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NON-ARCHIMEDEAN EBERLEIN-ŠMULIAN THEORY

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ABSTRACT. It is shown that, for a large class of non-archimedean normed spaces \( E \), a subset \( X \) is weakly compact as soon as \( f(X) \) is compact for all \( f \in E' \) (Theorem 2.1), a fact that has no analogue in Functional Analysis over the real or complex numbers. As a Corollary we derive a non-archimedean version of the Eberlein-Šmulian Theorem (2.2 and 2.3, for the 'classical' theorem, see [1], VIII, §2 Theorem 1 and Corollary, page 219).

KEY WORDS AND PHRASES. Non-archimedean Banach space, weak compactness.

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INTRODUCTION

Let \( E \) be a two-dimensional normed space over \( \mathbb{R} \) or \( \mathbb{C} \) and let \( X := \{ x \in E : 0 < \| x \| \leq 1 \} \). Each \( f \in E' \) has zeros on \( X \), so \( f(X) = f(\{0\} \cup X) \) is compact, while obviously \( X \) is not. The same story can be told when we replace \( \mathbb{R} \) or \( \mathbb{C} \) by a complete non-trivially valued non-archimedean field \( K \) that is locally compact. However, if \( K \) is not locally compact then, under reasonable conditions, for a subset \( X \) of a normed space \( E \) over \( K \) compactness of \( f(X) \) for all \( f \in E' \) implies weak compactness of \( X \) (we point out that if such an \( X \) has more than one point it cannot be convex). To prove this curious fact (in §2) we shall develop some machinery in §1.

PRELIMINARIES

Throughout \( K \) is a non-trivially non-archimedean valued field which is complete with respect to the metric induced by the valuation \( | \cdot | \), and \( E \) is a normed \( K \)-vector space, where we assume \( \| \cdot \| \) to satisfy the strong triangle inequality \( \| x + y \| \leq \max(\| x \|, \| y \|) \). We write \( |K^*| := \{ |\lambda| : \lambda \in K, \lambda \neq 0 \} \). Let \( B_E(0,r) := \{ x \in E : \| x \| \leq r \} \), \( B_E := B_E(0,1) \).

\( E' \) is the space of all linear continuous functions \( E \to K \). Equipped with the norm \( f \mapsto \sup \{|f(x)| : x \in B_E \} \) it is a Banach space (i.e. a complete normed space). \( E \) is called normpolar if the norm is polar i.e. if \( \| x \| = \sup \{|f(x)| : f \in E', |f| \leq \| \| \} \) \( \quad (x \in E) \), in other words, if \( \gamma : E \to E'' \) is an isometry. \( E' \) is always normpolar. We assume throughout this note that \( E \) is normpolar.

A subset \( A \) of a (normed) space \( E \) is absolutely convex if it is a module over \( B_K \). A set \( X \subset E \) is convex if it is either empty or an additive coset of an absolutely convex set. A subset \( A \) of \( E \) is called edged if it is absolutely convex and, in case the valuation of \( K \) is dense, \( A = \bigcap \{ \lambda A : \lambda \in K, |\lambda| > 1 \} \). The weak topology \( \omega = \sigma(E,E') \) is the weakest topology on \( E \) making all \( f \in E' \) continuous. The weak-star topology \( \omega' = \sigma(E',E) \) is the weakest topology...
on $E'$ making all evaluation maps $f \mapsto f(a)$ ($a \in E$) continuous. For $X \subset E'$ we denote its \( w'\)-closure by $X^{w'}$.

For other notions used in this paper we refer to [4].

1. SEPARATION OF \( w'\)-PRECOMPACT SETS

**Lemma 1.1**. Let $X$ be a bounded subset of $E'$. Then \( \{ x \in E : \inf_{f \in X} |f(x)| > 0 \} \) is open in $E$.

**Proof.** $X$ is equicontinuous, so for each $n \in \mathbb{N}$ the set $U_n := \{ x \in E : |f(x)| > \frac{1}{n} \}$ for all $f \in X$ is open. Then so is $\bigcup_n U_n = \{ x \in E : \inf_{f \in X} |f(x)| > 0 \}$.

**Lemma 1.2**. Let $K$ be not locally compact. Let $X \subset E'$ and $a \in E$ be such that $X(a) := \{ f(a) : f \in X \}$ is precompact. Suppose $X \subset g + U$ where $U$ is an edged zero neighbourhood in $E'$, $U$ \( w'\)-closed and where $g \in E' \setminus U$. Then for any $\epsilon > 0$ there exists a $b \in E$ for which $\|a - b\| < \epsilon$ and $\inf_{f \in X} |f(b)| > 0$.

**Proof.** There exists an $r \in |K|$ such that $B_{E'}(0, r) \subset U$. Choose $\delta \in K, 0 < |\delta| < 1$. The equivalence relation $\sim$ on $K^*$ given by $\alpha \sim \beta$ iff $|\alpha - \beta| < |\beta|$ yields an open partition of $C := \{ \lambda \in K : |\delta| \leq |\lambda| \leq r \}$ that is infinite because $K$ is not locally compact. By precompactness $X(a)$ cannot meet each equivalence class and there exists a $\gamma \in C$ such that \( |f(a) - \gamma| \geq |\gamma| \) (\( f \in X \)).

$U$ is \( w'\)-closed and edged, $g \not\in U$, so by [6], 4.8 there exists a $c \in E$ such that $g(c) = \gamma, |f(c)| < |\gamma|$ for all $f \in U$. Set $b := a - c$. We have $|f(c)| \leq |\gamma|$ for all $f \in B_{E'}(0, r)$ so $\|a - b\| = \|c\| = |b(c)| \leq |\gamma|r^{-1} \leq \epsilon$. For each $f \in X$, writing $f = g + u$ where $u \in U$, we obtain $|f(a) - \gamma| = |f(a) - g(a)| = |u(c)| < |\gamma|$. This, combined with (1), yields $|f(a) - \gamma| > |f(c) - \gamma|$ for all $f \in X$, so $|f(b)| = |f(a) - f(c)| = \max\{|f(a) - \gamma|, |f(c) - \gamma|\} = |f(a) - \gamma| \geq |\gamma|$. It follows that $\inf_{f \in X} |f(b)| > 0$.

**Corollary 1.3.** Let $K$ be not locally compact, let $E$ be a Banach space. Let $X \subset E'$ be \( w'\)-precompact. Suppose $X \subset g + U$ where $U$ is an edged zero neighbourhood in $E'$, $U$ \( w'\)-closed, $g \in E' \setminus U$. Then \( \{ x \in E : \inf_{f \in X} |f(x)| > 0 \} \) is open and dense in $E$.

**Proof.** Just combine Lemmas 1.1 (\( w'\)-precompactness implies \( w'\)-boundedness hence norm boundedness by completeness) and 1.2.

**Definition 1.4.** Let us call $X \subset E'$ \( \sigma \)-decomposable in $E'$ if for each $g \in E' \setminus X$ there exist $f_1, f_2, \ldots \in X$ and edged zero neighbourhoods $U_1, U_2, \ldots$ in $E'$ such that each $U_n$ is \( w'\)-closed and $X \subset \bigcup (f_n + U_n), g \not\in \bigcup (f_n + U_n)$.

**Theorem 1.5.** (Separation Theorem) Let $K$ be not locally compact, let $E$ be a Banach space, let $X \subset E'$ be \( w'\)-precompact and \( \sigma \)-decomposable in $E'$. Then for each $g \in E' \setminus X$ there exists an $a \in E$ such that $g(a) \neq f(a)$ for all $f \in X$.

**Proof.** Without loss, assume $g = 0$. Let $\{ f_n + U_n : n \in \mathbb{N} \}$ be a covering of $X$ like in Definition 1.4. By Corollary 1.3 for each $n \in \mathbb{N}$ the set $\{ x \in E : \inf_{f \in X} |f(x)| > 0 \}$ is open and dense in $E$, where $X_n := X \cap (f_n + U_n)$. By completeness and the Baire Category Theorem $\{ x \in E : f(x) \neq x \} = \bigcap_{n \in \mathbb{N}} \{ x \in E : \inf_{f \in X} |f(x)| > 0 \} \neq \emptyset$.

**Remark:** It is not hard, by modifying 1.1 - 1.5, to prove the following dual form of this separation theorem. Let $K$ be not locally compact, let $X \subset E$ be weakly precompact and \( \sigma \)-decomposable in $E$ (see below). Then for each $a \in E \setminus X$ there exists an $f \in E'$ such that $f(a) \neq f(X)$. Here, $X$ is called \( \sigma \)-decomposable in $E$ if for each $a \in E \setminus X$ there exist $x_1, x_2, \ldots \in X$ and edged zero neighbourhoods $U_1, U_2, \ldots$ in $E$ such that each $U_n$ is weakly closed and $X \subset \bigcup (x_n + U_n), a \not\in \bigcup (x_n + U_n)$.
COROLLARY 1.6. Let $K$ be not locally compact, let $E$ be a Banach space, let $X \subset E'$ be $\sigma$-decomposable in $E'$. Suppose $X(a) := \{ f(a) : f \in X \}$ is compact for all $a \in E$. Then $X$ is $w'$-compact.

Proof. The map $f \mapsto (f(a))_{a \in E}$ is a homeomorphism of $(E',w')$ onto a subspace of $K^E$. The image of $X$ lies in the compact subset $\prod_{a \in E} X(a)$ so $X$ is $w'$-precompact. Since $E'$ is $w'$-quasicomplete by the p-adic Alaoglu Theorem [8], 3.1, it suffices to show that $X$ is $w'$-closed. To this end, let $g \in E' \setminus X$. By Theorem 1.5 there exists an $a \in E$ such that $g(a) \notin X(a)$. Now $X(a) \subset \overline{X(a)} \subset X(a) = X(a)$, so $g(a) \notin \overline{X(a)}$ i.e. $g \notin \overline{X'}$.

To find examples of $\sigma$-decomposable sets (1.9-1.11) we need the following Lemmas.

LEMMA 1.7. Let $n \in \mathbb{N}$, let $D$ be an $n$-dimensional subspace of $E'$. Then for each $t \in (0,1)$ there exist $a_1,a_2,\ldots,a_n \in B_E$ such that $\max_{1 \leq i \leq n} \| f(a_i) \| \geq t \| f \|$ $(f \in D)$.

Proof. First assume that the valuation of $K$ is dense. The space $H := \{ x \in E : f(x) = 0 \text{ for all } f \in D \}$ has codimension $n$ in $E$. Choose $a \in (t,1)$ and let $g_1,\ldots,g_n$ be a $\sqrt{s}$-orthogonal base of $(E/H)'$ such that $s^{-1} \leq \| g_i \| \leq t^{-1}$ for $i \in \{1,\ldots,n\}$. There exist $b_1,\ldots,b_n \in E/H$ such that $g_i(b_j) = \delta_{ij}$ $(i,j \in \{1,\ldots,n\})$. Let $s \in \{1,\ldots,n\}$, let $g = \Sigma \lambda_i g_i \in (E/H)'$. Then $\| g \| \geq \sqrt{s} \max_j | \lambda_j | \| g_j \|$ and $| \langle g,b_j \rangle | = | \lambda_j |$ so $| \langle g,b_j \rangle | \leq \max_j | \lambda_j | \leq s \max_j | \lambda_j | \| g_j \| \leq \sqrt{s} \| g \|$. So $\| b_k \| < 1$. Thus, with $\pi : E \to E/H$ denoting the canonical quotient map, there exist $a_1,\ldots,a_n \in B_E$ with $\pi(a_i) = b_i$ for each $i$. The adjoint $\pi'$ of $\pi$ maps $(E/H)'$ isometrically onto $D$. Now let $f \in D$. Then $f = \pi'(g)$ where $g \in (E/H)'$, $\| g \| = \| f \|$. We have, writing $g = \Sigma_{j=1}^n \lambda_j g_j$, $\max_{1 \leq i \leq n} \| f(a_i) \| = \max_j | \langle g,b_j \rangle | = \max_j | \lambda_j | \geq t \max_j | \lambda_j | \| g_j \| \geq t \| \Sigma \lambda_j g_j \| = t \| g \| = t \| f \|$. Now, if the valuation is discrete we can modify the above proof by taking $s = t = 1$. Then the $b_i$ have norm $\leq 1$ (rather than $< 1$), but one can use that $E/H$ is a strict quotient i.e. there exist $a_1,\ldots,a_n \in E$ with $\| a_i \| = \| b_i \|$ and $\pi(a_i) = b_i$ for each $i$.

LEMMA 1.8. Let $D$ be a subspace of $E'$, $D$ of countable type. Then there is a sequence $a_1,a_2,\ldots\in B_E$ such that $\max_{1 \leq i \leq n} \| f(a_i) \| \geq t \| f \|$ for all $f \in D$.

Proof. Let $D_1 \subset D_2 \subset \ldots$ be finite-dimensional subspaces of $D$, $\bigcup D_n$ is dense in $D$. Let $t \in (0,1)$. By Lemma 1.7 there exists a finite set $F_n^* \subset B_E$ such that $\max_{1 \leq i \leq n} \| f(x) \| \geq t \| f \|$ for all $f \in D_n$.

So, for $F^* := \bigcup_{n \in \mathbb{N}} F_n^*$ we obtain

(*) $\| f \| \geq \sup_{a \in F^*} \| f(a) \| \geq t \| f \|$ $(f \in D_n)$.

Now $F := \bigcup_{n \in \mathbb{N}} F_n^*$ is countable and (*) implies $\| f \| = \sup_{a \in F^*} \| f(a) \| \geq t \| f \|$ for all $f \in \bigcup D_n$, hence, by continuity, for all $f \in D$.

PROPOSITION 1.9. Let $X \subset E'$ be such that $X(a) := \{ f(a) : f \in X \}$ is separable for each $a \in E$ and $|X|$ is of countable type. Then $X$ is $\sigma$-decomposable in $E'$.

Proof. Let $g \in E' \setminus X$. Then $D := \{ g \} \cup X$ is of countable type by Lemma 1.8 there exist $a_1,a_2,\ldots \in B_E$ such that

(*) $\| h \| = \sup_{a \in F^*} | h(a) |$ $(h \in D)$.

For each $m,n \in \mathbb{N}$ the set $U_{mn} := \{ h \in E' : \text{ | } h(a_n) | \leq 1/m \}$ is an edged $w'$-zero neighbourhood. Its cosets, except for $g + U_{mn}$, cover $X \setminus (g + U_{mn})$ and by separability of $X(a_n)$ there exists a countable subcovering $F_{mn}$ no member of which contains $g$. Then $\bigcup_{m,n} F_{mn}$ still avoids $g$; it remains to be shown that it covers $X$. Suppose $f \in X$ is not covered. Then $f \in g + U_{mn}$ for all
COROLLARY 1.10. Let $X \subset E'$. If $X$ is norm precompact, or $X$ is $w'$-precompact and $|X|$ is of countable type, then $X$ is $\sigma$-decomposable in $E'$.

PROPOSITION 1.11. Let $X \subset E'$ be such that $X(a)$ is separable for each $a \in E$. Suppose that for each $h \in \overline{X'}$ the set $X \cup \{h\}$ is $w'$-metrizable. Then $X$ is $\sigma$-decomposable in $E'$.

Proof. Let $g \in E' \setminus X$. If $g \notin \overline{X'}$ then there exists a $w'$-zero neighbourhood $U$ such that $(g + U) \cap X = \emptyset$. We may assume that $U$ is of the form $\{f \in E': |f(a_1)| \leq \epsilon, \ldots, |f(a_n)| \leq \epsilon\}$ for some $\epsilon > 0$, $n \in \mathbb{N}$. Then $U$ is $w'$-closed and edged. By separability of $X(a_1) \times \cdots \times X(a_n)$ only countably many of the cosets $f + U : f \in X$ cover $X$ and none of them contains $g$. Now let $g \in \overline{X'}$. By $w'$-metrizability there exist $w'$-neighbourhoods of zero $U_1 \supset U_2 \supset \cdots$ such that $X \cap \bigcap (g + U_n) = \emptyset$. We may suppose that the $U_n$ are $w'$-closed and edged. By separability, like above, for each $n$ the set $X \setminus (g + U_n)$ is covered by countably many additive cosets of $U_n$ none of them containing $g$. Their union is a countable covering of $X$ avoiding $g$.

2. EBERLEIN-ŠMULIAN THEORY

We now apply the theory of §1. Recall ([5], p. 57) that $E$ is said to have property $(\ast)$ if for each subspace $D$ of countable type, every $f \in D'$ has an extension $\overline{f} \in E'$. By the non-archimedean Hahn-Banach Theorem [4], 4.8 every normed space over a spherically complete $K$ has $(\ast)$. For general $K$, spaces with a base, in particular spaces of countable type, have $(\ast)$ ([5], p. 58), and so have strongly polar spaces ([6], 4.2). Recall that $E$ is assumed to be normpolar.

THEOREM 2.1. Let $K$ be not locally compact, let $X$ be a subset of $E$ such that $f(X)$ is compact for all $f \in E'$. Then each one of the following properties implies that $X$ is weakly compact and weakly metrizable.

(i) $E$ has property $(\ast)$.
(ii) $E'$ is of countable type.
(iii) $|X|$ is of countable type.

Moreover, in case (i) $X$ is norm compact and the weak and norm topology coincide on $X$.

Proof. The natural isometry $j : E \to E''$ is easily seen to be a homeomorphism of $E$ with the weak topology onto $\mathcal{J}(E)$ with the restriction of the $w'$-topology $\sigma(E'', E')$. We show that $\mathcal{J}(X)$ is $\sigma$-decomposable in $E''$. First note that the predual $E'$ is normpolar. In case (i), from weak precompactness of $X$ it follows that $X$ is norm precompact by [7], Th. 3 (the assumption made throughout [7] that $E$ is complete is easily seen to be superfluous here). So $\mathcal{J}(X)$ is norm precompact in $E''$ and therefore $\sigma$-decomposable by Corollary 1.10. For case (ii) observe that every $(w'$-) bounded subset of $E''$ is $w'$-metrizable ([8], 6.1) which applies to $\mathcal{J}(X) \cup \{\emptyset\}$ for any $\emptyset \in E''$. For each $f \in E'$ the set $\mathcal{J}(X)(f) = f(X)$ is compact hence separable so $\mathcal{J}(X)$ is $\sigma$-decomposable in $E''$ by Proposition 1.11. For case (iii) we can directly apply Corollary 1.10. Thus, $\mathcal{J}(X)$ is $\sigma$-decomposable, and from Corollary 1.6 we conclude that $\mathcal{J}(X)$ is $w'$-compact, so $X = \mathcal{J}^{-1}(\mathcal{J}(X))$ is $w$-compact. Observe that $X$ is $w$-bounded hence bounded by normpolarity ([6], 7.7).

We have seen in passing that $\mathcal{J}(X)$ is $w'$-metrizable in case (ii), so $X$ is weakly metrizable. Now let $X$ satisfy (iii). Then $\mathcal{J}(X)$ is of countable type so by Lemma 1.8 there exist $f_1, f_2, \ldots \in B_{E'}$ such that $\|\mathcal{J}(x)\| = \sup |f_n(x)|$ for all $x \in X$. The formula $d(x, y) = \sup |f_n(x) - f_n(y)|2^{-n}$ defines an ultrametric $d$ on $X$ (if $d(x, y) = 0$ then $|f_n(x) - f_n(y)| = 0$ for all $n$ so $|x - y| = 0$).

By boundedness of $X$ the induced topology is weaker than the weak topology on $X$, but by
weak compactness these topologies coincide and so \( X \) is weakly metrizable. Finally, in case (i) apply [6], 5.12 to conclude that on \( X \) the weak and norm topology coincide, and that therefore \( X \) is norm compact and \( w \)-metrizable.

**REMARKS.**

1. If \( K \) is not spherically complete the space \( \ell^\infty \) does not have property (*) ([4], 4.15 (\&) \( \Rightarrow \) (\&)) but since \( (\ell^\infty)' \simeq c_0 ([4], 4.17) \) it satisfies (ii) of the above Theorem, and so do the non-reflexive space \( \ell^\infty \otimes \ell^\infty \) ([3], 2.3) and the space \( D \) of [4], 4.1.

2. Let \( K \) be not spherically complete, let \( E := \ell^\infty \), let \( X := \{0\} \cup \{e_1, e_2, \ldots\} \subset \ell^\infty \), when \( e_1, e_2, \ldots \) are the unit vectors. Then (ii) and (iii) above hold. \( X \) is weakly compact (since \( \lim_{n \to \infty} e_n = 0 \) weakly) but is obviously not norm compact.

3. The following example indicates that extending Theorem 2.1 to, say, metrizable locally convex spaces is doubtful. Let \( E := K^N \) with the product topology. Then \( E' \simeq \bigoplus K \). Let \( X := \{e_1, e_2, \ldots\} \) where \( e_1, e_2, \ldots \) are the unit vectors of \( K^N \). Then \( E \) is of countable type so (i), (ii), (iii) of Theorem 2.1 are (formally) satisfied. For each \( f \in E' \) we have \( f(e_n) = 0 \) for large \( n \), so \( f(X) \) is finite (hence compact) and contains 0. Yet, \( X \) is not (weakly) compact as \( 0 = \lim_{n \to \infty} e_n \notin X \).

The following is now an almost trivial consequence of Theorem 2.1.

**COROLLARY 2.2.** (\( p \)-adic Eberlein-Smulian Theorem I) Let \( K \) be not locally compact and let \( X, E \) satisfy one of the conditions (i), (ii), (iii) of Theorem 2.1. Then the following are equivalent.

(a) \( X \) is weakly compact.

(b) \( X \) is weakly sequentially compact.

(c) \( X \) is weakly countably compact.

**Proof.** Each one of the properties (a), (b), (c) implies compactness of \( f(X) \) for all \( f \in E' \). By Theorem 2.1 \( X \) is weakly metrizable and from that the equivalence of (a), (b), (c) follows easily.

**NOTE.** In Corollary 2.2, (a), (b), (c) are obviously equivalent to: 'for all \( f \in E' \) the image \( f(X) \) is compact.'

We have seen in the Introduction that Theorem 2.1 fails if \( K \) is locally compact. We now investigate what happens to Corollary 2.2. Note that every normed space over \( K \) has (*).

**THEOREM 2.3.** (\( p \)-adic Eberlein-$\tilde{\Sigma}$mulian Theorem II) Let \( K \) be locally compact, let \( X \subset E \). Then each one of the above statements (a), (b), (c) is equivalent to '\( X \) is norm compact'.

**Proof.** We have (a) \( \Rightarrow \) (c), (b) \( \Rightarrow \) (c). It suffices to prove that (c) implies that \( X \) is a norm compactoid (then \( X \) is weakly metrizable since the norm and weak topology coincide on \( X \) ([6], 5.12]). Suppose not. Then by [7], Th. 2 there is a \( \tau \in (0,1] \) and a \( \tau \)-orthogonal sequence \( e_1, e_2, \ldots \) in \( X \) such that \( \inf_n \|e_n\| > 0 \). By (c) there is a weak accumulation point \( a \) of \( \{e_1, e_2, \ldots\} \). This \( a \) is in the weak closure \( D \) of \( \{e_1, e_2, \ldots\} \) which equals the norm closure, so \( a = \sum_{i=1}^\infty \lambda_i e_i \) where \( \|\lambda_i e_i\| \to 0 \). If \( \lambda_j \neq 0 \) for some \( j \), let \( U := \{x \in E : |\delta_j(x)| < |\lambda_j|\} \) where \( \delta_j \in E' \) is an extension of the \( j \)-th coordinate function \( \Sigma \xi_j e_i \mapsto \xi_j \) on \( D \). Then \( a + U \) is a weak neighbourhood of \( a \) but for each \( n \in \mathbb{N}, n \neq j \) we have \( |\delta_j(a - e_n)| = |\lambda_j| \) so \( e_n \notin a + U \), a contradiction. Hence, \( a = 0 \).

But then \( \{x \in E : |f(x)| < 1\} \) is a weak neighbourhood of \( a \) containing no \( e_n \) if \( f \in E' \) is such that \( f(e_n) = 1 \) for all \( n \). Contradiction, so \( X \) is a norm compactoid.

**REMARK.** Corollary 2.2 for strongly polar spaces \( E \) and Theorem 2.3 were first proved directly by the first author.
REMARK. The following 'relative' version of the Eberlein-Šmulian Theorem holds. (Compare [1], VIII §2, Theorem 1). Let \( X \subset E \). Suppose one of the conditions (i), (ii), (iii) of Theorem 2.1 is satisfied. Then the following are equivalent. (a) \( X \) is weakly relatively compact. (b) \( X \) is weakly relatively sequentially compact. (c) \( X \) is weakly relatively countably compact. We leave the easy proof to the reader.

COUNTEREXAMPLES. We show that the previous theory fails for certain subsets \( X \) of \( \ell^\infty(I) \) where \( I \) has at least the cardinality of the continuum, but is non-measurable, and where \( K \) is not spherically complete. The \( \ell^p \)-valued characteristic function of a subset \( S \subset I \), \( I \subset \mathbb{N} \), is denoted \( \xi_S \) and is given by \( \xi_S(x) := 1 \) if \( x \in S \), \( \xi_S(x) := 0 \) if \( x \in I \setminus S \).

1. Let \( X := \{ \xi_S : S \subset I \} \). Then \( X \) is a weakly compact but not weakly sequentially compact subset of \( \ell^\infty(I) \).

Proof. \( X \) is bounded and since \( \ell^\infty(I)' \simeq c_0(I) \) (cf. 4.21) the weak topology on \( X \) is the topology of pointwise convergence. Clearly the map \( f \mapsto (f(x))_{x \in I} \) is a homeomorphism \( X \to \{0,1\}^I \), hence \( X \) is weakly compact. To prove that \( X \) is not weakly sequentially compact, let \( \phi : I \to Y \) be a surjection where \( Y := \{ \xi_A : A \subset \mathbb{N} \} \subset \ell^\infty \). The formula \( \phi(x) = (\xi_{S_1}(x), \xi_{S_2}(x), \ldots) \) \( (x \in I) \) defines subsets \( S_1, S_2, \ldots \) of \( I \). If \( \xi_{S_1}, \xi_{S_2}, \ldots \) is a subsequence of \( \xi_{S_1}, \xi_{S_2}, \ldots \) then, by surjectivity of \( \phi \), there is an \( x \in I \) for which \( (\xi_{S_1}(x), \xi_{S_2}(x), \ldots) \neq (1,0,1,0,1,\ldots) \), so the subsequence is not weakly convergent.

2. Let \( Z := \{ \xi_S : S \subset I, S \text{ countable} \} \subset \ell^\infty(I) \). Then \( Z \) is weakly sequentially compact but not weakly compact.

Proof. Clearly the weak closure of \( Z \) equals \( X \) of above, so \( Z \) is not weakly compact. On the other hand, if \( \xi_{S_1}, \xi_{S_2}, \ldots \) is a sequence in \( Z \) then \( S := \cup S_n \) is countable and by a standard diagonal procedure one obtains a subsequence converging at all points of \( S \), hence at all points of \( I \), to an element of \( Z \).

3. Let \( T := \{ \xi_{i_n} : i \in I \} \subset \ell^\infty(I) \). Then \( f(T) \) is compact for all \( f \in \ell^\infty(I)' \) but \( T \) is not weakly countably compact.

Proof. Let \( f \in \ell^\infty(I)' \). As \( \ell^\infty(I)' \simeq c_0(I) \) we have that \( f(\xi_{i_n}) = 0 \) except for \( i \in \{ i_1, i_2, \ldots \} \) where we may assume the \( i_n \in I \) to be distinct. Then \( \xi_{i_n} \to 0 \) weakly so \( T_1 := \{0\} \cup \{ \xi_{i_n} : n \in \mathbb{N} \} \) is weakly compact and \( f(T_1) \) is compact. However the only weak accumulation point of \( \{ \xi_{i_1}, \xi_{i_2}, \ldots \} \) is \( 0 \), so that \( T \) is not weakly countably compact.

REFERENCES