NON-ARCHIMEDEAN EBERLEIN-ŠMULIAN THEORY

T. KIYOSAWA
Faculty of Education
Shizuoka University
Ohya, Shizuoka, 422 Japan

W.H. SCHIKHOF
Department of Mathematics
University of Nijmegen, Toernooiveld
6525 ED Nijmegen, The Netherlands

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ABSTRACT. It is shown that, for a large class of non-archimedean normed spaces $E$, a subset $X$ is weakly compact as soon as $f(X)$ is compact for all $f \in E'$ (Theorem 2.1), a fact that has no analogue in Functional Analysis over the real or complex numbers. As a Corollary we derive a non-archimedean version of the Eberlein-Šmulian Theorem (2.2 and 2.3, for the 'classical' theorem, see [1], VIII, §2 Theorem 1 and Corollary, page 219).

KEY WORDS AND PHRASES. Non-archimedean Banach space, weak compactness.

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INTRODUCTION

Let $E$ be a two-dimensional normed space over $\mathbb{R}$ or $\mathbb{C}$ and let $X := \{x \in E : 0 < ||x|| \leq 1\}$. Each $f \in E'$ has zeros on $X$, so $f(X) = f(0) \cup X$ is compact, while obviously $X$ is not. The same story can be told when we replace $\mathbb{R}$ or $\mathbb{C}$ by a complete non-trivially valued non-archimedean field $K$ that is locally compact. However, if $K$ is not locally compact then, under reasonable conditions, for a subset $X$ of a normed space $E$ over $K$ compactness of $f(X)$ for all $f \in E'$ implies weak compactness of $X$ (we point out that if such an $X$ has more than one point it cannot be convex). To prove this curious fact (in §2) we shall develop some machinery in §1.

PRELIMINARIES

Throughout $K$ is a non-trivially non-archimedean valued field which is complete with respect to the metric induced by the valuation $| |$, and $E$ is a normed $K$-vector space, where we assume $|| |$ to satisfy the strong triangle inequality $||x + y|| \leq \max(||x||, ||y||)$. We write $|K^*| := \{|\lambda| : \lambda \in K, \lambda \neq 0\}$, $B_E(0,r) := \{x \in E : ||x|| \leq r\}$, $B_E := B_E(0,1)$.

$E'$ is the space of all linear continuous functions $E \rightarrow K$. Equipped with the norm $f \mapsto \sup\{|f(x)| : x \in B_E\}$ it is a Banach space (i.e. a complete normed space). $E$ is called normpolar if the norm is polar i.e. if $||x|| = \sup\{|f(x)| : f \in E', ||f|| \leq ||\} \quad (x \in E)$, in other words, if $\gamma : E \rightarrow E''$ is an isometry. $E'$ is always normpolar. We assume throughout this note that $E$ is normpolar.

A subset $A$ of a (normed) space $E$ is absolutely convex if it is a module over $B_K$. A set $X \subset E$ is convex if it is either empty or an additive coset of an absolutely convex set. A subset $A$ of $E$ is called edged if it is absolutely convex and, in case the valuation of $K$ is dense, $A = \cap\{\lambda A : \lambda \in K, |\lambda| > 1\}$. The weak topology $w = \sigma(E, E')$ is the weakest topology on $E$ making all $f \in E'$ continuous. The weak-star topology $w' = \sigma(E', E)$ is the weakest topology
on $E'$ making all evaluation maps $f \mapsto f(a)$ ($a \in E$) continuous. For $X \subseteq E'$ we denote its
$w'$-closure by $\overline{X}_{w'}$.

For other notions used in this paper we refer to [4].

1. SEPARATION OF $w'$-PRECOMPACT SETS

**LEMMA 1.1.** Let $X$ be a bounded subset of $E'$. Then $\{x \in E : \inf_{f \in X} |f(x)| > 0\}$ is open in $E$.

**Proof.** $X$ is equicontinuous, so for each $n \in \mathbb{N}$ the set $U_n := \{x \in E : |f(x)| > \frac{1}{n} \text{ for all } f \in X\}$ is open. Then so is $\bigcup_n U_n = \{x \in E : \inf_{f \in X} |f(x)| > 0\}$.

**LEMMA 1.2.** Let $K$ be not locally compact. Let $X \subseteq E'$ and $a \in E$ be such that $X(a) := \{f(a) : f \in X\}$ is precompact. Suppose $X \subseteq g + U$ where $U$ is an edged zero neighbourhood in $E'$, $U$ $w'$-closed and where $g \in E' \setminus U$. Then for any $\varepsilon > 0$ there exists a $b \in E$ for which $\|a - b\| < \varepsilon$ and $\inf_{f(b)} |f(b)| > 0$.

**Proof.** There exists an $r \in |K|^*$ such that $B_E(0, r) \subseteq U$. Choose $\delta \in K, 0 < |\delta| < 1$. The equivalence relation $\sim$ on $K^*$ given by $\alpha \sim \beta$ iff $|\alpha - \beta| < |\beta|$ yields an open partition of $C := \{\lambda \in K : |\delta| \leq |\lambda| \leq r\}$ that is infinite because $K$ is not locally compact. By precompactness $X(a)$ cannot meet each equivalence class and there exists a $\gamma \in C$ such that

$$|f(a) - \gamma| \geq |\gamma| \quad (f \in X).$$

$U$ is $w'$-closed and edged, $g \notin U$, so by [8], 4.8 there exists a $c \in E$ such that $g(c) = \gamma, |f(c)| < |\gamma|$ for all $f \in U$. Set $b := a - c$. We have $|f(c)| \leq |\gamma|$ for all $f \in B_E(0, r)$ so $\|a - b\| = \|c\| = |b(c)| \leq |\gamma| r^{-1} < \varepsilon$. For each $f \in X$, writing $f = g + w$ where $w \in U$, we obtain $|f(c) - \gamma| = |f(c) - c| = |u(c)| < |\gamma|$. This, combined with (*), yields $|f(a) - \gamma| > |f(c) - \gamma|$ for all $f \in X$, so $|f(b)| = |f(a) - f(c)| = \max(|f(a) - \gamma|, |f(c) - \gamma|) = |f(a) - \gamma| \geq |\gamma|$. It follows that $\inf_{f \in X} |f(b)| > 0$.

**COROLLARY 1.3.** Let $K$ be not locally compact, let $E$ be a Banach space. Let $X \subseteq E'$ be $w'$-precompact. Suppose $X \subseteq g + U$ where $U$ is an edged zero neighbourhood in $E'$, $U$ $w'$-closed, $g \in E' \setminus U$. Then $\{x \in E : \inf_{f \in X} |f(x)| > 0\}$ is open and dense in $E$.

**Proof.** Just combine Lemmas 1.1 ($w'$-precompactness implies $w'$-boundedness hence norm boundedness by completeness) and 1.2.

**DEFINITION 1.4.** Let us call $X \subseteq E'$ $\sigma$-decomposable in $E'$ if for each $g \in E' \setminus X$ there exist $f_1, f_2, \ldots \in X$ and edged zero neighbourhoods $U_1, U_2, \ldots$ in $E'$ such that each $U_n$ is $w'$-closed and $X \subseteq \bigcup(f_n + U_n), g \notin \bigcup(f_n + U_n)$.

**THEOREM 1.5.** (SEPARATION THEOREM) Let $K$ be not locally compact, let $E$ be a Banach space, let $X \subseteq E'$ be $w'$-precompact and $\sigma$-decomposable in $E'$. Then for each $g \in E' \setminus X$ there exists an $a \in E$ such that $g(a) \neq f(a)$ for all $f \in X$.

**Proof.** Without loss, assume $g = 0$. Let $\{f_n + U_n : n \in \mathbb{N}\}$ be a covering of $X$ like in Definition 1.4. By Corollary 1.3 for each $n \in \mathbb{N}$ the set $\{x \in E : \inf_{f \in X} |f(x)| > 0\}$ is open and dense in $E$, where $X_n := X \cap (f_n + U_n)$. By completeness and the Baire Category Theorem $\{x \in E : f(x) \neq 0 \text{ for all } f \in X\} \cap \bigcap_n (x \in E : \inf_{f \in X} |f(x)| > 0) \neq \emptyset$.

**REMARK.** It is not hard, by modifying 1.1 - 1.5, to prove the following dual form of this separation theorem. Let $K$ be not locally compact, let $X \subseteq E$ be weakly precompact and $\sigma$-decomposable in $E$ (see below). Then for each $a \in E \setminus X$ there exists an $f \in E'$ such that $f(a) \neq f(X)$. Here, $X$ is called $\sigma$-decomposable in $E$ if for each $a \in E \setminus X$ there exist $x_1, x_2, \ldots \in X$ and edged zero neighbourhoods $U_1, U_2, \ldots$ in $E$ such that each $U_n$ is weakly closed and $X \subseteq \bigcup_n (x_n + U_n), a \notin \bigcup_n (x_n + U_n)$.
**COROLLARY 1.6.** Let \( K \) be not locally compact, let \( E \) be a Banach space, let \( X \subset E' \) be \( \sigma \)-decomposable in \( E' \). Suppose \( X(a) := \{ f(a) : f \in X \} \) is compact for all \( a \in E \). Then \( X \) is \( w' \)-compact.

**Proof.** The map \( f \mapsto \{ f(a) \}_{a \in E} \) is a homeomorphism of \( (E', w') \) onto a subspace of \( K^E \).

The image of \( X \) lies in the compact subset \( \bigcap_{a \in E} X(a) \) so \( X \) is \( w' \)-precompact. Since \( E' \) is \( w' \)-quasicomplete by the p-adic Alaoglu Theorem [8, 3.1], it suffices to show that \( X \) is \( w' \)-closed. To this end, let \( g \in E' \setminus X \). By Theorem 1.5 there exists an \( a \in E \) such that \( g(a) \notin X(a) \). Now \( X(a) \subset \overline{X}^{w'} \subset X(a) = X(a) \), so \( g(a) \notin \overline{X}^{w'}(a) \) i.e. \( g \notin \overline{X}^{w'} \).

To find examples of \( \sigma \)-decomposable sets (in 1.9-1.11) we need the following Lemmas.

**LEMMA 1.7.** Let \( n \in \mathbb{N} \), let \( D \) be an \( n \)-dimensional subspace of \( E' \). Then for each \( t \in (0, 1) \) there exist \( a_1, a_2, \ldots, a_n \in B_E \) such that \( \max_{1 \leq i \leq n} |f(a_i)| \geq t|f|| \) (\( f \in D \)).

**Proof.** First assume that the valuation of \( K \) is dense. The space \( H := \{ x \in E : f(x) = 0 \text{ for all } f \in D \} \) has codimension \( n \) in \( E \). Choose \( s \in (t, 1) \) and let \( g_1, \ldots, g_n \) be a \( \sqrt{s} \)-orthogonal basis of \( (E/H)' \) such that \( s^{-1} \leq \|g_i\| \leq t^{-1} \) for \( i \in \{1, \ldots, n\} \). There exist \( b_1, \ldots, b_n \in E/H \) such that \( g_i(b_j) = \delta_{ij} \) for \( i, j \in \{1, \ldots, n\} \). Let \( s \in \{1, \ldots, n\} \), let \( g = \Sigma \lambda_i g_i \in (E/H)' \). Then \( \|g\| \geq \sqrt{s} \max_j |\lambda_j| \|g_j\| \) and \( \|g(b_i)\| = |\lambda_i| \) so \( \|g(b_i)\| \leq \max_j |\lambda_j| \leq s \max_j |\lambda_j| \|g\| \leq \sqrt{s} \|g\| \).

So \( \|g\| < 1 \). Thus, with \( \pi : E \rightarrow E/H \) denoting the canonical quotient map, there exist \( a_1, \ldots, a_n \in B_E \) with \( \pi(a_i) = b_i \) for each \( i \). The adjoint \( \pi' \) maps \( (E/H)' \) isometrically onto \( D \). Now let \( f \in D \). Then \( f = \pi'(g) \) where \( g \in (E/H)' \), \( \|g\| = \|f\| \). We have, writing \( g = \Sigma \lambda_i g_i \), \( \max_{1 \leq i \leq n} |f(a_i)| = \max_j |\lambda_i| \|g(b_i)\| = \max_j |\lambda_i| \geq t \max_j |\lambda_j| \|g\| \geq t \|\Sigma \lambda_i g_i\| = t \|g\| = t \|f\| \).

Now, if the valuation is discrete we can modify the above proof by taking \( s = t = 1 \). Then the \( b_i \) have norm \( \leq 1 \) (rather than \( < 1 \)), but one can use that \( E/H \) is a strict quotient i.e. there exist \( a_1, \ldots, a_n \in E \) with \( \|a_i\| = \|b_i\| \) and \( \pi(a_i) = b_i \) for each \( i \).

**LEMMA 1.8.** Let \( D \) be a subspace of \( E' \), \( D \) of countable type. Then there is a sequence \( a_1, a_2, \ldots, a_n \in B_E \) such that \( \max_{1 \leq i \leq n} |f(a_i)| \geq t|f|| \) for all \( f \in D \).

**Proof.** Let \( D_1 \subset D_2 \subset \ldots \) be finite-dimensional subspaces of \( D \), \( \bigcup_{n \in \mathbb{N}} D_n \) is dense in \( D \). Let \( t \in (0, 1) \). By Lemma 1.7 there exists a finite set \( F_n^* \subset B_E \) such that \( \max_{a \in F_n^*} |f(a)| \geq t|f|| \) for all \( f \in D_n \).

So, for \( F^* := \bigcup_{n \in \mathbb{N}} F_n^* \) we obtain

\[
(*) \quad \|f\| \geq \sup_{a \in F^*} |f(a)| \geq t\|f\| \quad (f \in \bigcup_{n \in \mathbb{N}} D_n).
\]

Now \( F := \bigcup_{n \in \mathbb{Q} \cap (0, 1)} F_n^* \) is countable and \((*)\) implies \( \|f\| = \sup_{a \in F} |f(a)| \) for all \( f \in \bigcup_{n \in \mathbb{N}} D_n \), hence, by continuity, for all \( f \in D \).

**PROPOSITION 1.9.** Let \( X \subset E' \) be such that \( X(a) := \{ f(a) : f \in X \} \) is separable for each \( a \in E \) and \( |X| \) is of countable type. Then \( X \) is \( \sigma \)-decomposable in \( E' \).

**Proof.** Let \( g \in E' \setminus X \). Then \( D := \{g\} \cup X \) is of countable type so by Lemma 1.8 there exist \( a_1, a_2, \ldots \in B_E \) such that

\[
(*) \quad \|h\| = \sup_{a \in \mathbb{N}} |h(a)| \quad (h \in D).
\]

For each \( m, n \in \mathbb{N} \) the set \( U_{mn} := \{ h \in E' : |h(a_n)| \leq \frac{1}{m} \} \) is an edged \( w' \)-zero neighbourhood. Its cosets, except for \( g + U_{mn} \), cover \( X \setminus (g + U_{mn}) \) and by separability of \( X(a_n) \) there exists a countable subcovering \( F_m \) no member of which contains \( g \). Then \( \bigcup_{m, n} F_{mn} \) still avoids \( g \); it remains to be shown that it covers \( X \). Suppose \( f \in X \) is not covered. Then \( f \in g + U_{mn} \) for all
weak precompactness of \( j(X) \) is \( \sigma \)-decomposable in \( X = X' \) is \( \sigma \)-bounded hence bounded by norm-polarity; a contradiction since \( g \not\in X \).

**COROLLARY 1.10.** Let \( X \subset E' \). If \( X \) is norm precompact, or \( X \) is \( w' \)-precompact and \( |X| \) is of countable type, then \( X \) is \( \sigma \)-decomposable in \( E' \).

**PROPOSITION 1.11.** Let \( X \subset E' \) be such that \( X(a) \) is separable for each \( a \in E \). Suppose that for each \( h \in \overline{X}^{w'} \) the set \( X \cup \{ h \} \) is \( w' \)-metrizable. Then \( X \) is \( \sigma \)-decomposable in \( E' \).

**Proof.** Let \( g \in E' \setminus X \). If \( g \not\in \overline{X}^{w'} \) then there exists a \( w' \)-zero neighbourhood \( U \) such that \( (g + U) \cap X = \emptyset \). We may assume that \( U \) is of the form \( \{ f \in E' : |f(a_1)| \leq \varepsilon, \ldots , |f(a_n)| \leq \varepsilon \} \) for some \( \varepsilon > 0 \), \( n \in \mathbb{N} \), \( a_1, \ldots , a_n \in E \). Then \( U \) is \( w' \)-closed and edged. By separability of \( X(a_1) \times \ldots \times X(a_n) \) only countably many of the cosets \( f + U : f \in X \) cover \( X \) and none of them contains \( g \). Now let \( g \in \overline{X}^{w'} \). By \( w' \)-metrizability there exist \( w' \)-neighbourhoods of zero \( U_1 \supset U_2 \supset \ldots \) such that \( X \cap (g + U_1) = \emptyset \). We may suppose that the \( U_n \) are \( w' \)-closed and edged. By separability, like above, for each \( n \) the set \( X \setminus (g + U_n) \) is covered by countably many additive cosets of \( U_n \) none of them containing \( g \). Their union is a countable covering of \( X \) avoiding \( g \).

**2. EBERLEIN-ŠMULIAN THEORY**

We now apply the theory of §1. Recall ([5], p. 57) that \( E \) is said to have property (\( \ast \)) if for each subspace \( D \) of countable type, every \( f \in D' \) has an extension \( \overline{f} \in E' \). By the non-archimedean Hahn-Banach Theorem [4], 4.8 every normed space over a spherically complete \( K \) has (\( \ast \)). For general \( K \), spaces with a base, in particular spaces of countable type, have (\( \ast \)) ([5], p. 58), and so have strongly polar spaces ([6], 4.2). Recall that \( E \) is assumed to be normpolar.

**THEOREM 2.1.** Let \( K \) be not locally compact, let \( X \) be a subset of \( E \) such that \( f(X) \) is compact for all \( f \in E' \). Then each one of the following properties implies that \( X \) is weakly compact and weakly metrizable.

(i) \( E \) has property (\( \ast \)).

(ii) \( E' \) is of countable type.

(iii) \( [X] \) is of countable type.

Moreover, in case (i) \( X \) is norm compact and the weak and norm topology coincide on \( X \).

**Proof.** The natural isometry \( j : E \rightarrow E'' \) is easily seen to be a homeomorphism of \( E \) with the weak topology onto \( j(E) \) with the restriction of the \( w' \)-topology \( \sigma(E'', E') \). We show that \( j(X) \) is \( \sigma \)-decomposable in \( E'' \). First note that the predual \( E' \) is normpolar. In case (i), from weak precompactness of \( X \) it follows that \( X \) is norm precompact by [7], Th. 3 (the assumption made throughout [7] that \( E \) is complete is easily seen to be superfluous here). So \( j(X) \) is norm precompact in \( E'' \) and therefore \( \sigma \)-decomposable by Corollary 1.10. For case (ii) observe that every \( (w' \prime) \) bounded subset of \( E'' \) is \( w' \)-metrizable ([8], 6.1) which applies to \( j(X) \cup \{ \emptyset \} \) for any \( \emptyset \in E'' \). For each \( f \in E' \) the set \( j(X)(f) = f(X) \) is compact hence separable so \( j(X) \) is \( \sigma \)-decomposable in \( E'' \) by Proposition 1.11. For case (iii) we can directly apply Corollary 1.10. Thus, \( j(X) \) is \( \sigma \)-decomposable, and from Corollary 1.6 we conclude that \( j(X) \) is \( w' \)-compact, so \( X = j^{-1}(j(X)) \) is \( w' \)-compact. Observe that \( X \) is \( w \)-bounded hence bounded by normpolarity ([6], 7.7).

We have seen in passing that \( j(X) \) is \( w' \)-metrizable in case (ii), so \( X \) is weakly metrizable. Now let \( X \) satisfy (iii). Then \( j(X) \) is of countable type so by Lemma 1.8 there exist \( f_1, f_3, \ldots \in B_E \) such that \( \|j(x)\| = \sup_{n \in \mathbb{N}} |f_n(x)| \) for all \( x \in X \). The formula \( d(x, y) = \sup_{n \in \mathbb{N}} |f_n(x) - f_n(y)| 2^{-n} \) defines an ultrametric \( d \) on \( X \) (if \( d(x, y) = 0 \) then \( |f_n(x) - f_n(y)| = 0 \) for all \( n \) so \( \|x - y\| = 0 \)). By boundedness of \( X \) the induced topology is weaker than the weak topology on \( X \), but by
weak compactness these topologies coincide and so \( X \) is weakly metrizable. Finally, in case (i) apply [6], 5.12 to conclude that on \( X \) the weak and norm topology coincide, and that therefore \( X \) is norm compact and \( w \)-metrizable.

**REMARKS.**

1. If \( K \) is not spherically complete the space \( \ell^\infty \) does not have property \((\ast)\) ([4], 4.15 \((\delta) \Rightarrow (\gamma)\)) but since \( (\ell^\infty)' \simeq c_0 \) ([4], 4.17) it satisfies (ii) of the above Theorem, and so do the non-reflexive space \( \ell^\infty \otimes \ell^\infty \) ([3], 2.3) and the space \( D \) of [4], 4.1.

2. Let \( K \) be not spherically complete, let \( E := \ell^\infty \), let \( X := \{0\} \cup \{e_1, e_2, \ldots\} \subset \ell^\infty \), when \( e_1, e_2, \ldots \) are the unit vectors. Then (ii) and (iii) above hold. \( X \) is weakly compact (since \( \lim_{n \to \infty} e_n = 0 \) weakly) but is obviously not norm compact.

3. The following example indicates that extending Theorem 2.1 to, say, metrizable locally convex spaces is doubtful. Let \( E := K^N \) with the product topology. Then \( E' \cong \bigoplus_k K \). Let \( X := \{e_1, e_2, \ldots\} \) where \( e_1, e_2, \ldots \) are the unit vectors of \( K^N \). Then \( E \) is of countable type so (i), (ii), (iii) of Theorem 2.1 are (formally) satisfied. For each \( f \in E' \) we have \( f(e_n) = 0 \) for large \( n \), so \( f(X) \) is finite (hence compact) and contains \( 0 \). Yet, \( X \) is not (weakly) compact as \( 0 = w - \lim_{n \to \infty} e_n \not\in X \).

The following is now an almost trivial consequence of Theorem 2.1.

**COROLLARY 2.2.** (\( p \)-adic Eberlein-Smulian Theorem I) Let \( K \) be not locally compact and let \( X, E \) satisfy one of the conditions (i), (ii), (iii) of Theorem 2.1. Then the following are equivalent.

- \((a)\) \( X \) is weakly compact.
- \((\beta)\) \( X \) is weakly sequentially compact.
- \((\gamma)\) \( X \) is weakly countably compact.

**Proof.** Each one of the properties \((a), (\beta), (\gamma)\) implies compactness of \( f(X) \) for all \( f \in E' \). By Theorem 2.1 \( X \) is weakly metrizable and from that the equivalence of \((a), (\beta), (\gamma)\) follows easily.

**NOTE.** In Corollary 2.2, \((a), (\beta), (\gamma)\) are obviously equivalent to: ‘for all \( f \in E' \) the image \( f(X) \) is compact.’

We have seen in the Introduction that Theorem 2.1 fails if \( K \) is locally compact. We now investigate what happens to Corollary 2.2. Note that every normed space over \( K \) has \((\ast)\).

**THEOREM 2.3.** (\( p \)-adic Eberlein-Šmulian Theorem II) Let \( K \) be locally compact, let \( X \subset E \). Then each one of the above statements \((a), (\beta), (\gamma)\) is equivalent to ‘\( X \) is norm compact’.

**Proof.** We have \((a) \Rightarrow (\gamma), (\beta) \Rightarrow (\gamma)\). It suffices to prove that \((\gamma)\) implies that \( X \) is a norm compactoid (then \( X \) is weakly metrizable since the norm and weak topology coincide on \( X \) ([6], 5.12]). Suppose not. Then by [7], Th. 2 there is a \( t \in (0,1] \) and a \( t \)-orthogonal sequence \( e_1, e_2, \ldots \) in \( X \) such that \( \inf_n ||e_n|| > 0 \). By \((\gamma)\) there is a weak accumulation point \( a \) of \( \{e_1, e_2, \ldots\} \). This \( a \) is in the weak closure \( D \) of \( \{e_1, e_2, \ldots\} \) which equals the norm closure, so \( a = \sum_{i=1}^\infty \lambda_i e_i \) where \( ||\lambda_i e_i|| \to 0 \). If \( \lambda_j \neq 0 \) for some \( j \), let \( U := \{z \in E : |\delta_j(z)| < |\lambda_j|\} \) where \( \delta_j \in E' \) is an extension of the \( j \)th coordinate function \( \Sigma \xi_k e_k \mapsto \xi_j \) on \( D \). Then \( a + U \) is a weak neighbourhood of \( a \) but for each \( n \in \mathbb{N} \), \( n \neq j \) we have \( |\delta_j(a - e_n)| = |\lambda_j| \), so \( e_n \not\in a + U \), a contradiction. Hence, \( a = 0 \). But then \( \{z \in E : |f(z)| < 1\} \) is a weak neighbourhood of \( a \) containing no \( e_n \) if \( f \in E' \) is such that \( f(e_n) = 1 \) for all \( n \). Contradiction, so \( X \) is a norm compactoid.

**REMARK.** Corollary 2.2 for strongly polar spaces \( E \) and Theorem 2.3 were first proved directly by the first author.
REMARK. The following 'relative' version of the Eberlein-Šmulian Theorem holds. (Compare [1], VIII §2, Theorem 1). Let $X \subset E$. Suppose one of the conditions (i), (ii), (iii) of Theorem 2.1 is satisfied. Then the following are equivalent. (a) $X$ is weakly relatively compact. (b) $X$ is weakly relatively sequentially compact. (c) $X$ is weakly relatively countably compact. We leave the easy proof to the reader.

COUNTEREXAMPLES. We show that the previous theory fails for certain subsets $X$ of $\ell^\infty(I)$ where $I$ has at least the cardinality of the continuum, but is non-measurable, and where $K$ is not spherically complete. The $K$-valued characteristic function of a subset $S \subset I$ is denoted $\xi_S$ and is given by $\xi_S(x) := 1$ if $x \in S$, $\xi_S(x) := 0$ if $x \in I \setminus S$.

1. Let $X := \{\xi_S : S \subset I\}$. Then $X$ is a weakly compact but not weakly sequentially compact subset of $\ell^\infty(I)$.

Proof. $X$ is bounded and since $\ell^\infty(I)' \approx c_0(I)$ ([4], 4.21) the weak topology on $X$ is the topology of pointwise convergence. Clearly the map $f \mapsto (f(i))_{i \in I}$ is a homeomorphism $X \to \{0,1\}^I$, hence $X$ is weakly compact. To prove that $X$ is not weakly sequentially compact, let $\phi : I \to Y$ be a surjection where $Y := \{\xi_A : A \subset N\} \subset \ell^\infty$. The formula $\phi(x) = (\xi_{S_1}(x), \xi_{S_2}(x), \ldots)$ ($x \in I$) defines subsets $S_1, S_2, \ldots$ of $I$. If $\xi_{S_1}, \xi_{S_2}, \ldots$ is a subsequence of $\xi_{S_1}, \xi_{S_2}, \ldots$ then, by surjectivity of $\phi$, there is an $x \in I$ for which $(\xi_{S_1}(x), \xi_{S_2}(x), \ldots) = (1, 0, 1, 0, 1, \ldots)$, so the subsequence is not weakly convergent.

2. Let $Z := \{\xi_S : S \subset I, S \text{ countable}\} \subset \ell^\infty(I)$. Then $Z$ is weakly sequentially compact but not weakly compact.

Proof. Clearly the weak closure of $Z$ equals $X$ of above, so $Z$ is not weakly compact. On the other hand, if $\xi_{S_1}, \xi_{S_2}, \ldots$ is a sequence in $Z$ then $S := \cup S_n$ is countable and by a standard diagonal procedure one obtains a subsequence converging at all points of $S$, hence at all points of $I$, to an element of $Z$.

3. Let $T := \{\xi_{i_n} : i \in I\} \subset \ell^\infty(I)$. Then $f(T)$ is compact for all $f \in \ell^\infty(I)'$ but $T$ is not weakly countably compact.

Proof. Let $f \in \ell^\infty(I)'$. As $\ell^\infty(I)' \approx c_0(I)$ we have that $f(\xi_{i_n}) = 0$ except for $i \in \{i_1, i_2, \ldots\}$ where we may assume the $i_n \in I$ to be distinct. Then $\xi_{i_n} \rightharpoonup 0$ weakly so $T_* := \{0\} \cup \{\xi_{i_n} : n \in N\}$ is weakly compact and $f(T) = f(T_*)$ is compact. However the only weak accumulation point of $\{\xi_{i_1}, \xi_{i_2}, \ldots\}$ is $0 \notin T$ so that $T$ is not weakly countably compact.

REFERENCES