Cardinality and Mackey Topologies of Non-Archimedean Banach and Fréchet Spaces\(^*\)

by

Jerzy KAKOL, Cristina PEREZ-GARCIA and Wim SCHIKHOF

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Summary. Let \( K \) be a non-Archimedean complete non-trivially valued field. One obtains the cardinality of non-Archimedean Fréchet spaces of countable type over \( K \). This enables us to get a new characterization of the spherical completeness of \( K \) in terms of the Hahn-Banach theorem and the Mackey-Arens theorem.

1. Introduction. Throughout the paper \( K \) is a non-Archimedean valued field which is complete with respect to the metric induced by the non-trivial valuation \( |.| \). If \( E = (E, \tau) \) is a locally convex space (lcs) over \( K \) with topology \( \tau \) we denote by \( \mathcal{P}(E) \) or \( \mathcal{P} \) the family of (non-Archimedean) \( \tau \)-continuous seminorms. By \( E' \) and \( E^* \) we denote the topological and algebraic dual of \( E \), respectively. For the basic notions and properties about lcs we refer to [21] when \( K \) is spherically complete and to [17] when \( K \) is not spherically complete. We recall only the following. A non-empty subset \( B \) of a vector space \( E \) (over \( K \)) is called absolutely convex if \( ax + by \in B \) whenever \( x, y \in B \) and \( \alpha, \beta \in K \), \( |\alpha| \leq 1 \), \( |\beta| \leq 1 \). If \( B \) is absolutely convex then the edged hull \( B^e \) of \( B \) is defined as follows. If \( K \) has discrete valuation,
then $B^n = B$; if the valuation is dense then $B^n = \bigcap_{|\lambda| > 1} \lambda B$ (cf. [17]).

If $E$ is an lcs and $p \in \mathcal{P}(E)$ by $E_p$ we denote the quotient space $E / \text{Ker} p$ endowed with the natural norm. The space $E$ is said to be of **countable type** if, for every continuous seminorm $p$ on $E$ the normed space $E_p$ is of countable type. Recall that a normed space $E$ is of countable type if there exists a countable subset $X$ of $E$ such that the closure of the linear hull $[X]$ of $X$ in $E$ equals $E$; this means that either dim $E < \infty$ or the completion of $E$ is isomorphic to the space $c_0$ of null $K$-sequences (cf. [16]). The weak topology $\sigma(E, E')$ on $E$ is of countable type (cf. [17]). A seminorm $p$ on a $K$-vector space $E$ is **polar** if $p = \sup\{|f| : f \in E^*, |f| \leq p\}$. An lcs $E$ is **polar** if there is a basis of continuous seminorms consisting of polar seminorms (cf. [17] also for examples). An lcs $E$ is polar iff the polar neighbourhoods of zero form a neighbourhood basis of zero for the topology of $E$ (cf. [17], Proposition 5.2). A metrizable and complete lcs will be called a Fréchet space. Two locally convex topologies on $E$ will be called **compatible** if they define the same continuous linear functionals on $E$.

It is natural to ask if $E$ admits the Mackey topology $\mu(E, E')$, i.e. the finest locally convex topology on $E$ whose topological dual is still $E'$. 

In ([21], Theorem 4.17), van Tiel proved that the Mackey topology exists if $K$ is spherically complete; in this case $\mu(E, E')$ is determined by the seminorms $x \mapsto \sup\{|f(x)| : f \in A\}$, where $A$ runs through the collection of all subsets of $E'$ which are bounded and $c$-compact in the weak topology $\sigma(E', E)$. In [6] we proved that whenever $K$ is not spherically complete then the space $\ell^\infty$ of bounded $K$-sequences does not admit the Mackey topology; nevertheless, the weak topology $\sigma(\ell^\infty, c_0)$ is of countable type and $\ell^\infty$ admits the finest locally convex topology of countable type to be compatible with $\sigma(\ell^\infty, c_0)$, this follows from Theorem 2.1 of [18] (cf. also [5], Theorem 2) and our Corollary 3.2.

The following question is still open (cf. [17]). Let $E$ be a polar lcs over a non-spherically complete field. Does the polar Mackey topology exist on $E$ (i.e. the strongest polar topology on $E$ compatible with the original one)? We know only that the answer is positive for polarly barrelled and polarly bornological spaces (cf. [17], Corollary 7.9).

In this note (applying ideas of [7] and [8]) we extend results of [6] and [7] by showing the following.

**Theorem 1.1.** Let $E$ be a vector space over $K$ with $\dim E \geq K^\#$, where $K^\#$ denotes the cardinal number of $K$ and $\dim E$ denotes the algebraic dimension of $E$. Let $\Delta = \{(x, x) : x \in E\} \subset E \times E$. The following assertions are equivalent.

1. $K$ is spherically complete.
2. For every locally convex topology $\tau$ on $E$ the space $(E, \tau)$ admits the
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Mackey topology.

(3) There exists a locally convex topology $\tau$ on $E$ with $(E, \tau)' \neq E^*$ such that $(E, \tau)$ admits the Mackey topology.

(4) $(E, \tau)' \neq \{0\}$ for every (complete) Hausdorff locally convex topology $\tau$ on $E$.

(5) For every $f \in E^*$ and seminorms $p, q$ on $E$ such that $|f| \leq \max\{p, q\}$, there exist $f_1, f_2 \in E^*$ such that $f = f_1 + f_2$ and $f_1$ is $p$-continuous and $f_2$ is $q$-continuous.

(6) The diagonal $\Delta$ has the following Hahn-Banach property. For every seminorm $p$ on $E \times E$ and every $f \in \Delta^*$ such that $|f| \leq p$, there exists a linear extension $h$ of $f$ to $E \times E$ such that $|h| \leq p$.

We will see, however (see Remark 3.4 (a)) that (6) implies (1) for any vector space $E$ with $\dim E \geq 1$.

Since every linear functional on a vector space $E$ is continuous with respect to the finest locally convex topology, Theorem 1.1 yields also the following.

**Corollary 1.2.** Let $E$ be a lcs with $\dim E \geq K\#$. Then $E$ admits the Mackey topology iff $K$ is spherically complete or every linear functional on $E$ is continuous.

We will see (Proposition 2.2) that $\dim E \geq K\#$ for any infinite-dimensional Frechet space. This together with Theorem 1 yields the following.

**Corollary 1.3.** Let $K$ be not spherically complete. Then no infinite-dimensional Frechet space admits the Mackey topology.

We recall that by ([17], Corollary 7.9), every polar Frechet space admits the polar Mackey topology.

The problem seems to be much more complicated when we are looking for the polar Mackey topology for non-metrisable spaces. We prove the following.

**Theorem 1.4.** The following conditions on $K$ are equivalent.

1. For every (polar) lcs over $K$ the polar Mackey topology exists.
2. For every lcs $E$ over $K$ of finite type (i.e. each continuous seminorm on $E$ has finite-codimensional kernel) and bounded complete absolutely convex subsets $A$ and $B$ the closure $A + B$ of $A + B$ is complete.
3. For every index set $I$, every pair of closed, bounded, absolutely convex subsets $A$ and $B$ of $K^I$ we have $A + B \subset [A + B]$.
4. If $p, q$ are polar seminorms on some $K$-vector space $E$ and if $f \in E^*$ such that $|f| \leq \max\{p, q\}$, then $f = f_1 + f_2$, where $f_1, f_2 \in E^*$, $f_1$ is $p$-continuous, $f_2$ is $q$-continuous.
In Theorem 1.1 we used essentially the assumption "dim $E \geq K#". It is natural to ask whether there is a relation between the cardinality of $K$ and the linear dimension of any Fréchet space $E$ over $K$. We will see (Propositions 2.2 and 2.5) that dim $E \geq K#$ for any infinite-dimensional Fréchet space and that $E$ contains a closed subspace $F$ such that dim($E/F) = K#$.

2. Dimension of Fréchet spaces. It is known that dim $E \geq 2^{\aleph_0}$ for any infinite-dimensional real or complex Fréchet space; dim $E = 2^{\aleph_0}$, if $E$ is additionally separable (cf. [12]). In [13] Popov showed that for every non-atomic measure space $(\Omega, \Sigma, \mu)$ with $\Sigma# > 2^{\aleph_0}$ there exists a subset $\Omega_1$ of $\Omega$ with $\mu(\Omega_1) > 0$ such that the real space $L^p(\Omega_1)$, $0 < p < 1$, has no Hausdorff $2^{\aleph_0}$-dimensional quotient; recall that $L^p(\Omega_1)$ has trivial topological dual.

We start with the following observation.

**Lemma 2.1.** dim $\ell^\infty = K#$.

**Proof.** Since dim $K^\mathbb{N} = (K#)^{\aleph_0}$ (cf. [10], 5(3), p. 75) and $\ell^\infty \subset K^\mathbb{N}$, we have dim $\ell^\infty \leq (K#)^{\aleph_0}$. On the other hand, we can proceed as in the proof of 5(3) of ([10], p. 75) and obtain dim $\ell^\infty \geq K#$. Now the conclusion follows from $(K#)^{\aleph_0} = K#$ (cf. [15], Corollary 3.9). We give a direct proof of this fact: take any $x = (x_n) \in K^\mathbb{N}$. For every $n \in \mathbb{N}$ there exists a unique two-sided sequence $(a_{jn} : j \in \mathbb{Z})$ of a full set of representatives in $\{x \in K : |x| \leq 1\}$ modulo $\{x \in \mathbb{K} : |x| \leq |\pi|\}$, $0 < |\pi| < 1$, such that $a_{jn} = 0$ for large $j \in \mathbb{N}$ and $x_n = \sum_{j=-\infty}^{\infty} a_{jn} \pi^j$ (cf. [19], Theorems 12.1 and 12.3). Let $\sigma : \mathbb{N} \rightarrow \mathbb{Z} \times \mathbb{Z}$ be a bijection. Then the map $T : ((x_n)) \mapsto \sum_{n \in \mathbb{N}} a_{\sigma(n)} \pi^n$ of $K^\mathbb{N}$ into $K$ is (as easily seen) injective.

**Proposition 2.2.** Let $E$ be a sequentially complete lcs over $K$ containing an infinite-dimensional bounded set $B$. Then there exists a continuous injective linear map $T : \ell^\infty \rightarrow E$. In particular, dim $E \geq K#$ for any infinite-dimensional Fréchet space $E$.

**Proof.** Since the linear span of the closed absolutely convex hull of $B$ endowed with the Minkowski functional norm topology is an infinite-dimensional Banach space, we may assume that $E$ is an infinite-dimensional Banach space. Hence $E$ contains a subspace isomorphic to $c_0$. Let $(e_n)$ be a Schauder basis of this subspace; we may assume that $e_n \rightarrow 0$. Then the map $T : \ell^\infty \rightarrow E$, $T((b_n)) = \sum_{n=1}^{\infty} b_n e_n$ is continuous and injective.

**Examples 2.3.** By Proposition 2.2 infinite-dimensional Fréchet spaces have the dimension at least $K#$. Proposition 2.2 applies also to get the same conclusion for the following spaces. (1) Any $(LF)$-space (i.e. the inductive limit space of an increasing sequence of infinite-dimensional Fréchet
spaces, see [12], for the definition). (2) Any topological product of infinite-dimensional Fréchet spaces. (3) Any perfect sequence space \( \Lambda \neq \varphi \) endowed with its natural topology \( n(\Lambda, \Lambda^\ast) \), where \( \Lambda^\ast \) is the Köthe dual of \( \Lambda \) and \( \varphi \) is the space of all sequences in \( K \) with only finitely many of non-zero coordinates. By ([1], Proposition 7), where the assumption that \( K \) is spherically complete is not necessary, the space \( \Lambda \) is complete. Moreover, if \( (\alpha_n) \in \Lambda, \alpha_n \neq 0, n \in N \), then \( \{ (\beta_n) : |\beta_n| \leq |\alpha_n|, n \in N \} \) is an infinite-dimensional bounded set in \( \Lambda \).

**Corollary 2.4.** If \( E \) is an infinite-dimensional Fréchet space over \( K \) of countable type, then \( \dim E = K^\# \).

**Proof.** Obviously, \( \dim E \geq K^\# \). The space \( E \) is isomorphic to a subspace of a countable product of Banach spaces of countable type; hence \( E \) is isomorphic to a subspace of the product \( c_0^N \). Now \( \dim E \leq \dim c_0^N \leq (K^\#)^{N_0} = K^\# \) which completes the proof.

**Proposition 2.5.** Every infinite-dimensional Fréchet space \( E = (E, \tau) \) contains a closed subspace \( F \) such that \( \dim(E/F) = K^\# \).

**Proof.** If \( \tau = \sigma(E, E') \), then \( E \) is isomorphic to the space \( K^\mathbb{N} \) (cf. [20], Theorem 7). Now suppose that \( \tau \neq \sigma(E, E') \). Then there exists a continuous seminorm \( p \) on \( E \) such that \( \dim E_p = \infty \). Let \( \|\cdot\|_p \) be the corresponding norm on \( E_p \). Let, for some \( t \in (0,1) \), \( (e_n) \) be a \( t \)-orthogonal sequence in \( E_p \) with \( a < \|e_n\|_p \leq 1, n \in N, 0 < a < 1 \). Let \( (f_n) \) be a similar sequence in \( c_0^\infty /c_0 \). Define a continuous linear map \( T: [e_n : n \in N] \to [f_n : n \in N] \) by \( T(\sum_{n \in N} \lambda_n e_n) = \sum_{n \in N} \lambda_n f_n \). Since \( c_0^\infty /c_0 \) is spherically complete ([16], Theorem 4.1), there exists by Ingleton’s theorem (cf. [16], Theorem 4.8), a continuous linear extension \( T_0 : E_p \to c_0^\infty /c_0 \) of \( T \). Let \( F := \ker(T_0 \circ \pi_p) \), where \( \pi_p : E \to E_p \) is the quotient map. Then \( F \) is closed, so \( E/F \) is a Fréchet space, infinite-dimensional, and \( \dim E/F = \dim(\text{Im}T_0) \leq \dim(c_0^\infty/c_0) = K^\# \). By Proposition 2.2, \( \dim E/F \geq K^\# \).

**Remark 2.6.** (1) In ([4], Proposition 2.6) De Grande-De Kimpe proved that any non-normable Fréchet space \( E \) has a subspace isomorphic to \( K^\mathbb{N} \). Moreover, she showed that if \( K \) is spherically complete, this subspace is complemented in \( E \). The latter result is even true for non-spherically complete \( K \) if the dual \( E' \) separates the points of \( E \) (cf. [20], Corollary 9.1 (iv)).

(2) Proposition 2.5 suggests also the following question (still open for the real or complex Banach spaces, cf. [11,14]). Which infinite-dimensional Fréchet spaces over \( K \) admit infinite-dimensional quotients of countable type? We shall say that \( E \) has a *Quotient* if \( E \) has such a quotient. We have the following partial answer.
(a) A Banach space has a Quotient iff $E$ contains a complemented copy of $c_0$ (cf. [3]).

(b) If the valuation of $K$ is discrete, then any infinite dimensional Banach space has a Quotient. This follows from ([16], Corollary 4.14).

(c) The space $\ell^\infty$ does not have a Quotient if $K$ has a dense valuation. Indeed, by ([16], Corollary 5.19) every continuous linear map of $\ell^\infty$ onto $c_0$ is compact.

(d) If $E$ is spherically complete and $K$ has a dense valuation, then $E$ does not have a Quotient. This follows from ([16], Corollary 5.20).

3. Proof of Theorem 1.1. We start with the following Lemma.

**Lemma 3.1.** Let $(E, \tau)$ be a lcs over $K$. Let $\{\tau_\alpha\}_{\alpha \in A}$ be a family of locally convex topologies on $E$ compatible with $\tau$. Then the topology $\tau = \sup \tau_\alpha$ is compatible with $\tau$ provided $K$ is spherically complete or every topology $\tau_\alpha$ is of countable type.

**Proof.** If $K$ is spherically complete we proceed as in the proof of Theorem 1 of [7]. Now assume that every $\tau_\alpha$ is of countable type. Observe that $(E, \xi)$ is of countable type. In fact $(E, \xi)$ is isomorphic to the diagonal $\Delta$ of the product $\prod_{\alpha \in A}(E, \tau_\alpha)$. The last product and also $\Delta$ (endowed with the relative topology) are spaces of countable type by ([17], Proposition 4.12). Since spaces of countable type are strongly polar, Theorem 4.2 of [17] combined with our Lemma 1 of [7] (and its proof) yields the following. Let $f \in (E, \xi)$. $\varepsilon > 0$. There exist seminorms $p_{\alpha_1}, p_{\alpha_2}, \ldots, p_{\alpha_n}$ continuous in $\tau_{\alpha_1}, \tau_{\alpha_2}, \ldots, \tau_{\alpha_n}$, respectively, such that $|f| \leq \max\{p_{\alpha_1}, p_{\alpha_2}, \ldots, p_{\alpha_n}\}$. Then there are $f_1, f_2, \ldots, f_n \in E^*$ such that $f = f_1 + f_2 + \ldots + f_n$ and $|f_i| \leq (1 + \varepsilon)p_{\alpha_i}$, $1 \leq i \leq n$. Then (by assumption) $f \in (E, \tau)'$.

**Corollary 3.2** [5, 18]. Every lcs $(E, \tau)$ over $K$ admits the finest locally convex topology $\mu$ of countable type compatible with $\tau$.

**Proof.** Let $(\tau_\alpha)_{\alpha \in A}$ be the family of all locally convex topologies on $E$ of countable type finer than $\sigma(E, E')$ and compatible with $\tau$; recall that $(E, \sigma(E, E'))$ is one of them. Then using Lemma 3.1 one gets that $\mu = \sup_{\alpha \in A} \tau_\alpha$ is compatible with $\tau$ and $(E, \mu)$ is of countable type.

**Remark 3.3.** It is known (cf. [9]) that when $K$ is not spherically complete, the space $(\ell^\infty, \sigma(\ell^\infty, c_0))$ has a Schauder basis but $\ell^\infty$ (with respect to the norm topology) is not of countable type. Using Lemma 3.1 (and its proof) one deduces also that if $(E, \tau)$ is an lcs such that $(E, \sigma(E, E'))$ has a Schauder basis $(e_n)$, then $E$ admits the finest locally convex topology $\xi$ such that $\sigma(E, E') \leq \xi \leq \tau$ and $(e_n)$ is a Schauder basis for $(E, \xi)$. In fact, $\xi$...
is the finest compatible topology of countable type between $\sigma(E,E')$ and $\tau$ (observe that $\xi$ and $\sigma(E,E')$ have the same convergent sequences by ([17], Proposition 4.11)).

Now we are ready to prove our Theorem 1.1.

Proof of Theorem 1.1. (1) $\Rightarrow$ (2) follows from Lemma 3.1 and (2) $\Rightarrow$ (3) from the following observation. Let $(x_\alpha)$ be a Hamel basis for $E$. Then the topology $\tau$ defined by the norm $\|x\| = \max_{\alpha} |t_\alpha|$, $x = \sum_{\alpha} t_\alpha x_{\alpha}$, where $t_\alpha \in K$, is as in (3). Now we prove (3) $\Rightarrow$ (4). Assume that $(E,\tau)$ is a lcs as in (3) and $E$ admits a locally convex topology $\varphi_1$ such that $(E,\varphi_1)' = \{0\}$. Take $f_0 \in E^* \setminus E'$ and $x_0 \neq 0$ in $E$ such that $f_0(x_0) = 2$. Consider the map $T : E \rightarrow E$ defined by $T(x) = x - f_0(x)x_0$, then $T^2 = \text{id}$. Let $\varphi_2$ be the image by $T$ of $\varphi_1$. Consider the topologies $\psi_i = \sup\{\sigma(E,E'),\varphi_i\}$, $i = 1,2$. Observe that both topologies $\psi_i$ are compatible with the original one of $E$: fix $i \in \{1,2\}$. Let $f \in (E,\psi_i)'$. Then there exist $f_1, f_2, \ldots, f_n \in E'$ and a $\varphi_i$-continuous seminorm $p$ such that $|f| \leq \max\{|f_1|, \ldots, |f_n|, p\}$. Let $H = \bigcap_{k=1}^n \{x \in E : f_k(x) = 0\}$. Then $|f| \leq p$ on $H$. Since $H$ is a finite-codimensional subspace of $E$, there exists a continuous linear extension of $f$ to the space $(E,\varphi_1)$. Since $(E,\varphi_1)' = \{0\}$, we have $f = 0$ on $H$. Hence $f$ is a linear combination of $f_1, \ldots, f_n$, so $f \in E'$. On the other hand $f_0$ is continuous with respect to the topology sup$\{\varphi_1, \varphi_2\}$. In fact, if $(x_\alpha)$ is a null net in $E$ with respect to the topology $\varphi = \sup\{\varphi_1, \varphi_2\}$, then $T(x_\alpha) = x_\alpha - f_0(x_\alpha)x_0 \rightarrow 0$ in $\varphi_2$ and $x_\alpha \rightarrow 0$ in $\varphi_2$. Now $x_0 \neq 0$, so $f_0(x_\alpha) \rightarrow 0$. By assumption (the Mackey topology $\mu(E,E')$ exists) we have $\sigma(E,E') \leq \sup\{\psi_1, \psi_2\} \leq \mu(E,E')$. This implies that $f_0 \in E'$, a contradiction.

(4) $\Rightarrow$ (1). Assume that $K$ is not spherically complete. Since $\dim E \geq K^\#$ and also $\dim \ell^\infty/c_0 = K^\#$, there exists an index set $A$ and a family of vector subspaces $E_\alpha$ of $E$, $\alpha \in A$, such that $E = \bigoplus_{\alpha \in A} E_\alpha$ (algebraically) with $\dim E_\alpha = \dim \ell^\infty/c_0$ for all $\alpha \in A$. Endow every $E_\alpha$ with the isomorphic copy of the original topology of $\ell^\infty/c_0$. Let $\phi$ be the Hausdorff locally convex direct sum topology on $E = \bigoplus_{\alpha \in A} E_\alpha$. Then $(E,\phi)$ is a Hausdorff and complete lcs, cf. [21] (where the assumption that $K$ is spherically complete is not necessary). From $(\ell^\infty/c_0)' = \{0\}$ we obtain $(E,\phi)' = \{0\}$.

(1) $\Rightarrow$ (5). It is a direct consequence of our Lemma 1 of [7]. (5) $\Rightarrow$ (2). If $\mu(E,E')$ is the supremum topology of all locally convex topologies on $E$ compatible with the original one, then $\mu(E,E')$ is compatible with the original one; hence $\mu(E,E')$ is the Mackey topology of $E$. (1) $\Rightarrow$ (6). This follows from Ingleton's Theorem, (cf. [21], Theorem 3.5). (6) $\Rightarrow$ (5). We proceed similarly as in the proof of Lemma 1 of [7].

Remark 3.4. (1) Observe that the implication (6) $\Rightarrow$ (1) is true for any
vector space $E$ over $K$ with dim $E > 1$. In fact, assume that $K$ is not spherically complete. First, suppose that $\dim E = 1$, i.e. the space $E$ is algebraically isomorphic to $K$. According to ([16], p. 68) there exists a norm $\| \cdot \|$ on $K^2$, a one-dimensional space $D$ and $g \in D'$ with $\| g(x) \| \leq \| x \|$, $x \in D$. that does not admit an extension $g_0 \in (K^2, \| \cdot \|)'$ with $\| g_0(x) \| \leq \| x \|$, $x \in K^2$.

Now let $T : K \times K \to E \times E$ be a linear bijection such that $T(D) = \Delta$. Set $f = g \circ T^{-1} \Delta$ and $p(x) = \| T^{-1}(x) \|$, $x \in E \times E$. Then $f \in \Delta^*$, $\| f(z) \| \leq p(z)$ for all $z \in \Delta$. Suppose that there exists on $E \times E$ a linear extension $\hat{g}$ of $g$ satisfying $\| \hat{g}(y) \| \leq p(Ty) = \| y \|$ for all $y \in K^2$, which is a contradiction. Now suppose that $\dim E > 1$. Take a non-zero element $a \in E$ and an algebraic complement $S$ of $[a]$. Take on $E \times E$ the seminorm $(s, \lambda a) \times (s', \gamma a) \mapsto p(\lambda a, \gamma a)$, where $p : [a] \times [a] \to \mathbb{R}$ is as above, and a linear functional $h : \Delta \to K$ defined by $(s, \lambda a) \times (s, \lambda a) \mapsto f(\lambda a, \lambda a)$, where $f$ is as above. Clearly $\| h \| \leq p$ on $\Delta$ but $h$ cannot be extended to a linear functional $h_0$ on $E \times E$ with $\| h_0 \| \leq p$ by the first part.

(2) From the proof of Theorem 1.1, (3) $\Rightarrow$ (4), one deduces also the following. Let $(E, \tau)$ be an lcs with trivial topological dual. Then there exists a family $(\tau_\alpha)_{\alpha \in A}$ of locally convex topologies on $E$ such that every $(E, \tau_\alpha)$ is isomorphic to $(E, \tau)$ and $(E, \sup_{\alpha \in A} \tau_\alpha)' = E^*$. Therefore, if $E \neq \{0\}$, then $(E, \tau)$ does not admit the Mackey topology.

(3) We do not know any example of a lcs $E$ over a non-spherically complete $K$ with $\aleph_0 < \dim E < K^\#$ which admits the Mackey topology. Note that any lcs $(E, \tau)$ with dim $E = \aleph_0$ admits the Mackey topology. Indeed, if $(\tau_\alpha)_{\alpha \in A}$ is the family of all locally convex topologies on $E$ compatible with $\tau$, then $(E, \sup_{\alpha \in A} \tau_\alpha)$ is of countable type ([17], Examples 4.5). Now Lemma 3.1 completes the proof.

(4) Using Corollary 1 we deduce also that the spaces considered in Examples 2.3 admit the Mackey topology iff $K$ is spherically complete.

4. Proof of Theorem 1.4. Let $E$ be a polar lcs. By a special covering of $E'$ (cf. [17], Definition 7.3) we mean a covering $G$ of $E'$ such that

- each member of $G$ is edged, $\sigma(E', E)$-bounded, $\sigma(E', E)$-complete;
- for each $A, B \in G$ there is a $C \in G$ such that $A \cup B \subset C$;
- for each $A \in G$ and $\lambda \in K$ there is a $B \in G$ with $\lambda A \subset B$.

For a special covering $G$ of $E'$ the $G$-topology on $E$ is the topology induced by the seminorms $x \mapsto \sup\{\| f(x) \| : f \in A\}$, where $A$ runs through $G$.

In order to prove Theorem 2 we shall need the following.

Lemma 4.1. Let $E$ be a polar lcs and suppose that the polar Mackey topology $\mu(E, E')$ exists. Then the family $G$ of all edged and absolutely convex
complete compactoids in \((E',\sigma(E',E))\) is a special covering and \(\mu(E,E')\) equals the \(G\)-topology.

**Proof.** For each \(\mu(E,E')\)-continuous polar seminorm \(p\) on \(E\) set \(A_p = \{f \in E^* : |f| \leq p\}\). Then by ([17], Proposition 7.4 and its proof), the topology \(\mu(E,E')\) is the \(G\)-topology, where \(G = \{A_p : p\) is a \(\mu(E,E')\)-continuous seminorm on \(E\}\) is a special covering of \(E'\). Let \(A\) be an edged, absolutely convex complete compactoid in \((E\setminus (E\setminus E))\) is a special covering and \(\pi(E,El)\) equals the \(Q\)-topology.

By ([17], Proposition 7.4 and its proof), the \(\pi(E,El)\)-topology is the \(Q\)-topology, where \(Q = \{A_p : p\) is a \(\mu(E,E')\)-continuous seminorm on \(E\}\) is a special covering of \(E'\). Let \(A\) be an edged, absolutely convex complete compactoid in \((E\setminus (E\setminus E))\). The completeness of \((\lambda A + F)^e : \lambda \in K, F \subseteq E', F\) be absolutely convex and \(\sigma(E',E)\)-bounded, \(\dim[F] < \infty\). It is easy to check that \(G_A\) is a special covering of \(E'\); the completeness of \((\lambda A + F)^e\) follows from ([18], Theorem 4.1). By ([17], Proposition 7.4), the \(G_A\)-topology is compatible with the original one of \(E'\), hence the seminorm \(p : x \mapsto \sup\{|f(x)| : f \in A\}\) is \(\mu(E,E')\)-continuous and polar. Hence \(A_p\) is in \(G\) and \(A \subseteq A_p\). Since by ([17], Proposition 4.10), the set \(A\) is a polar set, we get equality.

Now we are ready to prove Theorem 1.4.

**Proof of Theorem 1.4.** (1) \(\Rightarrow\) (2): Let \((F,\tau)\) be a lcs of finite type; then \(\sigma(F,F') = \tau\). The space \(E := (F',\sigma(F,F'))\) admits the polar Mackey topology (by (1)) and by Lemma 4.1 it is the \(G\)-topology, where \(G\) equals the set of all edged, absolutely convex, complete compactoids in \((E',\sigma(E',E))\).

By (b) above, for \(A, B \subseteq G\) the set \(A + B\) is complete; (2) now follows because the natural map \((F,\sigma(F,F')) \rightarrow (E',\sigma(E',E))\) is an isomorphism.

(2) \(\Rightarrow\) (3): Note that \([A + B]\) is of finite type, being a subspace of \(K^I\). By (2) one gets that the closure of \(A + B\) in \([A + B]\) is complete; hence the closure of \(A + B\) in \(K^I\) lies in \([A + B]\).

(3) \(\Rightarrow\) (4): Let \((e_i : i \in I)\) be an algebraic base for \(E\) and endow \(E\) with the finest locally convex topology. Then \(E^* = E'\) and \((E',\sigma(E',E))\) is isomorphic to the product \(K^I\). Let \(p, q\) be two polar seminorms on \(E\), then \(A = \{f \in E' : |f| \leq p\}\) and \(B = \{f \in E' : |f| \leq q\}\) are edged, absolutely convex, bounded and complete. Hence we may apply (3) to conclude that \(A + B \subseteq [A + B]\). By polarity of \(p\) and \(q\) we have \((A + B)^e = (A + B)^{00} = (U \cap V)^0\), where \(U, V\) are the unit balls of \(p, q\), respectively, and so \(U \cap V\) is the unit ball in the normed topology defined by \(\max\{p, q\}\). Hence by (3) one gets \(\{f \in E' : |f| \leq \max\{p, q\}\} \subseteq (A + B)^e \subseteq [A + B] = [A] + [B]\). Thus, we can write every \(f \in E^*\) with \(|f| \leq \max\{p, q\}\) as \(g + h\), where \(g \in [A]\) and \(h \in [B]\); hence \(g\) is \(p\)-continuous and \(h\) is \(q\)-continuous.

(4) \(\Rightarrow\) (1): It suffices to show that if \(\tau_1\) and \(\tau_2\) are polar locally convex topologies on a lcs \((E,\tau)\) compatible with \(\tau\), then so is \(\sup\{\tau_1, \tau_2\}\). Let \(p\) be a \(\tau_1\)-continuous polar seminorm and \(q\) be a \(\tau_2\)-continuous polar seminorm on \(E\). Let \(f \in E^*\), \(|f| \leq \max\{p, q\}\). Then, by (4), \(f = g + h\), where \(g\) is
$\tau_1$-continuous, $h$ is $\tau_2$-continuous, i.e. $g, h \in (E, \tau)'$. Hence $f$ is $\tau$-continuous and the proof is complete.

REFERENCES


