NON-ARCHIMEDEAN EBERLEIN-ŠMULIAN THEORY

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ABSTRACT. It is shown that, for a large class of non-archimedean normed spaces $E$, a subset $X$ is weakly compact as soon as $f(X)$ is compact for all $f \in E'$ (Theorem 2.1), a fact that has no analogue in Functional Analysis over the real or complex numbers. As a Corollary we derive a non-archimedean version of the Eberlein-Šmulian Theorem (2.2 and 2.3, for the 'classical' theorem, see [1], VIII, §2 Theorem 1 and Corollary, page 219).

KEY WORDS AND PHRASES. Non-archimedean Banach space, weak compactness.

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INTRODUCTION

Let $E$ be a two-dimensional normed space over $\mathbb{R}$ or $\mathbb{C}$ and let $X := \{x \in E : 0 < \|x\| \leq 1\}$. Each $f \in E'$ has zeros on $X$, so $f(X) = f(\{0\} \cup X)$ is compact, while obviously $X$ is not. The same story can be told when we replace $\mathbb{R}$ or $\mathbb{C}$ by a complete non-trivially valued non-archimedean field $K$ that is locally compact. However, if $K$ is not locally compact then, under reasonable conditions, for a subset $X$ of a normed space $E$ over $K$ compactness of $f(X)$ for all $f \in E'$ implies weak compactness of $X$ (we point out that if such an $X$ has more than one point it cannot be convex). To prove this curious fact (in §2) we shall develop some machinery in §1.

PRELIMINARIES

Throughout $K$ is a non-trivially non-archimedean valued field which is complete with respect to the metric induced by the valuation $|\ |$, and $E$ is a normed $K$-vector space, where we assume $\|\|$ to satisfy the strong triangle inequality $\|x + y\| \leq \max(\|x\|, \|y\|)$. We write $|K^\times| := \{\lambda : \lambda \in K, \lambda \neq 0\}$, $B_E(0, r) := \{x \in E : \|x\| \leq r\}$, $B_E := B_E(0, 1)$.

$E'$ is the space of all linear continuous functions $E \to K$. Equipped with the norm $f \mapsto \sup\{|f(x)| : x \in B_E\}$ it is a Banach space (i.e. a complete normed space). $E$ is called normpolar if the norm is polar i.e. if $\|\| = \sup\{|f(x)| : f \in E', |f| \leq \|\|\}$ (in other words, if $f : E \to E''$ is an isometry). $E'$ is always normpolar. We assume throughout this note that $E$ is normpolar.

A subset $A$ of a (normed) space $E$ is absolutely convex if it is a module over $B_K$. A set $X \subset E$ is convex if it is either empty or an additive coset of an absolutely convex set. A subset $A$ of $E$ is called edged if it is absolutely convex and, in case the valuation of $K$ is dense, $A = \bigcap\{\lambda A : \lambda \in K, |\lambda| > 1\}$. The weak topology $w = \sigma(E, E')$ is the weakest topology on $E$ making all $f \in E'$ continuous. The weak-star topology $w' = \sigma(E', E)$ is the weakest topology.
on $E'$ making all evaluation maps $f \mapsto f(a)$ $(a \in E)$ continuous. For $X \subseteq E'$ we denote its $w'$-closure by $\bar{X}^{w'}$.

For other notions used in this paper we refer to [4].

1. SEPARATION OF $w'$-PRECOMPACT SETS

**Definition 1.4.** Let us call $X$ $\sigma$-decomposable in $E'$ then for each $x \in X$ such that $f(x)^{-1}(y) \neq \varnothing$ for all $y \in E'$ there exists an $a \in E$ such that $f(a) \in f(x)^{-1}(y)$.

**Theorem 1.5.** (SEPARATION THEOREM) Let $K$ be not locally compact, let $E$ be a Banach space, let $X \subseteq E'$ be $w'$-precompact and $\sigma$-decomposable in $E'$. Then for each $x \in X$ there exists an $a \in E$ such that $f(a) \notin f(x)$ for all $f \in E'$ such that $f(x)^{-1}(y) \neq \varnothing$ for all $y \in E'$.

**Proof.** Without loss, assume $g = 0$. Let $\{f_n + U_n : n \in \mathbb{N}\}$ be a covering of $X$ like in Definition 1.4. By Corollary 1.3 for each $n \in \mathbb{N}$ the set $\{x \in E : \inf_{f \in E} |f(x)| > 0\}$ is open and dense in $E$, where $X_n := X \cap (f_n + U_n)$. By completeness and the Baire Category Theorem $\{x \in E : f(x) \neq 0\}$ is open and dense in $E$.
COROLLARY 1.6. Let $K$ be not locally compact, let $E$ be a Banach space, let $X \subset E'$ be $\sigma$-decomposable in $E'$. Suppose $X(a) := \{f(a) : f \in X\}$ is compact for all $a \in E$. Then $X$ is $w'$-compact.

Proof. The map $f \mapsto \{f(a)\}_{a \in E}$ is a homeomorphism of $(E', w')$ onto a subspace of $K^E$. The image of $X$ lies in the compact subset $\prod_{a \in E} X(a)$ so $X$ is $w'$-precompact. Since $E'$ is $w'$-quasicomplete by the p-adic Alaoglu Theorem [8], 3.1, it suffices to show that $X$ is $w'$-closed. To this end, let $g \in E' \setminus X$. By Theorem 1.5 there exists an $a \in E$ such that $g(a) \notin X(a)$. Now $X(a) \subset \overline{X}'(a) \subset \overline{X(a)} = X(a)$, so $g(a) \notin \overline{X}'(a)$ i.e. $g \notin \overline{X}'$.

To find examples of $\sigma$-decomposable sets (in 1.9-1.11) we need the following Lemmas.

LEMMA 1.7. Let $n \in \mathbb{N}$, let $D$ be an $n$-dimensional subspace of $E'$. Then for each $t \in (0, 1)$ there exist $a_1, a_2, \ldots, a_n \in B_E$ such that $\max_{1 \leq i \leq n} |f(a_i)| \geq t ||f||$ $(f \in D)$.

Proof. First assume that the valuation of $K$ is dense. The space $H := \{x \in E : f(x) = 0$ for all $f \in D\}$ has codimension $n$ in $E$. Choose $s \in (t, 1)$ and let $g_1, \ldots, g_n$ be a $\sqrt{s}$-orthogonal base of $(E/H)'$ such that $s^{-1} \leq ||g_i|| \leq t^{-1}$ for $i \in \{1, \ldots, n\}$. There exist $b_1, \ldots, b_n \in E/H$ such that $g_i(b_i) = \delta_{i0}$, $(i, j \in \{1, \ldots, n\})$. Let $x \in \{1, \ldots, n\}$, let $g = \sum \lambda_i g_i \in (E/H)'$. Then $||g|| \geq \sqrt{s} \max_i ||\lambda_i|| ||g_i||$ and $||g(b_i)|| = ||\lambda_i||$ so $||g(b_i)|| \leq \max_i ||\lambda_i|| \leq s \max_i ||\lambda_i|| ||g_i|| \leq \sqrt{s} ||g||$. So $||b_i|| \leq 1$. Thus, with $\pi : E \rightarrow E/H$ denoting the canonical quotient map, there exist $a_1, \ldots, a_n \in B_E$ with $\pi(a_i) = b_i$ for each $i$. The adjoint $\pi'$ of $\pi$ maps $(E/H)'$ isometrically onto $D$. Now let $f \in D$. Then $f = \pi'(g)$ where $g \in (E/H)'$, $||g|| = ||f||$. We have, writing $g = \sum \lambda_i g_i$, $\max_{1 \leq i \leq n} |f(a_i)| = \max_{1 \leq i \leq n} |g(b_i)| = \max_i ||\lambda_i|| \geq \max_i ||\lambda_i|| ||g_i|| \geq t ||\Sigma \lambda_i g_i|| = t ||g|| = t ||f||$.

Now, if the valuation is discrete we can modify the above proof by taking $s = t = 1$. Then the $b_i$ have norm $\leq 1$ (rather than $< 1$), but one can use that $E/H$ is a strict quotient i.e. there exist $a_1, \ldots, a_n \in E$ with $||a_i|| = ||b_i||$ and $\pi(a_i) = b_i$ for each $i$.

LEMMA 1.8. Let $D$ be a subspace of $E'$, $D$ of countable type. Then there is a sequence $a_1, a_2, \ldots, a_n \in B_E$ such that $\max_{1 \leq i \leq n} |f(a_i)| \geq t ||f||$ for all $f \in D$.

Proof. Let $D_1 \subset D_2 \subset \ldots$ be finite-dimensional subspaces of $D$, $\bigcup D_n$ is dense in $D$. Let $t \in (0, 1)$. By Lemma 1.7 there exists a finite set $F_n' \subset B_E$ such that $\max_{a \in F_n'} |f(a)| \geq t ||f||$ for all $f \in D_n$. So, for $F^1 := \bigcup_{n \in \mathbb{N}} F_n'$ we obtain

(*) $||f|| \geq \sup_{a \in F^1} |f(a)| \geq t ||f||$ $(f \in \bigcup_{n \in \mathbb{N}} D_n)$.

Now $F := \bigcup_{n \in \mathbb{Q} \cap (0, 1)} F^1$ is countable and (*) implies $||f|| = \sup_{a \in F} |f(a)|$ for all $f \in \bigcup_{n \in \mathbb{N}} D_n$, hence, by continuity, for all $f \in D$.

PROPOSITION 1.9. Let $X \subset E'$ be such that $X(a) := \{f(a) : f \in X\}$ is separable for each $a \in E$ and $[X]$ is of countable type. Then $X$ is $\sigma$-decomposable in $E'$.

Proof. Let $g \in E' \setminus X$. Then $D := \{g\} \cup X$ is of countable type so by Lemma 1.8 there exist $a_1, a_2, \ldots, a_n \in B_E$ such that

(*) $||h|| = \sup_{a \in \mathbb{N}} |h(a_n)|$ $(h \in D)$.

For each $m, n \in \mathbb{N}$ the set $U_{mn} := \{h \in E' : ||h(a_n)|| \leq \frac{1}{m}\}$ is an edged $w'$-zero neighbourhood. Its cosets, except for $g + U_{mn}$, cover $X \setminus (g + U_{mn})$ and by separability of $X(a_n)$ there exists a countable subcovering $F_{mn}$ no member of which contains $g$. Then $\bigcup_{m, n} F_{mn}$ still avoids $g$; it remains to be shown that it covers $X$. Suppose $f \in X$ is not covered. Then $f \in g + U_{mn}$ for all
Thus, every (u/-) bounded subset of \( X \) is \( \sigma \)-decomposable. Observe that \( X \) is \( \sigma \)-bounded hence bounded by norm polarity. We may suppose that the \( u/- \)-neighbourhoods of zero \( U_1 \cup U_2 \cup \ldots \) such that \( X \cap \bigcap (g + U_n) = \emptyset \). We may assume that the \( U_n \) are \( u/- \)-closed and edged. By separability, like above, for each \( n \) the set \( X \setminus (g + U_n) \) is covered by countably many additive cosets of \( U_n \) none of them containing \( g \). Their union is a countable covering of \( X \) avoiding \( g \).

2. EBERLEIN-ŠMULIAN THEORY

We now apply the theory of §1. Recall ([5], p. 57) that \( E \) is said to have property (\( \ast \)) if for each subspace \( D \) of countable type, every \( f \in D' \) has an extension \( \overline{f} \in E' \). By the non-archimedean Hahn-Banach Theorem [4], 4.8 every normed space over a spherically complete \( K \) has (\( \ast \)). For general \( K \), spaces with a base, in particular spaces of countable type, have (\( \ast \)) ([5], p. 58), and so have strongly polar spaces ([6], 4.2). Recall that \( E \) is assumed to be normpolar.

**Theorem 2.1.** Let \( K \) be not locally compact, let \( X \) be a subset of \( E \) such that \( f(X) \) is compact for all \( f \in E' \). Then each one of the following properties implies that \( X \) is weakly compact and weakly metrizable.

(i) \( E \) has property (\( \ast \)).

(ii) \( E' \) is of countable type.

(iii) \( [X] \) is of countable type.

Moreover, in case (i) \( X \) is norm compact and the weak and norm topology coincide on \( X \).

**Proof.** The natural isometry \( j : E \rightarrow E'' \) is easily seen to be a homeomorphism of \( E \) with the weak topology onto \( j(E) \) with the restriction of the \( w' \)-topology \( \sigma(E'', E') \). We show that \( j(X) \) is \( \sigma \)-decomposable in \( E'' \). First note that the predual \( E' \) is normpolar. In case (i), from weak precompactness of \( X \) it follows that \( X \) is norm precompact by [7], Th. 3 (the assumption made throughout [7] that \( E \) is complete is easily seen to be superfluous here). So \( j(X) \) is norm precompact in \( E'' \) and therefore \( \sigma \)-decomposable by Corollary 1.10. For case (ii) observe that every \( (w'='-) \) bounded subset of \( E'' \) is \( w' \)-metrizable ([8], 6.1) which applies to \( j(X) \cup \{ \emptyset \} \) for any \( \emptyset \in E'' \). For each \( f \in E' \) the set \( j(X)(f) = f(X) \) is compact hence separable so \( j(X) \) is \( \sigma \)-decomposable in \( E'' \) by Proposition 1.11. For case (iii) we can directly apply Corollary 1.10. Thus, \( j(X) \) is \( \sigma \)-decomposable, and from Corollary 1.6 we conclude that \( j(X) \) is \( w' \)-compact, so \( X = j^{-1}(j(X)) \) is \( w' \)-compact. Observe that \( X \) is \( w' \)-bounded hence bounded by norm polarity ([6], 7.7).

We have seen in passing that \( j(X) \) is \( w' \)-metrizable in case (ii), so \( X \) is weakly metrizable. Now let \( X \) satisfy (iii). Then \( j(X) \) is of countable type so by Lemma 1.8 there exist \( f_1, f_2, \ldots \in B_{E'} \) such that \( \|j(z)\| = \sup_{n \in \mathbb{N}} |f_n(z)| \) for all \( z \in X \). The formula \( d(x, y) = \sup_{n \in \mathbb{N}} |f_n(x) - f_n(y)|2^{-n} \) defines an ultrametric \( d \) on \( X \) if \( d(x, y) = 0 \) then \( |f_n(x) - f_n(y)| = 0 \) for all \( n \) so \( \|x - y\| = 0 \). By boundedness of \( X \) the induced topology is weaker than the weak topology on \( X \), but by
weak compactness these topologies coincide and so \( X \) is weakly metrizable. Finally, in case (i) apply [6], 5.12 to conclude that on \( X \) the weak and norm topology coincide, and that therefore \( X \) is norm compact and \( w \)-metrizable.

**REMARKS.**

1. If \( K \) is not spherically complete the space \( \ell^\infty \) does not have property (*). ([4], 4.15 (\( \delta \)) \( \Rightarrow \) (\( \gamma \))) but since \( (\ell^\infty)' \cong c_0 \) ([4], 4.17) it satisfies (ii) of the above Theorem, and so do the non-reflexive space \( \ell^\infty \mathbb{S} \) ([3], 2.3) and the space \( D \) of [4], 4.1.

2. Let \( K \) be not spherically complete, let \( E := \ell^\infty \), let \( X := \{0\} \cup \{e_1, e_2, \ldots\} \subset \ell^\infty \), when \( e_1, e_2, \ldots \) are the unit vectors. Then (ii) and (iii) above hold. \( X \) is weakly compact (since \( \lim_{n \to \infty} e_n = 0 \) weakly) but is obviously not norm compact.

3. The following example indicates that extending Theorem 2.1 to, say, metrizable locally convex spaces is doubtful. Let \( E := \mathbb{N} \) with the product topology. Then \( E' \cong \bigoplus K \). Let \( X := \{e_1, e_2, \ldots\} \) where \( e_1, e_2, \ldots \) are the unit vectors of \( K^\mathbb{N} \). Then \( E \) is of countable type so (i), (ii), (iii) of Theorem 2.1 are (formally) satisfied. For each \( f \in E' \) we have \( f(e_n) = 0 \) for large \( n \), so \( f(X) \) is finite (hence compact) and contains 0. Yet, \( X \) is not (weakly) compact as \( 0 = w - \lim_{n \to \infty} e_n \in X \).

The following is now an almost trivial consequence of Theorem 2.1.

**COROLLARY 2.2.** \((p\text{-adic Eberlein-Smulian Theorem I)}\) Let \( K \) be not locally compact and let \( X, E \) satisfy one of the conditions (i), (ii), (iii) of Theorem 2.1. Then the following are equivalent.

\( \alpha \) \( X \) is weakly compact.
\( \beta \) \( X \) is weakly sequentially compact.
\( \gamma \) \( X \) is weakly countably compact.

**Proof.** Each one of the properties (\( \alpha \)), (\( \beta \)), (\( \gamma \)) implies compactness of \( f(X) \) for all \( f \in E' \). By Theorem 2.1 \( X \) is weakly metrizable and from that the equivalence of (\( \alpha \)), (\( \beta \)), (\( \gamma \)) follows easily.

**NOTE.** In Corollary 2.2, (\( \alpha \)), (\( \beta \)), (\( \gamma \)) are obviously equivalent to: ‘for all \( f \in E' \) the image \( f(X) \) is compact.’

We have seen in the Introduction that Theorem 2.1 fails if \( K \) is locally compact. We now investigate what happens to Corollary 2.2. Note that every normed space over \( K \) has (*).

**THEOREM 2.3.** \((p\text{-adic Eberlein-Šmulian Theorem II)}\) Let \( K \) be locally compact, let \( X \subset E \). Then each one of the above statements (\( \alpha \)), (\( \beta \)), (\( \gamma \)) is equivalent to ‘\( X \) is norm compact’.

**Proof.** We have (\( \alpha \)) \( \Rightarrow \) (\( \gamma \)), (\( \beta \)) \( \Rightarrow \) (\( \gamma \)). It suffices to prove that (\( \gamma \)) implies that \( X \) is a norm compactoid (then \( X \) is weakly metrizable since the norm and weak topology coincide on \( X \) ([6], 5.12)). Suppose not. Then by [7], Th. 2 there is a \( t \in (0,1] \) and a \( t \)-orthogonal sequence \( e_1, e_2, \ldots \) in \( X \) such that \( \inf_n ||e_n|| > 0 \). By (\( \gamma \)) there is a weak accumulation point \( a \) of \( \{e_1, e_2, \ldots\} \). This \( a \) is in the weak closure \( D \) of \( \{e_1, e_2, \ldots\} \) which equals the norm closure, so \( a = \sum_{i=1}^{\infty} \lambda_i e_i \) where \( ||\lambda_i e_i|| \to 0 \). If \( \lambda_j \neq 0 \) for some \( j \), let \( U := \{x \in E : |\delta_j(x)| < |\lambda_j| \} \) where \( \delta_j \in E' \) is an extension of the \( j \)th coordinate function \( \Sigma \xi e_i \mapsto \xi_j \) on \( D \). Then \( a + U \) is a weak neighbourhood of \( a \) but for each \( n \in \mathbb{N}, n \neq j \) we have \( |\delta_j(a - e_n)| = |\lambda_j| \) so \( e_n \not\in a + U \), a contradiction. Hence, \( a = 0 \). But then \( \{x \in E : |f(x)| < 1\} \) is a weak neighbourhood of \( a \) containing no \( e_n \) if \( f \in E' \) is such that \( f(e_n) = 1 \) for all \( n \). Contradiction, so \( X \) is a norm compactoid.

**REMARK.** Corollary 2.2 for strongly polar spaces \( E \) and Theorem 2.3 were first proved directly by the first author.
REMARK. The following 'relative' version of the Eberlein-Šmulian Theorem holds. (Compare [1], VIII §2, Theorem 1). Let $X \subset E$. Suppose one of the conditions (i), (ii), (iii) of Theorem 2.1 is satisfied. Then the following are equivalent. (a) $X$ is weakly relatively compact. (b) $X$ is weakly relatively sequentially compact. (c) $X$ is weakly relatively countably compact. We leave the easy proof to the reader.

COUNTEREXAMPLES. We show that the previous theory fails for certain subsets $X$ of $\ell^\infty(I)$ where $I$ has at least the cardinality of the continuum, but is non-measurable, and where $K$ is not spherically complete. The $K$-valued characteristic function of a subset $S \subset I$ is denoted $\xi_S$ and is given by $\xi_S(x) := 1$ if $x \in S$, $\xi_S(x) := 0$ if $x \in I \setminus S$.

1. Let $X := \{\xi_S : S \subset I\}$. Then $X$ is a weakly compact but not weakly sequentially compact subset of $\ell^\infty(I)$.

Proof. $X$ is bounded and since $\ell^\infty(I)' \simeq c_0(I)$ ([4], 4.21) the weak topology on $X$ is the topology of pointwise convergence. Clearly the map $f \mapsto (f(i))_{i \in I}$ is a homeomorphism $X \to \{0, 1\}^I$, hence $X$ is weakly compact. To prove that $X$ is not weakly sequentially compact, let $\phi : I \to Y$ be a surjection where $Y := \{\xi_A : A \subset I\} \subset \ell^\infty$. The formula $\phi(x) = (\xi_{S_1}(x), \xi_{S_2}(x), \ldots) (x \in I)$ defines subsets $S_1, S_2, \ldots$ of $I$. If $\xi_{S_{n_1}}, \xi_{S_{n_2}}, \ldots$ is a subsequence of $\xi_{S_1}, \xi_{S_2}, \ldots$, then, by surjectivity of $\phi$, there is an $x \in I$ for which $(\xi_{S_{n_1}}(x), \xi_{S_{n_2}}(x), \ldots) = (1, 0, 1, 0, 1, \ldots)$, so the subsequence is not weakly convergent.

2. Let $Z := \{\xi_S : S \subset I, S \text{ countable}\} \subset \ell^\infty(I)$. Then $Z$ is weakly sequentially compact but not weakly compact.

Proof. Clearly the weak closure of $Z$ equals $X$ of above, so $Z$ is not weakly compact. On the other hand, if $\xi_{S_1}, \xi_{S_2}, \ldots$ is a sequence in $Z$ then $S := \cup S_n$ is countable and by a standard diagonal procedure one obtains a subsequence converging at all points of $S$, hence at all points of $I$, to an element of $Z$.

3. Let $T := \{\xi_i : i \in I\} \subset \ell^\infty(I)$. Then $f(T)$ is compact for all $f \in \ell^\infty(I)'$ but $T$ is not weakly countably compact.

Proof. Let $f \in \ell^\infty(I)'$. As $\ell^\infty(I)' \simeq c_0(I)$ we have that $f(\xi_i) = 0$ except for $i \in \{i_1, i_2, \ldots\}$ where we may assume the $i_n \in I$ to be distinct. Then $\xi_{i_n} \to 0$ weakly so $T_i := \{0\} \cup \{\xi_{i_n} : n \in \mathbb{N}\}$ is weakly compact and $f(T) = f(T_i)$ is compact. However the only weak accumulation point of $\{\xi_{i_1}, \xi_{i_2}, \ldots\}$ is $0 \not\in T$ so that $T$ is not weakly countably compact.

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