mathcal{N}$ not reducing to X such that $\mathbf{MonPath}_{01}$ is a subset of Y and Y is a subset of $\overline{\mathbf{MonPath}_{01}}$.
(ii) Let X be a positively Borel subset of $\mathcal{N}$ containing $\mathbf{MonPath}_{01}$. One may construct a positively Borel subset Y of $\mathcal{N}$ containing $\mathbf{MonPath}_{01}$, properly contained in X and also not reducing to X. Y might be called a "better" approximation of $\mathbf{MonPath}_{01}$ than X.

For every subset X of $\mathcal{N}$ we denote the complement of X, that is, the set of all α in $\mathcal{N}$ that do not belong to X, by $X^\neg$.
We shall prove, see Theorem 2.3, that the closure $\overline{\mathbf{MonPath}_{01}}$ of the set $\mathbf{MonPath}_{01}$ coincides with the double complement $(\mathbf{MonPath}_{01})^{\neg\neg}$ of the set $\mathbf{MonPath}_{01}$.

0.9.2 *Second result:*
(i) Let X be a positively Borel subset of $\mathcal{N}$. One may construct a positively Borel subset Y of $\mathcal{N}$ not reducing to X such that **Fin** is a subset of Y and Y is a subset of **Almost*****Fin**.
(ii) Let X be a positively Borel subset of $\mathcal{N}$ that is contained in **Almost*****Fin**. One may construct a positively Borel subset Y of $\mathcal{N}$ that is contained in **Almost*****Fin** and properly contains X and does not reduce to X. Y might be called a "better" approximation of **Almost*****Fin** than X.

For all α, β in $\mathcal{N}$ we say that α *is apart from* β, notation: $\alpha \# \beta$, if and only if, for some n, $\alpha(n)$ is unequal to $\beta(n)$. It is well-known that this relation is *co-transitive*, that is, for all α, β, γ, if $\alpha \# \beta$ then either $\alpha \# \gamma$ or $\gamma \# \beta$. Also, for all α, β, $\alpha = \beta$ if and only if not: $\alpha \# \beta$.
For every subset X of $\mathcal{N}$ we let X^c be the set of all α in $\mathcal{N}$ such that, for every β in X, $\alpha \# \beta$. We call X^c the *constructive complement* of X.
The set **Almost*****Fin** coincides with the double constructive complement $\mathbf{Fin}^{cc}$ of the set **Fin**. We shall prove, using Brouwer's Thesis 1.6.1, that the set **Almost*****Fin** is contained in the double complement $\mathbf{Fin}^{\neg\neg}$ of the set **Fin**, see Theorem 3.19. It is shown in [20] that the statement that $\mathbf{Fin}^{\neg\neg}$ is a subset of **Almost*****Fin** is equivalent to *Markov's Principle*. Markov's Principle claims that, for every α, if $\neg\neg\exists n[\alpha(n) = 0]$, then $\exists n[\alpha(n) = 0]$.
We shall repeat the argument in Subsection 3.20 of this paper.

Apart from this introductory Section the paper is divided into three Sections. In the first Section we shortly introduce the axioms of intuitionistic analysis. In the second section we study the set $\mathbf{MonPath}_{01}$ and in the third Section we study the set **Almost*****Fin**.

I want to express my profound thanks to an anonymous referee of an earlier version of this paper who noticed a number of inaccuracies and found a mistake in one of the proofs. He also suggested various changes in the style of the presentation.

1 The axioms of intuitionistic analysis

1.1 In intuitionistic analysis, the logical constants have their constructive meaning, and one follows the well-known rules of intuitionistic logic. In particular, a statement of the form $A \vee B$ is considered proven only if one has a proof of A or a proof of B and a proof of an existential statement $\exists x \in V[A(x)]$ has to provide one with a particular element x_0 from the set V and a proof of the corresponding statement $A(x_0)$.

1.2 The intuitionistic mathematician treats infinite objects with care. He is alive to the fact that the growing sequence $0, 1, 2, \ldots$ of the natural numbers is always unfinished and he prefers to call it a *project* rather than a completed *object*. It is the first and foremost example of an infinite sequence, and it sets a pattern. Following this paradigm, every infinite sequence α of natural numbers is a step-by-step construction $\alpha(0), \alpha(1), \alpha(2), \ldots$ of the individual members of the sequence, and never finished. Sometimes, the step-by-step construction is governed by a law or an algorithm, and then the values of the sequences will be found as a result of patient and obedient calculation. At other occasions, the intuitionistic mathematician likes to imagine that he creates the sequence himself. He then does not follow any instruction imposed at the start, but chooses the successive values of the sequence freely, to his own preference, one by one. He believes the following axioms to be plausible.

1.3 We first mention three axioms of countable choice.

1.3.1 *First Axiom of Countable Choice:*

> Let R be a subset of $\mathbb{N} \times \mathbb{N}$ such that, for every m, there exists n such that mRn.
> (We write "mRn" intending "$(m, n) \in R$".)
> Then there exists α in $\mathcal{N}$ such that, for every m, $mR\alpha(m)$.

One may create an infinite sequence α satisfying the requirements step by step, first choosing $\alpha(0)$ such that $0R\alpha(0)$, then choosing $\alpha(1)$ such that $1R\alpha(1)$, and so on. There is no obligation to formulate a law or a rule or an algorithm foretelling the course of the sequence α. Observe that the simple rule: "take, for every m, the least n such that mRn", is no good, as it is possible that we know $0R1$ but are unable to decide $0R0$.

1.3.2 *First Axiom of Dependent Choices:*

> Let X be a subset of $\mathbb{N}$ and R a subset of $X \times X$ such that for every m in X there exists n in X such that mRn.
> Then, given any m in X, there exists α such that $\alpha(0) = m$ and, for each n, $\alpha(n)R\alpha(n+1)$.

Again, given some m in X, one may construct α step by step, first choosing $\alpha(1)$ in X such that $mR\alpha(1)$, then choosing $\alpha(2)$ in X such that $\alpha(1)R\alpha(2)$, and so on.
We let $*$ be the binary function on $\mathbb{N}$ corresponding to the operation of concatenating finite sequences of natural numbers, that is, for all $(m_0, \ldots, m_{k-1})$ and $(n_0, \ldots n_{l-1})$ in $\mathbb{N}^*$, $\langle m_0, \ldots, m_{k-1}\rangle * \langle n_0, \ldots, n_{l-1}\rangle = \langle m_0, \ldots, m_{k-1}, n_0, \ldots, n_{l-1}\rangle$. Given any α in $\mathcal{N}$, n in $\mathbb{N}$, we let α^n be the element of $\mathcal{N}$ satisfying: for all m, $\alpha^n(m) := \alpha(\langle n\rangle * m)$.

1.3.3 *Second Axiom of Countable Choice:*

> Let R be a subset of $\mathbb{N} \times \mathcal{N}$ such that, for every m, there exists α such that $mR\alpha$.
> Then there exists α such that, for every m, $mR\alpha^m$.

One builds the promised sequence α step by step. One first starts a project for an infinite sequence suitable for 0, say α^0, and calculates $\alpha^0(0)$. One then starts a project for an infinite sequence suitable for 1, say α^1, and calculates $\alpha^1(0)$, and, resuming the

project started earlier, also $\alpha^0(1)$. One then starts a project for an infinite sequence suitable for 2, say α^2, and calculates $\alpha^2(0)$, $\alpha^1(1)$ and $\alpha^0(2)$, and so on. In this way one continues.
We will not make use of the so-called Second Axiom of Dependent Choices, see [24], and do not mention it here.

1.4 Recall, from Section 0.5, that, for every α in $\mathcal{N}$, n in $\mathbb{N}$, we denote $\langle \alpha(0), \ldots \alpha(n-1) \rangle$ by $\overline{\alpha}n$. Also, given any α in $\mathcal{N}$, a in $\mathbb{N}$, we say that α *passes through* a if and only if, for some n, $\overline{\alpha}n = a$.
The following axiom does not stand a classical reading of the quantifiers.

1.4.1 *Brouwer's Continuity Principle:*

> Let R be a subset of $\mathcal{N} \times \mathbb{N}$ such that, for every α, there exists n such that $\alpha R n$. Then, for every α, there exist m, n such that, for every β, if β passes through $\overline{\alpha}m$, then $\beta R n$.

This postulate may be justified along the following lines. Every infinite sequence α of natural numbers, even if it is given by an algorithm, may be imagined to result from a free step-by-step-construction. If α is the result of a sequence of free choices, and at some moment of time a number suitable for α is found, only finitely many values of α will have been chosen and the number found for α will suit any sequence β that has these finitely many values the same as α.
Brouwer's Continuity Principle revolutionizes analysis. It is the main tool for proving the very non-classical results of this paper. We want to enhance its applicability and formulate it more generally.

1.4.2 Suppose that γ belongs to **Bar**. For every α in $\mathcal{N}$ we let $\gamma(\alpha)$ be the natural number p such that, for some n, $\gamma(\overline{\alpha}n) = p+1$ and for all $i < n$, $\gamma(\overline{\alpha}i) = 0$. In this way every γ in **Bar** acts as a code for a continuous function from $\mathcal{N}$ to $\mathbb{N}$. For this reason, the set **Bar** is sometimes called **Fun**.

Suppose that γ belongs to **Fun** and $\gamma(\langle\,\rangle) = 0$. Observe that, for each n, γ^n belongs to **Fun**. We let $\gamma|\alpha$ be the infinite sequence β such that, for all n, $\beta(n) = \gamma^n(\alpha)$. In this way every γ in **Fun** such that $\gamma(\langle\,\rangle) = 0$ acts as a code for a continuous function from $\mathcal{N}$ to $\mathcal{N}$. We sometimes say: "let γ be a (continuous) function from $\mathcal{N}$ to $\mathcal{N}$" meaning: "let γ belong to **Fun** and $\gamma(\langle\,\rangle) = 0$".
Let γ be a function from $\mathcal{N}$ to $\mathcal{N}$. The set of all sequences of the form $\gamma|\beta$, where β belongs to $\mathcal{N}$ is called the *range* of γ.
Recall from Section 0.8.0 that a subset X of $\mathcal{N}$ is called a *spread* if and only if (i) for every α in $\mathcal{N}$, if for each n there exists β in X passing through $\overline{\alpha}n$, then α belongs to X and (ii) there exists β such that, for every a in $\mathbb{N}$, $\beta(a) = 1$ if and only if a contains an element of X.

1.4.3 Lemma:

> Let X be a spread. There exists a continuous function r from $\mathcal{N}$ to $\mathcal{N}$ with the

following two properties:
(i) for every α in $\mathcal{N}$, $r|\alpha$ belongs to X, and
(ii) for every α in X, $r|\alpha = \alpha$.
(A function r satisfying these two conditions is called a *retraction* of $\mathcal{N}$ onto X.)

Proof: Let X be a spread. Let β be an element of $\mathcal{N}$ such that, for every n in $\mathbb{N}$, $\beta(n) = 1$ if and only if n contains an element of X. We define a (continuous) function r from $\mathcal{N}$ to $\mathcal{N}$ such that for all α, for all n, $(r|\alpha)(n) = \alpha(n)$ if $\beta((\overline{r|\alpha})n * \langle\alpha(n)\rangle) = 1$ and $(r|\alpha)(n) = p$ where p is the least k such that $\beta((\overline{r|\alpha})n * \langle k\rangle) = 1$ if not. One easily verifies that r satisfies the requirements. $\square$

1.4.4 *Brouwer's Continuity Principle*, general formulation:
Let X be a spread and let R be a subset of $X \times \mathbb{N}$ such that, for every α in X, there exists n such that αRn.
Then, for every α in X there exist m, n such that for every β in X, if β passes through $\overline{\alpha}m$, then βRn.

Proof: Let r be a retraction of $\mathcal{N}$ onto X. Let T be the subset of $\mathcal{N} \times \mathbb{N}$ consisting of all pairs (α, n) such that $(r|\alpha)Rn$. Observe that for every α in $\mathcal{N}$ there exists n such that αTn. Applying the Continuity Principle 1.4.1 we see that for every α in X there exist m, n such that for every β in X, if β passes through $\overline{\alpha}m$, then $(r|\beta)Rn$, that is βRn. $\square$

1.5 The collection of the positively Borel subsets of $\mathcal{N}$ has been introduced by an inductive definition. Such definitions make sense although it seems quite fantastic to imagine the construction of all the members of such a collection as completed, much more so than it seems to imagine the construction of the infinite sequence $0, 1, 2, \dots$ as completed. Nevertheless, useful observations may be made on every "possible" positively Borel set, and we have some general understanding of this notion. As long as we do not lay down general principles on the construction of sets, the question if the collection of all objects falling under this notion deserves to be called a set, is not important.
Brouwer himself, although he, like Borel, see [2] and [12], refused to accept Cantor's first uncountable ordinal, does not shy at a similar inductively defined notion in the proof of his bar theorem.

1.5.1 For every n, for every subset A of $\mathbb{N}$ we let $n * A$ be the set of all numbers $n * a$, where a belongs to A.
$\langle\, \rangle$ is the code number of the empty sequence.
We now introduce *stumps*. Stumps are certain decidable subsets of $\mathbb{N}$. (The word "stump" is used in [4] in a sense slightly different from ours.)
The definition of the collection of all stumps is by induction:

(i) The empty set is a stump.

(ii) Given any infinite sequence $S_0, S_1, \ldots$ of stumps, we may form another stump S in the following way: $S := \{\langle\ \rangle\} \cup \bigcup_{n\in\mathbb{N}} \langle n\rangle * S_n$. The stumps $S_0, S_1, \ldots$ are called the *immediate substumps* of the stump S.
(iii) Every stump is obtained from the empty stump by the repeated application of the construction step (ii).

One may give definitions and proofs by induction on the set of stumps.

1.5.2 *Principle of induction on the set of stumps:*
Let P be a collection of stumps such that every stump belongs to P as soon as each one of its immediate substumps belongs to P.
Then every stump belongs to P.

Stumps may be used for labeling the members of other inductively defined collection like the collection of the positively Borel sets. They then play the role fulfilled by countable ordinals in classical set theory.
We will apply them in this way in Section 3 of this paper. In Section 2, however, we treat some inductively defined classes of subsets of $\mathcal{N}$ without mentioning stumps.
In [25] there is a number of applications of the principle of induction on the set of stumps.

¿From now on we want to use $\sigma, \tau, \ldots$ as variables on the set of stumps.

We define two binary relations $<$ and $\leq$ on the set of stumps, as follows.
For all stumps σ, τ we pronounce: "$\sigma < \tau$" as: "σ is *strictly-smaller* than τ", and: "$\sigma \leq \tau$" as: "σ is *smaller* than τ". We define for all stump σ, τ : σ is strictly-smaller than τ if and only if σ is smaller than some substump of τ, and σ is smaller than τ if and only if every immediate substump of σ is strictly-smaller than τ. Observe that no stump is strictly-smaller than the empty stump and that the empty stump is smaller than any stump.

1.5.3 Theorem
(i) For all stumps σ, τ, ρ:
Not: $\sigma < \sigma$ and: $\sigma \leq \sigma$.
If $\sigma < \tau$, then $\sigma \leq \tau$.
If $\sigma < \tau$ and $\tau \leq \rho$, then $\sigma < \rho$.
If $\sigma \leq \tau$ and $\tau < \rho$, then $\sigma < \rho$.
If $\sigma \leq \tau$ and $\tau \leq \rho$, then $\sigma \leq \rho$.
(ii) Let P be a collection of stumps such that every stump σ belongs to P as soon as every stump strictly-smaller than σ belongs to P.
Then P coincides with the set of stumps.

Proof: The proof is straightforward and left to the reader. □

1.6 The axiom we are about to mention now is perhaps Brouwer's most daring proposal. We want to present it in a simplified form, that suffices for the purposes of this paper. A stronger version occurs in [25]. More details and an exposition of the

considerations that led Brouwer to enunciate this axiom may be found in [3], [9] and [24].
We let **Stump** be the set of all α in $\mathcal{N}$ such that there exists a stump σ coinciding with the set of all a in $\mathbb{N}$ such that for every initial segment b of a, $\alpha(b) = 0$.

1.6.1 *Brouwer's Thesis:*
Bar coincides with **Stump**.

The argument Brouwer brings forward in support of his thesis is a metamathematical one. Brouwer maintains that, if I know some α to belong to **Bar**, I must have some kind of "canonical" proof for this fact, and consideration of this proof will enable me to conclude that α also belongs to **Stump**.
On the other hand, the proof that **Stump** is a subset of **Bar** is by straightforward induction on the set of stumps.

2 The analytic example: $\mathbf{MonPath}_{01}$

2.1 In Section 0.8.1 we introduced the set $\mathbf{MonPath}_{01}$ consisting of all α in $\mathcal{N}$ such that for some β in $\mathcal{N}$, for every n, $\beta(n) \leq \beta(n{+}1) \leq 1$ and $\alpha(\overline{\beta}n) = 0$. Our aim in this section is to prove that this analytic set is not positively Borel.
We will reach this aim not at one stroke but in a number of steps that we describe in subsection 2.1.2. In Subsection 2.1.1 we introduce some notation and mention some facts concerning positively Borel sets.

2.1.1 We define a sequence $\mathbf{\Sigma}^0_1, \mathbf{\Pi}^0_1, \mathbf{\Sigma}^0_2, \mathbf{\Pi}^0_2, \ldots$ of classes of subsets of $\mathcal{N}$, as follows.

A subset X of $\mathcal{N}$ belongs to $\mathbf{\Sigma}^0_1$ or $\mathbf{\Pi}^0_1$, respectively if and only if X is *open* or *closed*, respectively.
For each positive n, for every subset X of $\mathcal{N}$, we define: X belongs to $\mathbf{\Sigma}^0_{n+1}$ if and only if there is a sequence $X_0, X_1, \ldots$ of members of $\mathbf{\Pi}^0_n$ such that $X = \bigcup\limits_{i\in\mathbb{N}} X_i$, and: X belongs to $\mathbf{\Pi}^0_{n+1}$ if and only if there is a sequence $X_0, X_1, \ldots$ of members of $\mathbf{\Sigma}^0_n$ such that $X = \bigcap\limits_{i\in\mathbb{N}} X_i$.
It is easy to see that, for each n, $\mathbf{\Pi}^0_n$ and $\mathbf{\Sigma}^0_n$ are subclasses of both $\mathbf{\Sigma}^0_{n+1}$ and $\mathbf{\Pi}^0_{n+1}$.
It follows from the (Finite) Borel Hierarchy Theorem proved in [24] that, for each positive n, there are sets that belong to $\mathbf{\Pi}^0_n$ but not to $\mathbf{\Sigma}^0_n$ and also sets that belong to $\mathbf{\Sigma}^0_n$ but not to $\mathbf{\Pi}^0_n$.
We define a sequence $E_1, A_1, E_2, A_2, \ldots$ of subsets of $\mathcal{N}$, as follows: E_1 is the set of all α such that, for some n, $\alpha(n) \neq 0$ and A_1 is the set with $\underline{0}$ as its one and only member. For each positive n, we let E_{n+1} be the set of all α such that, for some i, α^i belongs to A_n and we let A_{n+1} be the set of all α such that, for every i, α^i belongs to E_n.
Using the Second Axiom of Countable Choice one may verify that, for every positive n, $\mathbf{\Sigma}^0_n$ is the class of all subsets of $\mathcal{N}$ reducing to E_n and $\mathbf{\Pi}^0_n$ is the class of all subsets

of $\mathcal{N}$ reducing to A_n.
We sometimes say: "X is $\mathbf{\Sigma}^0_n$", or: "X is $\mathbf{\Pi}^0_n$" while intending: "the set X belongs to the class $\mathbf{\Sigma}^0_n$", "the set X belongs to the class $\mathbf{\Pi}^0_n$", respectively.

Let X, Y be subsets of $\mathcal{N}$. We say that X *is a proper subset of* Y if X is a subset of Y and the assumption that Y is a subset of X leads to a contradiction. Observe that this notion is a rather weak one. In general, we are unable, given subsets X, Y, of $\mathcal{N}$ such that X is a proper subset of Y, to indicate an element of Y that does not belong to X.

2.1.2 Our first steps on the way towards the main result of this Section are cautious ones. We first show, in Theorem 2.5, that the set $\mathbf{MonPath}_{01}$ is not $\mathbf{\Pi}^0_1$, and then, in Theorem 2.7 that it is not $\mathbf{\Pi}^0_2$. We continue and show, in Theorem 2.8, that it is not $\mathbf{\Sigma}^0_2$ and not $\mathbf{\Sigma}^0_3$ and, in Theorem 2.10, that it is not $\mathbf{\Pi}^0_3$ and not $\mathbf{\Sigma}^0_4$. We then observe the following. The closure $\overline{\mathbf{MonPath}_{01}}$ of the set $\mathbf{MonPath}_{01}$, *firstly,* contains but does not coincide with the set $\mathbf{MonPath}_{01}$, according to Theorem 2.5, and, *secondly,* is contained in every closed subset of $\mathcal{N}$ that contains the set $\mathbf{MonPath}_{01}$. So there exists a *best approximation of the set* $\mathbf{MonPath}_{01}$ *in the class* $\mathbf{\Pi}^0_1$ and this best approximation still does not coincide with the set $\mathbf{MonPath}_{01}$. Similarly, according to Theorem 2.10, there exists a $\mathbf{\Pi}^0_3$ subset P of $\mathcal{N}$ that, *firstly,* contains but does not coincide with the set $\mathbf{MonPath}_{01}$ and, *secondly,* is contained in every $\mathbf{\Pi}^0_3$ subset of $\mathcal{N}$ that contains the set $\mathbf{MonPath}_{01}$. So there exists a *best approximation of the set* $\mathbf{MonPath}_{01}$ *in the class* $\mathbf{\Pi}^0_3$ and this best approximation still does not coincide with the set $\mathbf{MonPath}_{01}$.

Our strategy for proving the general result is the following. We define a class $\mathcal{B}$ of subsets of $\mathcal{N}$ called *Borel-approximations of the set* $\mathbf{MonPath}_{01}$, and prove, *firstly,* that the set $\mathbf{MonPath}_{01}$ is properly contained in any one of its Borel-approximations and, *secondly,* that every Borel set containing the set $\mathbf{MonPath}_{01}$ also contains a Borel-approximation of the set $\mathbf{MonPath}_{01}$.
We first introduce, in Subsection 2.11, a *monotone operator* $X \mapsto X^-$ on the class of all subsets of $\mathcal{N}$. We then define $\mathcal{B}$ inductively, in Section 2.15, as follows. The first Borel-approximation of $\mathbf{MonPath}_{01}$ and the *initial* set of $\mathcal{B}$ is the closure $\overline{\mathbf{MonPath}_{01}}$ of the set $\mathbf{MonPath}_{01}$. The further members of $\mathcal{B}$ are obtained by applying the operator to an earlier obtained approximation or by taking the countable intersection of a decreasing sequence of earlier obtained approximations.
It turns out that he set $\mathbf{MonPath}_{01}$ is a fixed point of the monotone operator $X \mapsto X^-$, but none of its Borel-approximations is. In fact, the set $\mathbf{MonPath}_{01}$ is the only fixed point of the operator $X \mapsto X^-$ that contains $\mathbf{MonPath}_{01}$.

Theorem 2.14 shows how one may obtain the set P considered in Theorem 2.10 from $\overline{\mathbf{MonPath}_{01}}$ by means of the operator introduced in Subsection 2.11.
Theorem 2.18 proves that the set $\mathbf{MonPath}_{01}$ is a proper subset of every one of its Borel-approximations. We will show this by constructing, for every such Borel approximation X, a continuous function from $\mathcal{N}$ to $\mathcal{N}$ mapping $\mathcal{N}$ into X but not into $\mathbf{MonPath}_{01}$. The subtle Lemma 2.17 prepares the way for this construction and

establishes that the Borel-approximations of $\mathbf{MonPath}_{01}$ are *proof against procrastination.* The meaning of this expression will be given in the course of the proof of Lemma 2.17.
Theorem 2.20 proves that every Borel set containing the set $\mathbf{MonPath}_{01}$ also contains a Borel-approximation of $\mathbf{MonPath}_{01}$. We obtain an apparently stronger result: for every positively Borel set X there exists a Borel-approximation Y of $\mathbf{MonPath}_{01}$ such that every continuous function from $\mathcal{N}$ into $\mathcal{N}$ mapping $\mathbf{MonPath}_{01}$ into X also maps Y into X. If Y shows such behaviour towards X we will say that Y *testifies against* X.

At this point we will have convinced ourselves that $\mathbf{MonPath}_{01}$ is not positively Borel but we want to add some refinements. We introduce so-called *special* Borel-approximations of $\mathbf{MonPath}_{01}$ and show, in Theorem 2.22, that every special Borel-approximation Y of $\mathbf{MonPath}_{01}$ is a subset of every Borel set containing $\mathbf{MonPath}_{01}$ and reducing to Y, that is, Y is indeed the *best possible approximation of* $\mathbf{MonPath}_{01}$ *in the class of all sets reducing to* Y. This conclusion follows from the slightly stronger statement that we actually prove: every special Borel-approximation of $\mathbf{MonPath}_{01}$ testifies against itself.
Theorem 2.23 proves the result announced in Subsection 0.9.1 and also shows that, for every special Borel-approximation Y of $\mathbf{MonPath}_{01}$, Y^- is a proper subset of Y that does not reduce to Y.
The final Theorem 2.25 will establish, among other things, that the set E_1 and a fortiori the set E_1^1 do not reduce to the set $\mathbf{MonPath}_{01}$. In this sense, $\mathbf{MonPath}_{01}$ is a "simple" subset of $\mathcal{N}$.

2.2 Our first aim is to show that the closure $\overline{\mathbf{MonPath}_{01}}$ of the set $\mathbf{MonPath}_{01}$ coincides with the double complement $(\mathbf{MonPath}_{01})^{\neg\neg}$ of the set $\mathbf{MonPath}_{01}$.
We let $S_{01\mathrm{mon}}$ be the set of all β in $\mathcal{N}$ such that, for every n, $\beta(n) \leq \beta(n+1) \leq 1$. The set $S_{01\mathrm{mon}}$ is a spread. Observe that $\mathbf{MonPath}_{01}$ is the set of all α admitting an element of $S_{01\mathrm{mon}}$. It will be useful to make some observations on $S_{01\mathrm{mon}}$.
For every n, we let $\underline{n}$ be the element of $\mathcal{N}$ with the constant value n, so for all n, i, $\underline{n}(i) := n$.
For every a in $\mathbb{N}$, for every α in $\mathcal{N}$, we let $a * \alpha$ be the element of $\mathcal{N}$ that we obtain by putting the infinite sequence α behind the finite sequence coded by a.
We define an infinite sequence of $0^*, 1^*, \ldots$ of elements of $\mathcal{N}$: for each n, $n^* := \overline{\underline{0}}n * \underline{1}$.
¿From a classical point of view, the sequence $\underline{0}, 0^*, 1^*, \ldots$ is a complete enumeration of the elements of $S_{01\mathrm{mon}}$. Intuitionistically, however, the set $S_{01\mathrm{mon}}$ also contains elements for which we can not make out which is their place in the sequence $\underline{0}, 0^*, 1^*, \ldots$.
Let us consider an example. Let $d : \mathbb{N} \rightarrow \{0, 1, \ldots, 9\}$ be the decimal expansion of π, that is, $\pi = 3 + \sum\limits_{n=0}^{\infty} d(n) \cdot 10^{-n-1}$. Let β be the element of Cantor space $\mathcal{C}$ such that for every n, $\beta(n) = 1$ if and only if for some $i < n$, for each $j < 10$, $d(i+j) = 9$. If β coincides with $\underline{0}$, there is no uninterrupted sequence of 10 9's in the decimal expansion of π, and if β coincides with some other member of the sequence $\underline{0}, 0^*, 1^*, \ldots$, there is. This shows that finding the place of β in the sequence $\underline{0}, 0^*, 1^*, \ldots$ is a task beyond

our capacities.
Using Brouwer's Continuity Principle one may even derive a contradiction from the assumption that, for all α in $S_{01\text{mon}}$ either $\alpha = \underline{0}$ or, for some n, $\alpha = n^*$. For suppose this assumption holds. Applying the Continuity Principle we find m such that every α in $S_{01\text{mon}}$ passing through $\overline{\underline{0}}m$ equals $\underline{0}$. This conclusion is false.
Therefore, the sequence $\underline{0}, 0^*, 1^*, \ldots$ is not a complete enumeration of the elements of the set $S_{01\text{mon}}$.

Let X be a subset of $\mathcal{N}$. We say that X is *enumerable* if and only if there exists β such that β *enumerates* X, that is, (i) for all n, β^n belongs to X and (ii) for every α in X there exists n such that $\alpha = \beta^n$.
The set $S_{01\text{mon}}$ is not enumerable. For suppose β enumerates $S_{01\text{mon}}$. Applying the Continuity Principle we find m, p, such that every α in $S_{01\text{mon}}$ passing through $\overline{\underline{0}}m$ equals β^p. This conclusion is false.

It will be clear that the set $S_{01\text{mon}}$ is the closure of the enumerable set $\{\underline{0}, 0^*, 1^*, \ldots\}$. The set $S_{01\text{mon}}$ also coincides with the double complement $\{\underline{0}, 0^*, 1^*, \ldots\}^{\neg\neg}$ of the set $\{\underline{0}, 0^*, 1^*, \ldots\}$. We only show that $S_{01\text{mon}}$ is a subset of this double complement, leaving the proof of the converse inclusion to the reader. Let α belong to $S_{01\text{mon}}$. Remark that, *if* $\alpha \# \underline{0}$ or $\alpha = \underline{0}$, then α belongs to $\{\underline{0}, 0^*, 1^*, \ldots\}$. As $\neg\neg(\alpha \# \underline{0}$ or $\alpha = \underline{0})$, α belongs in any case to $\{\underline{0}, 0^*, 1^*, \ldots\}^{\neg\neg}$.
We are applying the fact that for all propositions A, B, if B follows from A, then $\neg A$ follows from $\neg B$, and $\neg\neg B$ from $\neg\neg A$. Slightly more generally, it is true that for all propositions A, B, C, if C follows from A, B, then $\neg\neg C$ follows from $\neg\neg A, \neg\neg B$.
Let X be a subset of $\mathcal{N}$ and a spread. X is called a *finitary spread* or a *fan* if and only if, for each s, there are only finitely many n such that $s * \langle n \rangle$ contains an element of X.
Let X be a subset of $\mathcal{N}$ and let P be a subset of $\mathbb{N}$. P is a *bar in* X if and only if for every β in X there exists n such that $\overline{\beta}n$ belongs to P.
The *Fan Theorem* claims that, for every spread X, every bar in X has a finite subset that is also a bar in X. Brouwer used his Thesis on bars, a statement slightly stronger than our 1.6.1, for proving the Fan Theorem.
The set $S_{01\text{mon}}$ is an easy example of a fan. We may prove that this fan satisfies the conclusion of the fan theorem without using Brouwer's thesis on bars. This observation forms the first item of the next theorem. The last item is the statement that the closure of the set $\mathbf{MonPath}_{01}$ coincides with the double complement of the set $\mathbf{MonPath}_{01}$.

2.3 Theorem:

(i) For every subset P of $\mathbb{N}$, if for every β in $S_{01\text{mon}}$ there exists n such that $\overline{\beta}n$ belongs to P, then there exists m such that for every β in $S_{01\text{mon}}$ there exists $n \leq m$ such that $\overline{\beta}n$ belongs to P.
(ii) For every subset Q of $\mathbb{N}$, if, for every n, $\neg\neg Q(n)$, then, for every n, $\neg\neg\forall k < n[Q(k)]$.
(iii) For every subset P of $\mathbb{N}$, if, for every β in $S_{01\text{mon}}$, $\neg\neg$(there exists n such that

$\overline{\beta}n$ belongs to P), then $\neg\neg$ (there exists m such that for every β in $S_{01\text{mon}}$ there exists $n \leq m$ such that $\overline{\beta}n$ belongs to P).

(iv) The closure $\overline{\mathbf{MonPath}_{01}}$ of the set $\mathbf{MonPath}_{01}$ coincides with its double complement $(\mathbf{MonPath}_{01})^{\neg\neg}$.

Proof: (i) First calculate n_0 such that $\overline{\underline{0}}n_0$ belongs to P. Then find, for each $i < n_0$, a number p_i such that $\overline{i^*}p_i$ belongs to P. Now let m be the greatest of the numbers $n_0, p_0, \ldots, p_{n_0-1}$.
(ii) Observe that for all propositions A, B, if both $\neg\neg A$ and $\neg\neg B$, then $\neg\neg(A \wedge B)$. Apply this law and use complete induction.
(iii) Assume first that there exists n such that $\overline{\underline{0}}n$ belongs to P. Find n_0 such that $\overline{\underline{0}}n_0$ belongs to P. Using (ii), observe that $\neg\neg$ (for every $i < n_0$ there exists p such that $\overline{i^*}p$ belongs to P), therefore $\neg\neg$ (there exists m such that for every β in $S_{01\text{mon}}$ there exists $n \leq m$ such that $\overline{\beta}n$ belongs to P).
Now observe that we know that $\neg\neg$ (there exists n such that $\overline{\underline{0}}n$ belongs to P). Therefore our conclusion is valid anyhow.
We are applying the fact that for all propositions A, B, if B follows from A, then $\neg\neg B$ from $\neg\neg A$. Also, for every proposition A, $\neg A$ is equivalent to $\neg\neg\neg A$.
(iv) Observe that the closure $\overline{\mathbf{MonPath}_{01}}$ of the set $\mathbf{MonPath}_{01}$ is the set of all α such that, for every n, there exists β in $S_{01\text{mon}}$ such that, for every $m \leq n$, $\alpha(\overline{\beta}m) = 0$.
Remark that for every s one may decide if s contains an element of $\mathbf{MonPath}_{01}$ or not. It follows that, for every α in $(\mathbf{MonPath}_{01})^{\neg\neg}$, for every n, $\overline{\alpha}n$ contains an element of $\mathbf{MonPath}_{01}$, and therefore $(\mathbf{MonPath}_{01})^{\neg\neg}$ is a subset of $\overline{\mathbf{MonPath}_{01}}$.
Now assume that α belongs to $\overline{\mathbf{MonPath}_{01}}$. Then $\neg$ (there exists m such that for every β in $S_{01\text{mon}}$ there exists $n < m$ such that $\alpha(\overline{\beta}n) \neq 0$), therefore, according to (iii), $\neg$ (for every β in $S_{01\text{mon}}$, $\neg\neg$ (there exists n such that $\alpha(\overline{\beta}n) \neq 0$)), and therefore $\neg\neg$ (there exists β in $S_{01\text{mon}}$ such that for every n, $\alpha(\overline{\beta}n) = 0$), that is, α belongs to $(\mathbf{MonPath}_{01})^{\neg\neg}$. □

2.4 We now want to show that the set $\mathbf{MonPath}_{01}$ is a proper subset of its closure $\overline{\mathbf{MonPath}_{01}}$. We will do so by exhibiting a nonclosed set reducing to $\mathbf{MonPath}_{01}$.

Recall that we defined, in Section 1.3, for every α in $\mathcal{N}$, for every i in $\mathbb{N}$, an element α^i of $\mathcal{N}$. For each m, $\alpha^i(m) = \alpha(\langle i \rangle * m)$. α^i is called the *i-th subsequence of* α.
Now let X be a subset of $\mathcal{N}$. For every positive n, we let $D^n(X)$ be the set of all α in $\mathcal{N}$ such that, for some $i < n$, α^i belongs to X. We call $D^n(X)$ the *n-fold disjunction of* X.
Recall that A_1 is the set that has the sequence $\underline{0}$ as its one and only member. The following theorem establishes, among other things, that the set $D^2(A_1)$ is not closed and reduces to $\mathbf{MonPath}_{01}$.

2.5 Theorem:

(i) For all subsets X, Y of $\mathcal{N}$, for each positive n, Y reduces to $D^n(X)$ if and only if there are subsets $Z_0, \ldots, Z_{n-1}$ of $\mathcal{N}$, each of them reducing to X, such that $Y = \bigcup_{i<n} Z_i$.

(ii) For all subsets X of $\mathcal{N}$, X is closed if and only if X reduces to A_1.
(iii) For each positive n, the set $D^n(A_1)$ reduces to the set $\mathbf{MonPath_{01}}$.
(iv) For each positive n, the set $D^{n+1}(A_1)$ does not reduce to the set $D^n(A_1)$.
(v) For each positive n, the set $\mathbf{MonPath_{01}}$ does not reduce to the set $D^n(A_1)$.
(vi) The set $\mathbf{MonPath_{01}}$ is not closed and does not coincide with the set $(\mathbf{MonPath_{01}})^{\neg\neg}$.

Proof: We only prove (iii) and (iv) and leave the proof of the other statements to the reader.
(iii) For each positive n, we may construct a function γ from $\mathcal{N}$ to $\mathcal{N}$ such that, for every α, $(\gamma|\alpha)(\overline{\underline{0}}n) = 1$ and for each $i < n$, $\alpha^i = \underline{0}$ if and only if, for each k, $(\gamma|\alpha)(\overline{i^*}k) = 0$, and such a function reduces the set $D^n(A_1)$ to the set $\mathbf{MonPath_{01}}$.
(iv) We first prove that the set $D^2(A_1)$ does not reduce to the set A_1. Observe that the closure $\overline{D^2(A_1)}$ of the set $D^2(A_1)$ is the set of all α such that, for each n, either α^0 passes through $\overline{\underline{0}}n$ or α^1 passes through $\overline{\underline{0}}n$. Also remark that the set $\overline{D^2(A_1)}$ is a spread containing $\underline{0}$. Now assume that $\overline{D^2(A_1)}$ forms part of $D^2(A_1)$.
Applying the Continuity Principle we find i, m such that either $i = 0$ and for all α in $\overline{D^2(A_1)}$, if α passes through $\overline{\underline{0}}m$, then $\alpha^0 = \underline{0}$, or $i = 1$ and for all α in $\overline{D^2(A_1)}$, if α passes through $\overline{\underline{0}}m$, then $\alpha^1 = \underline{0}$. Both alternatives are obviously false, as there exist β_0, β_1 in $D^2(A_1)$ such that both β_0 and β_1 pass through $\overline{\underline{0}}m$ and $(\beta_0)^0$ and $(\beta_1)^1$ fail to coincide with $\underline{0}$.
Now assume that n is a nonzero natural number and γ is a function from $\mathcal{N}$ to $\mathcal{N}$ reducing $D^{n+1}(A_1)$ to $D^n(A_1)$.
Observe that, for each $i < n+1$, the set of all α such that $\alpha^i = \underline{0}$ is a spread containing $\underline{0}$, and, using the Continuity Principle, find m_i, k_i such that $k_i < n$ and, for all α passing through $\overline{\underline{0}}m_i$, if $\alpha^i = \underline{0}$, then $(\gamma|\alpha)^{k_i} = 0$. Observe that, by the pigeonhole principle, there exist i, $j < n + 1$ such that $i < j$ and $k_i = k_j$. Without loss of generality we may assume $k_0 = k_1 = 0$. We let m be the greatest one of the numbers $m_0, m_1, \ldots, m_n$. We let δ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that for all α, $\overline{(\delta|\alpha)}m = \overline{\underline{0}}m$ and $(\delta|\alpha)^0 = \overline{\underline{0}}m * \alpha^0$ and $(\delta|\alpha)^1 = \overline{\underline{0}}m * \alpha^1$ and, for each i, if $1 < i < n + 1$, then $(\delta|\alpha)^i = \overline{\underline{0}}m * \underline{1}$. (These requirements do not collide: we assume that we coded the finite sequences of the natural numbers in such a way that for each positive m, $m < \overline{\underline{0}}m$ and for all s, t, $s \leq s * t$.) Let ε be the function from $\mathcal{N}$ to $\mathcal{N}$ such that, for all α, $\varepsilon|\alpha = (\gamma|(\delta|\alpha))^0$. Observe that the function ε reduces $D^2(A_1)$ to A_1. We saw that $D^2(A_1)$ does not reduce to A_1. □

2.6 Theorem 2.5 not only makes sure that the set $\mathbf{MonPath_{01}}$ is not a closed subset of $\mathcal{N}$ but also gives us some information on the fine structure of the intuitionistic Borel hierarchy. The class of all sets that are the union of two closed sets properly contains the class of all closed sets, and the class of all sets that are the union of three closed sets properly contains the class of all sets that are the union of two closed sets, and so on. On the other hand, every union of finitely many closed sets reduces to the set $\mathbf{MonPath_{01}}$.

Our next result will be that the set $\mathbf{MonPath_{01}}$ is not a countable intersection of open sets, that is, it does not belong to the class $\mathbf{\Pi}^0_2$.

For every α in $\mathcal{N}$, every β in $S_{01\text{mon}}$, we let α_β be the element of $\mathcal{N}$ such that, for each n, $\alpha_\beta(\overline{\beta}n) = 0$ and for every a, if β does not pass through a, then $\alpha_\beta(a) = a$. Observe that for every α, α belongs to $\mathbf{MonPath}_{01}$ if and only if, for some β in $S_{01\text{mon}}$, α coincides with α_β.

2.7 Theorem:

(i) Every open set containing $\mathbf{MonPath}_{01}$ contains $(\mathbf{MonPath}_{01})^{\neg\neg}$.
(ii) Every $\mathbf{\Pi}^0_2$-set containing $\mathbf{MonPath}_{01}$ contains $(\mathbf{MonPath}_{01})^{\neg\neg}$.
(iii) The set $\mathbf{MonPath}_{01}$ is not $\mathbf{\Pi}^0_2$.

Proof: (i) Let X be an open subset of $\mathcal{N}$ containing $\mathbf{MonPath}_{01}$. Find β such that for every α, α belongs to X if and only if, for some n, $\beta(\overline{\alpha}n) \neq 0$.
Let α belong to $\mathcal{N}$. Observe that for every γ in $S_{01\text{mon}}$, α_γ belongs to $\mathbf{MonPath}_{01}$ and therefore to X, so for some n, $\beta(\overline{\alpha_\gamma}n) \neq 0$. Applying Theorem 2.3(i), find m such that for every γ in $S_{01\text{mon}}$ there exists $n \leq m$ such that $\beta(\overline{\alpha_\gamma}n) \neq 0$. Now assume that α belongs to $(\mathbf{MonPath}_{01})^{\neg\neg}$; then there exists γ in $S_{01\text{mon}}$ such that $\overline{\alpha_\gamma}m = \overline{\alpha}m$ and therefore also $n \leq m$ such that $\beta(\overline{\alpha_\gamma}n) = \beta(\overline{\alpha}n) \neq 0$, so α belongs to X.
(ii) and (iii) easily follow from (i), in view of Theorem 2.5. □

It follows from Theorem 2.7 that the set $\overline{\mathbf{MonPath}_{01}}$ is not only contained in every $\mathbf{\Pi}^0_1$-set containing $\mathbf{MonPath}_{01}$ but also in every $\mathbf{\Pi}^0_2$-set containing $\mathbf{MonPath}_{01}$.
We now prove that the set $\mathbf{MonPath}_{01}$ is not $\mathbf{\Sigma}^0_3$.

2.8 Theorem:

(i) For every sequence $X_0, X_1, \ldots$ of subsets of $\mathcal{N}$, if $\mathbf{MonPath}_{01}$ reduces to $\bigcup\limits_{n\in\mathbb{N}} X_n$, then, for some n, $\mathbf{MonPath}_{01}$ reduces to X_n.
(ii) The set $\mathbf{MonPath}_{01}$ is not $\mathbf{\Sigma}^0_2$.
(iii) The set $\mathbf{MonPath}_{01}$ is not $\mathbf{\Sigma}^0_3$.

Proof: (i) Let $X_0, X_1, \ldots$ be a sequence of subsets of $\mathcal{N}$ and γ a function from $\mathcal{N}$ to $\mathcal{N}$ such that for each α, α belongs to $\mathbf{MonPath}_{01}$ if and only if $\gamma|\alpha$ belongs to $\bigcup\limits_{n\in\mathbb{N}} X_n$. Observe that for every α, for every β in $S_{01\text{mon}}$, the sequence α_β belongs to $\mathbf{MonPath}_{01}$, and, using the Continuity Principle, find m, n such that $m > 0$ and for all α, for all β in $S_{01\text{mon}}$, if both α, β pass through $\underline{\overline{0}}m$, then $\gamma|(\alpha_\beta)$ belongs to X_n. Now build a function δ from $\mathcal{N}$ to $\mathcal{N}$ such that for each α, for each a, $(\delta|\alpha)(\underline{\overline{0}}m * a) = \alpha(a)$ and for each $i < m$, $(\delta|\alpha)(\overline{i^*}m) \neq 0$, and $\delta|\alpha$ passes through $\underline{\overline{0}}m$. (We assume that we defined our coding of finite sequences of natural numbers in such a way that these requirements do not collide.) Observe that, for each α, α belongs to $\mathbf{MonPath}_{01}$ if and only if there exists β in $S_{01\text{mon}}$ passing through $\underline{\overline{0}}m$ admitted by $\delta|\alpha$ if and only if $\gamma|(\delta|\alpha)$ belongs to X_n. Therefore $\mathbf{MonPath}_{01}$ reduces to X_n.
(ii) is an easy consequence of (i) and Theorem 2.5.
(iii) is an easy consequence of (i) and Theorem 2.7. □

2.9 It is not true that every $\mathbf{\Sigma}^0_2$-set containing $\mathbf{MonPath}_{01}$ contains $(\mathbf{MonPath}_{01})^{\neg\neg}$. We show this by the following example. Observe that $\mathbf{MonPath}_{01}$ is contained in the union of the set E consisting of all α admitting $0^* = \underline{1}$ and the set F consisting of all α such that $\neg\neg$(there exists β in $S_{01\text{mon}}$ such that $\beta(0) = 0$ and α admits β). Observe that both E and F are closed. The set $(\mathbf{MonPath}_{01})^{\neg\neg}$ is not a subset of $E \cup F$. In order to see this, observe that $(\mathbf{MonPath}_{01})^{\neg\neg}$ is a spread containing $\underline{0}$. Assume that $(\mathbf{MonPath}_{01})^{\neg\neg}$ is a subset of $E \cup F$. Using the Continuity Principle we find m such that *either* every α in $(\mathbf{MonPath}_{01})^{\neg\neg}$ passing through $\overline{\underline{0}}m$ belongs to E *or* every α in $(\mathbf{MonPath}_{01})^{\neg\neg}$ passing through $\overline{\underline{0}}m$ belongs to F. Both alternatives are clearly false.
Our next goal is to prove that the set $\mathbf{MonPath}_{01}$ is not $\mathbf{\Pi}^0_3$.
We first extend our terminology. In Section 0.5 we defined: for all α, β, α *admits* β if and only if, for each n, $\alpha(\overline{\beta}n) = 0$. We now define: for all α in $\mathcal{N}$, for all s in $\mathbb{N}$, α *admits* s if and only if, for each $i \leq length(s)$, $\alpha(\overline{s}i) = 0$.
We now let P be the set consisting of all α in $\mathcal{C}$ such that, for every n, *either*, for some $i < n$, α admits i^*, *or* $\neg\neg$ (there exists β in $S_{01\text{mon}}$ passing through $\overline{\underline{0}}n$ such that α admits β). The second of these two alternatives is equivalent to: for each j there exists β in $S_{01\text{mon}}$ passing through $\overline{\underline{0}}n$ such that α admits $\overline{\beta}j$.

It is not difficult to see that the set P belongs to the class $\mathbf{\Pi}^0_3$. One also verifies easily that $\mathbf{MonPath}_{01}$ is a subset of P and that P is a subset of the double complement $(\mathbf{MonPath}_{01})^{\neg\neg}$ of $\mathbf{MonPath}_{01}$. We now are going to prove that P is the best $\mathbf{\Pi}^0_3$-approximation of $\mathbf{MonPath}_{01}$.

For every α, for every a, we let ${}^a\alpha$ be the element of $\mathcal{N}$ such that, for each q, ${}^a\alpha(q) = \alpha(a * q)$.

Subsets of $\mathcal{N}$ coinciding with their own double complement are sometimes called *stable* subsets of $\mathcal{N}$. The first item of the next theorem is the observation that every closed subset of $\mathcal{N}$ is a stable subset of $\mathcal{N}$.

2.10 Theorem:
(i) For every $\mathbf{\Pi}^0_1$-set X, $X^{\neg\neg}$ coincides with X.
(ii) For every $\mathbf{\Pi}^0_3$-set X, if $\mathbf{MonPath}_{01}$ is a subset of X, then P is a subset of X.
(iii) There exists a function from $\mathcal{N}$ to $\mathcal{N}$ mapping $\mathcal{N}$ into P but not into $\mathbf{MonPath}_{01}$.
(iv) The set $\mathbf{MonPath}_{01}$ does not belong to the class $\mathbf{\Pi}^0_3$.
(v) The set $\mathbf{MonPath}_{01}$ does not belong to the class $\mathbf{\Sigma}^0_4$.

Proof:(i) Observe that for every closed subset X of $\mathcal{N}$ there exists an open subset Y of $\mathcal{N}$ such that X coincides with $Y^\neg$, and therefore $X^{\neg\neg} = Y^{\neg\neg\neg} = Y^\neg = X$.
(ii) It clearly suffices to show that for every $\mathbf{\Sigma}^0_2$-set Y, if $\mathbf{MonPath}_{01}$ is a subset of Y, then P is a subset of Y.
Suppose that $Y_0, Y_1, \ldots$ is a sequence of closed sets such that $\mathbf{MonPath}_{01}$ is included in $\bigcup\limits_{n\in\mathbb{N}} Y_n$. Assume that α belongs to P. As in Section 2.6, we consider, for each β in $S_{01\text{mon}}$, the element α_β of $\mathcal{C}$ satisfying, for each n, $\alpha_\beta(\overline{\beta}n) = 0$ and for each a, if β

does not pass through a, then $\alpha_\beta(a) = \alpha(a)$. Observe that, for each β in $S_{01\text{mon}}$, α_β belongs to $\mathbf{MonPath}_{01}$ and therefore to $\bigcup\limits_{n\in\mathbb{N}} Y_n$. Applying the Continuity Principle we find m, p such that for every β, if β passes through $\overline{\underline{0}}m$, then α_β belongs to Y_p.
We now distinguish two cases.
First case: for some $i < m$, α admits i^*. Then α belongs to $\mathbf{MonPath}_{01}$ and therefore also to $\bigcup\limits_{n\in\mathbb{N}} Y_n$.
Second case: $\neg\neg$ (for some β in $S_{01\text{mon}}$ passing through $\overline{\underline{0}}m$, α admits β), therefore $\neg\neg$ (for some β passing through $\overline{\underline{0}}m$, $\alpha = \alpha_\beta$) and α belongs to $Y_p^{\neg\neg} = Y_p$ and therefore also to $\bigcup\limits_{n\in\mathbb{N}} Y_n$.
We thus see that P is a subset of $\bigcup\limits_{n\in\mathbb{N}} Y_n$.

(iii) We first make a preparatory remark. The closure $\overline{\mathbf{MonPath}_{01}}$ of the set $\mathbf{MonPath}_{01}$ is a *spread* in the sense of Section 0.8.0. Using Lemma 1.4.3 we let r_0 be a function from $\mathcal{N}$ to $\mathcal{N}$ that is a *retraction* of $\mathcal{N}$ onto $\overline{\mathbf{MonPath}_{01}}$. So, for every α, $r_0|\alpha$ belongs to $\overline{\mathbf{MonPath}_{01}}$ and, for every α in $\overline{\mathbf{MonPath}_{01}}$, $r_0|\alpha = \alpha$.
We now construct a function γ from $\mathcal{N}$ to $\mathcal{N}$ such that, for every α, the following conditions are satisfied:

- $(*)$ for each n, if $\overline{\alpha}n = \overline{\underline{0}}n$, then $(\gamma|\alpha)(\overline{0^*}n) = (\gamma|\alpha)(\overline{1^*}n) = (\gamma|\alpha)(\overline{\underline{0}}n) = 0$ and, for each i, if $1 < i < n$, then $\alpha(\overline{i^*}n) = 1$, and
- $(**)$ for each n, if n is the least j such that $\alpha(j) \neq 0$, then $(\gamma|\alpha)(\overline{\underline{0}}n * \langle 1\rangle) = (\gamma|\alpha)(\overline{n^*}(n+1) = 1$, and
  in case n is *odd*, $(\gamma|\alpha)(\overline{0^*}(n+1) = 1$ and $\gamma|\alpha$ admits $1^* = \langle 0\rangle * \underline{1}$ and,
  in case p is *even*, $(\gamma|\alpha)(\overline{1^*}(n+1) = 1$ and $\gamma|\alpha$ admits $0^* = \underline{1}$, and,
  in both cases, ${}^{\overline{\underline{0}}(n+1)}(\gamma|\alpha) = r_0|({}^{\overline{\underline{0}}(n+1)}\alpha)$.

Observe that for every α, for every n, if α passes through $\overline{\underline{0}}n$, then for all i, if $1 \le i \le n$, $\gamma|\alpha$ does not admit i^*, and therefore, if $\gamma|\alpha$ admits some β in $S_{01\text{mon}}$ passing through $\overline{\underline{0}}2$, then $\gamma|\alpha$ admits some β in $S_{01\text{mon}}$ passing through $\overline{\underline{0}}(n+1)$.

We first verify that γ maps $\mathcal{N}$ into P.
Let α belong to $\mathcal{N}$ and n to $\mathbb{N}$. We distinguish two cases.

First case: there exists $p < n$ such that $\alpha(p) \neq 0$. Then either $\gamma|\alpha$ admits 0^* or $\gamma|\alpha$ admits 1^*.

Second case: α passes through $\overline{\underline{0}}n$.
First make the further assumption: for some $p \ge n$, $\alpha(p) \neq 0$. Let p_0 be the least such p. Observe that, now, for each k, there exists β passing through $\overline{\underline{0}}(p_0+1)$ such that $\gamma|\alpha$ admits $\overline{\beta}k$, therefore $\neg\neg$ (for some β in $S_{01\text{mon}}$ passing through $\overline{\underline{0}}(p_0+1)$ and therefore also through $\overline{\underline{0}}(n+1)$, $\gamma|\alpha$ admits β).
Then make the alternative further assumption: for all p, $\alpha(p) = 0$. Observe that, now, for each j, $(\gamma|\alpha)(\overline{\underline{0}}j) = 0$, and also $\neg\neg$ (for some β in $S_{01\text{mon}}$ passing through $\overline{\underline{0}}(n+1)$, for all j, $(\gamma|\alpha)(\overline{\beta}j) = 0$).

Finally observe that, as $\neg\neg$ (for some $p > n$, $\alpha(p) \neq 0$, or: for all p, $\alpha(p) = 0$), we may conclude, without any further assumption: $\neg\neg$ (for some β in $S_{01\text{mon}}$ passing through $\overline{\underline{0}}(n+1)$, $\gamma|\alpha$ admits β).

In both cases we may affirm the conclusion: $\gamma|\alpha$ admits 0^* or $\gamma|\alpha$ admits 1^* or $\neg\neg$ (for some β in $S_{01\text{mon}}$ passing through $\overline{\underline{0}}(n+1)$, $\gamma|\alpha$ admits β).
It follows that γ maps $\mathcal{N}$ to P.

We now prove that γ does not map $\mathcal{N}$ into $\mathbf{MonPath}_{01}$.
Assume that γ does map $\mathcal{N}$ into $\mathbf{MonPath}_{01}$. Then, for every α there exists β in $S_{01\text{mon}}$ such that $\gamma|\alpha$ admits β. We apply the Continuity Principle and find n, i_0, i_1 with the property: $n > 2$ and for every α passing through $\overline{\underline{0}}n$ there exists β in $S_{01\text{mon}}$ passing through $\langle i_0, i_1\rangle$ such that $\gamma|\alpha$ admits β.
We now distinguish three cases.
First case: $i_0 = i_1 = 1$. Then, for every α passing through $\overline{\underline{0}}n$, $\gamma|\alpha$ admits $0^* = \underline{1}$. Therefore, for every α passing through $\overline{\underline{0}}n$, if there exists j such that $\alpha(j) \neq 0$, then the first such j is even. Considering $\alpha = \overline{\underline{0}}(2n{+}1) * \underline{1}$ we find a contradiction.
Second case: $i_0 = 0$ and $i_1 = 1$. Then for every α passing through $\overline{\underline{0}}n$, $\gamma|\alpha$ admits 1^*. Therefore, for every α passing through $\overline{\underline{0}}n$, if there exists j such that $\alpha(j) \neq 0$, then the first such j is odd. Considering $\alpha = \overline{\underline{0}}(2n) * \underline{1}$ we find a contradiction.
Third case: $i_0 = i_1 = 0$. Then for every α passing through $\overline{\underline{0}}n$ there exists β in $S_{01\text{mon}}$ passing through $\overline{\underline{0}}2$ such that $\gamma|\alpha$ admits β. Reconsidering the definition of the function γ, we obtain the stronger conclusion that for every α passing through $\overline{\underline{0}}n$ there exists β in $S_{01\text{mon}}$ passing through $\overline{\underline{0}}(n+1)$ such that $\gamma|\alpha$ admits β, and therefore ${}^{\overline{\underline{0}}(n+1)}(\gamma|\alpha)$ belongs to $\mathbf{MonPath}_{01}$. We now build a function δ from $\mathcal{N}$ to $\mathcal{N}$ such that, for each α, $\delta|\alpha$ passes through $\overline{\underline{0}}n * \langle 1\rangle$ and ${}^{\overline{\underline{0}}(n+1)}(\delta|\alpha) = \alpha$.
(These two requirements do not collide. We assume that we defined the function coding finite sequences of natural numbers in such a way that, for every m, $m < \overline{\underline{0}}(m+1)$ and for all s, t, $s \leq s * t$.)
It now follows that, for each α, the sequence ${}^{\overline{\underline{0}}(n+1)}(\gamma|(\delta|\alpha)) = r_0|({}^{\overline{\underline{0}}(n+1)}(\delta|\alpha)) = r_0|\alpha$ belongs to $\mathbf{MonPath}_{01}$. As r_0 is a retraction of $\mathcal{N}$ onto $(\mathbf{MonPath}_{01})^{\neg\neg}$, we conclude that $(\mathbf{MonPath}_{01})^{\neg\neg}$ coincides with $\mathbf{MonPath}_{01}$, but this is false according to Theorem 2.5.
It follows that γ does not map $\mathcal{N}$ into $\mathbf{MonPath}_{01}$.
(iv) is an easy consequence of (ii) and (iii).
(v) follows from (iv), by Theorem 2.8(i). $\square$

2.11 Let X be a subset of $\mathcal{N}$. We let X^- be the set of all α such that *either* α admits $0^* = \underline{1}$ *or* $\alpha(\langle\,\rangle) = 0$ and ${}^{\langle 0\rangle}\alpha$ belongs to X.
Let X, Y be subsets of $\mathcal{N}$. We let $D(X, Y)$ be the set of all α such that either α^0 belongs to X or α^1 belongs to Y. We call the set $D(X, Y)$ the *disjunction* of the sets X, Y.
Recall that A_1 is the set containing $\underline{0}$ as its one and only member.
The following Lemma helps us to understand the complexity of the operation $X \mapsto X^-$.

2.12 Lemma: For every subset X of $\mathcal{N}$, if at least one element of $\mathcal{N}$ does not belong to X, then the set X^- is of the same degree of reducibility as the set $D(A_1, X)$, that is: X^- reduces to $D(A_1, X)$ and $D(A_1, X)$ reduces to X^-.

Proof: Let X be a subset of $\mathcal{N}$ and let β be some element of $\mathcal{N}$ not belonging to X. We first construct a function γ from $\mathcal{N}$ to $\mathcal{N}$ reducing X^- to $D(A_1, X)$. It suffices to ensure that for all α in $\mathcal{N}$, for each j in $\mathbb{N}$, $(\gamma|\alpha)^0(j) = \alpha(\overline{\underline{0}}j)$ and, *if* $\alpha(\langle\ \rangle) = 0$, then $(\gamma|\alpha)^1 = {}^{\langle 0 \rangle}\alpha$, and, *if not*, then $(\gamma|\alpha)^1 = \beta$. We leave it to the reader to verify that γ reduces X^- to $D(A_1, X)$.
We now construct a function δ from $\mathcal{N}$ to $\mathcal{N}$ reducing $D(A_1, X)$ to X^-. It suffices to ensure that for all α in $\mathcal{N}$, $(\delta|\alpha)(\langle\ \rangle) = 0$ and for each j in $\mathbb{N}$, $(\delta|\alpha)(\overline{0^*}(j+1)) = (\delta|\alpha)(\overline{\underline{1}}(j+1)) = \alpha^0(j)$ and ${}^{\langle 0 \rangle}(\delta|\alpha) = \alpha^1$.
We leave it to the reader to verify that δ reduces $D(A_1, X)$ to X^-. $\square$.

2.13 For every subset X of $\mathcal{N}$ we define a sequence $X^{(0)}, X^{(1)}, \ldots$ of subsets of $\mathcal{N}$ as follows: $X^{(0)} := X$, and, for each n, $X^{(n+1)} := (X^{(n)})^-$. We also define, for each n, $Q_n := ((\mathbf{MonPath}_{01})^{\neg\neg})^{(n)}$.
The next Theorem establishes that the operation $X \mapsto X^-$ is a monotone operator on the class of all subsets of $\mathbb{N}$, and that the set $\mathbf{MonPath}_{01}$ is the only fixed point of this operator containing the set $\mathbf{MonPath}_{01}$. The theorem also characterizes the set P occurring in Theorem 2.10 in terms of this operator.

2.14 Theorem:

(i) For all subsets X, Y of $\mathcal{N}$, if X is a subset of Y, then X^- is a subset of Y^-.
(ii) For each n, Q_{n+1} is a subset of Q_n.
(iii) For each n, the set Q_{n+1} does not reduce to the set Q_n, and Q_{n+1} is a proper subset of Q_n.
(iv) The set P coincides with $\bigcap\limits_{n\in\mathbb{N}} Q_n$.
(v) The set $(\mathbf{MonPath}_{01})^-$ coincides with the set $\mathbf{MonPath}_{01}$.
(vi) For every subset X of $\mathcal{N}$, if $\mathbf{MonPath}_{01}$ is a subset of X, and X^- coincides with X, then X coincides with $\mathbf{MonPath}_{01}$.

Proof: The proof of the statements (i), (ii) and (v) is left to the reader.
(iii) Using Lemma 2.12, the reader may verify that, for each n, the sets Q_n and $D^{n+1}(A_1)$ reduce to each other. The result then follows by Theorem 2.5(iv).
(iv) Observe that, for each n, the set Q_{n+1} coincides with the set of all α such that *either* there exists $i \le n$ such that α admits i^* *or* α admits $\overline{\underline{0}}n$ and ${}^{\underline{0}(n+1)}\alpha$ belongs to $(\mathbf{MonPath}_{01})^{\neg\neg}$.
(vi) Suppose that X is a subset of $\mathcal{N}$ containing $\mathbf{MonPath}_{01}$ such that X^- coincides with X. Assume that α belongs to X. We will construct β in $S_{01\mathrm{mon}}$ such that, for each j, $\alpha(\overline{\beta}j) = 0$ and we do so step by step.
In order to determine $\beta(0)$ we use the fact that α belongs to X^- and we distinguish two cases.
First case: α admits $0^* = \underline{1}$. We now define $\beta(0) = 1$, and thus commit ourselves to define also for every $j > 0$, $\beta(j) = 1$.

Second case: $\alpha(\langle\,\rangle) = 0$ and ${}^{\langle 0\rangle}\alpha$ belongs to X. We now define $\beta(0) = 0$.
Once we are in the second case we see that ${}^{\langle 0\rangle}\alpha$ belongs to X and therefore to X^- and we repeat the procedure in order to find $\beta(1)$. Observe that the two cases do not exclude each other, so sometimes we will have to choose. We make the proof rigorous in the following way.
We first define a binary relation R on $\mathbb{N}$. For all a, b in $\mathbb{N}$, aRb if and only if *either* $b = a * \langle 1\rangle$ and α admits $a * \underline{1}$, *or* there exists n such that $a = \overline{\underline{0}}n$ and $b = \overline{\underline{0}}(n+1)$ and α admits b and ${}^{b}\alpha$ belongs to X. Let Y be the set of all a in $\mathbb{N}$ such that, for some b, aRb. Observe that for all a in Y there exists b in Y such that aRb, and that the empty sequence $\langle\,\rangle$ belongs to Y. Now apply the First Axiom of Dependent Choices and find δ such that $\delta(0) = \langle\,\rangle$ and, for each n, $\delta(n)R\delta(n+1)$. Consider the element β of $S_{01\text{mon}}$ that passes through every $\delta(n)$. Observe that α admits β and belongs to $\mathbf{MonPath}_{01}$. $\square$

2.15 We want to study the positively Borel sets that we obtain from the set $(\mathbf{MonPath}_{01})^{\neg\neg}$, by applying the operation $X \mapsto X^-$ introduced in Section 2.11 repeatedly and forming countable intersections of decreasing sequences of sets obtained earlier. We may expect these sets to resemble more and more the set the $\mathbf{MonPath}_{01}$ that is a fixed point of the operator $X \mapsto X^-$.
Thus we define a class of $\mathcal{B}$ of subsets of $\mathcal{N}$ that we want to call *Borel-approximations* of $\mathbf{MonPath}_{01}$, as follows, by induction:

(i) $(\mathbf{MonPath}_{01})^{\neg\neg}$ belongs to $\mathcal{B}$. We sometimes call this set the *initial* set of the class $\mathcal{B}$.
(ii) For every element X of $\mathcal{B}$, also X^- belongs to $\mathcal{B}$.
(iii) For every sequence $X_0, X_1, \ldots$ of elements of $\mathcal{B}$, if, for each n, X_{n+1} is a subset of X_n, then the set $\bigcap_{n\in\mathbb{N}} X_n$ belongs to $\mathcal{B}$.
(iv) Every element of $\mathcal{B}$ is obtained from the initial set $(\mathbf{MonPath}_{01})^{\neg\neg}$ by the repeated application of (ii) and (iii).

2.16 Theorem:
(i) For every X in $\mathcal{B}$, X^- is a subset of X.
(ii) For every X in $\mathcal{B}$, $\mathbf{MonPath}_{01}$ is a subset of X and X is a subset of $(\mathbf{MonPath}_{01})^{\neg\neg}$.
(iii) For all subsets X, Y of $\mathbb{N}$, the set $(X \cap Y)^-$ coincides with the set $X^- \cap Y^-$.
(iv) For all X, Y in $\mathcal{B}$, the intersection $X \cap Y$ belongs to $\mathcal{B}$.

Proof: (i) We use induction on the class $\mathcal{B}$.
Our starting point is the easy fact that $((\mathbf{MonPath}_{01})^{\neg\neg})^-$ is a subset of $(\mathbf{MonPath}_{01})^{\neg\neg}$.
Now assume that X belongs to $\mathcal{B}$ and that X^- is a subset of X. It follows from the monotonicity of the operator $X \mapsto X^-$, see Theorem 2.14(i), that $(X^-)^-$ is a subset of X^-.
Finally, assume that $X_0, X_1, \ldots$ is a sequence of elements of $\mathcal{B}$ such that, for each n,

X_{n+1} is a subset of X_n and X_n^- is a subset of X_n. It follows, by Theorem 2.14(i) that for each n, $(\bigcap_{j\in\mathbb{N}} X_j)^-$ is a subset of $(X_n)^-$ and also of X_n. Therefore $(\bigcap_{j\in\mathbb{N}} X_j)^-$ is a subset of $\bigcap_{j\in\mathbb{N}} X_j$.
(ii) We again use induction on the class $\mathcal{B}$. The only nontrivial point is the following. Suppose that X belongs to $\mathcal{B}$ and $\mathbf{MonPath}_{01}$ is a subset of X. We have to show that $\mathbf{MonPath}_{01}$ is a subset of X^-. This conclusion may be drawn from the fact that the operator $X \mapsto X^-$ is monotone and has $\mathbf{MonPath}_{01}$ as a fixed point, see Theorem 2.14(i) and (v).
(iii) The proof is straightforward and left to the reader.
(iv) Let us say that a member X of $\mathcal{B}$ *has the intersection property* if and only if for each Y in $\mathcal{B}$, the intersection $X \cap Y$ belongs to $\mathcal{B}$. We have to show that each member of $\mathcal{B}$ has the intersection property and do so by induction on $\mathcal{B}$.
It follows from (ii) that the initial set $(\mathbf{MonPath}_{01})^{\neg\neg}$ has the intersection property. Suppose that X belongs to $\mathcal{B}$ and has the intersection property. We want to show that X^- has the intersection property, that is, for each Y in $\mathcal{B}$, the intersection $X^- \cap Y$ belongs to $\mathcal{B}$. We again use induction on $\mathcal{B}$.

> It follows from (ii) that the statement holds if Y is the initial set $(\mathbf{MonPath}_{01})^{\neg\neg}$.
> Now assume that Y belongs to $\mathcal{B}$ and also $X^- \cap Y$ belongs to $\mathcal{B}$. It follows that $X^- \cap Y^- = (X \cap Y)^-$ belongs to $\mathcal{B}$.
> Finally, assume that $Y_0, Y_1, \ldots$ is a sequence of elements of $\mathcal{B}$ such that, for each n, Y_{n+1} is a subset of Y_n and $X^- \cap Y_n$ belongs to $\mathcal{B}$. It follows easily that $X^- \cap \bigcap_{n\in\mathbb{N}} Y_n = \bigcap_{n\in\mathbb{N}} (X^- \cap Y_n)$ belongs to $\mathcal{B}$.

Finally assume that $X_0, X_1, \ldots$ is a sequence of elements of $\mathcal{B}$, each of them having the intersection property, such that for each n, X_{n+1} is a subset of X_n. We have to show that $\bigcap_{n\in\mathbb{N}} X_n$ has the intersection property. Observe that for every Y in $\mathcal{B}$, $(\bigcap_{n\in\mathbb{N}} X_n) \cap Y$ coincides with $\bigcap_{n\in\mathbb{N}} (X_n \cap Y)$ and belongs to $\mathcal{B}$. □

2.17 Lemma:(*The members of $\mathcal{B}$ are proof against procrastination*)

(i) For every X in $\mathcal{B}$, for every α, if $\alpha(\langle\ \rangle) = 0$ and ${}^{\langle 0\rangle}\alpha$ belongs to X, then α belongs to X.
(ii) For every X in $\mathcal{B}$, for every α, for every n,
if, for some n, α admits $\overline{\underline{0}}n$ and ${}^{\underline{0}(n+1)}\alpha$ belongs to X, then α belongs to X.
(iii) For every X in $\mathcal{B}$, for every α, for every β, *if firstly* for each j, if $\overline{\beta}j = \overline{\underline{0}}j$ then $\alpha(\overline{\underline{0}}j) = 0$ and *secondly* for each n, if n is the least j such that $\beta(j) \neq 0$, then ${}^{\underline{0}(n+1)}\alpha$ belongs to X, *then* α belongs to X.

Proof: (i) We use Theorem 2.16(i). For every X in $\mathcal{B}$, X^- is a subset of X. Observe that for every α, if $\alpha(\langle\ \rangle) = 0$ and ${}^{\langle 0\rangle}\alpha$ belongs to X, then α belongs to X^- and therefore to X.
(ii) follows from (i), by induction.
We still have to prove (iii).

We first point out that, *from a classical point of view*, (iii) is an easy consequence of (ii). Suppose that X belongs to $\mathcal{B}$, and α, β to $\mathcal{N}$, and (i) for each j, if $\overline{\beta}j = \overline{\underline{0}}j$ then $\alpha(\overline{\underline{0}}j) = 0$ and (ii) for each n, if n is the least j such that $\beta(j) \neq 0$, then ${}^{\overline{\underline{0}}(n+1)}\alpha$ belongs to X. We distinguish two cases.
First case: $\beta = \underline{0}$. Then α admits $\underline{0}$ and belongs to $\mathbf{MonPath}_{01}$ and therefore to X.
Second case: for some j, $\beta(j) \neq 0$. We let n be the least j such that $\beta(j) \neq 0$, and observe that α admits $\overline{\underline{0}}n$ and ${}^{\overline{\underline{0}}(n+1)}\alpha$ belongs to X. Using (ii), we see that α itself belongs to X.
This argument is unacceptable from an intuitionistic point of view, as it asks us to take a decision we sometimes are unable to take: *either* for all j, $\beta(j) = 0$, *or* for some j, $\beta(j) \neq 0$.

We now give a correct proof of (iii) and do so by induction on $\mathcal{B}$.
If X in $\mathcal{B}$ has the property mentioned in (iii), we will say that X is *proof against procrastination*. We do so because (iii) seems to say something like: even if we indefinitely procrastinate giving conclusive evidence that α belongs to X, not mentioning n such that ${}^{\overline{\underline{0}}(n+1)}\alpha$ belongs to X, but only giving, for the successive values of n, the meager assurance that α admits $\overline{\underline{0}}n$, we may be sure that α belongs to X.

We first consider $X = (\mathbf{MonPath}_{01})^{\neg\neg}$ and show that this set is proof against procrastination. In this special case we may use the classical argument that does not work in general.
Suppose that X belongs to $\mathcal{B}$, and α, β to $\mathcal{N}$, and (i) for each j, if $\overline{\beta}j = \overline{\underline{0}}j$ then $\alpha(\overline{\underline{0}}j) = 0$ and (ii) for each n, if n is the least j such that $\beta(j) \neq 0$, then ${}^{\overline{\underline{0}}(n+1)}\alpha$ belongs to X.
Reasoning as above we find that, *if either* for all j, $\beta(j) = 0$,*or* for some j, $\beta(j) \neq 0$, *then* α belongs to X.
Observing that $\neg\neg$ (for all j, $\beta(j) = 0$ *or* for some j, $\beta(j) \neq 0$), we conclude that α will belong to $X^{\neg\neg}$ and thus to X.

Now assume that X belongs to $\mathcal{B}$ and is proof against procrastination. We have to show that X^- is proof against procrastination. So assume that α, β belong to $\mathcal{N}$ and (i) for each j, if $\overline{\beta}j = \overline{\underline{0}}j$ then $\alpha(\overline{\underline{0}}j) = 0$ and (ii) for each n, if n is the least j such that $\beta(j) \neq 0$, then ${}^{\overline{\underline{0}}(n+1)}\alpha$ belongs to X^-. We make a *decidable* case distinction.
First case: $\beta(0) \neq 0$. Then ${}^{\langle 0\rangle}\alpha$ belongs to X^- and, as $\alpha(\langle\ \rangle) = 0$, also α itself belongs to X^-.
Second case: $\beta(0) = 0$. We now consider $\gamma := {}^{\langle 0\rangle}\alpha$ and observe: for each j, (i) if $\overline{\beta}(j+1) = \overline{\underline{0}}(j+1)$, then $\gamma(\overline{\underline{0}}j) = 0$ and (ii) if n is the least j such that $\beta(j+1) \neq 0$, then ${}^{\overline{\underline{0}}(n+1)}\gamma$ belongs to X^- and therefore to X. As X is proof against procrastination, $\gamma = {}^{\langle 0\rangle}\alpha$ belongs to X. As $\alpha(\langle\ \rangle) = 0$, α itself belongs to X^-.

Finally, assume that $X_0, X_1, \ldots$ is a sequence of elements of $\mathcal{B}$, each of them proof against procrastination. We have to show that $\bigcap_{k\in\mathbb{N}} X_k$ is proof against procrastination.

So assume that α, β belong to $\mathcal{N}$ and (i) for each j, if $\overline{\beta}j = \underline{\overline{0}}j$ then $\alpha(\underline{\overline{0}}j) = 0$ and (ii) for each n, if n is the least j such that $\beta(j) \neq 0$, then ${}^{\underline{\overline{0}}(n+1)}\alpha$ belongs to $\bigcap\limits_{k \in \mathbb{N}} X_k$.
The proof that, for each k, α belongs to X_k, is straightforward. $\square$

2.18 Theorem:

For each X in $\mathcal{B}$ there exists a function from $\mathcal{N}$ to $\mathcal{N}$ mapping $\mathcal{N}$ into X but not into $\mathbf{MonPath}_{01}$.
Therefore $\mathbf{MonPath}_{01}$ does not belong to $\mathcal{B}$, and $\mathbf{MonPath}_{01}$ is a proper subset of every element of $\mathcal{B}$.

Proof: We use induction on $\mathcal{B}$. We first have to define a function γ from $\mathcal{N}$ to $\mathcal{N}$ mapping $\mathcal{N}$ into the initial set $(\mathbf{MonPath}_{01})^{\neg\neg}$ but not into **MonPath**. For this purpose we use a *retraction* of $\mathcal{N}$ onto the *spread* $(\mathbf{MonPath}_{01})^{\neg\neg}$. We mentioned the existence of such a retraction in the first lines of the proof of Theorem 2.10(ii).
Now assume that X is a member of $\mathcal{B}$ and γ is a function from $\mathcal{N}$ to $\mathcal{N}$ mapping $\mathcal{N}$ into X but not into $\mathbf{MonPath}_{01}$. We have to construct a function δ from $\mathcal{N}$ to $\mathcal{N}$ mapping $\mathcal{N}$ into X^{-} but not into $\mathbf{MonPath}_{01}$. When defining δ, we take care that for each α, $(\delta|\alpha)(\langle\,\rangle) = 0$, and ${}^{\langle 0\rangle}(\delta|\alpha) = \gamma|\alpha$ and $(\delta|\alpha)(\langle 1\rangle) = 1$. It will be clear that δ then satisfies the requirements.
Finally, assume that $X_0, X_1, \ldots$ is a sequence of elements of $\mathcal{B}$ such that, for each n, X_{n+1} is a subset of X_n, and that $\gamma_0, \gamma_1, \ldots$ is a sequence of functions from $\mathcal{N}$ to $\mathcal{N}$ such that, for each n, γ_n maps $\mathcal{N}$ into X_n but not into $\mathbf{MonPath}_{01}$.
We now build a function δ from $\mathcal{N}$ to $\mathcal{N}$ such that, for every α, the following conditions are satisfied:

($*$) for each n, if $\overline{\alpha}n = \underline{\overline{0}}n$, then $(\delta|\alpha)(\overline{0^*}n) = (\delta|\alpha)(\overline{1^*}n) = (\delta|\alpha)(\underline{\overline{0}}n) = 0$, and, for each i, if $1 < i < n$, then $\alpha(\overline{i^*}n) = 1$, and

($**$) for each n, if n is the least j such that $\alpha(j) \neq 0$, then $(\delta|\alpha)(\underline{\overline{0}}n * \langle 1\rangle) = (\delta|\alpha)(\overline{n^*}(n+1)) = 1$ and,
if n is *odd*, $(\delta|\alpha)(\overline{0^*}n) = 1$ and $\delta|\alpha$ admits 1^* and,
if n is *even*, $(\delta|\alpha)(\overline{1^*}n) = 1$ and $\delta|\alpha$ admits 0^*,
and *in both cases*, ${}^{\underline{\overline{0}}(n+1)}(\delta|\alpha) = \gamma_n|({}^{\underline{\overline{0}}(n+1)}\alpha)$.

Observe that for every α, for every n, if α passes through $\underline{\overline{0}}n$, then for all i, if $1 \leq i \leq n$, $\delta|\alpha$ does not admit i^*, and therefore, if $\delta|\alpha$ admits some β in $S_{01\text{mon}}$ passing through $\underline{\overline{0}}2$, then $\delta|\alpha$ admits some β in $S_{01\text{mon}}$ passing through $\underline{\overline{0}}(n+1)$.
Observe that for all α, for all n, if n is the least j such that $\alpha(j) \neq 0$, then ${}^{\underline{\overline{0}}(n+1)}(\delta|\alpha)$ belongs to X_n.

We first verify that δ maps $\mathcal{N}$ into $\bigcap\limits_{k \in \mathbb{N}} X_k$.
Let α belong to $\mathcal{N}$ and k to $\mathbb{N}$. We distinguish two cases.
First case: $\overline{\alpha}k \neq \underline{\overline{0}}k$. Then, for some j, $\alpha(j) \neq 0$, therefore either $\delta|\alpha$ admits 0^* or $\delta|\alpha$ admits 1^*. It follows that $\delta|\alpha$ belongs to $\mathbf{MonPath}_{01}$, and therefore also to X_k.
Second case: $\overline{\alpha}k = \underline{\overline{0}}k$. Consider $\beta := {}^{\underline{\overline{0}}k}(\delta|\alpha)$. Observe that (i) for each j, if $\overline{\alpha}(k+j) = \overline{0}(k+j)$, then $\beta(\underline{\overline{0}}(j)) = 0$ and (ii) for each n, if n is the least j such that

$\alpha(k+j) \neq 0$, then $^{\overline{0}(n+1)}\beta = {}^{\overline{0}(k+n+1)}(\delta|\alpha)$ belongs to X_{k+n} and therefore also to X_k. As, according to Theorem 2.17, X_k is proof against procrastination, $\beta = {}^{\overline{0}k}(\delta|\alpha)$ belongs to X_k. As $\delta|\alpha$ admits $\overline{0}k$, $\delta|\alpha$ itself belongs to X_k.

We now want to prove that δ does not map $\mathcal{N}$ into $\mathbf{MonPath_{01}}$.
So assume that δ does map $\mathcal{N}$ into $\mathbf{MonPath_{01}}$.
Then, for every α there exists β in $S_{01\mathrm{mon}}$ such that $\delta|\alpha$ admits β. We apply the Continuity Principle and find n, i_0, i_1 with the following properties: $n > 2$ and for every α passing through $\overline{0}n$ there exists β in $S_{01\mathrm{mon}}$ such that $\delta|\alpha$ admits β and $\beta(0) = i_0$ and $\beta(1) = i_1$. We now distinguish three cases.
First case: $i_0 = i_1 = 1$. Then for all α passing through $\overline{0}n$, $\delta|\alpha$ admits $0^* = \underline{1}$. Therefore, for every α passing through $\overline{0}n$, if there exists j such that $\alpha(j) \neq 0$ then the first such j is even. This is false.
Second case: $i_0 = 0$ and $i_1 = 1$. Then for all α passing through $\overline{0}n$, $\delta|\alpha$ admits $1^* = \langle 0 \rangle * \underline{1}$. Therefore, for every α passing through $\overline{0}n$, if there exists j such that $\alpha(j) \neq 0$, then the first such j is odd. This is false.
Third case: $i_0 = i_1 = 0$. Then for every α passing through $\overline{0}n$ there exists β in $S_{01\mathrm{mon}}$ passing through $\overline{0}2$ such that $\delta|\alpha$ admits β. Reconsidering the definition of δ we see that for every α passing through $\overline{0}n$ there exists β in $S_{01\mathrm{mon}}$ passing through $\overline{0}(n+1)$ such that $\delta|\alpha$ admits β and therefore $^{\overline{0}(n+1)}(\delta|\alpha)$ belongs to $\mathbf{MonPath_{01}}$.
We now construct a function ε from $\mathcal{N}$ to $\mathcal{N}$ such that, for each ζ, n is the least j such that $(\varepsilon|\zeta)(j) \neq 0$ and $^{\overline{0}(n+1)}(\varepsilon|\zeta) = \zeta$. (As in the proof of Theorem 2.10, we may assume that these two requirements do not collide.) Observe that, for each ζ, $^{\overline{0}(n+1)}(\delta|(\varepsilon|\zeta)) = \gamma_n|\zeta$ and $^{\overline{0}(n+1)}(\delta|(\varepsilon|\zeta))$ belongs to $\mathbf{MonPath_{01}}$. As, for each ζ, $\varepsilon|\zeta$ passes through $\overline{0}n$, we conclude that γ_n maps $\mathcal{N}$ into $\mathbf{MonPath_{01}}$. This is false.
It follows that δ does not map $\mathcal{N}$ into $\mathbf{MonPath_{01}}$. □

2.19 Let X be positively Borel set and let Y belong to the set $\mathcal{B}$ of all Borel-approximations of $\mathbf{MonPath_{01}}$. Y *testifies against* X if and only if for every function δ from $\mathcal{N}$ to $\mathcal{N}$, if δ maps $\mathbf{MonPath_{01}}$ into X, then δ also maps Y into X.
It follows from Theorem 2.18 that the set $\mathbf{MonPath_{01}}$ does not coincide with any set Y in $\mathcal{B}$. If we can show, for some positively Borel set X, that there exists Y in $\mathcal{B}$ testifying against X, we may conclude that X does not coincide with $\mathbf{MonPath_{01}}$. This will be our strategy for showing that the set $\mathbf{MonPath_{01}}$ is not positively Borel.

We now introduce a partial infinitary operation, $\mathcal{L}$, on the class of subsets of $\mathcal{N}$. For every sequence $X_0, X_1, \ldots$ of subsets of $\mathcal{N}$ such that, for each n, X_{n+1} forms part of X_n, we let $\mathcal{L}_{n\in\mathbb{N}} X_n$ be the set of all α such that, for every n, *either*, for some $i < n$, α admits i^* , *or* α admits $\overline{0}n$ and the sequence $^{\overline{0}(n+1)}\alpha$ belongs to X_n.
Observe that the set $\mathcal{L}_{n\in\mathbb{N}} X_n$ coincides with the set $\bigcap_{n\in\mathbb{N}} (X_n)^{(n+1)}$.
Using Theorem 2.16 we see that, if $X_0, X_1, \ldots$ is a sequence of elements of $\mathcal{B}$ such that, for each n, X_{n+1} is a subset of X_n, then the set $\mathcal{L}_{n\in\mathbb{N}} X_n$ also belongs to $\mathcal{B}$, as, for each n, $(X_n)^{(n+1)}$ belongs to $\mathcal{B}$ and $(X_{n+1})^{(n+2)}$ is a subset of $(X_n)^{(n+1)}$. Also,

$\mathcal{L}_{n\in\mathbb{N}} X_n$ is a subset of $\bigcap_{n\in\mathbb{N}} X_n$.
Finally, observe that the set P, introduced in Section 2.9, coincides with $\mathcal{L}_{n\in\mathbb{N}} X_n$ if we choose every X_n equal to the initial set $(\mathbf{MonPath}_{01})^{\neg\neg}$.

2.20 Theorem:

(i) The initial set $(\mathbf{MonPath}_{01})^{\neg\neg}$ testifies against every closed subset of $\mathcal{N}$.
(ii) For every sequence $X_0, X_1, \dots$ of positively Borel sets, for every sequence $Y_0, Y_1, \dots$ of elements of $\mathcal{B}$, if, for each n, Y_n testifies against X_n, then $\bigcap_{n\in\mathbb{N}} Y_n$ testifies against $\bigcap_{n\in\mathbb{N}} X_n$.
(iii) For every sequence $X_0, X_1, \dots$ of positively Borel sets, for every sequence $Y_0, Y_1, \dots$ of elements of $\mathcal{B}$, if, for each n, Y_n testifies against X_n, then $\mathcal{L}_{n\in\mathbb{N}}(\bigcap_{k\le n} Y_k)$ testifies against $\bigcup_{n\in\mathbb{N}} X_n$.
(iv) For every positively Borel set X there exists Y in $\mathcal{B}$ that testifies against X.

Proof: We leave the easy proof of (i) and (ii) to the reader.
(iii) Assume that $X_0, X_1, \dots$ is a sequence of positively Borel sets and $Y_0, Y_1, \dots$ is a sequence of elements of $\mathcal{B}$ such that, for each n, Y_n testifies against X_n.
According to Theorem 2.16(iv), for each n, the set $\bigcap_{k\le n} Y_k$ belongs to $\mathcal{B}$, and as, for each n, the set $\bigcap_{k\le n+1} Y_k$ is a subset of the set $\bigcap_{k\le n} Y_k$, also the set $\mathcal{L}_{n\in\mathbb{N}}(\bigcap_{k\le n} Y_k)$ belongs to $\mathcal{B}$. We claim that the set $Y := \mathcal{L}_{n\in\mathbb{N}}(\bigcap_{k\le n} Y_k)$ testifies against $\bigcup_{n\in\mathbb{N}} X_n$.
Assume that δ is a function from $\mathcal{N}$ to $\mathcal{N}$ mapping $\mathbf{MonPath}_{01}$ into $\bigcup_{n\in\mathbb{N}} X_n$.
Let α belong to $\mathcal{N}$ and β to $S_{01\mathrm{mon}}$. As in Section 2.6, we let α_β be the element of $\mathcal{N}$ such that, for every n, $\alpha_\beta(\overline{\beta}n) = 0$, and, for every a, if β does not pass through a, then $\alpha_\beta(a) = \alpha(a)$. Observe that α belongs to $\mathbf{MonPath}_{01}$ if and only if there exists β in $S_{01\mathrm{mon}}$ such that α coincides with α_β. Therefore, for every β in $S_{01\mathrm{mon}}$, the sequence $\delta|(\alpha_\beta)$ belongs to $\bigcup_{n\in\mathbb{N}} X_n$.
Now let α belong to $\mathcal{N}$. Applying the Continuity Principle we find m, n such that for every γ in $\mathcal{N}$, for every β in $S_{01\mathrm{mon}}$, if γ passes through $\overline{\alpha}m$ and β passes through $\overline{\underline{0}}m$, then $\delta|(\gamma_\beta)$ belongs to X_n. Without loss of generality we may assume that $m \ge n$.
Now assume *in addition* that α belongs to $Y := \mathcal{L}_{n\in\mathbb{N}}(\bigcap_{k\le n} Y_k)$. In particular, α belongs to $(\bigcap_{k\le m} Y_k)^{(m+1)}$. We distinguish two cases.
First case: there exists $i < m$ such that α admits i^*. Then α belongs to $\mathbf{MonPath}_{01}$ and therefore $\delta|\alpha$ belongs to $\bigcup_{k\in\mathbb{N}} X_k$.
Second case: α admits $\overline{\underline{0}}m$ and and ${}^{\overline{\underline{0}}(m+1)}\alpha$ belongs to $\bigcap_{k\le m} Y_k$. Construct a function ε from $\mathcal{N}$ to $\mathcal{N}$ such that, for each ζ, (i) the sequence ${}^{\overline{\underline{0}}(m+1)}(\varepsilon|\zeta)$ coincides with ζ,

and (ii) for each a, if there is no b such that $a = \underline{\overline{0}}(m+1) * b$, then $(\varepsilon|\zeta)(a) = \alpha(a)$. (As in earlier such cases, these requirements do not collide.) Observe that for each ζ, if ζ belongs to $\mathbf{MonPath}_{01}$, then there exist γ passing through $\overline{\alpha}m$ and β in $S_{01\mathrm{mon}}$ passing through $\underline{\overline{0}}m$ such that $\varepsilon|\zeta$ coincides with γ_β and therefore $\delta|(\varepsilon|\zeta)$ belongs to X_n. Therefore, as Y_n testifies against X_n, for each ζ, if ζ belongs to Y_n, then $\delta|(\varepsilon|\zeta)$ belongs to X_n. Observe that α itself coincides with $\varepsilon|({}^{\underline{\overline{0}}(m+1)}\alpha)$ and ${}^{\underline{\overline{0}}(m+1)}\alpha$ belongs to Y_n. Therefore, $\delta|\alpha$ belongs to X_n and also to $\bigcup\limits_{k\in\mathbb{N}} X_k$.

It follows that δ maps $Y = \mathop{\mathcal{L}}\limits_{n\in\mathbb{N}}(\bigcap\limits_{k\leq n} Y_k)$ into $\bigcup\limits_{k\in\mathbb{N}} X_k$.

(iv) As every positively Borel set may be obtained from closed sets by means of the operations of countable union and countable intersection, the conclusion follows from (i), (ii) and (iii). □

2.21 We have established the fact that the set $\mathbf{MonPath}_{01}$ is not positively Borel. We want to prove a refinement of Theorem 2.20.

To this end we introduce we introduce a class $\mathcal{B}_0$ of so-called *special Borel-approximations of* $\mathbf{MonPath}_{01}$ by means of the following inductive definition:

(i) The *initial* set $(\mathbf{MonPath}_{01})^{\neg\neg}$ belongs to $\mathcal{B}_0$.

(ii) For every sequence $Y_0, Y_1, \ldots$ of elements of $\mathcal{B}_0$, if for every n, Y_{n+1} is a subset of Y_n then the set set $\mathop{\mathcal{L}}\limits_{n\in\mathbb{N}} Y_n$ belongs to $\mathcal{B}_0$.

(iii) Every element of $\mathcal{B}_0$ is obtained from the initial set by repeatedly applying the operation mentioned in (ii).

2.22 Theorem:

(i) For all X, Y in $\mathcal{B}_0$, also $X \cap Y$ belongs to $\mathcal{B}_0$.

(ii) For every Y in $\mathcal{B}_0$, Y testifies against Y.

Proof: (i). Let us say, given any X in $\mathcal{B}_0$, that X *has the intersection property* if and only if for each Y in $\mathcal{B}_0$, $X \cap Y$ belongs to $\mathcal{B}_0$. We have to show that each X in $\mathcal{B}_0$ has the intersection property and we do so by induction on $\mathcal{B}_0$.

It follows from Theorem 2.16(ii) that the initial set $(\mathbf{MonPath}_{01})^{\neg\neg}$ has the intersection property.

Now assume that $X_0, X_1, \ldots$ is a sequence of elements of $\mathcal{B}_0$ such that for each n, X_n has the intersection property and X_{n+1} is a subset of X_n. We now consider $X := \mathop{\mathcal{L}}\limits_{n\in\mathbb{N}} X_n$ and show that, for each Y in $\mathcal{B}_0$, $X \cap Y$ belongs to $\mathcal{B}_0$. We again use induction on $\mathcal{B}$. It follows from Theorem 2.16(iii) that $X \cap (\mathbf{MonPath}_{01})^{\neg\neg}$ belongs to $\mathcal{B}_0$. Now assume that $Y_0, Y_1, \ldots$ is a sequence of elements of $\mathcal{B}_0$ and that, for each n, $X \cap Y_n$ belongs to $\mathcal{B}_0$ and Y_{n+1} is a subset of Y_n. It follows from Theorem 2.16(iii) that $(\mathop{\mathcal{L}}\limits_{n\in\mathbb{N}} X_n) \cap (\mathop{\mathcal{L}}\limits_{n\in\mathbb{N}} Y_n)$, coinciding with $(\bigcap\limits_{n\in\mathbb{N}} (X_n)^{(n+1)}) \cap (\bigcap\limits_{n\in\mathbb{N}} (Y_n)^{(n+1)})$, also coincides with $\bigcap\limits_{n\in\mathbb{N}} (X_n \cap Y_n)^{(n+1)}$ and $\mathop{\mathcal{L}}\limits_{n\in\mathbb{N}} (X_n \cap Y_n)$ and belongs to $\mathcal{B}_0$.

(ii) We use induction on $\mathcal{B}_0$. The closed set $(\mathbf{MonPath}_{01})^{\neg\neg}$ testifies against every closed subset of $\mathcal{N}$ and in particular against itself. Now assume that $Y_0, Y_1, \ldots$ is a sequence of special Borel-approximations of $\mathbf{MonPath}_{01}$ and that, for every n, Y_{n+1} is a subset of Y_n and Y_n testifies against Y_n. We have to show that the set $\underset{n\in\mathbb{N}}{\mathcal{L}} Y_n$ testifies against itself.
Let δ be a function from $\mathcal{N}$ to $\mathcal{N}$ mapping the set $\mathbf{MonPath}_{01}$ into the set $\underset{n\in\mathbb{N}}{\mathcal{L}} Y_n$. We must prove that δ maps also $\underset{n\in\mathbb{N}}{\mathcal{L}} Y_n$ into $\underset{n\in\mathbb{N}}{\mathcal{L}} Y_n$. So we must prove that, for each k, δ maps the set $\underset{n\in\mathbb{N}}{\mathcal{L}} Y_n$ into the set $Y_k^{(k+1)}$.
Let α belong to $\mathcal{N}$ and β to $S_{01\text{mon}}$. As in Section 2.6, we let α_β be the element of $\mathcal{N}$ such that, for every n, $\alpha_\beta(\overline{\beta}n) = 0$, and, for every a, if β does not pass through a, then $\alpha_\beta(a) = \alpha(a)$. Observe that α belongs to $\mathbf{MonPath}_{01}$ if and only if there exists β in $S_{01\text{mon}}$ such that α coincides with α_β. Observe that, for every β in $S_{01\text{mon}}$, for every k,v the sequence $\delta|(\alpha_\beta)$ belongs to $Y_k^{(k+1)}$.
Now let α belong to $\mathcal{N}$ and k to $\mathbb{N}$. Using the Continuity Principle we find m, n such that $n \le k$ and $m \ge k$ and for every γ in $\mathcal{N}$, for every β in $S_{01\text{mon}}$, if γ passes through $\overline{\alpha}m$ and β passes through $\underline{\overline{0}}m$, then, *either* $n < k$ and $\delta|(\gamma_\beta)$ admits n^*, *or* $n = k$ and $\delta|(\gamma_\beta)$ admits $\underline{\overline{0}}k$ and ${}^{\underline{\overline{0}}(k+1)}(\delta|(\gamma_\beta))$ belongs to Y_k.
Now make the *additional assumption* that α belongs to $\underset{n\in\mathbb{N}}{\mathcal{L}} Y_n$ and in particular to $(Y_m)^{m+1}$. We distinguish two cases.
First case: for some $j < m$, α admits j^*. Then α belongs to $\mathbf{MonPath}_{01}$, and therefore $\delta|\alpha$ belongs to $\underset{n\in\mathbb{N}}{\mathcal{L}} Y_n$ and in particular to $Y_k^{(k+1)}$.
Second case: α admits $\underline{\overline{0}}m$ and ${}^{\underline{\overline{0}}(m+1)}\alpha$ belongs to Y_m. We distinguish two subcases.
First subcase: $n < k$. Observe that Y_m is a subset of the closure $(\mathbf{MonPath}_{01})^{\neg\neg}$ of the set $\mathbf{MonPath}_{01}$. It follows that for every p there exist γ passing through $\overline{\alpha}m$ and β passing through $\underline{\overline{0}}m$ such that γ_β passes through $\overline{\alpha}p$ and $\delta|(\gamma_\beta)$ admits n^*. Therefore, for each i, $(\delta|\alpha)(\overline{n^* i}) = 0$, that is $\delta|\alpha$ admits n^* so $\delta|\alpha$ belongs to $\mathbf{MonPath}_{01}$ and in particular to $Y_k^{(k+1)}$.
Second subcase: $n = k$. Construct a function ε from $\mathcal{N}$ to $\mathcal{N}$ such that, for each ζ, (i) the sequence ${}^{\underline{\overline{0}}(m+1)}(\varepsilon|\zeta)$ coincides with ζ, and (ii) for each a, if there is no b such that $a = \underline{\overline{0}}(m+1) * b$, then $(\varepsilon|\zeta)(a) = \alpha(a)$. (As in previous such cases, the requirements do not collide.) Observe that for each ζ, if ζ belongs to $\mathbf{MonPath}_{01}$, then there exist γ passing through $\overline{\alpha}m$ and β in $S_{01\text{mon}}$ passing through $\underline{\overline{0}}m$ such that $\varepsilon|\zeta$ coincides with γ_β and therefore ${}^{\underline{\overline{0}}(k+1)}(\delta|(\varepsilon|\zeta))$ belongs to Y_k. Therefore, as Y_k testifies against itself, for each ζ, if ζ belongs to Y_k, then ${}^{\underline{\overline{0}}(k+1)}(\delta|(\varepsilon|\zeta))$ belongs to Y_k. Observe that α coincides with $\varepsilon|({}^{\underline{\overline{0}}(m+1)}\alpha)$ and ${}^{\underline{\overline{0}}(m+1)}\alpha$ belongs to Y_m and Y_m is a subset of Y_k. Therefore, ${}^{\underline{\overline{0}}(k+1)}(\delta|\alpha)$ belongs to Y_k, and $\delta|\alpha$ belongs to $Y_k^{(k+1)}$.
We may conclude that for every α in $\underset{n\in\mathbb{N}}{\mathcal{L}} Y_n$, for each k, $\delta|\alpha$ belongs to $(Y_k)^{k+1}$, that is, δ maps $\underset{n\in\mathbb{N}}{\mathcal{L}} Y_n$ into $\underset{n\in\mathbb{N}}{\mathcal{L}} Y_n$. □

2.23 Theorem:

(i) For each Y in $\mathcal{B}$ there exists X in $\mathcal{B}_0$ such that X is a subset of Y.

(ii) For each Y in $\mathcal{B}$, Y^- is a proper subset of Y.

(iii) For all positively Borel sets X, Z, for all Y in $\mathcal{B}$, if Y testifies against X and Z reduces to X, then Y testifies against Z.

(iv) For each Y in $\mathcal{B}_0$, the set Y^- does not reduce to the set Y.

(v) For each positively Borel set X there exists a Borel-approximation Y of $\mathbf{MonPath}_{01}$ testifying against X and not reducing to X.

(vi) For each positively Borel set X containing $\mathbf{MonPath}_{01}$ there exists a Borel-approximation Y of $\mathbf{MonPath}_{01}$ properly contained in X and not reducing to X.

Proof: (i) We use induction on $\mathcal{B}$.
The statement is obviously true if Y is the initial set $(\mathbf{MonPath}_{01})^{\neg\neg}$.
Suppose that Y belongs to $\mathcal{B}$ and X to $\mathcal{B}_0$ and X is a subset of Y. Define, for each n, $X_n := X$ and observe that $\underset{n\in\mathbb{N}}{\mathcal{L}} X_n$ belongs to $\mathcal{B}_0$ and coincides with $\bigcap\limits_{n\in\mathbb{N}}(X_n)^{(n+1)}$ and is a subset of Y^-.
Now assume that $Y_0, Y_1, \ldots$ is a sequence of elements of $\mathcal{B}$ and $X_0, X_1, \ldots$ is a sequence of elements of $\mathcal{B}_0$ such that for each n, Y_{n+1} is a subset of Y_n and X_n is a subset of Y_n. Using Theorem 2.22(i) we observe that $\underset{n\in\mathbb{N}}{\mathcal{L}}(\bigcap\limits_{k\le n} X_k)$ belongs to $\mathcal{B}_0$ and is a subset of $\bigcap\limits_{n\in\mathbb{N}} Y_n$.
(ii) is a consequence of Theorem 2.16(i), Theorem 2.18 and Theorem 2.14(vi).
(iii) Assume that X, Z are positively Borel sets such that Z reduces to X, and that Y in $\mathcal{B}$ testifies against X. Let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ reducing Z to X. Let δ be a function from $\mathcal{N}$ to $\mathcal{N}$ mapping $\mathbf{MonPath}_{01}$ into Z. Observe that for every α, if α belongs to $\mathbf{MonPath}_{01}$, then $\gamma|(\delta|\alpha)$ belongs to X. Therefore for every α, if α belongs to Y, then $\gamma|(\delta|\alpha)$ belongs to X, and also: $\delta|\alpha$ belongs to Z. Thus we see that Y testifies against Z.
(iv) Observe that, for each Y in $\mathcal{B}_0$, Y testifies against Y, and because of (i), Y does not testify against Y^-. It now follows from (iii) that Y^- does not reduce to Y.
(v) Let X be a positively Borel set. Using Theorem 2.17 and (i), find Y in $\mathcal{B}_0$ such that Y testifies against X. It follows from (i) that Y does not testify against Y^-. It follows from (iii) that Y^- does not reduce to X. As Y testifies against X and Y^- is a subset of Y, also Y^- testifies against X.
(vi) Let X be a positively Borel set containing $\mathbf{MonPath}_{01}$. Using (v), find a Borel-approximation Y of $\mathbf{MonPath}_{01}$ testifying against X and not reducing to X. As Y testifies against X and the identity function maps $\mathbf{MonPath}_{01}$ into X, Y is a subset of X. As Y does not reduce to X, Y is a proper subset of X. $\square$

2.24 Observe that Theorem 2.19(v) and (vi) imply the statements we promised to prove in Section 0.9.1.
It follows from Theorem 2.23(iv) and Lemma 2.12 that for each special Borel-approximation Y of $\mathbf{MonPath}_{01}$, the set $D(A_1, Y)$ does not reduce to the set Y. This fact may be compared to the result established in [23] that for each positive

n, the set $D(A_1, A_n)$ does not reduce to the set A_{n+1}. A generalization of this result to all levels of the positive Borel hierarchy is contained in [24].
Finally, we prove that the set $\mathbf{MonPath}_{01}$ has two properties in common with the set A_1: not every open set reduces to $\mathbf{MonPath}_{01}$ and the set $\mathbf{MonPath}_{01}$ is *disjunctively productive* in the sense that the set $D^2(\mathbf{MonPath}_{01})$ does not reduce to the set $\mathbf{MonPath}_{01}$ itself. It follows that $\mathbf{MonPath}_{01}$ is in a certain sense a "simple" set.

2.25 Theorem: ($\mathbf{MonPath}_{01}$ *is "simple"*)

(i) The set E_1 does not reduce to the set $\mathbf{MonPath}_{01}$.
(ii) The set $D^2(\mathbf{MonPath}_{01})$ does not reduce to the set $\mathbf{MonPath}_{01}$.
(iii) The set $D^2(E_1^1)$ reduces to the set E_1^1.
(iv) The set E_1^1 does not reduce to the set $\mathbf{MonPath}_{01}$, that is, $\mathbf{MonPath}_{01}$ is not a complete element of the class of the analytic subsets of $\mathcal{N}$.

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Proof: (i) Assume that γ is a function from $\mathcal{N}$ to $\mathcal{N}$ reducing E_1 to $\mathbf{MonPath}_{01}$. Observe that $\mathcal{N}$ coincides with the closure $\overline{E_1}$ of E_1. Therefore, for every α, $\gamma|\alpha$ belongs to the closure $\overline{\mathbf{MonPath}_{01}}$ of $\mathbf{MonPath}_{01}$. As $\overline{\mathbf{MonPath}_{01}}$ coincides with $(\mathbf{MonPath}_{01})^{\neg\neg}$, we find: $\neg\neg(\gamma|\underline{0}$ belongs to $\mathbf{MonPath}_{01})$, and therefore $\neg\neg\exists n[\underline{0}(n) \neq 0]$. Contradiction.
(iii) Let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ reducing $D^2(\mathbf{MonPath}_{01})$ to $\mathbf{MonPath}_{01}$. We claim that for all α, if α^0 admits $\underline{0}$, then $\gamma|\alpha$ admits $\underline{0}$. We prove this claim as follows.
Assume that α belongs to $\mathcal{N}$ and that α^0 admits $\underline{0}$ and that, for some k, $(\gamma|\alpha)(\overline{\underline{0}}k) \neq 0$. We let n be the least such k and determine m such that for all β, if $\overline{\beta}m = \overline{\alpha}m$, then $(\gamma|\beta)(\overline{\underline{0}}n) = (\gamma|\alpha)(\overline{\underline{0}}n)$. We now construct a function δ from $\mathcal{N}$ to $\mathcal{N}$ such that for every ζ, $(\overline{\delta|\zeta})m = \overline{\alpha}m$ and $(\delta|\zeta)^1$ does not belong to $\mathbf{MonPath}_{01}$ and $^{\overline{\underline{0}}(m+1)}((\delta|\zeta)^0) = \zeta$ and $(\delta|\zeta)^0$ admits $\overline{\underline{0}}m$ and, for all $i \leq m$, $(\delta|\zeta)^0$ does not admit i^*. Observe that for all ζ, ζ belongs to $\mathbf{MonPath}_{01}$ if and only if $(\delta|\zeta)^0$ belongs to $\mathbf{MonPath}_{01}$ if and only if $\delta|\zeta$ belongs to $D^2(\mathbf{MonPath}_{01})$ if and only if $\gamma|(\delta|\zeta)$ belongs to $\mathbf{MonPath}_{01}$. As, for every ζ, $(\gamma|(\delta|\zeta))(\overline{\underline{0}}n) \neq 0$, it follows that for every ζ, ζ belongs to $\mathbf{MonPath}_{01}$ if and only if, for some $i < n$, $\gamma|(\delta|\zeta)$ admits i^*. Therefore, $\mathbf{MonPath}_{01}$ reduces to $D^n(A_1)$. But, according to Theorem 2.5, $\mathbf{MonPath}_{01}$ does not reduce to $D^n(A_1)$. Thus we see, that, for all α, if α^0 admits $\underline{0}$, then $\gamma|\alpha$ admits $\underline{0}$.
In the same way one proves that for all α, if α^1 admits $\underline{0}$, then $\gamma|\alpha$ admits $\underline{0}$.
We now construct a function η from $\mathcal{N}$ to $\mathcal{N}$ such that, for all ζ, $(\zeta|\eta)^0$ admits $\underline{0}$ if and only if $\eta^0 = \underline{0}$, and $(\zeta|\eta)^1$ admits $\underline{0}$ if and only if $\eta^1 = \underline{0}$ and for all i, both $(\zeta|\eta)^0$ and $(\zeta|\eta)^1$ do not admit i^*. Observe that, for all η, η belongs to $D^2(A_1)$ if and only if $\gamma|(\zeta|\eta)$ admits $\underline{0}$. Therefore, $D^2(A_1)$ reduces to A_1. But, according to Theorem 2.5, $D^2(A_1)$ does not reduce to A_1.
(iii) Let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that for every α, for every s, $(\gamma|\alpha)(\langle 0\rangle * s) = (\gamma|\alpha)^0(s) = \alpha^0(s)$ and $(\gamma|\alpha)(\langle 1\rangle * s) = (\gamma|\alpha)^1(s) = \alpha^1(s)$ and, for every $n > 1$, $(\gamma|\alpha)(\langle n\rangle * s) \neq 0)$. It will be clear that γ reduces the set $D^2(E_1^1)$ to the set E_1^1.
(iv) is an easy consequence of (i) and also of (ii) and (iii). □

2.26 Theorem 2.25(ii) may be taken as the starting point for establishing large

hierarchies of analytic sets that are not positively Borel, see [18].
We may conclude from Theorem 2.25(ii) that the set $\mathbf{Path}_{01}$ does not reduce to the set $\mathbf{MonPath}_{01}$, as we mentioned in Subsection 0.8.1. In order to see this one has to supply a proof that $D^2(\mathbf{Path}_{01})$ reduces to the set $\mathbf{Path}_{01}$ itself. This proof is similar to the proof of Theorem 2.25(iii) and may be left to the reader.

3 The co-analytic example: $\mathbf{Almost^*Fin}$

3.1 In Subsection 0.8.2 we introduced the set $\mathbf{Almost^*Fin}$ consisting of all α in $\mathcal{C}$ such that for every one-to-one sequence γ in $\mathcal{N}$ there exists n such that $\alpha(\gamma(n)) = 0$. Our aim in this section is to prove that this co-analytic set is not positively Borel.
In the next subsection we sketch the background of this result and we describe how we intend to prove it.

3.2 In Subsection 0.8.2 we also introduced the set $\mathbf{Fin}$ consisting of all α in Cantor space $\mathcal{C}$ such that for some m, for every $n > m$, $\alpha(n) = 0$. For every α in $\mathcal{C}$, α belongs to $\mathbf{Fin}$ if and only if α is the characteristic function of a finite subset of $\mathbb{N}$. We now let $\mathbf{Inf}$ be the set of all α in $\mathcal{C}$ such that for every m, for some $n > m$, $\alpha(n) = 1$. So, for every α in $\mathcal{C}$, α belongs to $\mathbf{Inf}$ if and only if α is the characteristic function of an infinite and decidable subset of $\mathbb{N}$.
The set $\mathbf{Fin}$ is countable and belongs to the class $\mathbf{\Sigma}^0_2$ and the set $\mathbf{Inf}$ belongs to the class $\mathbf{\Pi}^0_2$.
Recall that we defined, in Subsection 0.9, for every subset X of $\mathcal{N}$, the constructive complement X^c of X as the set of all α that are apart from every element of X. The set $\mathbf{Inf}$ is easily seen to be the constructive complement $\mathbf{Fin}^c$ of the set $\mathbf{Fin}$.
The set $\mathbf{Almost^*Fin}$, in its turn, coincides with the set of all α such that for every β in $\mathbf{Inf}$ there exists n such that $\alpha(n) \neq \beta(n)$, that is, $\mathbf{Almost^*Fin}$ is the constructive complement $\mathbf{Inf}^c$ of $\mathbf{Inf}$, and $\mathbf{Almost^*Fin}$ coincides with the double constructive complement $\mathbf{Fin}^{cc}$ of $\mathbf{Fin}$.
It is not difficult to prove that for every subset X of $\mathcal{N}$, X is a subset of X^{cc}. In particular, $\mathbf{Fin}$ is a subset of $\mathbf{Almost^*Fin}$. It will follow from the results of this section that $\mathbf{Fin}$ is a *proper* subset of $\mathbf{Almost^*Fin}$ and should be thought of as a first Borel-approximation of $\mathbf{Almost^*Fin}$.
The fact that $\mathbf{Almost^*Fin}$ is not positively Borel shows that the constructive complement of a positively Borel set is not always positively Borel itself.
In Subsection 3.2.1 we give some more information on positively Borel sets. In Subsection 3.2.2 we outline the further contents of this section.

3.2.1 We introduce the notion of a *complementary pair of positively Borel sets*. This notion is defined by induction, as follows:

(i) For every open subset X of $\mathcal{N}$, the ordered pairs $(X, X^\neg)$ and $(X^\neg, X)$ are complementary pairs of positively Borel sets. These are the *initial* complementary pairs of positively Borel sets.

(ii) For every sequence $(X_0, Y_0), (X_1, Y_1), \ldots$ of complementary pairs of positively Borel sets, the ordered pairs $(\bigcap_{n \in \mathbb{N}} X_n, \bigcup_{n \in \mathbb{N}} Y_n)$ and $(\bigcup_{n \in \mathbb{N}} X_n, \bigcap_{n \in \mathbb{N}} Y_n)$ are complementary pairs of positively sets.

(iii) Every complementary pair of positively Borel sets is obtained from initial complementary pairs of positively Borel sets by the repeated application of (ii).

Clearly, for all positively Borel sets X, Y, if (X, Y) is a complementary pair, then (Y, X) is a complementary pair.
Also, for all positively Borel sets X, Y, if (X, Y) is a complementary pair, then for every α in X, for every β in Y, α is apart from β.
For each n, (E_n, A_n) is a complementary pair of positively Borel sets. One may wonder in which sense the union $E_n \cup A_n$ comes close to the whole set $\mathcal{N}$. It follows from the Continuity Principle that for every n, the set $E_n \cup A_n$ is a proper subset of $\mathcal{N}$.
(E.R. Bishop calls the statement that $E_1 \cup A_1$ coincides with $\mathcal{N}$ the *limited principle of omniscience*. If one, unwisely, decides to use this principle and also applies the First Axiom of Countable Choice 1.3.1 one may "prove" that for every complementary pair (X, Y) of positively Borel sets, the set $X \cup Y$ coincides with $\mathcal{N}$. The assumption that the set $E^1_1 \cup A^1_1$ coincides with $\mathcal{N}$ seems to be a stronger "principle".)
The set $(E_1 \cup A_1)^{\neg\neg}$ coincides with $\mathcal{N}$. Only if one uses a semi-classical assumption like Markov's Principle one may prove that also the set $(E_2 \cup A_2)^{\neg\neg}$ coincides with $\mathcal{N}$. Markov's Principle does not seem to enable one to prove that $(E_3 \cup A_3)^{\neg\neg}$ coincides with $\mathcal{N}$, as is observed in [24].
It seems useful to mention a different but related problem. The *positively arithmetical* subsets of the set $\mathbb{N}$ of natural numbers are obtained from the recursive subsets of $\mathbb{N}$ by the repeated application of the operations of existential and universal projection. In [13] it is shown, by a clever argument due to R. Solovay and J.R. Moschovakis, that these sets form a proper hierarchy upon the assumption of both Markov's Principle and Brouwer's Thesis. The same assumptions also guarantee that for all positively arithmetical subsets X of $\mathbb{N}$, $(X \cup X^{\neg})^{\neg\neg}$ coincides with $\mathbb{N}$.

3.2.2 For each s, the ordered pair $(\{s * \underline{0}\}, \{s * \underline{0}\}^c)$ is an initial pair of positively Borel sets and **Fin** and **Inf** coincide with $\bigcup_{s \in \mathbb{N}} \{s * \underline{0}\}$ and $\bigcap_{s \in \mathbb{N}} \{s * \underline{0}\}^c$, respectively. Therefore, $(\mathbf{Fin}, \mathbf{Inf})$ is a complementary pair of positively Borel sets.
We will see that **Fin** is not the only positively Borel set X such that $(X, \mathbf{Inf})$ is a complementary pair.
We first show, in Theorem 3.3, that every set in $\mathbf{\Pi}^0_2$ reduces to **Inf**, that is **Inf** is a *complete* element of the class $\mathbf{\Pi}^0_2$, and also that, surprisingly, not every set in $\mathbf{\Sigma}^0_2$ reduces to **Fin**, that is, **Fin** is not a complete element of the class $\mathbf{\Sigma}^0_2$.
Nevertheless, the set **Fin** does not reduce to the set **Inf**, or, equivalently, to the set A_2. In Theorem 3.5 we prove a very strong statement implying this fact: every function from $\mathcal{N}$ to $\mathcal{N}$ mapping **Fin** into A_2 also maps an element of **Inf** into A_2. The argument is elementary and does not use the Continuity Principle.
In Subsection 3.6 we will introduce the set Perhaps(**Fin**). It turns out that this set

properly extends **Fin** and belongs to $\mathbf{\Sigma}^0_4$ but not to $\mathbf{\Pi}^0_4$. The pair (Perhaps(**Fin**), **Inf**) is a complementary pair of positively Borel sets. It is not very easy to show that Perhaps(**Fin**) does not belong to $\mathbf{\Pi}^0_4$. In Theorem 3.8 we obtain the strong result implying this assertion: every function from $\mathcal{N}$ to $\mathcal{N}$ mapping Perhaps(**Fin**) into A_4 also maps an element of **Inf** into A_4. One should compare this result to the fact, proven in [24], that every function from $\mathcal{N}$ to $\mathcal{N}$ mapping E_4 into A_4 also maps an element of A_4 into A_4. This fact forms part of the intuitionistic Borel Hierarchy Theorem proven in [24]. Unlike the proof of Theorem 3.5, the proof of Theorem 3.8 is not elementary: it uses the Continuity Principle.

In Subsection 3.9 we extend and generalize this first discovery. For all subsets X, Y of $\mathcal{N}$ such that X is a subset of Y we introduce a subset Perhaps(X, Y) of $\mathcal{N}$. So we are using one and the same name both for a unary and for a binary operation on subsets of $\mathcal{N}$. The set Perhaps(**Fin**) introduced in Subsection 3.6 coincides with the set Perhaps(**Fin**, **Fin**). We then consider the operation $X \mapsto X^+ :=$ Perhaps(**Fin**, X) that is defined for every set X containing **Fin** and contained in **Almost*****Fin**.

We obtain the *Borel-approximations of* **Almost*****Fin** by starting from the *initial* set **Fin** and repeatedly applying this monotone operator while allowing forming the union of a countable sequence of earlier obtained approximations.

For every Borel-approximation X of **Almost*****Fin**, the pair $(X, \mathbf{Inf})$ is a complementary pair of positively Borel sets. We use stumps in order to label some of these Borel-approximations of **Almost*****Fin**, introducing for each stump σ a set $\mathbb{P}(\sigma, \mathbf{Fin})$, called the *$\sigma$-th perhapsive extension of* **Fin**. We also introduce, for each stump σ, a positively Borel set $\mathbb{K}(\sigma)$ such that every positively Borel set reduces to one of these sets. The crucial Theorem 3.14 generalizes Theorems 3.5 and 3.8 in the following way: every function from $\mathcal{N}$ to $\mathcal{N}$ mapping $\mathbb{P}(\sigma, \mathbf{Fin})$ into $\mathbb{K}(\sigma)$ also maps an element of **Inf** into $\mathbb{K}(\sigma)$.

In Theorem 3.17 we collect some consequences. We will see that for every Borel-approximation X of **Almost*****Fin**, X is a proper subset of X^+ whereas the set **Almost*****Fin** is a fixed point of the operator $X \mapsto X^+$. We also obtain the conclusions announced in Subsection 0.9.2. In Theorem 3.19 we prove, using Brouwer's Thesis, that the set **Almost*****Fin** is the (uncountable) union of its Borel approximations, and that $\mathbf{Almost}^*\mathbf{Fin} = \mathbf{Fin}^{cc}$ is a subset of $\mathbf{Fin}^{\neg\neg}$. It also will become clear that **Almost*****Fin** is the least fixed point of the operator $X \mapsto X^+$ containing **Fin**.

In Subsection 3.20 we observe that Markov's Principle would imply and is equivalent to the statement that $\mathbf{Fin}^{\neg\neg}$ is a subset of **Almost*****Fin**.

We then turn to the problem of showing that **Almost*****Fin**, although not positively Borel, is, in some sense, "simple". Theorems 3.3 and 3.23 make sure that the sets $D^2(A_1)$ and **Fin**, respectively, do not reduce to **Almost*****Fin**. Our final Theorem 3.24 secures that also **Inf** does not reduce to **Almost*****Fin** and thus that **Almost*****Fin** is not a complete element of the class of the co-analytic subsets of $\mathcal{N}$. The theorem, similar in formulation to Theorems 3.5, 3.8 and 3.14, establishes the stronger fact that every function from $\mathcal{N}$ into $\mathcal{N}$ mapping **Inf** into **Almost*****Fin** also maps some element of **Almost*****Fin** into **Almost*****Fin**. In the proof we use both the Continuity Principle and Brouwer's Thesis.

We first prove that **Inf** is a complete element of the class $\mathbf{\Pi}^0_2$ and that, surprisingly, **Fin** is not a complete element of the class $\mathbf{\Sigma}^0_2$.

3.3 Theorem:

(i) For every subset X of $\mathcal{N}$, if X belongs to $\mathbf{\Pi}^0_2$, then X reduces to **Inf**.

(ii) For every function γ from $\mathcal{N}$ to $\mathcal{N}$, if γ maps the set $D^2(A_1)$ into **Fin**, then γ also maps the closure $\overline{D^2(A_1)}$ of $D^2(A_1)$ into **Fin**.

(iii) The set $D^2(A_1)$ belongs to $\mathbf{\Sigma}^0_2$ and does not reduce to **Fin**.

(iv) For every function γ from $\mathcal{N}$ to $\mathcal{N}$, if γ maps the set $D^2(A_1)$ into the set A^1_1, then γ also maps the closure $\overline{D^2(A_1)}$ of $D^2(A_1)$ into A^1_1.

(v) The set $D^2(A_1)$ does not reduce to the set A^1_1 and thus is not a co-analytic subset of $\mathcal{N}$. It follows that $D^2(A_1)$ also does not reduce to **Inf** or to **Almost*Fin**.

Proof: (i) As every set in the class $\mathbf{\Pi}^0_2$ reduces to the set A_2, it suffices to show that the set A_2 reduces to the set **Inf**. Recall that A_2 is the set of all α in $\mathcal{N}$ such that for every m, for some n, $\alpha^m(n) \neq 0$. Let δ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that for every α, for every n, $(\delta|\alpha)(n)$ is the greatest number $k < n$ with the property that for every $i < k$ there exists $j < n$ such that $\alpha^i(j) \neq 0$.
Let γ be function from $\mathcal{N}$ to $\mathcal{N}$ such that, for every α, for every n, $(\gamma|\alpha)(n) := 1$ if $(\delta|\alpha)(n{+}1) > (\delta|\alpha)(n)$ and $(\gamma|\alpha)(n) := 0$ if not. γ is easily seen to reduce the set A_2 to the set **Inf**.
(ii) Let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ mapping the set $D^2(A_1)$ into the set **Fin**. Observe that the set $D^2(A_1)$ contains the set $\{\alpha \mid \alpha \in \mathcal{N} \mid \alpha^0 = \underline{0}\}$ and that this set is a spread. Given any α in $\mathcal{N}$, we let α' be the element of $\mathcal{N}$ such that $(\alpha')^0 = \underline{0}$ and for each j, if there is no k such that $j = \langle 0\rangle * k$, then $(\alpha')(j) := \alpha(j)$. Let m be a natural number. Applying the Continuity Principle we find p, q such that, for every β, if β passes through $\overline{\alpha'}p$ and $\beta^0 = \underline{0}$, then $(\gamma|\beta)^m(q) \neq 0$. We determine $r > p$ such that for every β, if β passes through $\overline{\alpha'}r$, then $(\gamma|\beta)^m(q) = (\gamma|\alpha')^m(q)$. Now assume that α belongs to the closure $\overline{D^2(A_1)}$ of the set $D^2(A_1)$ and distinguish two cases. *Either* α passes through $\overline{\alpha'}r$ and therefore $(\gamma|\alpha)^m(q) \neq 0$ *or* α does not pass through $\overline{\alpha'}r$ and therefore $\alpha^0 \# \underline{0}$ and $\alpha^1 = \underline{0}$, and α belongs to $D^2(A_1)$, and $(\gamma|\alpha)^m \# \underline{0}$. In both cases we find that $(\gamma|\alpha)^m$ is apart from $\underline{0}$.
We conclude that, for every α, if α belongs $\overline{D^2(A_1)}$, then, for each m, $(\gamma|\alpha)^m \# \underline{0}$ and therefore $\gamma|\alpha$ belongs to A_2.
(iii) This follows from (ii) and the fact that the sets $\overline{D^2(A_1)}$ and $D^2(A_1)$ do not coincide, as the set $D^2(A_1)$ is not closed, see Theorem 2.5(iv) and Theorem 2.5(ii).
(iv) The proof is similar to the proof of (ii) but does not use the Continuity Principle. Let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ mapping the set $D^2(A_1)$ into the set A^1_1. Given any α in $\mathcal{N}$, we let α' be the element of $\mathcal{N}$ such that $(\alpha')^0 = \underline{0}$ and for each j, if there is no k such that $j = \langle 0\rangle * k$, then $(\alpha')(j) := \alpha(j)$. Let δ belong to $\mathcal{N}$. We determine p, q such that, for every β, if β passes through $\overline{\alpha'}p$, then $(\gamma|\beta)(\overline{\delta}q) = (\gamma|\alpha')(\overline{\delta}q) \neq 0$. Now assume that α belongs to the closure $\overline{D^2(A_1)}$ of the set $D^2(A_1)$ and distinguish two cases. *Either* α passes through $\overline{\alpha'}p$ and therefore $(\gamma|\alpha)(\overline{\delta}q) \neq 0$ *or* α does not pass through $\overline{\alpha'}r$ and therefore $\alpha^0 \# \underline{0}$ and $\alpha^1 = \underline{0}$, and α belongs to $D^2(A_1)$, and $\gamma|\alpha$ belongs to A^1_1 and, for some r, $(\gamma|\alpha)(\overline{\delta}r) \neq 0$. In both cases we find that, for some t, $(\gamma|\alpha)(\overline{\delta}t) \neq 0$.

It follows that $\gamma|\alpha$ belongs to A^1_1.
Thus we see that γ maps $\overline{D^2(A_1)}$ into A^1_1.
(v) follows from (iv) and Theorem 2.5(iv) and Theorem 2.5(ii). We also need the observation that every set in the class $\mathbf{\Pi}^0_2$ is co-analytic. $\square$

3.4 We want to compare the result contained in Theorem 3.3(iv) that the set $D^2(A_1)$ is not co-analytic and not $\mathbf{\Pi}^0_2$ with a result mentioned in [13]. We first prove that the set F consisting of all α such that either $\alpha \# \underline{0}$ or $\alpha = \underline{0}$ is not $\mathbf{\Pi}^0_2$.
Let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ reducing F to A_2. We claim that, for every α, $\gamma|\alpha$ belongs to A_2 and prove this claim as follows. Let α belong to $\mathcal{N}$ and i to $\mathbb{N}$. Find m such that $(\gamma|\underline{0})^i(m) \neq 0$. Find n such that, for every β, if $\overline{\beta}n = \overline{\underline{0}}n$, then $(\gamma|\beta)(m) = (\gamma|\underline{0})(m)$ and distinguish two cases. *Either* $\overline{\alpha}n = \overline{\underline{0}}n$ and $(\gamma|\alpha)(m) \neq 0$, *or* $\overline{\alpha}n \neq \overline{\underline{0}}n$, and $\alpha\#\underline{0}$ and $\gamma|\alpha$ belongs to A_2 and there exists p sucht that $(\gamma|\alpha)(p) \neq 0$. In both cases $(\gamma|\alpha)^i \# \underline{0}$.
It follows that $\mathcal{N}$ coincides with F. The Continuity Principle now implies that, for some q, if $\overline{\alpha}q = \overline{\underline{0}}q$, then $\alpha = \overline{0}$. This is absurd.
We may conclude that also the sets $D(A_1, E_1)$ and the set of all α such that either $\alpha = \underline{0}$ or, for some n, $\alpha(n) = 3$ are not $\mathbf{\Pi}^0_2$ as the set F reduces to both of them. The set of all α such that either $\alpha = \underline{0}$ or, for some n, $\alpha(n) = 3$ is the set mentioned in [13].
In Subsection 0.5, we introduced a binary operation $(a, i) \mapsto a(i)$ on the set $\mathbb{N}$ of the natural numbers such that for all a, for all k, if $\text{length}(a) = k$, then $a = \langle a(0), a(1), \ldots, a(k-1)\rangle$.
For all a in $\mathbb{N}$ we let $\sharp(a)$ be the number of elements of the set $\{i \mid i < \text{length}(a) \mid a(i) \neq 0\}$.
For all a, b in $\mathbb{N}$ we say that b *extends* a or: a *is an initial part of* b if and only if, for some c, b equals $a * c$.
The next theorem shows that, like the set E_2, the set **Fin** does not reduce to the set A_2. The fact that E_2 does not reduce to the set A_2 is the starting point for the Borel Hierarchy Theorem proved in [24].

3.5 Theorem:
For every function γ from $\mathcal{N}$ to $\mathcal{N}$, if γ maps the set **Fin** into the set A_2, then γ also maps some element of **Inf** into A_2.

Proof: Suppose that γ is a function from $\mathcal{N}$ to $\mathcal{N}$ mapping the set **Fin** into the set A_2. We define a sequence β, as follows. The values $\beta(0), \beta(1), \ldots$ of β will be regarded as code numbers of finite sequences of natural numbers. Let $\beta(0) := \langle\ \rangle$. Now suppose that m belongs to $\mathbb{N}$ and that we defined $\beta(m)$. Consider the infinite sequence $\beta(m) * \underline{0}$ and observe that this sequence belongs to **Fin**. Find n such that $(\gamma|(\beta(m) * \underline{0}))^m(n) \neq 0$ and then find $p > 0$ such that, for every α if α passes through $\beta(m) * \overline{\underline{0}}p$, then $(\gamma|\alpha)^m(n) = (\gamma|(\beta(m) * \underline{0}))^m(n)$. Now define $\beta(m+1) := \beta(m) * \overline{\underline{0}}p * \langle 1\rangle$. Observe that, for each m, $\beta(m)$ is an initial part of $\beta(m+1)$ and $\sharp(\beta(m+1)) = \sharp(\beta(m)) + 1$. Let α be the element of $\mathcal{N}$ passing through every $\beta(m)$. Observe that α belongs to **Inf** while $\gamma|\alpha$ belongs to A_2. $\square$

3.6 We want to show that the set **Fin** is but a first approximation of the set **Almost*****Fin**, and that there are positively Borel sets that come closer to **Almost*****Fin**. We let Perhaps(**Fin**) be the set of all α in $\mathcal{N}$ such that for some n, for all $m > n$, if $\alpha(m) > 0$, then α belongs to **Fin**. Observe that the set Perhaps(**Fin**) belongs to the class $\mathbf{\Sigma}^0_4$, and that **Fin** is a subset of Perhaps(**Fin**).
One may prove that, for all α, α belongs to Perhaps(**Fin**) if and only if there exists β in **Fin** such that, if α is apart from β, then α belongs to **Fin**.
More generally, given any subset X of $\mathcal{N}$ we may introduce the subset Perhaps(X) of $\mathcal{N}$ consisting of all α such that, for some β in X, if α is apart from β, then α belongs to X.
This unary operation on the class of subsets of $\mathcal{N}$ is studied in [21] and [24].
In order to obtain some sense of what it means, for a given infinite sequence α, to belong to the set Perhaps(**Fin**), one should realize that infinite sequences in general are created step by step. If we are given the information that α belongs to Perhaps(**Fin**), we can not, in general, immediately determine the number of elements of the set $\{n \mid n \in \mathbb{N} \mid \alpha(n) \neq 0\}$. We are given a natural number m and the assurance that the number of elements of the set $\{n \mid n \in \mathbb{N} \mid \alpha(n) \neq 0\}$ is likely to be $\sharp(\overline{\alpha}m)$. We have to wait and see if this assurance comes true. The successive further elements of the sequence α are disclosed one by one: $\alpha(m), \alpha(m{+}1), \ldots$ As soon as one of these numbers turns out to be positive, we have to correct our opinion on the number of elements of the set $\{n \mid n \in \mathbb{N} \mid \alpha(n) \neq 0\}$ but, fortunately, at this very moment we are informed about the exact number of its elements. Should none of these numbers turn out to be positive, our opinion, based upon the initial assurance, is right, but, paradoxically, we will never be sure of this fact ourselves.
It follows from Theorem 3.5 that the set **Fin**, while belonging to the class $\mathbf{\Sigma}^0_2$, does not belong to the class $\mathbf{\Pi}^0_2$. We now want to prove that the set Perhaps(**Fin**), while belonging to the class $\mathbf{\Sigma}^0_4$, does not belong to the class $\mathbf{\Pi}^0_4$.

3.7 Let X be a subset of $\mathcal{N}$. X is *strictly analytic* if and only if there exists a function γ from $\mathcal{N}$ to $\mathcal{N}$ such that X coincides with the range of γ, that is, for all α, α belongs to X if and only if, for some β, α coincides with $\gamma|\beta$.
Every strictly analytic subset of $\mathcal{N}$ is analytic but the converse fails, see [24].
The sets **Fin** and Perhaps(**Fin**) both are strictly analytic subsets of $\mathcal{N}$. In order to prove these highly important facts we introduce functions f_1 and f_2 from $\mathcal{N}$ to $\mathcal{N}$, as follows.
We construct f_1 in such a way that for all m, a, for all α, if length$(a) = m$ then $f_1|(\langle m\rangle * a * \alpha) = a * \underline{0}$.
We construct f_2 in such a way that for all m, a, p, k, for all α, if length$(a) = m$ and $k \neq 0$, then $f_2|(\langle m\rangle * a * \overline{0}p * \langle k\rangle * \alpha) = a * \overline{0}p * \langle k\rangle * (f_1|\alpha)$.

3.7.1 Lemma:

(i) For all α, α belongs to **Fin** if and only if, for some β, α coincides with $f_1|\beta$.

(ii) For all α, α belongs to Perhaps(**Fin**) if and only if, for some β, α coincides with $f_2|\beta$.

Proof: The proof is left to the reader. $\square$

3.7.2 Lemma:

(i) Let X be an open subset of $\mathcal{N}$ and a a natural number such that every α passing through a and belonging to **Fin** belongs to X.
Then there exists b such that b extends a and $\sharp(b) = \sharp(a) + 1$ and every α passing through b belongs to X.

(ii) Let $X_0, X_1, \ldots$ be a sequence of subsets of $\mathcal{N}$ and a a natural number such that every α passing through a and belonging to Perhaps(**Fin**) belongs to $\bigcup_{n \in \mathbb{N}} X_n$.
Then there exist b, n such that b extends a and $\sharp(b) = \sharp(a) + 1$ and every α passing through b and belonging to **Fin** belongs to X_n.

Proof: (i) Let X be an open subset of $\mathcal{N}$. Let β be such that, for every α, α belongs to X if and only if, for some n, $\beta(\overline{\alpha}n) \neq 0$. Now assume a is a natural number and every α passing through a and belonging to **Fin** belongs to X. Consider the sequence $a * \underline{0}$ and find p such that $\beta(a * \overline{\underline{0}}p) \neq 0$. Then $b := a * \overline{\underline{0}}p * \langle 1 \rangle$ satisfies the requirements.
(ii) Let $X_0, X_1, \ldots$ be a sequence of subsets of $\mathcal{N}$ and a a natural number such that every α passing through a and belonging to Perhaps(**Fin**) belongs to $\bigcup_{n \in \mathbb{N}} X_n$. Observe that, for every β, the sequence $f_2|(\langle \text{length}(a) \rangle * a * \beta)$ passes through a and belongs to Perhaps(**Fin**) and therefore to $\bigcup_{n \in \mathbb{N}} X_n$. Apply the Continuity Principle and find p, n such that for every β, if β passes through $\overline{\underline{0}}p$, then $f_2|(\langle \text{length}(a) \rangle * a * \beta)$ belongs to X_n. Now $b := a * \overline{\underline{0}}p * \langle 1 \rangle$ satisfies the requirements, because, for every α passing through b and belonging to **Fin** there exists β passing through $\overline{\underline{0}}p$ such that α coincides with $f_2|(\langle \text{length}(a) \rangle * a * \beta)$. $\square$

We let K_1 be the set of all α such that, for every m, there exists n such that $\alpha(\langle m, n \rangle) \neq 0$. K_1 is but slightly different from A_2, the set of all α such that for every m, there exists n such that $\alpha(\langle m \rangle * n) \neq 0$.
For all m, n, for all α, we will denote $(\alpha^m)^n$ by $\alpha^{m,n}$. Observe that, for all m, n, $(\alpha^m)^n$ coincides with $\alpha^{\langle m,n \rangle}$. We let K_2 be the set of all α such that, for every m, there exists n such that $\alpha^{m,n}$ belongs to K_1. The set K_2 is but slightly different from the set A_4.

3.7.3 Lemma:

(i) For every α, α belongs to K_1 if and only if there exists δ *securing* that α belongs to K_1, that is, for all n, $\alpha(\langle n, \delta(n) \rangle) \neq 0$.

(ii) For every α, α belongs to K_2 if and only if there exists δ *securing* that α belongs to K_2, that is, for all m, n, $\alpha(\langle m, \delta(\langle m \rangle), n, \delta(\langle m, n \rangle) \rangle) \neq 0$.

(iii) The sets A_2 and K_1 reduce to each other, and the sets A_4 and K_2 also reduce to each other.

Proof: The proof is left to the reader. In the proof of (ii) one has to use the Second Axiom of Countable Choice. □

3.8 Theorem:

For every function γ from $\mathcal{N}$ to $\mathcal{N}$, if γ maps the set Perhaps(**Fin**) into the set K_2, then γ also maps some element of **Inf** into K_2.

Proof: Suppose that γ is a function from $\mathcal{N}$ to $\mathcal{N}$ mapping the set Perhaps(**Fin**) into the set K_2. We now construct two sequences β, δ, as follows, by induction. The *values* $\beta(0), \beta(1), \dots$ of β as well as the *arguments* of δ will be regarded as code numbers of finite sequences of natural numbers. Our aim is that the following properties hold:

(i) $\beta(0) := \langle\ \rangle$ and for each m, $\beta(m)$ is an initial part of $\beta(m+1)$ and $\sharp(\beta(m+1)) = (\sharp\beta(m)) + 1$.
(ii) For each m, j, if $m = \langle j \rangle$, then for every α, if α passes through $\beta(m+1)$ and belongs to **Fin**, then $(\gamma|\alpha)^{\langle j, \delta(\langle j \rangle)\rangle})$ belongs to A_2.
(iii) For each m, j, k, if $m = \langle j, k \rangle$, then for every α, if α passes through $\beta(m+1)$, then $(\gamma|\alpha)(\langle j, \delta(\langle j \rangle), k, \delta(\langle j, k \rangle)\rangle) \neq 0$.

It is not difficult to see that one may indeed build such sequences, repeatedly applying Lemma 3.7.2. At step $m+1$ of the construction one defines the numbers $\beta(m+1)$ and $\delta(m)$.
We let α be the element of $\mathcal{N}$ passing through every $\beta(m)$. Observe that α belongs to **Inf** and that δ secures that $\gamma|\alpha$ belongs to K_2. □

3.9 We want to extend and generalize Theorem 3.8.
We first introduce a partial binary operation called Perhaps on the class of subsets of $\mathcal{N}$. For all subsets X, Y of $\mathcal{N}$ such that X is inhabited and a subset of Y, we let Perhaps(X, Y) be the set of all α with the the property that, for some β in X, if α is apart from β, then α belongs to Y. Observe that, for every inhabited subset X of $\mathcal{N}$, the set Perhaps(X) coincides with the set Perhaps(X, X).
Next, we introduce a partial unary operation on the class of subsets of $\mathcal{N}$. For every subset X of $\mathcal{N}$ such that **Fin** is a subset of X, we let X^+ be the set Perhaps(**Fin**, X).
We then introduce the class $\mathcal{D}$ of the *Borel-approximations of* **Almost*Fin** by means of the following inductive definition:

(i) **Fin** is the *initial* Borel-approximation of **Almost*Fin**.
(ii) For every Borel-approximation X of **Almost*Fin**, also $X^+ =$ Perhaps(**Fin**, X) is a Borel-approximation of **Almost*Fin**.
(iii) For every sequence $X_0, X_1, \dots$ of Borel-approximations of **Mon** $*$ **Fin**, also $\bigcup_{n \in \mathbb{N}} X_n$ is a Borel-approximation of **Mon** $*$ **Fin**.
(iv) Every Borel-approximation of **Almost*Fin** is obtained from **Fin** by the repeated application of steps (ii) and (iii).

For every non-empty stump σ, for every n, we denote the n-th immediate substump of σ by σ^n.
We introduce, for each stump σ, a subset $\mathbb{P}(\sigma, \mathbf{Fin})$ of $\mathcal{N}$ that belongs to the class $\mathcal{D}$ of the Borel-approximations of **Almost*Fin** and that we want to call the *σ-th perhapsive extension of* **Fin**. We do so by induction on the set of stumps, as follows:

(i) $\mathbb{P}(\emptyset, \mathbf{Fin}) := \mathbf{Fin}$.
(ii) For every non-empty stump σ, $\mathbb{P}(\sigma, \mathbf{Fin})$ is the set of all α in $\mathcal{N}$ such that for some m, for all $n > m$, if $\alpha(n) \neq 0$, then there exists p such that α belongs to $\mathbb{P}(\sigma^p, \mathbf{Fin})$.

Observe that the set $\mathbb{P}(\{\langle\,\rangle\}, \mathbf{Fin})$ coincides with the set Perhaps(**Fin**).
Also observe that, for every non-empty stump σ, the set $\mathbb{P}(\sigma, \mathbf{Fin})$ coincides with the set Perhaps($\mathbf{Fin}, \bigcup\limits_{n\in\mathbb{N}} \mathbb{P}(\sigma_n, \mathbf{Fin})$).

We also define, for each stump σ, a function f_σ from $\mathcal{N}$ to $\mathcal{N}$ and again do so by induction on the set of stumps. We take care that

(i) $f_\emptyset := f_1$.
(The function f_1 has been introduced in Section 3.6.
Recall that the set **Fin** coincides with the range of the function f_1.)

and
(ii) for each non-empty stump σ, for all m, a, p, k, l, for all α, if length$(a) = m$ and $k \neq 0$, then $f_\sigma|(\langle m\rangle * a * \underline{\overline{0}}p * \langle k, l\rangle * \alpha) := a * \underline{\overline{0}}p * \langle k\rangle * (f_{(\sigma^l)}|\alpha)$.

3.10 Theorem:
(i) For every X in $\mathcal{D}$, X is a subset of X^+.
(ii) For all X, Y in $\mathcal{D}$, if X is a subset of Y, then X^+ is a subset of Y^+.
(iii) For every X in $\mathcal{D}$, **Fin** is a subset of X.
(iv) For every X in $\mathcal{D}$, X is a subset of $\mathbf{Fin}^{\neg\neg}$.
(v) For every X in $\mathcal{D}$, X is a subset of **Almost*Fin**.
(vi) For every X in $\mathcal{D}$, for every α, for every a, α belongs to X if and only if $a * \alpha$ belongs to X.
(vii) For every α, for every a, α belongs to **Almost*Fin** if and only if $a * \alpha$ belongs to **Almost*Fin**.
(viii) For each stump σ, $\mathbb{P}(\sigma, \mathbf{Fin})$ belongs to $\mathcal{D}$.
(ix) For each non-empty stump σ, for each n, the set $\mathbb{P}(\sigma^n, \mathbf{Fin})$ is a subset of the set $\mathbb{P}(\sigma, \mathbf{Fin})$.
For all stumps σ, τ, if $\sigma \leq \tau$, then $\mathbb{P}(\sigma, \mathbf{Fin})$ is a subset of $\mathbb{P}(\tau, \mathbf{Fin})$.
(x) For each stump σ, the set $\mathbb{P}(\sigma, \mathbf{Fin})$ coincides with the range of the function f_σ.

Proof: We leave most of the straightforward proof to the reader and only prove (iv) and (v).
(iv). We use induction on $\mathcal{D}$. Observe that **Fin**, the initial set of $\mathcal{D}$, is a subset of $\mathbf{Fin}^{\neg\neg}$. Now assume that X belongs to $\mathcal{D}$ and is a subset of $\mathbf{Fin}^{\neg\neg}$. We want to prove

that Perhaps($\mathbf{Fin}, X$) is also a subset of $\mathbf{Fin}^{\neg\neg}$. Let α belong to Perhaps($\mathbf{Fin}, X$). Find m such that for all $n > m$, if $\alpha(n) \neq 0$, then α belongs to X. We distinguish the following two cases.
First case: for all $n > m$, $\alpha(n) = 0$. Then α belongs to $\mathbf{Fin}$ and also to $\mathbf{Fin}^{\neg\neg}$.
Second case: for some $n > m$, $\alpha(n) \neq 0$. Then α belongs to X and therefore also to $\mathbf{Fin}^{\neg\neg}$.
We observe: $\neg\neg$ (For all $n > m$, $\alpha(n) = 0$ or for some $n > m$, $\alpha(n) \neq 0$). Therefore α belongs to $\mathbf{Fin}^{\neg\neg\neg\neg}$, that is, to $\mathbf{Fin}^{\neg\neg}$. Finally, assume that $X_0, X_1, \ldots$ is a sequence of elements of $\mathcal{D}$ and that for each n, X_n is a subset of $\mathbf{Fin}^{\neg\neg}$. Obviously, also $\bigcup_{n\in\mathbb{N}} X_n$ is a subset of $\mathbf{Fin}^{\neg\neg}$.
(v) We use induction on $\mathcal{D}$. Observe that $\mathbf{Fin}$, the initial set of $\mathcal{D}$, is a subset of $\mathbf{Almost}^*\mathbf{Fin}$. Now assume that X belongs to $\mathcal{D}$ and is a subset of $\mathbf{Almost}^*\mathbf{Fin}$. We want to prove that Perhaps($\mathbf{Fin}, X$) is also a subset of $\mathbf{Almost}^*\mathbf{Fin}$. Let α belong to Perhaps($\mathbf{Fin}, X$) and let γ be a one-to-one sequence in $\mathcal{N}$. Find m such that for all $n > m$, if $\alpha(n) \neq 0$, then α belongs to X. Find i such that $\gamma(i) > m$ and distinguish two cases. *Either* $\alpha(\gamma(i)) = 0$ *or* $\alpha(\gamma(i)) \neq 0$ and α belongs to X. In the latter case α belongs to $\mathbf{Almost}^*\mathbf{Fin}$ and there exists k such that $\alpha(\gamma(k)) = 0$. So in both cases, there exists k such that $\alpha(\gamma(k)) = 0$. So α belongs to $\mathbf{Almost}^*\mathbf{Fin}$.
Finally, assume that $X_0, X_1, \ldots$ is a sequence of elements of $\mathcal{D}$ and that for each n, X_n is a subset of $\mathbf{Almost}^*\mathbf{Fin}$. Obviously, also $\bigcup_{n\in\mathbb{N}} X_n$ is a subset of $\mathbf{Almost}^*\mathbf{Fin}$.
□

3.11 Lemma:

Let σ be a non-empty stump.
Let $X_0, X_1, \ldots$ be a sequence of subsets of $\mathcal{N}$ and a a natural number such that every α passing through a and belonging to $\mathbb{P}(\sigma, \mathbf{Fin})$ belongs to $\bigcup_{n\in\mathbb{N}} X_n$.
Then there exist b, n such that b extends a and $\sharp(b) = \sharp(a) + 1$ and every α passing through b and belonging to $\mathbb{P}(\sigma^n, \mathbf{Fin})$ belongs to X_n.

Proof: Observe that under the given circumstances, for every β, the sequence $f_\sigma|(\langle\text{length}(a)\rangle * a * \beta)$ passes through a and belongs to $\mathbb{P}(\sigma, \mathbf{Fin})$ and therefore to $\bigcup_{n\in\mathbb{N}} X_n$. Apply the Continuity Principle and find p, n such that, for every β, if β passes through $\overline{0}p$, then $f_\sigma|(\langle\text{length}(a)\rangle * a * \beta)$ belongs to X_n. Now $b = a * \overline{0}p * \langle 1\rangle$ satisfies the requirements of the Lemma, because, for every α passing through b and belonging to $\mathbb{P}(\sigma^n, \mathbf{Fin})$ there exists β passing through $\langle\text{length}(a)\rangle * a * \overline{0}p * \langle 1, n\rangle$ such that $f_\sigma|\beta$ coincides with α. □

3.12 We introduce, for each stump σ, a subset $\mathbb{K}(\sigma)$ of $\mathcal{N}$. We do so by induction on the set of stumps, as follows.

(i) $\mathbb{K}(\emptyset) := K_1 =$ the set of all α such that, for every m, there exists n such that $\alpha(\langle m, n\rangle) \neq 0$.
(ii) For every non-empty stump σ, $\mathbb{K}(\sigma)$ is the set of all α in $\mathcal{N}$ such that for every m there exists n such that $\alpha^{m,n}$ belongs to $\mathbb{K}(\sigma^n)$.

In order to obtain a clear understanding of the sets $\mathbb{K}(\sigma)$ we introduce some game-theoretic terminology.
Let $a = \langle a(0), a(1), \ldots, a(2n-1)\rangle$ be a natural number coding a finite sequence of natural numbers of even length. We imagine ourselves that this finite sequence is the result of a play of $2n$ moves in which players I, II alternatively choose a natural number.

$$\begin{array}{ccccccccc} \text{player } I & & & & & & \text{player } I & & \\ \text{chooses } a(0) & & & & & & \text{chooses } a(2n-2) & & \\ & \searrow & & \nearrow & \cdots & \nearrow & & \searrow & \\ & & \text{player } II & & & & & & \text{player } II \\ & & \text{chooses } a(1) & & & & & & \text{chooses } a(2n-1) \end{array}$$

We denote the sequences of moves made by player I, player II respectively by a_I, a_{II} respectively. So $a_I := \langle a(0), a(2), \ldots, a(2n-2)\rangle$ and $a_{II} := \langle a(1), a(3), \ldots, a(2n-1)\rangle$.
Now let δ belong to $\mathcal{N}$. We say that *a II-obeys* δ if and only if, for each $k < n$, $a(2k+1) = \delta(\langle a(0), a(2), \ldots, a(2k)\rangle)$. Suppose that d belongs to $\mathbb{N}$ and has length greater than $\langle a(0), a(2), \ldots, a(2n-2)\rangle$. We say that *a II-obeys* d if and only if, for each $k < n$, $a(2k+1) = d(\langle a(0), a(2), \ldots, a(2k)\rangle)$. It will be clear that we are interpreting δ, d as strategies for player II in the just-mentioned game.
We now introduce, for each stump σ and each b in $\mathbb{N}$, the notion: b is *just outside of* σ. We do so by induction on the set of stumps, as follows: (i) For each b in $\mathbb{N}$, b is *just outside of* the empty stump if and only if $b = \langle\ \rangle$ and (ii) for each non-empty stump σ, for each b in $\mathbb{N}$, b is *just outside of* σ if and only if there exists n, a such that $b = \langle n\rangle * a$ and a is just outside of σ^n.
We also introduce, for each stump σ and each b in $\mathbb{N}$, the notion: *b is one step outside of* σ. We do so by induction on the set of stumps, as follows: (i) for each b in $\mathbb{N}$, b is *one step outside of the empty stump* if and only if there exists n such that $b = \langle n\rangle$ and (ii) for each non-empty stump σ, for each b in $\mathbb{N}$, b is *one step outside of* σ if and only if there exist n, a such that $b = \langle n\rangle * a$ and a is one step outside of σ^n.
Let σ be a stump and let α, δ belong to $\mathcal{N}$. We say that *δ secures that α belongs to* $\mathbb{K}(\sigma)$ if and only if for every a, if a II-obeys δ and a_{II} is just outside of σ, then $\alpha(a) \neq 0$.

3.13 Theorem:

(i) For each stump σ, the class of subsets of $\mathcal{N}$ reducing to $\mathbb{K}(\sigma)$ is closed under the operation of countable intersection.

(ii) For each non-empty stump σ, for every sequence $X_0, X_1, \ldots$ of subsets of $\mathcal{N}$, if, for each n, the set X_n reduces to the set $\mathbb{K}(\sigma_n)$, then the countable union $\bigcup_{n\in\mathbb{N}} X_n$ reduces to the set $\mathbb{K}(\sigma)$.

(iii) For each positively Borel set X there exists a stump σ such that X reduces to $\mathbb{K}(\sigma)$.

(iv) For each stump σ, for each α, α belongs to $\mathbb{K}(\sigma)$ if and only if there exists δ securing that α belongs to $\mathbb{K}(\sigma)$.

Proof: (i) Let σ be a stump and $X_0, X_1, \ldots$ a sequence of subsets of $\mathcal{N}$, each of them reducing to $\mathbb{K}(\sigma)$. Let $\delta_0, \delta_1, \ldots$ be a sequence of functions from $\mathcal{N}$ to $\mathcal{N}$ such that,

for each n, δ_n reduces X_n to $\mathbb{K}(\sigma)$. Observe that, if σ is empty, then α belongs to $\bigcap_{n\in\mathbb{N}} X_n$ if and only if for all p, m, for some n, $(\delta_p|\alpha)^m(\langle n\rangle) \neq 0$. Observe also that, if σ is non-empty, then α belongs to $\bigcap_{n\in\mathbb{N}} X_n$ if and only if, for all p, m, for some n, $(\delta_p|\alpha)^{m,n}$ belongs to $\mathbb{K}(\sigma^n)$. Let ε be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for every α, for all p, m, $(\varepsilon|\alpha)^{\langle p,m\rangle}$ equals $(\delta_p|\alpha)^m$, and for all j, if there are no p, m such that $j = \langle p, m\rangle$, then, if σ is empty, $(\varepsilon|\alpha)^j(0) \neq 0$ and if σ is non-empty, then $(\varepsilon|\alpha)^{j,0}$ belongs to $\mathbb{K}(\sigma^0)$. ε is easily seen to reduce $\bigcap_{n\in\mathbb{N}} X_n$ to $\mathbb{K}(\sigma)$.
(ii) Assume that σ is a non-empty stump and $X_0, X_1, \ldots$ is a sequence of subsets of $\mathcal{N}$ such that, for each n, X_n reduces to $\mathbb{K}(\sigma^n)$. Let $\delta_0, \delta_1, \ldots$ be a sequence of functions from $\mathcal{N}$ to $\mathcal{N}$ such that, for each n, δ_n reduces X_n to $\mathbb{K}(\sigma^n)$. Now consider a function ε from $\mathcal{N}$ to $\mathcal{N}$ such that, for each α, for all m, n, $(\varepsilon|\alpha)^{m,n} = \delta_n|\alpha$. Observe that ε reduces $\bigcup_{n\in\mathbb{N}} X_n$ to $\mathbb{K}(\sigma)$.
(iii) Observe that every subset of $\mathcal{N}$ that belongs to the class $\mathbf{\Pi}^0_2$ reduces to $\mathbb{K}(\emptyset)$. Using (i) and (ii) and the Second Axiom of Countable Choice one proves by induction that for every positively Borel set X there exists a stump σ such that X reduces to $\mathbb{K}(\sigma)$.
(iv) We use induction on the set of stumps.
Observe that, for each α, α belongs to $\mathbb{K}(\emptyset)$ if and only if there exists δ such that for each m, $\alpha(\langle m, \delta(m)\rangle) \neq 0$, that is, δ secures that α belongs to $\mathbb{K}(\emptyset)$.
Now assume that σ is a non-empty stump and that the statement has been verified for every immediate substump σ^n of σ. Using the First and the Second Axiom of Countable Choice we observe that for each α, α belongs to $\mathbb{K}(\sigma)$ if and only if there exists δ such that for each m, $\alpha^{m,\delta(m)}$ belongs to $\mathbb{K}(\sigma^{\delta(m)})$ if and only if there exists δ such that for each m there exists ε securing that $\alpha^{m,\delta(m)}$ belongs to $\mathbb{K}(\sigma^{\delta(m)})$ if and only if there exist δ, ε such that for each m, ε^m secures that $\alpha^{m,\delta(m)}$ belongs to $\mathbb{K}(\sigma^{\delta(m)})$. Now suppose we are given $\alpha, \delta, \varepsilon$ such that, for each m, ε^m secures that $\alpha^{m,\delta(m)}$ belongs to $\mathbb{K}(\sigma^{\delta(m)})$. Consider ζ in $\mathcal{N}$ such that, for each m, $\zeta(\langle m\rangle) = \delta(m)$ and for each m, a, $\zeta(\langle m\rangle * a) = \varepsilon^m(a)$ and observe that ζ secures that α belongs to $\mathbb{K}(\sigma)$. $\square$

3.14 For all natural numbers d, a we define a natural number $d`a$. The definition is by induction on length$(a)$. We define $d`\langle\ \rangle := \langle\ \rangle$ and for each a, p, *if* $a * \langle p\rangle <$ length(d), then $d`(a * \langle p\rangle) = (d`a) * \langle p, d(a * \langle p\rangle)\rangle$ and *if not* $a * \langle p\rangle <$ length(d), then $d`(a * \langle p\rangle) = d`a$. So $d`a$ codes the longest finite sequence b of even length such that b_I is an initial part of a and b II-obeys d.

3.15 Theorem:

For every stump σ, for every function γ from $\mathcal{N}$ to $\mathcal{N}$, if γ maps the set $\mathbb{P}(\sigma, \mathbf{Fin})$ into the set $\mathbb{K}(\sigma)$, then γ also maps some element of $\mathbf{Inf}$ into $\mathbb{K}(\sigma)$.

Proof: Suppose that σ is a stump and γ is a function from $\mathcal{N}$ to $\mathcal{N}$ mapping the set $\mathbb{P}(\sigma, \mathbf{Fin})$ into the set $\mathbb{K}(\sigma)$. We construct three sequences β, ζ and η in $\mathcal{N}$. The values of these sequences will be considered as code numbers of finite sequences of natural numbers. Our aim is that the following properties hold:

(i) $\beta(0) := \langle\ \rangle$ and, for each m, $\beta(m)$ is an initial part of $\beta(m+1)$ and $\sharp(\beta(m+1)) = \sharp(\beta(m)) + 1$.

(ii) $\eta(0) := \langle 0 \rangle$ and for each m, there exists n such that $\eta(m+1) = \eta(m) * \langle n \rangle$ and $\mathrm{length}(\zeta(m)) = 2 \cdot \mathrm{length}(m)$ and $(\zeta(m))_I = m$ and $\zeta(m)$ II-obeys $\eta(m)$, so $\zeta(m) = (\eta(m))\text{‘}m$.

(iii) For each m, if $(\zeta(m))_{II}$ is inside σ, then for all α passing through $\beta(m)$, if α belongs to $\mathbb{P}({}^{(\zeta(m))_{II}}\sigma, \mathbf{Fin})$, then ${}^{\zeta(m)}(\gamma|\alpha)$ belongs to $\mathbb{K}({}^{(\zeta(m))_{II}}\sigma)$.

(iv) For each m, if $(\zeta(m))_{II}$ is just outside of σ, then for all α passing through $\beta(m)$, if α belongs to $\mathbf{Fin}$, then ${}^{\zeta(m)}(\gamma|\alpha)$ belongs to K_1, that is, for each p there exists q such that $(\gamma|\alpha)((\zeta(m)) * \langle p, q \rangle) \neq 0$.

(v) For each m, if $(\zeta(m))_{II}$ is one step outside of σ, then for every α passing through $\beta(m)$, $(\gamma|\alpha)(\zeta(m)) \neq 0$.

The definition of the three sequences $\beta(0), \beta(1), \ldots$ and $\zeta(0), \zeta(1), \ldots$ and $\eta(0), \eta(1), \ldots$ goes by induction.
We define $\beta(0) := \langle\ \rangle$ and $\zeta(0) := \langle\ \rangle$ and $\eta(0) := \langle 0 \rangle$.
Now assume m is a positive natural number and we defined, for each $i < m$, the numbers $\beta(i)$, $\zeta(i)$ and $\eta(i)$. We indicate how to obtain the numbers $\beta(m)$, $\zeta(m)$ and $\eta(m)$.
Find $k := \mathrm{length}(m)$ and consider $m = \langle m(0), \ldots, m(k-1) \rangle$. Observe that m does not code the empty sequence and k is positive. Find the immediate predecessor m' of m, that is, $m' := \langle m(0), \ldots, m(k-2) \rangle$ and consider $e := ((\eta(m))\text{‘}m')$ and $f := ((\eta(m))\text{‘}(m'))_{II}$. We distinguish several cases:

First case: f is inside σ, that is, f belongs to σ.

Then: for every α, if α passes through $\beta(m-1)$ and belongs to $\mathbb{P}({}^{f}\sigma, \mathbf{Fin})$, then ${}^{e}(\gamma|\alpha)$ belongs to $\mathbb{K}({}^{f}\sigma)$. Applying Lemma 3.11 we find natural numbers c, n such that c extends $\beta(m-1)$ and $\sharp(c) = \sharp(\beta(m-1)) + 1$ and for every α, if α passes through c and belongs to $\mathbb{P}({}^{f*\langle n \rangle}\sigma, \mathbf{Fin})$ then ${}^{e*\langle m(k-1), n \rangle}(\gamma|\alpha)$ belongs to $\mathbb{K}({}^{f*\langle n \rangle}\sigma)$.
We define $\beta(m) := c$ and $\zeta(m) := e * \langle m(k-1), n \rangle$ and $\eta(m) := (\eta(m-1)) * \langle n \rangle$.

Second case: f is just outside of σ.

Then: for every α, if α passes through $\beta(m-1)$ and belongs to $\mathbf{Fin} = \mathbb{P}({}^{f}\sigma,\ \mathbf{Fin})$, then ${}^{e}(\gamma|\alpha)$ belongs to $K_1 = \mathbb{K}({}^{f}\sigma)$. Applying Lemma 3.7.2 we find c, n such that c extends $\beta(m-1)$ and $\sharp(c) = \sharp(\beta(m-1)) + 1$ and for every α, if α passes through c, then $(\gamma|\alpha)(e * \langle m(k-1), n \rangle) \neq 0$. We define $\beta(m) := c$ and $\zeta(m) := e * \langle m(k-1), n \rangle$ and $\eta(m) = (\eta(m-1)) * \langle n \rangle$.

Third case: some proper initial part of f is just outside of σ.

We now define: $\beta(m) := (\beta(m-1)) * \langle 1 \rangle$, $\zeta(m) := e * \langle m(k-1), 0 \rangle$ and $\eta(m) := (\eta(m-1)) * \langle 0 \rangle$.

We leave it to the reader to verify that, proceeding in this way, we will satisfy the requirements (i)-(v).

We now let α be the element of $\mathcal{N}$ passing through every $\beta(m)$ and δ the element of $\mathcal{N}$ passing through every $\eta(m)$. α will belong to **Inf**, because of (i). Also observe that for every b, if b II-obeys δ, and b_{II} is one step outside of σ, then $(\gamma|\alpha)(b) \neq 0$. Therefore δ secures that $\gamma|\alpha$ belongs to $\mathbb{K}(\sigma)$. $\square$

3.16 For every stump σ, we let σ^+, the *successor* of σ be the stump such that for each n, $(\sigma^+)^n = \sigma$.

3.17 Theorem:

(i) For every stump σ, the set $\mathbb{P}(\sigma, \textbf{Fin})$ does not reduce to the set $\mathbb{K}(\sigma)$.

(ii) For every positively Borel set X there exists σ such that the set $\mathbb{P}(\sigma, \textbf{Fin})$ does not reduce to the set X.

(iii) For every positively Borel set X that is a subset of **Almost*****Fin** there exists a positively Borel set Y not reducing to X such that both X and **Fin** are subsets of Y and Y is a subset of **Almost*****Fin**.

(iv) For every subset X of $\mathcal{N}$, if **Fin** is a subset of X and $X = X^+$, then every Borel-approximation of **Almost*****Fin** is a subset of X.

(v) For every Borel-approximation X of **Almost*****Fin**, X is a proper subset of X^+.

(vi) The set $(\textbf{Almost}^*\textbf{Fin})^+$ coincides with the set **Almost*****Fin**.

(vii) The set $(\textbf{Fin}^{\neg\neg})^+$ coincides with the set $\textbf{Fin}^{\neg\neg}$.

(viii) For each stump σ, the set $\mathbb{P}(\sigma, \textbf{Fin})$ is a proper subset of the set $\mathbb{P}(\sigma^+, \textbf{Fin})$.

(ix) For all stumps σ, τ, if $\sigma < \tau$, then $\mathbb{P}(\sigma, \textbf{Fin})$ is a proper subset of $\mathbb{P}(\tau, \textbf{Fin})$.

Proof: (i) is an immediate consequence of Theorem 3.15.
(ii) Given any positively Borel subset X of $\mathcal{N}$ we determine a stump σ such that X reduces to $\mathbb{K}(\sigma)$. The set $\mathbb{P}(\sigma, \textbf{Fin})$ does not reduce to $\mathbb{K}(\sigma)$ and therefore not to X.
(iii) Given any positively Borel subset X of **Almost*****Fin** we determine a stump σ such that X reduces to $\mathbb{K}(\sigma)$ and then consider the set $Y := X \cup \mathbb{P}(\sigma, \textbf{Fin})$. According to Theorem 3.15, every function from $\mathcal{N}$ to $\mathcal{N}$ mapping Y into $\mathbb{K}(\sigma)$ will also map some element of **Inf** into $\mathbb{K}(\sigma)$, therefore Y does not reduce to $\mathbb{K}(\sigma)$ and not to X, and Y does not coincide with X. Observe that it follows that $\mathbb{P}(\sigma, \textbf{Fin})$ is not a subset of X.
(iv) The proof is by straightforward induction on the class $\mathcal{D}$ of the Borel-approximations of **Almost*****Fin**.
(v) Assume that X belongs to $\mathcal{D}$ and that X^+ coincides with X. By the argument for (iii), there exists a stump σ such that $\mathbb{P}(\sigma, \textbf{Fin})$ is not a subset of X. According to (iv), every Borel-approximation of **Almost*****Fin** is a subset of X. Contradiction.
(vi) We have to show that the set Perhaps(**Fin**, **Almost*****Fin**) coincides with the set **Almost*****Fin**, in particular, that the first set is a subset of the second one as the other inclusion is obvious. Let α belong to Perhaps(**Fin**, **Almost*****Fin**) and let γ be a one-to-one sequence in $\mathcal{N}$. Find m such that for every $p > m$, if $\alpha(p) \neq 0$, then α belongs to **Almost*****Fin**. Find i such that $\gamma(i) > m$ and distinguish two cases.
First case: $\alpha(\gamma(i)) = 0$.
Second case: $\alpha(\gamma(i)) \neq 0$. Then α belongs to **Almost*****Fin** and there will exist k such that $\alpha(\gamma(k)) = 0$.
In both cases, there exists k such that $\alpha(\gamma(k)) = 0$.

We may conclude that α belongs to **Almost*Fin**.
(vii) The proof is similar to the proof of (vi) and left to the reader.
Also the proof of (viii) is left to the reader.
(ix) For all stumps σ, τ, if $\sigma < \tau$, then $\sigma < \sigma^+ \le \tau$, and therefore, because of (viii), $\mathbb{P}(\sigma$, **Fin**) is a proper subset of $\mathbb{P}(\sigma^+$,**Fin**) and also a proper subset of $\mathbb{P}(\tau$, **Fin**) . $\square$

3.18 Observe that the statements (ii) and (iii) of Theorem 3.17 imply the statements we promised to prove in Section 0.9.2.
For all stumps σ, τ we define: $\sigma \le^* \tau$ ("σ embeds into τ") if and only if there exists a γ in $\mathcal{N}$ such that for every a, if a belongs to σ, then $\gamma(a)$ belongs to τ, and for all a, b, if a is a proper initial segment of b, then $\gamma(a)$ is a proper initial segement of b.

The next Theorem mentions some consequences of Brouwer's Thesis. The first item, however, does not depend on Brouwer's Thesis. It offers an alternative characterization of the relation $\le$ on the set of stumps.

3.19 Theorem:
(i) For all stumps σ, τ, $\sigma \le \tau$ if and only if $\sigma \le^* \tau$.
(ii) (*The set* **Almost*Fin** *is the union of its own Borel-approximations:*) For every α, if α belongs to **Almost*Fin**, then there exists a stump σ such that α belongs to $\mathbb{P}(\sigma$, **Fin**).
(iii) **Almost*Fin** is a subset of **Fin**$^{\neg\neg}$.
(iv) (*A Boundedness Result:*) For every strictly analytic subset X of **Almost*Fin** there exists a stump σ such that X is a subset of $\mathbb{P}(\sigma$, **Fin**).

(v) the set **Almost***Fin is not strictly analytic.
(vi) For every subset X of $\mathcal{N}$, if **Fin** is a subset of X and X coincides with X^+, then **Almost*Fin** is a subset of X, that is, **Almost*Fin** is the least fixed point of the operator $X \mapsto X^+$ containing **Fin**.

Proof: (i) The proof is straightforward and left to the reader.
(ii) Let σ be a stump and α an element of $\mathcal{N}$. We say that σ *proves* α *to belong to* **Almost*Fin** if and only if for every one-to-one sequence γ in $\mathcal{N}$ there exists n such that $\overline{\gamma}n$ belongs to σ and $\alpha(\gamma(n-1)) = 0$. Brouwer's Thesis implies that, for every α, if α belongs to **Almost*Fin**, then there exists a stump σ proving that α belongs to **Almost*Fin**. We now show, by induction on the set of stumps, that for every stump σ, for every α, if σ proves that α belongs to **Almost*Fin**, then α belongs to $\mathbb{P}(\sigma$, **Fin**). This statement is true if σ is the empty stump, as no α is proved to belong to **Almost*Fin** by the empty stump. Now assume that σ is a non-empty stump and, for each n, every α proved by σ^n to belong to **Almost*Fin** belongs to $\mathbb{P}(\sigma^n$, **Fin**). Suppose that α belongs $\mathcal{N}$ and is proved by σ to belong to **Almost*Fin**
Remark that, for every n, if n is the least p such that $\alpha(p) \neq 0$, then σ^n proves that the sequence $\lambda i \in \mathbb{N}.\alpha(i+n)$ belongs to **Almost*Fin**, and therefore the sequence $\lambda i \in \mathbb{N}.\alpha(i+n)$ belongs to $\mathbb{P}(\sigma^n$, **Fin**) and also α itself belongs to $\mathbb{P}(\sigma^n$, **Fin**), see Theorem 3.10(vi). Therefore, for every p, if $\alpha(p) \neq 0$, then there exists n such that α belongs to $\mathbb{P}(\sigma^n$, **Fin**), that is: α belongs to $\mathbb{P}(\sigma$, **Fin**).

(iii) Use (i) and Theorem 3.10(ii).
(iv) Let X be a strictly analytic subset of **Almost*Fin**. Let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that X coincides with the range of γ. Observe that, for every α, for every one-to-one sequence β in $\mathcal{N}$ there exists n such that $(\gamma|\alpha)(\beta(n)) = 0$.
For every α in $\mathcal{N}$, we α_I, α_{II} be the elements of $\mathcal{N}$ satisfying, for all n, $\alpha_I(n) = \alpha(2n)$ and $\alpha_{II}(n) = \alpha(2n+1)$. Observe that, for every α there exists n such that $(\gamma|\alpha_I)(\alpha_{II}(n)) = 0$ or there exist i, j such that $i < j$ and $\alpha_{II}(i) = \alpha_{II}(j)$.
Let $a = \langle a(0), a(1), \ldots, a(2n-1)\rangle$ be an element of $\mathbb{N}$ coding a finite sequence of even length $2n$. We want to call a *safe* if and only if there exist $i < n$, $j < n$ such that $\gamma(\langle a(2j+1), a(0), a(2), \ldots, a(2i)\rangle) = 1$ and, for all $p < i$, $\gamma(\langle a(2j+1), a(0), a(2), \ldots, a(2p)\rangle) = 0$, or $i < j$ and $a(2i+1) = a(2j+1)$. Observe that for all a, if $a = \langle a(0), \ldots, a(2n-1)\rangle$ is safe, then for every α passing through $a_I = \langle a(0), a(2), \ldots, a(2n-2)\rangle$, for every β passing through $a_{II} = \langle a(1), a(3), \ldots, a(2m-1)\rangle$ there exists $j < n$ such that $(\gamma|\alpha)(\beta(j)) = 0$ or there exist $i, j < n$ such that $i < j$ and $\beta(i) = \beta(j)$. Remark that every infinite sequence α has a safe initial part. Using Brouwer's Thesis, we find a stump σ such that every α in $\mathcal{N}$ has a safe initial part that belongs to σ. For every infinite sequence α we let τ be the set of all $b = \langle b(0), \ldots, b(m-1)\rangle$ such that $\langle \alpha(0), b(0), \ldots, \alpha(m-1), b(m-1)\rangle$ belongs to σ. τ is easily see to be a stump embedding into σ and proving that $\gamma|\alpha$ belongs to **Almost*Fin**. Using (i) and the argument developed in the proof of (ii), we conclude that $\gamma|\alpha$ belongs to $\mathbb{P}(\tau, \textbf{Fin})$ and then, using (i) and Theorem 3.10(ix), that $\gamma|\alpha$ belongs to $\mathbb{P}(\sigma, \textbf{Fin})$. Therefore X is a subset of $\mathbb{P}(\sigma, \textbf{Fin})$.
(v) is a consequence of (iv) and Theorem 3.17. (vi) This follows from (ii) and and Theorem 3.17(iv) and (vi). □

3.20 In [24] **AlmostFin** is defined as the set of all α such that, for some stump σ, α belongs to $\mathbb{P}(\sigma, \textbf{Fin})$. The previous Theorem shows that Brouwer's Thesis implies that the sets **AlmostFin** and **Almost*Fin** are the same.
One sometimes considers the following extended version of Markov's Principle:

($*$) For every α, if the assumption that for each n, $\alpha(n) \neq 0$ leads to a contradiction, then there exists n such that $\alpha(n) = 0$.

The set **Almost*Fin**$^{\neg\neg}$ coincides with the set of all α such that, for every one-to-one sequence γ, $\neg\neg($ there exists n such that $\alpha(\gamma(n)) = 0)$. Observe that ($*$) implies that **Almost*Fin**$^{\neg\neg}$ is a subset of **Almost*Fin**, and therefore also that **Fin**$^{\neg\neg}$ is a subset of **Almost*Fin**. Conversely, if we assume that **Fin**$^{\neg\neg}$ is a subset of **Almost*Fin**, we may derive ($*$), as follows.
Let α belong to $\mathcal{N}$ and suppose $\neg\neg\exists n[\alpha(n) = 0]$. Let β in $\mathcal{C}$ be such that, for each n, $\beta(n) = 0$ if and only if for some $m < n$, $\beta(m) = 0$. Then β belongs to **Fin**$^{\neg\neg}$, and therefore to **Almost*Fin**, and there exists n such that $\beta(n) = 0$, and there exists n such that $\alpha(n) = 0$.
There seems to be no good reason to adopt ($*$) as an axiom.

We want to show that, in some sense, the set **Almost*Fin** is "simple", that is, there exist many and also not too difficult sets that do not reduce to **Almost*Fin**. We first prove that the set **Fin** does not reduce to **Almost*Fin**, and, in fact, we prove more.

The first item of the next theorem is about the binary operation Perhaps introduced in Subsection 3.9.

3.21 Theorem:

(i) Let X, Y and Z be subsets of $\mathcal{N}$ such that X has at least one element and is a subset of both Y and Z. Every function from $\mathcal{N}$ into $\mathcal{N}$ that maps X into X and Y into Z also maps $\text{Perhaps}(X, Y)$ into $\text{Perhaps}(X, Z)$.

(ii) Let γ be a function from $\mathcal{N}$ to $\mathcal{N}$. If γ maps **Fin** into **Fin**, then, for every Borel-approximation Y of **Almost*****Fin**, γ maps Y into Y.

(iii) Let γ be a function from $\mathcal{N}$ to $\mathcal{N}$. If γ maps **Fin** into **Almost*****Fin**, then, for every Borel-approximation Y of **Almost*****Fin**, γ maps Y into **Almost*****Fin**. Also, if γ maps **Fin** into **Fin**$^{\neg\neg}$, then, for every Borel-approximation Y of **Almost*****Fin**, γ maps Y into **Fin**$^{\neg\neg}$.

(iv) Let γ be a function from $\mathcal{N}$ to $\mathcal{N}$.
(Under the assumption of Brouwer's Thesis:) If γ maps **Fin** into **Almost*****Fin**, then γ also maps **Almost*****Fin** into **Almost*****Fin**.
(Without the assumption of Brouwer's Thesis:) If γ maps **Fin** into **Fin**$^{\neg\neg}$, then γ also maps **Fin**$^{\neg\neg}$ into **Fin**$^{\neg\neg}$.

(v) **Fin** does not reduce to **Almost*****Fin**. More generally, for every Borel-approximation Y of **Almost*****Fin**, Y does not reduce to **Almost*****Fin**.

Proof: (i) Let X, Y and Z be subsets of $\mathcal{N}$ such that X has at least one element and is a subset of both Y and Z. Let γ be a function from $\mathcal{N}$ into $\mathcal{N}$ that maps X into X and Y into Z. Assume that α belongs to $\text{Perhaps}(X, Y)$. We have to show that $\gamma|\alpha$ belongs to $\text{Perhaps}(X, Z)$, First determine β in X such that, if $\alpha \# \beta$, then α belongs to Y. Observe that $\gamma|\beta$ belongs to X and that, if $\gamma|\alpha \# \gamma|\beta$, then $\alpha \# \beta$, and therefore α belongs to Y and $\gamma|\alpha$ to Z.
(ii) The proof uses (i) and is by induction on the class $\mathcal{D}$ of the Borel-approximations of **Almost*****Fin**. It may be left to the reader.
(iii) The proof uses (i) and Theorem 3.17(vi) and is by induction on the class $\mathcal{D}$ of the Borel-approximations of **Almost*****Fin**. It may be left to the reader.
(iv) The first fact is an immediate consequence of (iv) and Theorem 3.19(ii).
The second fact is by an easy application of intuitionistic logic.
One might ask for a more elementary proof of the first fact, for instance one that would reduce it to the observation that a function from $\mathcal{N}$ into $\mathcal{N}$ mapping X into Y also maps X^{cc} into Y^{cc}. We did not find such a proof.
(v) Let Y be a Borel-approximation of **Almost*****Fin** and assume that γ is a function from $\mathcal{N}$ to $\mathcal{N}$ reducing Y to **Almost*****Fin**. Observe that γ maps **Fin**, and therefore, according to (ii), every Borel-approximation of **Almost*****Fin** into **Almost*****Fin**. Therefore, every Borel-approximation of **Almost*****Fin** would be a subset of Y. But, according to Theorem 3.17(v), Y is a proper subset of Y^+. □

The fact that **Fin** does not reduce to **Almost*****Fin** is a characteristically *intuitionistic* fact, as, from a classical point of view, the two sets coincide. We now want to show that also **Inf** does not reduce to **Almost*****Fin**. From a classical point of view, this fact is less surprising but the proof is rather subtle. Observe that **Inf**, like **Almost*****Fin**, is a co-analytic subset of $\mathcal{N}$.

3.22 Lemma:
(i) **Inf** is a strictly analytic subset of $\mathcal{N}$.
(ii) Let R be a subset of $\mathcal{N} \times \mathbb{N}$ such that, for every α in **Inf**, there exists n such that $\alpha R n$.
Then, for every α in **Inf**, there exist m, n such that, for every β in **Inf**, if β passes through $\overline{\alpha}m$, then $\beta R n$.

Proof: (i) We define a function δ from $\mathcal{N}$ to $\mathcal{N}$ such that, for every γ, for every p, *if* there exists $j \leq p$ such that $p = j + \sum_{i=1}^{j} \gamma(i)$, then $(\delta|\gamma)(p) = 1$, and *if not*, then $(\delta|\gamma)(p) = 0$. One verifies easily that, for every α, α belongs to **Inf** if and only if, for some γ, α coincides with $\delta|\gamma$.
The function δ has the following important property: for every α in **Inf**, for every γ, if $\alpha = \delta|\gamma$, then for every p, for every β in **Inf** passing through $\overline{\alpha}(p + \sum_{i=1}^{p-1} \gamma(i))$ there exists ζ in $\mathcal{N}$ passing through $\overline{\gamma}p$ such that β coincides with $\delta|\zeta$.
(ii) Let R be a subset of $\mathcal{N} \times \mathbb{N}$ such that, for every α in **Inf**, there exists n such that $\alpha R n$. Let δ be the function we defined in (i). Observe that for every γ there exists n such that $(\delta|\gamma) R n$. Let α belong to **Inf** and let γ be such that α coincides with $\delta|\gamma$. Using Brouwer's Continuity Principle we find m, n such that for every ζ, if ζ passes through $\overline{\gamma}m$, then $(\delta|\zeta) R n$. Observe that, for every β in **Inf**, if β passes through $\overline{\alpha}(m + \sum_{i=1}^{m-1} \gamma(i))$, then there exists ζ passing through $\overline{\gamma}m$ such that $\delta|\zeta$ coincides with β, and, therefore, $\beta R n$. □

Let X be a subset of **Almost*Fin**. We say that X *shows proper regard for* **Inf** if and only if every function δ from $\mathcal{N}$ to $\mathcal{N}$ mapping **Inf** into X also maps some element of X into X.

3.23 Theorem:
(i) **Fin** shows proper regard for **Inf**.
(ii) For every X in the class $\mathcal{D}$ of the Borel-approximations of **Almost*Fin**, if X shows proper regard for **Inf**, then X^+ shows proper regard for **Inf**.
(iii) For every sequence $X_0, X_1, \ldots$ of elements of $\mathcal{D}$, if , for each n, X_n shows proper regard for **Inf**, then $\bigcup_{n \in \mathbb{N}} X_n$ shows proper regard for **Inf**.
(iv) Every Borel-approximation of **Almost*Fin** shows proper regard for **Inf**.
(v) The set **Inf** does not reduce to any one of the Borel-approximations of **Almost*Fin**.

Proof: (i) Let δ be a function from $\mathcal{N}$ to $\mathcal{N}$ mapping **Inf** into **Fin**. Observe that for every α in **Inf** there exists m such that, for every $n > m$, $\alpha(n) = 0$, and that $\underline{1}$ belongs to **Inf**. Applying Lemma 3.22, we find p, m such that for every α in **Inf**, if $\overline{\alpha}p = \overline{\underline{1}}p$, then for every $n > m$, $(\delta|\alpha)(n) = 0$. Observe that now for every β passing through $\overline{\underline{1}}p$, for every $n > m$, $(\delta|\beta)(n) = 0$, as for every m there exists α in **Inf** passing through $\overline{\beta}m$. It follows that both $\overline{1}p * \underline{0}$ and $\delta|(\overline{1}p * \underline{0})$ belongs to **Fin**.
(ii) Let X be a subset of $\mathcal{N}$ showing proper regard for **Inf**. We have to prove that $X^+ = \text{Perhaps}(\mathbf{Fin}, X)$ also shows proper regard for **Inf**. Let δ be a function from

$\mathcal{N}$ to $\mathcal{N}$ mapping **Inf** into X^+. Observe that for every α in **Inf** there exists m such that, for every $n > m$, if $(\delta|\alpha)(n) \neq 0$, then $\delta|\alpha$ belongs to X, and that $\underline{1}$ belongs to **Inf**. Applying Lemma 3.22, we find p, m such that for every α in **Inf**, if $\overline{\alpha}p = \overline{\underline{1}}p$, then for every $n > m$, if $(\delta|\alpha)(n) \neq 0$, then $\delta|\alpha$ belongs to X.
We now define β in $\mathcal{N}$ as follows:

- for each $i < p$, $\beta(i) = 1$,
- for each $i \geq p$, if, for each j such that $m < j \leq i$, $(\delta|(\overline{\underline{1}}p * \underline{0}))(j) = 0$, then $\beta(i) = 0$,
- for each $i \geq p$, if i is the least j such that $m < j$ and $(\delta|(\overline{\underline{1}}p * \underline{0}))(j) \neq 0$, we determine q such that for every α, if α passes through $\overline{\underline{1}}p * \overline{\underline{0}}q$, then $\delta|\alpha$ passes through $\overline{(\delta|\underline{1})}(i+1)$. We let ζ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for every γ, $\zeta|\gamma$ equals $\delta|(\overline{\underline{1}}p * \overline{\underline{0}}q * \gamma)$ and observe that ζ maps **Inf** into X. We determine γ such that both γ and $\zeta|\gamma$ belong to X, and define, for each n, $\beta(p+q+n-1) := \gamma(n)$.

We claim that both β and $\delta|\beta$ belong to X^+.
The sequence $\overline{\underline{1}}p * \underline{0}$ belongs to **Fin**. Suppose that β is apart from $\overline{\underline{1}}p * \underline{0}$. Considering the definition of β we see that there exists c in $\mathbb{N}$ and γ in X such that $\beta = c * \gamma$, and therefore β itself belongs to X. We conclude that, if $\beta \# \overline{\underline{1}}p * \underline{0}$, then β belongs to X. Therefore, β belongs to Perhaps(**Fin**, X) $= X^+$.
Suppose that $\delta|\beta$ is apart from $\overline{\underline{1}}p * \underline{0}$. Considering the definition of β we see that there exists c in $\mathbb{N}$ and γ in X such that $\beta = c * \gamma$ and $\delta|\beta$ belongs to X.We conclude that, if $(\delta|\beta) \# \overline{\underline{1}}p * \underline{0}$, then β belongs to X. Therefore, $\delta|\beta$ belongs to Perhaps(**Fin**, X) $= X^+$.
(iii) Let $X_0, X_1, \ldots$ be a sequence of subsets of $\mathcal{N}$, each of them showing proper regard for **Inf**. We have to prove that also $\bigcup_{i \in \mathbb{N}} X_i$ shows proper regard for **Inf**. Let δ be a function from $\mathcal{N}$ to $\mathcal{N}$ mapping **Inf** into $\bigcup_{i \in \mathbb{N}} X_i$. Observe that $\underline{1}$ belongs to **Inf**. Using lemma 3.22 we find m, n such that every α in **Inf** passing through $\overline{\alpha}m$ belongs to X_n. Let ζ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that for every β, $\zeta|\beta$ coincides with $\delta|\overline{\underline{1}}m * \beta$. Clearly, ζ maps **Inf** into X_n. We determine γ such that both γ and $\zeta|\gamma$ belong to X_n. Define $\alpha := \overline{\underline{1}}m * \gamma$ and observe that both α and $\delta|\alpha$ belong to X_n and therefore to $\bigcup_{\imath \in \mathbb{N}} X_i$.
(iv) Immediate from (i), (ii), (iii) and the definition in Section 3.9.
(v) Immediate from (iv). □

3.24 Theorem: *(Under the assumption of Brouwer's Thesis)*
(**Almost*Finite** *is "simple"*)

(i) **Almost*Finite** shows proper regard for **Inf**, that is, every function from $\mathcal{N}$ to $\mathcal{N}$ mapping **Inf** into **Almost*Finite** also maps some element of **Almost*Finite** into **Almost*Finite**.
(ii) The set **Inf** does not reduce to the set **Almost*Finite**.
(iii) The set **Almost*Finite** is not a complete element of the class of the co-analytic subsets of $\mathcal{N}$.

Proof: (i) Let ζ be a function from $\mathcal{N}$ to $\mathcal{N}$ mapping **Inf** into **Almost*Finite**. We

have seen, in Lemma 3.22 that the set **Inf** is strictly analytic. Let δ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for all α, α belongs to **Inf** if and only if, for some β, α coincides with $\delta|\beta$. Using Theorem 3.19(iv) we find a stump σ such that for all β, $\zeta|(\delta|\beta)$ belongs to $\mathbb{P}(\sigma, \mathbf{Fin})$. Apparently, there exists a Borel-approximation X of **Almost*****Finite** such that ζ maps **Inf** into X. Using Theorem 3.23 we find α such that both α and $\zeta|\alpha$ belong to X and therefore to **Almost*****Finite**.
(ii) Immediate from (i).
(iii) Immediate from (ii). □

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