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NON-ARCHIMEDEAN EBERLEIN-ŠMULIAN THEORY

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ABSTRACT. It is shown that, for a large class of non-archimedean normed spaces $E$, a subset $X$ is weakly compact as soon as $f(X)$ is compact for all $f \in E'$ (Theorem 2.1), a fact that has no analogue in Functional Analysis over the real or complex numbers. As a Corollary we derive a non-archimedean version of the Eberlein-Šmulian Theorem (2.2 and 2.3, for the 'classical' theorem, see [1], VIII, §2 Theorem 1 and Corollary, page 219).

KEY WORDS AND PHRASES. Non-archimedean Banach space, weak compactness.

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INTRODUCTION

Let $E$ be a two-dimensional normed space over $\mathbb{R}$ or $\mathbb{C}$ and let $X := \{x \in E : 0 < \|x\| \leq 1\}$. Each $f \in E'$ has zeros on $X$, so $f(X) = f(\{0\} \cup X)$ is compact, while obviously $X$ is not. The same story can be told when we replace $\mathbb{R}$ or $\mathbb{C}$ by a complete non-trivially valued non-archimedean field $K$ that is locally compact. However, if $K$ is not locally compact then, under reasonable conditions, for a subset $X$ of a normed space $E$ over $K$ compactness of $f(X)$ for all $f \in E'$ implies weak compactness of $X$ (we point out that if such an $X$ has more than one point it cannot be convex). To prove this curious fact (in §2) we shall develop some machinery in §1.

PRELIMINARIES

Throughout $K$ is a non-trivially non-archimedean valued field which is complete with respect to the metric induced by the valuation $| |$, and $E$ is a normed $K$-vector space, where we assume $\| |$ to satisfy the strong triangle inequality $\|x + y\| \leq \max(\|x\|, \|y\|)$. We write $|K^*| := \{\lambda | \lambda \in K, \lambda \neq 0\}$, $B_E(0, r) := \{x \in E : \|x\| \leq r\}$, $B_E := B_E(0, 1)$.

$E'$ is the space of all linear continuous functions $E \rightarrow K$. Equipped with the norm $f \mapsto \sup\{|f(x)| : x \in B_E\}$ it is a Banach space (i.e. a complete normed space). $E$ is called normpolar if the norm is polar i.e. if $\|f\| = \sup\{|f(x)| : f \in E', |f| \leq 1\}$ (x $\in E$), in other words, if $\gamma : E \rightarrow E''$ is an isometry. $E'$ is always normpolar. We assume throughout this note that $E$ is normpolar.

A subset $A$ of a (normed) space $E$ is absolutely convex if it is a module over $B_K$. A set $X \subset E$ is convex if it is either empty or an additive coset of an absolutely convex set. A subset $A$ of $E$ is called edged if it is absolutely convex and, in case the valuation of $K$ is dense, $A = \bigcap\{\lambda A : \lambda \in K, |\lambda| > 1\}$. The weak topology $w = \sigma(E, E')$ is the weakest topology on $E$ making all $f \in E'$ continuous. The weak-star topology $w' = \sigma(E', E)$ is the weakest topology.
on $E'$ making all evaluation maps $f \mapsto f(a)$ ($a \in E$) continuous. For $X \subseteq E'$ we denote its $w'$-closure by $\overline{X}^{w'}$.

For other notions used in this paper we refer to [4].

1. SEPARATION OF $w'$-PRECOMPACT SETS

Lemma 1.1. Let $X$ be a bounded subset of $E'$. Then \( \{ x \in E : \inf_{f \in X} |f(x)| > 0 \} \) is open in $E$.

Proof. $X$ is equicontinuous, so for each $n \in \mathbb{N}$ the set $U_n := \{ x \in E : |f(x)| > \frac{1}{n} \text{ for all } f \in X \}$ is open. Then so is $\bigcup_n U_n = \{ x \in E : \inf_{f \in X} |f(x)| > 0 \}$.

Lemma 1.2. Let $K$ be not locally compact. Let $X \subseteq E'$ and $a \in E$ be such that $X(a) := \{ f(a) : f \in X \}$ is precompact. Suppose $X \subseteq g + U$ where $U$ is an edged zero neighbourhood in $E'$, $U$ $w'$-closed and where $g \in E' \setminus U$. Then for any $e > 0$ there exists a $b \in E$ for which $\|a - b\| \leq e$ and $\inf_{f \in X} |f(b)| > 0$.

Proof. There exists an $r \in |K^*|$ such that $B_r(0, r) \subseteq U$. Choose $\delta \in K, 0 < |\delta| < 1$. The equivalence relation $\sim$ on $K^*$ given by 'a $\sim$ b iff $|a - b| < |\delta|$' yields an open partition of $C := \{ \lambda \in K : |\delta| \leq |\lambda| \leq r \}$ that is infinite because $K$ is not locally compact. By precompactness $X(a)$ cannot meet each equivalence class and there exists a $\gamma \in C$ such that

\[
|f(a) - \gamma| \geq |\gamma| \quad (f \in X).
\]

$U$ is $w'$-closed and edged, $g \notin U$, so by [8], 4.8 there exists a $c \in E$ such that $g(c) = \gamma, |f(c)| < |\gamma|$ for all $f \in U$. Set $b := a - c$. We have $|f(c)| \leq |\gamma|$ for all $f \in B_r(0, r)$ so $\|a - b\| = \|a\| = \|c\| \leq |\gamma|^r \leq e$. For each $f \in X$, writing $f = g + u$ where $u \in U$, we obtain $|f(c) - \gamma| = |f(c) - g(c)| = |u(c)| < |\gamma|$. This, combined with (*), yields $|f(a) - \gamma| > |f(c) - \gamma|$ for all $f \in X$, so $|f(b)| = |f(a) - f(c)| = \max(|f(a) - \gamma|, |f(c) - \gamma|) = |f(a) - \gamma| \geq |\gamma|$. It follows that $\inf_{f \in X} |f(b)| > 0$.

Corollary 1.3. Let $K$ be not locally compact, let $E$ be a Banach space. Let $X \subseteq E'$ be $w'$-precompact. Suppose $X \subseteq g + U$ where $U$ is an edged zero neighbourhood in $E'$, $U$ $w'$-closed, $g \in E' \setminus U$. Then $\{ x \in E : \inf_{f \in X} |f(x)| > 0 \}$ is open and dense in $E$.

Proof. Just combine Lemmas 1.1 ($w'$-precompactness implies $w'$-boundedness hence norm boundedness by completeness) and 1.2.

Definition 1.4. Let us call $X \subseteq E'$ $\sigma$-decomposable in $E'$ if for each $g \in E' \setminus X$ there exist $f_1, f_2, \ldots \in X$ and edged zero neighbourhoods $U_1, U_2, \ldots$ in $E'$ such that each $U_n$ is $w'$-closed and $X \subseteq \bigcup_n (f_n + U_n), g \notin \bigcup_n (f_n + U_n)$.

Theorem 1.5. (Separation Theorem) Let $K$ be not locally compact, let $E$ be a Banach space, let $X \subseteq E'$ be $w'$-precompact and $\sigma$-decomposable in $E'$. Then for each $g \in E' \setminus X$ there exists an $a \in E$ such that $g(a) \neq f(a)$ for all $f \in X$.

Proof. Without loss, assume $g = 0$. Let $\{ f_n + U_n : n \in \mathbb{N} \}$ be a covering of $X$ like in Definition 1.4. By Corollary 1.3 for each $n \in \mathbb{N}$ the set $\{ x \in E : \inf_{f \in X} |f(x)| > 0 \}$ is open and dense in $E$, where $X_n := X \cap (f_n + U_n)$. By completeness and the Baire Category Theorem $\{ x \in E : f(x) \neq 0 \text{ for all } f \in X \} \supseteq \bigcap_n \{ x \in E : \inf_{f \in X} |f(x)| > 0 \} \neq \emptyset$.

Remark. It is not hard, by modifying 1.1 - 1.5, to prove the following dual form of this separation theorem. Let $K$ be not locally compact, let $X \subseteq E$ be weakly precompact and $\sigma$-decomposable in $E$ (see below). Then for each $a \in E \setminus X$ there exists an $f \in E'$ such that $f(a) \neq f(X)$. Here, $X$ is called $\sigma$-decomposable in $E$ if for each $a \in E \setminus X$ there exist $x_1, x_2, \ldots \in X$ and edged zero neighbourhoods $U_1, U_2, \ldots$ in $E$ such that each $U_n$ is weakly closed and $X \subseteq \bigcup_n (x_n + U_n), a \notin \bigcup_n (x_n + U_n)$. 


COROLLARY 1.6. Let K be not locally compact, let E be a Banach space, let \( X \subseteq E' \) be \( \sigma \)-decomposable in \( E' \). Suppose \( X(a) := \{ f(a) : f \in X \} \) is compact for all \( a \in E \). Then \( X \) is \( w' \)-compact.

Proof. The map \( f \mapsto (f(a))_{a \in E} \) is a homeomorphism of \( (E', w') \) onto a subspace of \( K^E \). The image of \( X \) lies in the compact subset \( \bigcap_{a \in E} X(a) \) so \( X \) is \( w' \)-precompact. Since \( E' \) is \( w' \)-quasicomplete by the p-adic Alaoglu Theorem [8], 3.1, it suffices to show that \( X \) is \( w' \)-closed. To this end, let \( g \in E' \setminus X \). By Theorem 1.5 there exists an \( a \in E \) such that \( g(a) \notin X(a) \). Now \( X(a) \subseteq \overline{X(a)} = X(a) \), so \( g(a) \notin \overline{X(a)} \) i.e. \( g \notin \overline{X(a)} \).

To find examples of \( \sigma \)-decomposable sets (in 1.9-1.11) we need the following Lemmas.

**LEMMA 1.7.** Let \( n \in \mathbb{N} \), let \( D \) be an \( n \)-dimensional subspace of \( E' \). Then for each \( t \in (0,1) \) there exist \( a_1, a_2, \ldots, a_n \in B_E \) such that \( \max_{1 \leq i \leq n} |f(a_i)| \geq t ||f|| \) \((f \in D)\).

Proof. First assume that the valuation of \( K \) is dense. The space \( H := \{ x \in E : f(x) = 0 \text{ for all } f \in D \} \) has codimension \( n \) in \( E \). Choose \( s \in (t,1) \) and let \( g_1, \ldots, g_n \) be a \( s \)-orthogonal base of \( (E/H)' \) such that \( s^{-1} \leq ||g_i|| \leq s^{-1} \) for \( i = 1, \ldots, n \). There exist \( b_1, \ldots, b_n \in E/H \) such that \( g_i(b_j) = \delta_{ij}, \) for \( i, j \in \{1, \ldots, n\} \). Let \( s \in \{1, \ldots, n\} \), let \( g = \Sigma \lambda_j g_j \in (E/H)' \). Then \( ||g|| \geq \sqrt{s} \text{max} |\lambda_j| ||g_j|| \) and \( |g(b_i)| = |\lambda_i| \) so \( |g(b_i)| \leq s \text{max} |\lambda_j| \leq s \text{max} |\lambda_j| ||g|| \leq \sqrt{s} ||g|| \). So \( b_i \) is \( < 1 \). Thus, with \( \pi : E \to E/H \) denoting the canonical quotient map, there exist \( a_1, \ldots, a_n \in B_E \) with \( f(a_i) = b_i \) for each \( i \). The adjoint \( \pi' \) of \( \pi \) maps \( (E/H)' \) isometrically onto \( D \). Now let \( f \in D \). Then \( f = \pi'(g) \) where \( g \in (E/H)' \), \( ||g|| = ||f|| \). We have, writing \( g = \Sigma_i \lambda_j g_j ||f(a_i)|| = \text{max} |\lambda_i| ||g|| \geq t ||\Sigma \lambda_j g_j|| = t ||g|| = ||f|| \).

Now, if the valuation is discrete we can modify the above proof by taking \( s = t = 1 \). Then the \( b_i \) have norm \( \leq 1 \) (rather than \( < 1 \)), but one can use that \( E/H \) is a strict quotient i.e. there exist \( a_1, \ldots, a_n \in E \) with \( ||a_i|| = ||b_i|| \) and \( \pi(a_i) = b_i \) for each \( i \).

**LEMMA 1.8.** Let \( D \) be a subspace of \( E', D \) of countable type. Then there is a sequence \( a_1, a_2, \ldots \in B_E \) such that \( \max_{1 \leq i \leq n} |f(a_i)| \geq t ||f|| \) for all \( f \in D \).

Proof. Let \( D_1 \subseteq D_2 \subseteq \ldots \) be finite-dimensional subspaces of \( D \), \( \bigcup D_n \) is dense in \( D \). Let \( t \in (0,1) \). By Lemma 1.7 there exists a finite set \( F_n \subseteq B_E \) such that \( \max_{1 \leq i \leq n} |f(a_i)| \geq t ||f|| \) for all \( f \in D_n \).

So, for \( F^t := \bigcup_{n \in \mathbb{N}} F_n^t \) we obtain

\[
(*) \quad ||f|| \geq \sup_{a \in F^t} |f(a)| \geq t ||f|| \quad (f \in \bigcup_{n \in \mathbb{N}} D_n).
\]

Now \( F := \bigcup_{t \in (0,1)} F^t \) is countable and \((*) \) implies \( ||f|| = \sup_{a \in F} |f(a)| \) for all \( f \in \bigcup_{n \in \mathbb{N}} D_n \), hence, by continuity, for all \( f \in D \).

**PROPOSITION 1.9.** Let \( X \subseteq E' \) be such that \( X(a) := \{ f(a) : f \in X \} \) is separable for each \( a \in E \) and \( [X] \) is of countable type. Then \( X \) is \( \sigma \)-decomposable in \( E' \).

Proof. Let \( g \in E' \setminus X \). Then \( D := \{ g \} \cup X \) is of countable type so by Lemma 1.8 there exist \( a_1, a_2, \ldots \in B_E \) such that

\[
(*) \quad ||h|| = \sup_{a \in F} |h(a)| \quad (h \in D).
\]

For each \( m, n \in \mathbb{N} \) the set \( U_{mn} := \{ h \in E' : ||h(a_n)|| \leq \frac{1}{m} \} \) is an edged \( w' \)-zero neighbourhood. Its cosets, except for \( g + U_{mn} \), cover \( E \setminus (g + U_{mn}) \) and by separability of \( X(a_n) \) there exists a countable subcovering \( F_{mn} \) no member of which contains \( g \). Then \( \bigcup_{mn} F_{mn} \) still avoids \( g \) it remains to be shown that it covers \( X \). Suppose \( f \in X \) is not covered. Then \( f \in g + U_{mn} \) for all
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weak precompactness of \( j(X) \) is \( \sigma \)-decomposable in \( X = \) precompact in \( X \).

COROLLARY 1.10. Let \( X \subseteq E' \). If \( X \) is norm precompact, or \( X \) is \( w' \)-precompact and \( \|X\| \) is of countable type, then \( X \) is \( \sigma \)-decomposable in \( E' \).

PROPOSITION 1.11. Let \( X \subseteq E' \) be such that \( X(a) \) is separable for each \( a \in E \). Suppose that for each \( h \in \overline{X}^{w'} \) the set \( X \cup \{h\} \) is \( w' \)-metrizable. Then \( X \) is \( \sigma \)-decomposable in \( E' \).

Proof. Let \( g \in E' \setminus X \). If \( g \notin \overline{X}^{w'} \) then there exists a \( w' \)-zero neighbourhood \( U \) such that \( (g + U) \cap X = \emptyset \). We may assume that \( U \) is of the form \( \{f \in E' : |f(a_1)| \leq \varepsilon , \ldots , |f(a_n)| \leq \varepsilon \} \) for some \( \varepsilon > 0 \), \( n \in \mathbb{N} \), \( a_1 , \ldots , a_n \in E \). Then \( U \) is \( w' \)-closed and edged. By separability of \( \langle X(a_1) \times \cdots \times X(a_n) \rangle \) only countably many of the cosets \( f + U : f \in X \) cover \( X \) and none of them contains \( g \). Now let \( g \in \overline{X}^{w'} \). By \( w' \)-metrizability there exist \( w' \)-neighbourhoods of zero \( U_1 \supset U_2 \supset \cdots \) such that \( X \cap \bigcap (g + U_n) = \emptyset \). We may suppose that the \( U_n \) are \( w' \)-closed and edged. By separability, like above, for each \( n \) the set \( X \setminus (g + U_n) \) is covered by countably many additive cosets of \( U_n \) none of them containing \( g \). Their union is a countable covering of \( X \) avoiding \( g \).

2. EBERLEIN-ŠMULIAN THEORY

We now apply the theory of \( \S 1 \). Recall ([5], p. 57) that \( E \) is said to have property \( (\ast) \) if for each subspace \( D \) of countable type, every \( f \in E' \) has an extension \( \overline{f} \in E' \). By the non-archimedean Hahn-Banach Theorem [4], 4.8 every normed space over a spherically complete \( K \) has \( (\ast) \). For general \( K \), spaces with a base, in particular spaces of countable type, have \( (\ast) \) ([5], p. 58), and so have strongly polar spaces ([6], 4.2). Recall that \( E \) is assumed to be normpolar.

THEOREM 2.1. Let \( K \) be not locally compact, let \( X \) be a subset of \( E \) such that \( f(X) \) is compact for all \( f \in E' \). Then each one of the following properties implies that \( X \) is weakly compact and weakly metrizable.

(i) \( E \) has property \( (\ast) \).
(ii) \( E' \) is of countable type.
(iii) \( \|X\| \) is of countable type.

Moreover, in case (i) \( X \) is norm compact and the weak and norm topology coincide on \( X \).

Proof. The natural isometry \( \hat{j} : E \to E'' \) is easily seen to be a homeomorphism of \( E \) with the weak topology onto \( j(E) \) with the restriction of the \( w' \)-topology \( \sigma(E'', E') \). We show that \( j(X) \) is \( \sigma \)-decomposable in \( E' \). First note that the predual \( E' \) is normpolar. In case (i), from weak precompactness of \( X \) it follows that \( X \) is norm precompact by [7], Th. 3 (the assumption made throughout [7] that \( E \) is complete is easily seen to be superfluous here). So \( j(X) \) is norm precompact in \( E'' \) and therefore \( \sigma \)-decomposable by Corollary 1.10. For case (ii) observe that every \( (w' \) bounded subset of \( E'' \) is \( w' \)-metrizable ([8], 6.1) which applies to \( j(X) \cup \{0\} \) for any \( \theta \in E'' \). For each \( f \in E' \) the set \( j(X)(f) = f(X) \) is compact hence separable so \( j(X) \) is \( \sigma \)-decomposable in \( E'' \) by Proposition 1.11. For case (iii) we can directly apply Corollary 1.10. Thus, \( j(X) \) is \( \sigma \)-decomposable, and from Corollary 1.6 we conclude that \( j(X) \) is \( w' \)-compact, so \( X = j^{-1}(j(X)) \) is \( w' \)-compact. Observe that \( X \) is \( w' \)-bounded hence bounded by normpolarity ([6], 7.7).

We have seen in passing that \( j(X) \) is \( w' \)-metrizable in case (ii), so \( X \) is weakly metrizable. Now let \( X \) satisfy (iii). Then \( j(X) \) is of countable type so by Lemma 1.8 there exist \( f_1 , f_2 , \ldots \in B_E \) such that \( \|j(x)\| = \sup \|f_n(x)\| \) for all \( x \in X \). The formula \( d(x, y) = \sup \|f_n(x) - f_n(y)\| 2^{-n} \) defines an ultrametric \( d \) on \( X \) (if \( d(x, y) = 0 \) then \( \|f_n(x) - f_n(y)\| = 0 \) for all \( n \neq \|x - y\| = 0 \)). By boundedness of \( X \) the induced topology is weaker than the weak topology on \( X \), but by
weak compactness these topologies coincide and so $X$ is weakly metrizable. Finally, in case (i) apply [6], 5.12 to conclude that on $X$ the weak and norm topology coincide, and that therefore $X$ is norm compact and $w$-metrizable.

**REMARKS.**

1. If $K$ is not spherically complete the space $\ell^\infty$ does not have property $(\ast)$ ([4], 4.15 (6) $\Rightarrow$ (7)) but since $(\ell^\infty') \simeq c_0$ ([4], 4.17) it satisfies (ii) of the above Theorem, and so do the non-reflexive space $\ell^\infty \otimes \ell^\infty$ ([3], 2.3) and the space $l^0 t_00$ ([3], 2.3) and the space $D$ of [4], 4.7.

2. Let $K$ be not spherically complete, let $E := \ell^\infty$, let $X := \{0\} \cup \{e_1, e_2, \ldots\} \subset \ell^\infty$, when $e_1, e_2, \ldots$ are the unit vectors. Then (ii) and (iii) above hold. $X$ is weakly compact (since $\lim e_n = 0$ weakly) but is obviously not norm compact.

3. The following example indicates that extending Theorem 2.1 to, say, metrizable locally convex spaces is doubtful. Let $E := K^N$ with the product topology. Then $E' \cong \bigoplus K$. Let $X := \{e_1, e_2, \ldots\}$ where $e_1, e_2, \ldots$ are the unit vectors of $K^N$. Then $E$ is of countable type so (i), (ii), (iii) of Theorem 2.1 are (formally) satisfied. For each $f \in E'$ we have $f(e_n) = 0$ for large $n$, so $f(X)$ is finite (hence compact) and contains 0. Yet, $X$ is not (weakly) compact as $0 = \lim_{n \to \infty} e_n \notin X$.

The following is now an almost trivial consequence of Theorem 2.1.

**COROLLARY 2.2.** (p-adic Eberlein-Smulian Theorem I) Let $K$ be not locally compact and let $X, E$ satisfy one of the conditions (i), (ii), (iii) of Theorem 2.1. Then the following are equivalent.

(a) $X$ is weakly compact.

(b) $X$ is weakly sequentially compact.

(c) $X$ is weakly countably compact.

**Proof.** Each one of the properties (a), (b), (c) implies compactness of $f(X)$ for all $f \in E'$.

By Corollary 2.2, (a), (b), (c) are obviously equivalent to: 'for all $f \in E'$ the image $f(X)$ is compact.'

We have seen in the Introduction that Theorem 2.1 fails if $K$ is locally compact. We now investigate what happens to Corollary 2.2. Note that every normed space over $K$ has $(\ast)$.

**THEOREM 2.3.** (p-adic Eberlein-Šmulian Theorem II) Let $K$ be locally compact, let $X \subset E$. Then each one of the above statements (a), (b), (c) is equivalent to 'X is norm compact'.

**Proof.** We have (a) $\Rightarrow$ (c), (b) $\Rightarrow$ (c). It suffices to prove that (c) implies that $X$ is a norm compactoid (then $X$ is weakly metrizable since the norm and weak topology coincide on $X$ ([6], 5.12)). Suppose not. Then by [7], Th. 2 there is a $t \in (0, 1]$ and a $t$-orthogonal sequence $e_1, e_2, \ldots$ in $X$ such that $\inf_n \|e_n\| > 0$. By (c) there is a weak accumulation point $a$ of $\{e_1, e_2, \ldots\}$. This $a$ is in the weak closure $D$ of $\{e_1, e_2, \ldots\}$ which equals the norm closure, so $a = \sum_{n=1}^\infty \lambda_n e_n$ where $\|\lambda_n e_n\| \to 0$. If $\lambda_j \neq 0$ for some $j$, let $U := \{z \in E : |\delta_j(z)| < |\lambda_j|\}$ where $\delta_j \in E'$ is an extension of the $j$th coordinate function $\Sigma \xi_j e_i \mapsto \xi_j$ on $D$. Then $a + U$ is a weak neighbourhood of $a$ but for each $n \in \mathbb{N}$, $n \neq j$ we have $|\delta_j(a - c_n)| = |\lambda_j|$ so $e_n \notin a + U$, a contradiction. Hence, $a = 0$.

But then $\{z \in E : |f(z)| < 1\}$ is a weak neighbourhood of $a$ containing no $e_n$ if $f \in E'$ is such that $f(e_n) = 1$ for all $n$. Contradiction, so $X$ is a norm compactoid.

**REMARK.** Corollary 2.2 for strongly polar spaces $E$ and Theorem 2.3 were first proved directly by the first author.
**Remark.** The following 'relative' version of the Eberlein-Šmulian Theorem holds. (Compare [1], VIII §2, Theorem 1). Let \( X \subset E \). Suppose one of the conditions (i), (ii), (iii) of Theorem 2.1 is satisfied. Then the following are equivalent, (a) \( X \) is weakly relatively compact. (b) \( X \) is weakly relatively sequentially compact. (c) \( X \) is weakly relatively countably compact. We leave the easy proof to the reader.

**Counterexamples.** We show that the previous theory fails for certain subsets \( X \) of \( \ell^\infty(I) \) where \( I \) has at least the cardinality of the continuum, but is non-measurable, and where \( K \) is not spherically complete. The \( K \)-valued characteristic function of a subset \( S \subset I \) is denoted \( \xi_S \) and is given by \( \xi_S(x) := 1 \) if \( x \in S \), \( \xi_S(x) := 0 \) if \( x \in I \setminus S \).

1. Let \( X := \{ \xi_S : S \subset I \} \). Then \( X \) is a weakly compact but not weakly sequentially compact subset of \( \ell^\infty(I) \).

**Proof.** \( X \) is bounded and since \( \ell^\infty(I)' \simeq c_0(I) \) (4.21) the weak topology on \( X \) is the topology of pointwise convergence. Clearly the map \( f \mapsto (f(i))_{i \in I} \) is a homeomorphism \( X \to \{0,1\}^I \), hence \( X \) is weakly compact. To prove that \( X \) is not weakly sequentially compact, let \( \phi : I \to Y \) be a surjection where \( Y := \{ \xi_A : A \subset N \} \subset \ell^\infty \). The formula \( \phi(x) = (\xi_{s_1}(x), \xi_{s_2}(x), \ldots) \) \((x \in I)\) defines subsets \( S_1, S_2, \ldots \) of \( I \). If \( s_{n_1}, s_{n_2}, \ldots \) is a subsequence of \( s_1, s_2, \ldots \) then, by surjectivity of \( \phi \), there is an \( x \in I \) for which \( (\xi_{s_{n_1}}(x), \xi_{s_{n_2}}(x), \ldots) = (1,0,1,0,1,\ldots) \), so the subsequence is not weakly convergent.

2. Let \( Z := \{ \xi_S : S \subset I, S \text{ countable} \} \subset \ell^\infty(I) \). Then \( Z \) is weakly sequentially compact but not weakly compact.

**Proof.** Clearly the weak closure of \( Z \) equals \( X \) of above, so \( Z \) is not weakly compact. On the other hand, if \( s_1, s_2, \ldots \) is a sequence in \( Z \) then \( S := \bigcup_n S_n \) is countable and by a standard diagonal procedure one obtains a subsequence converging at all points of \( S \), hence at all points of \( I \), to an element of \( Z \).

3. Let \( T := \{ \xi_{i_1} : i \in I \} \subset \ell^\infty(I) \). Then \( f(T) \) is compact for all \( f \in \ell^\infty(I)' \) but \( T \) is not weakly countably compact.

**Proof.** Let \( f \in \ell^\infty(I)' \). As \( \ell^\infty(I)' \simeq c_0(I) \) we have that \( f(\xi_{i_1}) = 0 \) except for \( i \in \{i_1, i_2, \ldots\} \) where we may assume the \( i_n \in I \) to be distinct. Then \( \xi_{i_n} \to 0 \) weakly so \( T := \{0\} \cup \{ \xi_{i_n} : n \in N \} \) is weakly compact and \( f(T) = f(T_1) \) is compact. However the only weak accumulation point of \( \{ \xi_{i_1}, \xi_{i_2}, \ldots\} \) is \( 0 \notin T \) so that \( T \) is not weakly countably compact.

**References**


