NON-ARCHIMEDEAN EBERLEIN-ŠMULIAN THEORY

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ABSTRACT. It is shown that, for a large class of non-archimedean normed spaces $E$, a subset $X$ is weakly compact as soon as $f(X)$ is compact for all $f \in E'$ (Theorem 2.1), a fact that has no analogue in Functional Analysis over the real or complex numbers. As a Corollary we derive a non-archimedean version of the Eberlein-Šmulian Theorem (2.2 and 2.3, for the 'classical' theorem, see [1], VIII, §2 Theorem 1 and Corollary, page 219).

KEY WORDS AND PHRASES. Non-archimedean Banach space, weak compactness.

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INTRODUCTION

Let $E$ be a two-dimensional normed space over $\mathbb{R}$ or $\mathbb{C}$ and let $X := \{x \in E : 0 < \|x\| \leq 1\}$. Each $f \in E'$ has zeros on $X$, so $f(X) = f(\{0\} \cup X)$ is compact, while obviously $X$ is not. The same story can be told when we replace $\mathbb{R}$ or $\mathbb{C}$ by a complete non-trivially valued non-archimedean field $K$ that is locally compact. However, if $K$ is not locally compact then, under reasonable conditions, for a subset $X$ of a normed space $E$ over $K$ compactness of $f(X)$ for all $f \in E'$ implies weak compactness of $X$ (we point out that if such an $X$ has more than one point it cannot be convex). To prove this curious fact (in §2) we shall develop some machinery in §1.

PRELIMINARIES

Throughout $K$ is a non-trivially non-archimedean valued field which is complete with respect to the metric induced by the valuation $|\cdot|$, and $E$ is a normed $K$-vector space, where we assume $\|\cdot\|$ to satisfy the strong triangle inequality $\|x + y\| \leq \max(\|x\|, \|y\|)$. We write $|K^*| := \{|\lambda| : \lambda \in K, \lambda \neq 0\}$, $B_E(0, r) := \{x \in E : \|x\| \leq r\}$, $B_E := B_E(0, 1)$.

$E'$ is the space of all linear continuous functions $E \to K$. Equipped with the norm $f \mapsto \sup\{|f(x)| : x \in B_E\}$ it is a Banach space (i.e. a complete normed space). $E$ is called normpolar if the norm is polar i.e. if $\|x\| = \sup\{|f(x)| : f \in E', |f| \leq \|\cdot\|\}$ for $x \in E$, in other words, if $\gamma : E \to E''$ is an isometry. $E'$ is always normpolar. We assume throughout this note that $E$ is normpolar.

A subset $A$ of a (normed) space $E$ is absolutely convex if it is a module over $B_K$. A set $X \subset E$ is convex if it is either empty or an additive coset of an absolutely convex set. A subset $A$ of $E$ is called edged if it is absolutely convex and, in case the valuation of $K$ is dense, $A = \bigcap\{\lambda A : \lambda \in K, |\lambda| > 1\}$. The weak topology $w = \sigma(E, E')$ is the weakest topology on $E$ making all $f \in E'$ continuous. The weak-star topology $w' = \sigma(E', E)$ is the weakest topology...
on $E'$ making all evaluation maps $f \mapsto f(a)$ \hspace{1em} (a \in E) \hspace{1em} \text{continuous. For } X \subseteq E' \text{ we denote its } w^*\text{-closure by } X^{w^*}.

For other notions used in this paper we refer to [4].

1. SEPARATION OF $w^*$-PRECOMPACT SETS

**Definition 1.4.**

**Corollary 1.3.**

**Theorem 1.5.** (Separation Theorem) Let $K$ be not locally compact, let $E$ be a Banach space, let $X \subseteq E'$ be $w^*$-precompact and $\sigma$-decomposable in $E'$. Then for each $g \in E' \setminus X$ there exists an $a \in E$ such that $g(a) \neq f(a)$ for all $f \in E$.

**Proof.** Without loss, assume $g = 0$. Let \{$f_n + u_n : n \in \mathbb{N}$\} be a covering of $X$ like in Definition 1.4. By Corollary 1.3 for each $n \in \mathbb{N}$ the set \{$x \in E : \inf_{f \in X} f(x) > 0$\} is open and dense in $E$, where $X_n := X \cap (f_n + u_n)$. By completeness and the Baire Category Theorem \{$x \in E : f(x) \neq 0 \text{ for all } f \in E \setminus X$\} \supset \bigcap_{n \in \mathbb{N}} \{x \in E : \inf_{f \in X_n} f(x) > 0\} \neq \emptyset.

**Remark:** It is not hard, by modifying 1.1 - 1.5, to prove the following dual form of this separation theorem. Let $K$ be not locally compact, let $X \subseteq E$ be weakly precompact and $\sigma$-decomposable in $E$ (see below). Then for each $a \in E \setminus X$ there exists an $f \in E'$ such that $f(a) \neq f(X)$. Here, $X$ is called $\sigma$-decomposable in $E$ if for each $a \in E \setminus X$ there exist $x_1, x_2, \ldots \in X$ and edged zero neighbourhoods $U_1, U_2, \ldots$ in $E$ such that each $U_n$ is weakly closed and $X \subseteq \bigcup_{n \in \mathbb{N}} (x_n + U_n)$, $a \notin \bigcup_{n \in \mathbb{N}} (x_n + U_n)$. 


COROLLARY 1.6. Let $K$ be not locally compact, let $E$ be a Banach space, let $X \subset E'$ be $\sigma$-decomposable in $E'$. Suppose $X(a) := \{f(a) : f \in X\}$ is compact for all $a \in E$. Then $X$ is $w'$-compact.

Proof. The map $f \mapsto (f(a))_{a \in E}$ is a homeomorphism of $(E', w')$ onto a subspace of $K^E$. The image of $X$ lies in the compact subset $\prod_{a \in E} X(a)$ so $X$ is $w'$-precompact. Since $E'$ is $w'$-quasicomplete by the $p$-adic Alaoglu Theorem [8, 3.1], it suffices to show that $X$ is $w'$-closed. To this end, let $g \in E' \setminus X$. By Theorem 1.5 there exists an $a \in E$ such that $g(a) \not\in X(a)$. Now $X(a) \subset X'(w') \subset X(a)$ = $X(a)$, so $g(a) \not\in X'(w')$ i.e. $g \not\in X'(w')$.

To find examples of $\sigma$-decomposable sets (in 1.9-1.11) we need the following Lemmas.

LEMMA 1.7. Let $n \in \mathbb{N}$, let $D$ be an $n$-dimensional subspace of $E'$. Then for each $t \in (0, 1)$ there exist $a_1, a_2, \ldots, a_n \in B_E$ such that $\max_{1 \leq i \leq n} |f(a_i)| > t ||f||$ ($f \in D$).

Proof. First assume that the valuation of $K$ is dense. The space $H := \{x \in E : f(x) = 0 \text{ for all } f \in D\}$ has codimension $n$ in $E$. Choose $n$ in $(t, 1)$ and let $g_1, \ldots, g_n$ be a $\sqrt{s}$-orthogonal base of $(E/H)'$ such that $s^{-1} ||g_i|| \leq t^{-1}$ for $i \in \{1, \ldots, n\}$. There exist $b_1, \ldots, b_n \in E/H$ such that $g_i(b_j) = \delta_{ij}$ ($i, j \in \{1, \ldots, n\}$). Let $s \in \{1, \ldots, n\}$, let $g = \sum \lambda_j g_j \in (E/H)'$. Then $||g|| \geq \sqrt{s} \max \lambda_j ||g_j||$ and $||g(b_i)|| = \lambda_i$ so $||g(b_i)|| \leq \max \lambda_j \leq t \max \lambda_j ||g_j|| \leq \sqrt{s}||g||$. So $b_i K$ $< 1$. Thus, with $\pi : E \to E/H$ denoting the canonical quotient map, there exist $a_1, \ldots, a_n \in B_E$ with $||a_i|| = \|b_i\|$ and $\pi(a_i) = b_i$ for each $i$. The adjoint $\pi'$ maps $(E/H)'$ isometrically onto $D$. Now let $f \in D$. Then $f = \pi'(g)$ where $g \in (E/H)'$, $||g|| = ||f||$. We have, writing $g = \sum_{j=1}^n \lambda_j g_j$, $\max_{1 \leq i \leq n} |f(a_i)| = \max |g(b_i)| \geq \max |\lambda_i| \geq t \max |\lambda_i| ||g_j|| \geq t ||\sum \lambda_j g_j|| = t ||f||$.

Now, if the valuation is discrete we can modify the above proof by taking $s = t = 1$. Then the $b_i$ have norm $\leq 1$ (rather than $< 1$), but one can use that $E/H$ is a strict quotient i.e. there exist $a_1, \ldots, a_n \in E$ with $||a_i|| = ||b_i||$ and $\pi(a_i) = b_i$ for each $i$.

LEMMA 1.8. Let $D$ be a subspace of $E'$, $D$ of countable type. Then there is a sequence $a_1, a_2, \ldots \in B_E$ such that $\max_{1 \leq i \leq n} |f(a_i)| > t ||f||$ for all $f \in D$.

Proof. Let $D_1 \subset D_2 \subset \ldots$ be finite-dimensional subspaces of $D$, $\bigcup D_n$ is dense in $D$. Let $t \in (0, 1)$. By Lemma 1.7 there exists a finite set $F_n \subset B_E$ such that $\max_{a \in F_n} |f(a)| > t ||f||$ for all $f \in D_n$.

So, for $F^1 := \bigcup_{n \in \mathbb{N}} F_n$ we obtain

(*) \[ ||f|| \geq \sup_{a \in F^1} |f(a)| \geq t ||f|| \quad (f \in \bigcup_{n \in \mathbb{N}} D_n). \]

Now $F := \bigcup_{a \in Q(0, 1)^c} F^1$ is countable and (*) implies $||f|| = \sup_{a \in F} |f(a)|$ for all $f \in \bigcup_{n \in \mathbb{N}} D_n$, hence, by continuity, for all $f \in D$.

PROPOSITION 1.9. Let $X \subset E'$ be such that $X(a) := \{f(a) : f \in X\}$ is separable for each $a \in E$ and $[X]$ is of countable type. Then $X$ is $\sigma$-decomposable in $E'$.

Proof. Let $g \in E' \setminus X$. Then $D := \{g\} \cup X$ is of countable type so by Lemma 1.8 there exist $a_1, a_2, \ldots \in B_E$ such that

(*) \[ ||h|| = \sup_{n \in \mathbb{N}} |h(a_n)| \quad (h \in D). \]

For each $m, n \in \mathbb{N}$ the set $U_{mn} := \{h \in E' : ||h(a_n)|| \leq \frac{1}{m}\}$ is an edged $w'$-zero neighbourhood. Its cosets, except for $g + U_{mn}$, cover $X \setminus (g + U_{mn})$ and by separability of $X(a_n)$ there exists a countable subcovering $F_{mn}$ no member of which contains $g$. Then $\bigcup_{m, n} F_{mn}$ still avoids $g$; it remains to be shown that it covers $X$. Suppose $f \in X$ is not covered. Then $f \in g + U_{mn}$ for all
Let $X \subseteq E'$. If $X$ is norm precompact, or $X$ is $w'$-precompact and $|X|$ is of countable type, then $X$ is $\sigma$-decomposable in $E'$.

**Proposition 1.11.** Let $X \subseteq E'$ be such that $X(a)$ is separable for each $a \in E$. Suppose that for each $h \in \overline{X}'$ the set $X \cup \{h\}$ is $w'$-metrizable. Then $X$ is $\sigma$-decomposable in $E'$.

**Proof.** Let $g \in E' \setminus X$. If $g \notin \overline{X}'$ then there exists a $w'$-zero neighbourhood $U$ such that $(g + U) \cap X = \emptyset$. We may assume that $U$ is of the form \[ \{ f \in E' : |f(a_1)| \leq \varepsilon, \ldots, |f(a_n)| \leq \varepsilon \} \] for some $\varepsilon > 0$, $n \in \mathbb{N}$, $a_1, \ldots, a_n \in E$. Then $U$ is $w'$-closed and edged. By separability of $X(a_1) \times \ldots \times X(a_n)$ only countably many of the cosets $f + U : f \in X$ cover $X$ and none of them contains $g$. Now let $g \in \overline{X}'$. By $w'$-metrizability there exist $w'$-neighbourhoods of zero $U_1 \supset U_2 \supset \ldots$ such that $X \cap \bigcap (g + U_n) = \emptyset$. We may suppose that the $U_n$ are $w'$-closed and edged. By separability, like above, for each $n$ the set $X \setminus (g + U_n)$ is covered by countably many additive cosets of $U_n$ none of them containing $g$. Their union is a countable covering of $X$ avoiding $g$.

**2. EBERLEIN-ŠMULIAN THEORY**

We now apply the theory of §1. Recall ([5], p. 57) that $E$ is said to have property (\*) if for each subspace $D$ of countable type, every $f \in D'$ has an extension $\overline{f} \in E'$. By the non-archimedean Hahn-Banach Theorem [4], 4.8 every normed space over a spherically complete $K$ has (\*). For general $K$, spaces with a base, in particular spaces of countable type, have (\*) ([5], p. 58), and so have strongly polar spaces ([6], 4.2). Recall that $E$ is assumed to be normpolar.

**Theorem 2.1.** Let $K$ be not locally compact, let $X$ be a subset of $E$ such that $f(X)$ is compact for all $f \in E'$. Then each one of the following properties implies that $X$ is weakly compact and weakly metrizable.

(i) $E$ has property (\*).

(ii) $E'$ is of countable type.

(iii) $|X|$ is of countable type.

Moreover, in case (i) $X$ is norm compact and the weak and norm topology coincide on $X$.

**Proof.** The natural isometry $j : E \rightarrow E''$ is easily seen to be a homeomorphism of $E$ with the weak topology onto $j(E)$ with the restriction of the $w'$-topology $\sigma(E'', E')$. We show that $j(X)$ is $\sigma$-decomposable in $E'$. First note that the predual $E'$ is normpolar. In case (i), from weak precompactness of $X$ it follows that $X$ is norm precompact by [7], Th. 3 (the assumption made throughout [7] that $E$ is complete is easily seen to be superfluous here). So $j(X)$ is norm precompact in $E''$ and therefore $\sigma$-decomposable by Corollary 1.10. For case (ii) observe that every $w'$-bounded subset of $E''$ is $w'$-metrizable ([8], 6.1) which applies to $j(X) \cup \{\emptyset\}$ for any $\emptyset \in E''$. For each $f \in E'$ the set $j(X)(f) = f(X)$ is compact hence separable so $j(X)$ is $\sigma$-decomposable in $E''$ by Proposition 1.11. For case (iii) we can directly apply Corollary 1.10. Thus, $j(X)$ is $\sigma$-decomposable, and from Corollary 1.6 we conclude that $j(X)$ is $w'$-compact, so $X = j^{-1}(j(X))$ is $w$-compact. Observe that $X$ is $w$-bounded hence bounded by normpolarity ([6], 7.7).

We have seen in passing that $j(X)$ is $w'$-metrizable in case (ii), so $X$ is weakly metrizable. Now let $X$ satisfy (iii). Then $j(X)$ is of countable type so by Lemma 1.8 there exist $f_1, f_2, \ldots \in B_E$ such that $\|j(x)\| = \sup_n |f_n(x)|$ for all $x \in X$. The formula $d(x, y) = \sup_n |f_n(x) - f_n(y)| 2^{-n}$ defines an ultrametric $d$ on $X$ (if $d(x, y) = 0$ then $|f_n(x) - f_n(y)| = 0$ for all $n$ so $\|x - y\| = 0$). By boundedness of $X$ the induced topology is weaker than the weak topology on $X$, but by
weak compactness these topologies coincide and so $X$ is weakly metrizable. Finally, in case (i) apply [6], 5.12 to conclude that on $X$ the weak and norm topology coincide, and that therefore $X$ is norm compact and $w$-metrizable.

**REMARKS.**

1. If $K$ is not spherically complete the space $l^\infty$ does not have property $(\star)$ ([4], 4.15 $(\delta) \Rightarrow (\gamma)$) but since $(l^\infty)' \cong c_0$ ([4], 4.17) it satisfies (ii) of the above Theorem, and so do the non-reflexive space $l^\infty \otimes l^\infty$ ([3], 2.3) and the space $D$ of [4], 4.1.

2. Let $K$ be not spherically complete, let $E := l^\infty$, let $X := \{0\} \cup \{e_1, e_2, \ldots\} \subset l^\infty$, when $e_1, e_2, \ldots$ are the unit vectors. Then (ii) and (iii) above hold. $X$ is weakly compact (since $\lim_{n \to \infty} e_n = 0$ weakly) but is obviously not norm compact.

3. The following example indicates that extending Theorem 2.1 to, say, metrizable locally convex spaces is doubtful. Let $E := K^N$ with the product topology. Then $E' \cong \bigoplus K$. Let $X := \{e_1, e_2, \ldots\}$ where $e_1, e_2, \ldots$ are the unit vectors of $K^N$. Then $E$ is of countable type so (i), (ii), (iii) of Theorem 2.1 are (formally) satisfied. For each $f \in E'$ we have $f(e_n) = 0$ for large $n$, so $f(X)$ is finite (hence compact) and contains 0. Yet, $X$ is not (weakly) compact as $0 = \lim_{n \to \infty} e_n \notin X$.

The following is now an almost trivial consequence of Theorem 2.1.

**COROLLARY 2.2.** (p-adic Eberlein-Smulian Theorem I) Let $K$ be not locally compact and let $X, E$ satisfy one of the conditions (i), (ii), (iii) of Theorem 2.1. Then the following are equivalent.

$(a)$ $X$ is weakly compact.

$(\beta)$ $X$ is weakly sequentially compact.

$(\gamma)$ $X$ is weakly countably compact.

**Proof.** Each one of the properties $(a), (\beta), (\gamma)$ implies compactness of $f(X)$ for all $f \in E'$.

By Theorem 2.1 $X$ is weakly metrizable and from that the equivalence of $(a), (\beta), (\gamma)$ follows easily.

**NOTE.** In Corollary 2.2, $(a), (\beta), (\gamma)$ are obviously equivalent to: 'for all $f \in E'$ the image $f(X)$ is compact.'

We have seen in the Introduction that Theorem 2.1 fails if $K$ is locally compact. We now investigate what happens to Corollary 2.2. Note that every normed space over $K$ has $(\star)$.

**THEOREM 2.3.** (p-adic Eberlein-Šmulian Theorem II) Let $K$ be locally compact, let $X \subset E$. Then each one of the above statements $(a), (\beta), (\gamma)$ is equivalent to ‘$X$ is norm compact’.

**Proof.** We have $(a) \Rightarrow (\gamma), (\beta) \Rightarrow (\gamma)$. It suffices to prove that $(\gamma)$ implies that $X$ is a norm compactoid (then $X$ is weakly metrizable since the norm and weak topology coincide on $X$ ([6], 5.12)). Suppose not. Then by [7], Th. 2 there is a $t \in (0, 1]$ and a $t$-orthogonal sequence $e_1, e_2, \ldots$ in $X$ such that $\inf_n \|e_n\| > 0$. By $(\gamma)$ there is a weak accumulation point $a$ of $\{e_1, e_2, \ldots\}$. This $a$ is in the weak closure $D$ of $\{e_1, e_2, \ldots\}$ which equals the norm closure, so $a = \sum_{i=1}^\infty \lambda_i e_i$ where $\|\lambda_i e_i\| \to 0$. If $\lambda_j \neq 0$ for some $j$, let $U := \{z \in E : |\delta_j(z)| < |\lambda_j|\}$ where $\delta_j \in E'$ is an extension of the $j$th coordinate function $\Sigma \xi_j e_i \mapsto \xi_j$ on $D$. Then $a + U$ is a weak neighbourhood of $a$ but for each $n \in N, n \neq j$ we have $|\delta_j(a - e_n)| = |\lambda_j| \Rightarrow e_n \notin a + U$, a contradiction. Hence, $a = 0$. But then $\{z \in E : |f(z)| < 1\}$ is a weak neighbourhood of $a$ containing no $e_n$ if $f \in E'$ is such that $f(e_n) = 1$ for all $n$. Contradiction, so $X$ is a norm compactoid.

**REMARK.** Corollary 2.2 for strongly polar spaces $E$ and Theorem 2.3 were first proved directly by the first author.
REMARK. The following 'relative' version of the Eberlein-Šmulian Theorem holds. (Compare [1], VIII §2, Theorem 1). Let \( X \subset E \). Suppose one of the conditions (i), (ii), (iii) of Theorem 2.1 is satisfied. Then the following are equivalent. (a) \( X \) is weakly relatively compact. (b) \( X \) is weakly relatively sequentially compact. (c) \( X \) is weakly relatively countably compact. We leave the easy proof to the reader.

COUNTEREXAMPLES. We show that the previous theory fails for certain subsets \( X \) of \( \ell^\infty(I) \) where \( I \) has at least the cardinality of the continuum, but is non-measurable, and where \( K \) is not spherically complete. The \( \ell^\infty \)-valued characteristic function of a subset \( S \subset I \) is denoted \( \xi_S \) and is given by \( \xi_S(x) := 1 \) if \( x \in S \), \( \xi_S(x) := 0 \) if \( x \in I \setminus S \).

1. Let \( X := \{ \xi_S : S \subset I \} \). Then \( X \) is a weakly compact but not weakly sequentially compact subset of \( \ell^\infty(I) \).

Proof. \( X \) is bounded and since \( \ell^\infty(I)' \cong \mathcal{C}_0(I) \) (cf.4.21) the weak topology on \( X \) is the topology of pointwise convergence. Clearly the map \( f \mapsto (f(i))_{i \in I} \) is a homeomorphism \( X \rightarrow \{0,1\}^I \), hence \( X \) is weakly compact. To prove that \( X \) is not weakly sequentially compact, let \( \phi : I \rightarrow Y \) be a surjection where \( Y := \{ \xi_A : A \subset N \} \subset \ell^\infty \). The formula \( \phi(x) = (\xi_{S_1}(x), \xi_{S_2}(x), \ldots) \) \((x \in I)\) defines subsets \( S_1, S_2, \ldots \) of \( I \). If \( \xi_{S_1}, \xi_{S_2}, \ldots \) is a subsequence of \( \xi_{S_1}, \xi_{S_2}, \ldots \), then, by surjectivity of \( \phi \), there is an \( x \in I \) for which \( (\xi_{S_1}(x), \xi_{S_2}(x), \ldots) = (1,0,1,0,1,\ldots) \), so the subsequence is not weakly convergent.

2. Let \( Z := \{ \xi_S : S \subset I, S \text{ countable} \} \subset \ell^\infty(I) \). Then \( Z \) is weakly sequentially compact but not weakly compact.

Proof. Clearly the weak closure of \( Z \) equals \( X \) of above, so \( Z \) is not weakly compact. On the other hand, if \( \xi_{S_1}, \xi_{S_2}, \ldots \) is a sequence in \( Z \) then \( S := \bigcup S_n \) is countable and by a standard diagonal procedure one obtains a subsequence converging at all points of \( S \), hence at all points of \( I \), to an element of \( Z \).

3. Let \( T := \{ \xi_{i_n} : i \in I \} \subset \ell^\infty(I) \). Then \( f(T) \) is compact for all \( f \in \ell^\infty(I)' \) but \( T \) is not weakly countably compact.

Proof. Let \( f \in \ell^\infty(I)' \). As \( \ell^\infty(I)' \cong \mathcal{C}_0(I) \) we have that \( f(\xi_{i_n}) = 0 \) except for \( i \in \{i_1,i_2,\ldots\} \) where we may assume the \( i_n \in I \) to be distinct. Then \( \xi_{i_n} \rightharpoonup 0 \) weakly so \( T_1 := \{0\} \cup \{\xi_{i_n} : n \in N\} \) is weakly compact and \( f(T) = f(T_1) \) is compact. However the only weak accumulation point of \( \{\xi_{i_1},\xi_{i_2},\ldots\} \) is 0 \( \not \in T \) so that \( T \) is not weakly countably compact.

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