FUNCTIONAL ANALYSIS

Cardinality and Mackey Topologies of Non-Archimedean Banach and Fréchet Spaces\(^(*)\)

by

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Summary. Let \(K\) be a non-Archimedean complete non-trivially valued field. One obtains the cardinality of non-Archimedean Fréchet spaces of countable type over \(K\). This enables us to get a new characterization of the spherical completeness of \(K\) in terms of the Hahn-Banach theorem and the Mackey-Arens theorem.

1. Introduction. Throughout the paper \(K\) is a non-Archimedean valued field which is complete with respect to the metric induced by the non-trivial valuation \(|.|\). If \(E = (E, \tau)\) is a locally convex space (lcs) over \(K\) with topology \(\tau\) we denote by \(\mathcal{P}(E)\) or \(\mathcal{P}\) the family of (non-Archimedean) \(\tau\)-continuous seminorms. By \(E'\) and \(E^*\) we denote the topological and algebraic dual of \(E\), respectively. For the basic notions and properties about lcs we refer to [21] when \(K\) is spherically complete and to [17] when \(K\) is not spherically complete. We recall only the following. A non-empty subset \(B\) of a vector space \(E\) (over \(K\)) is called absolutely convex if \(ax + \beta y \in B\) whenever \(x, y \in B\) and \(\alpha, \beta \in K\), \(|\alpha| \leq 1\), \(|\beta| \leq 1\). If \(B\) is absolutely convex then the edged hull \(B^e\) of \(B\) is defined as follows. If \(K\) has discrete valuation,
then $B^n = B$; if the valuation is dense then $B^n = \bigcap_{|\lambda| > 1} \lambda B$ (cf. [17]).

If $E$ is an lcs and $p \in \mathcal{P}(E)$ by $E_p$ we denote the quotient space $E/\text{Ker } p$ endowed with the natural norm. The space $E$ is said to be of countable type if, for every continuous seminorm $p$ on $E$ the normed space $E_p$ is of countable type. Recall that a normed space $E$ is of countable type if there exists a countable subset $X$ of $E$ such that the closure of the linear hull $[X]$ of $X$ in $E$ equals $E$; this means that either $\dim E < \infty$ or the completion of $E$ is isomorphic to the space $c_0$ of null $K$-sequences (cf. [16]). The weak topology $\sigma(E, E')$ on $E$ is of countable type (cf. [17]). A seminorm $p$ on a $K$-vector space $E$ is polar if $p = \sup\{|f| : f \in E^*, |f| \leq p\}$. An lcs $E$ is polar if there is a basis of continuous seminorms consisting of polar seminorms (cf. [17] also for examples). An lcs $E$ is polar iff the polar neighbourhoods of zero form a neighbourhood basis of zero for the topology of $E$ (cf. [17], Proposition 5.2). A metrizable and complete lcs will be called a Fréchet space. Two locally convex topologies on $E$ will be called compatible if they define the same continuous linear functionals on $E$.

It is natural to ask if $E$ admits the Mackey topology $\mu(E, E')$, i.e. the finest locally convex topology on $E$ whose topological dual is still $E'$. In ([21], Theorem 4.17), van Tiel proved that the Mackey topology exists if $K$ is spherically complete; in this case $\mu(E, E')$ is determined by the seminorms $x \mapsto \sup\{|f(x)| : f \in A\}$, where $A$ runs through the collection of all subsets of $E'$ which are bounded and $c$-compact in the weak topology $\sigma(E', E)$. In [6] we proved that whenever $K$ is not spherically complete then the space $\ell^\infty$ of bounded $K$-sequences does not admit the Mackey topology; nevertheless, the weak topology $\sigma(\ell^\infty, c_0)$ is of countable type and $\ell^\infty$ admits the finest locally convex topology of countable type to be compatible with $\sigma(\ell^\infty, c_0)$, this follows from Theorem 2.1 of [18] (cf. also [5], Theorem 2) and our Corollary 3.2.

The following question is still open (cf. [17]). Let $E$ be a polar lcs over a non-spherically complete field. Does the polar Mackey topology exist on $E$ (i.e. the strongest polar topology on $E$ compatible with the original one)? We know only that the answer is positive for polarly barrelled and polarly bornological spaces (cf. [17], Corollary 7.9).

In this note (applying ideas of [7] and [8]) we extend results of [6] and [7] by showing the following.

**Theorem 1.1.** Let $E$ be a vector space over $K$ with $\dim E \geq K^\#$, where $K^#$ denotes the cardinal number of $K$ and $\dim E$ denotes the algebraic dimension of $E$. Let $\Delta = \{(x, x) : x \in E\} \subset E \times E$. The following assertions are equivalent.

1. $K$ is spherically complete.
2. For every locally convex topology $\tau$ on $E$ the space $(E, \tau)$ admits the
Mackey topology.

(3) There exists a locally convex topology $\tau$ on $E$ with $(E, \tau)' \neq E^*$ such that $(E, \tau)$ admits the Mackey topology.

(4) $(E, \tau)' \neq \{0\}$ for every (complete) Hausdorff locally convex topology $\tau$ on $E$.

(5) For every vector space $E$, there exist $f_1, f_2 \in E^*$ such that $f = f_1 + f_2$ and $f_1$ is $p$-continuous and $f_2$ is $q$-continuous.

(6) The diagonal $\Delta$ has the following Hahn-Banach property. For every seminorm $p$ on $E$ and every $f \in \Delta^*$ such that $|f| \leq p$, there exists a linear extension $h$ of $f$ to $E \times E$ such that $|h| \leq p$.

We will see, however (see Remark 3.4 (a)) that (6) implies (1) for any vector space $E$ with $\dim E \geq 1$.

Since every linear functional on a vector space $E$ is continuous with respect to the finest locally convex topology, Theorem 1.1 yields also the following.

**COROLLARY 1.2.** Let $E$ be a lcs with $\dim E \geq K^\#$. Then $E$ admits the Mackey topology iff $K$ is spherically complete or every linear functional on $E$ is continuous.

We will see (Proposition 2.2) that $\dim E \geq K^\#$ for any infinite-dimensional Fréchet space. This together with Theorem 1 yields the following.

**COROLLARY 1.3.** Let $K$ be not spherically complete. Then no infinite-dimensional Fréchet space admits the Mackey topology.

We recall that by ([17], Corollary 7.9), every polar Fréchet space admits the polar Mackey topology.

The problem seems to be much more complicated when we are looking for the polar Mackey topology for non-metrizable spaces. We prove the following.

**THEOREM 1.4.** The following conditions on $K$ are equivalent.

1. For every (polar) lcs over $K$ the polar Mackey topology exists.

2. For every lcs $E$ over $K$ of finite type (i.e. each continuous seminorm on $E$ has finite-codimensional kernel) and bounded complete absolutely convex subsets $A$ and $B$ the closure $A + B$ of $A + B$ is complete.

3. For every index set $I$, every pair of closed, bounded, absolutely convex subsets $A$ and $B$ of $K^I$ we have $A + B \subseteq [A + B]$.

4. If $p, q$ are polar seminorms on some $K$-vector space $E$ and if $f \in E^*$ such that $|f| \leq \max\{p, q\}$, then $f = f_1 + f_2$, where $f_1, f_2 \in E^*$, $f_1$ is $p$-continuous, $f_2$ is $q$-continuous.
In Theorem 1.1 we used essentially the assumption "dim \( E \geq K^\# \)." It is natural to ask whether there is a relation between the cardinality of \( K \) and the linear dimension of any Fréchet space \( E \) over \( K \). We will see (Propositions 2.2 and 2.5) that dim \( E \geq K^\# \) for any infinite-dimensional Fréchet space and that \( E \) contains a closed subspace \( F \) such that dim\( (E/F) = K^\# \).

2. Dimension of Fréchet spaces. It is known that dim \( E \geq 2^{\aleph_0} \) for any infinite-dimensional real or complex Fréchet space; dim \( E = 2^{\aleph_0} \), if \( E \) is additionally separable (cf. [12]). In [13] Popov showed that for every nonatomic measure space \((\Omega, \Sigma, \mu)\) with \( \mu(\Omega_1) > 2^{\aleph_0} \) there exists a subset \( \Omega_1 \) of \( \Omega \) with \( \mu(\Omega_1) > 0 \) such that the real space \( L^p(\Omega_1) \), \( 0 < p < 1 \), has no Hausdorff \( 2^{\aleph_0} \)-dimensional quotient; recall that \( L^p(\Omega_1) \) has trivial topological dual.

We start with the following observation.

Lemma 2.1. dim \( \ell^\infty = \aleph^\# \).

Proof. Since dim \( K^N = (K^\#)^{\aleph_0} \) (cf. [10], 5(3), p. 75) and \( \ell^\infty \subset K^N \), we have dim \( \ell^\infty \leq (K^\#)^{\aleph_0} \). On the other hand, we can proceed as in the proof of 5(3) of ([10], p. 75) and obtain dim \( \ell^\infty \geq K^\# \). Now the conclusion follows from \( (K^\#)^{\aleph_0} = K^\# \) (cf. [15], Corollary 3.9). We give a direct proof of this fact: take any \( x = (x_n) \in K^N \). For every \( n \in N \) there exists a unique two-sided sequence \( (a_{jn})_j \in \mathbb{Z} \) of a full set of representatives in \( \{ x \in K : |x| \leq 1 \} \) modulo \( \{ x \in K : |x| \leq |\pi| \}, \) \( 0 < |\pi| < 1 \), such that \( a_{-jn} = 0 \) for large \( j \in N \) and \( x_n = \sum_{j=-\infty}^{\infty} a_{jn} \pi^j \) (cf. [19], Theorems 12.1 and 12.3). Let \( \sigma : N \to \mathbb{Z} \times \mathbb{Z} \) be a bijection. Then the map \( T : ((x_n)) \mapsto \sum_{n \in N} a_{\sigma(n)} \pi^n \) of \( K^N \) into \( K \) is (as easily seen) injective.

Proposition 2.2. Let \( E \) be a sequentially complete lcs over \( K \) containing an infinite-dimensional bounded set \( B \). Then there exists a continuous injective linear map \( T : \ell^\infty \to E \). In particular, dim \( E \geq K^\# \) for any infinite-dimensional Fréchet space \( E \).

Proof. Since the linear span of the closed absolutely convex hull of \( B \) endowed with the Minkowski functional norm topology is an infinite-dimensional Banach space, we may assume that \( E \) is an infinite-dimensional Banach space. Hence \( E \) contains a subspace isomorphic to \( c_0 \). Let \( (e_n) \) be a Schauder basis of this subspace; we may assume that \( e_n \to 0 \). Then the map \( T : \ell^\infty \to E, T((b_n)) = \sum_{n=1}^{\infty} b_n e_n \) is continuous and injective.

Examples 2.3. By Proposition 2.2 infinite-dimensional Fréchet spaces have the dimension at least \( K^\# \). Proposition 2.2 applies also to get the same conclusion for the following spaces. (1) Any \((LF)\)-space (i.e. the inductive limit space of an increasing sequence of infinite-dimensional Fréchet
spaces, see [12], for the definition). (2) Any topological product of infinite-dimensional Fréchet spaces. (3) Any perfect sequence space \( A \neq \varphi \) endowed with its natural topology \( n(A, \Lambda^*) \), where \( \Lambda^* \) is the Köthe dual of \( \Lambda \) and \( \varphi \) is the space of all sequences in \( K \) with only finitely many of non-zero coordinates. By ([1], Proposition 7), where the assumption that \( K \) is spherically complete is not necessary, the space \( A \) is complete. Moreover, if \((\alpha_n) \in A, \alpha_n \neq 0, n \in \mathbb{N}\), then \( \{ (\beta_n) : |\beta_n| \leq |\alpha_n|, n \in \mathbb{N}\} \) is an infinite-dimensional bounded set in \( A \).

**Corollary 2.4.** If \( E \) is an infinite-dimensional Fréchet space over \( K \) of countable type, then \( \dim E = K^\# \).

**Proof.** Obviously, \( \dim E \geq K^\# \). The space \( E \) is isomorphic to a subspace of a countable product of Banach spaces of countable type; hence \( E \) is isomorphic to a subspace of the product \( c_0^\mathbb{N} \). Now \( \dim E \leq \dim c_0^\mathbb{N} \leq (K^\#)^\mathbb{N} = K^\# \) which completes the proof.

**Proposition 2.5.** Every infinite-dimensional Fréchet space \( E = (E, \tau) \) contains a closed subspace \( F \) such that \( \dim(E/F) = K^\# \).

**Proof.** If \( \tau = \sigma(E, E') \), then \( E \) is isomorphic to the space \( K^\mathbb{N} \) (cf. [20], Theorem 7). Now suppose that \( \tau \neq \sigma(E, E') \). Then there exists a continuous seminorm \( p \) on \( E \) such that \( \dim E_p = \infty \). Let \( \| . \|_p \) be the corresponding norm on \( E_p \). Let, for some \( t \in (0,1), (e_n) \) be a \( t \)-orthogonal sequence in \( E_p \) with \( a < \| e_n \|_p \leq 1, n \in \mathbb{N}, 0 < a < 1 \). Let \( (f_n) \) be a similar sequence in \( \ell^\infty/c_0 \). Define a continuous linear map \( T : [e_n : n \in \mathbb{N}] \to [f_n : n \in \mathbb{N}] \) by \( T(\sum_{n \in \mathbb{N}} \lambda_n e_n) = \sum_{n \in \mathbb{N}} \lambda_n f_n \). Since \( \ell^\infty/c_0 \) is spherically complete ([16], Theorem 4.1), there exists by Ingleton's theorem (cf. [16], Theorem 4.8), a continuous linear extension \( T_0 : E_p \to \ell^\infty/c_0 \) of \( T \). Let \( F := \ker(T_0 \circ \pi_p) \), where \( \pi_p : E \to E_p \) is the quotient map. Then \( F \) is closed, so \( E/F \) is a Fréchet space, infinite-dimensional, and \( \dim E/F = \dim(\ker T_0) \leq \dim(\ell^\infty/c_0) \leq K^\# \). By Proposition 2.2, \( \dim E/F \geq K^\# \).

**Remark 2.6.** (1) In ([4], Proposition 2.6) De Grande-De Kimpe proved that any non-normable Fréchet space \( E \) has a subspace isomorphic to \( K^\mathbb{N} \). Moreover, she showed that if \( K \) is spherically complete, this subspace is complemented in \( E \). The latter result is even true for non-spherically complete \( K \) if the dual \( E' \) separates the points of \( E \) (cf. [20], Corollary 9.1 (iv)).

(2) Proposition 2.5 suggests also the following question (still open for the real or complex Banach spaces, cf. [11,14]). Which infinite-dimensional Fréchet spaces over \( K \) admit infinite-dimensional quotients of countable type? We shall say that \( E \) has a **Quotient** if \( E \) has such a quotient. We have the following partial answer.
(a) A Banach space has a Quotient iff $E$ contains a complemented copy of $c_0$ (cf. [3]).

(b) If the valuation of $K$ is discrete, then any infinite dimensional Banach space has a Quotient. This follows from ([16], Corollary 4.14).

(c) The space $\ell^\infty$ does not have a Quotient if $K$ has a dense valuation. Indeed, by ([16], Corollary 5.19) every continuous linear map of $\ell^\infty$ onto $c_0$ is compact.

(d) If $E$ is spherically complete and $K$ has a dense valuation, then $E$ does not have a Quotient. This follows from ([16], Corollary 5.20).

3. Proof of Theorem 1.1. We start with the following Lemma.

**Lemma 3.1.** Let $(E, \tau)$ be a lcs over $K$. Let $\{\tau_\alpha\}_{\alpha \in A}$ be a family of locally convex topologies on $E$ compatible with $\tau$. Then the topology $\xi = \sup \tau_\alpha$ is compatible with $\tau$ provided $K$ is spherically complete or every topology $\tau_\alpha$ is of countable type.

**Proof.** If $K$ is spherically complete we proceed as in the proof of Theorem 1 of [7]. Now assume that every $\tau_\alpha$ is of countable type. Observe that $(E, \xi)$ is of countable type. In fact $(E, \xi)$ is isomorphic to the diagonal $\Delta$ of the product $\prod_{\alpha \in A} (E, \tau_\alpha)$. The last product and also $\Delta$ (endowed with the relative topology) are spaces of countable type by ([17], Proposition 4.12). Since spaces of countable type are strongly polar, Theorem 4.2 of [17] combined with our Lemma 1 of [7] (and its proof) yields the following. Let $f \in (E, \xi)'$, $\varepsilon > 0$. There exist seminorms $p_1, p_2, \ldots, p_n$ continuous in $\tau_1, \tau_2, \ldots, \tau_n$, respectively, such that $|f| \leq \max\{p_1, p_2, \ldots, p_n\}$. Then there are $f_1, f_2, \ldots, f_n \in E^*$ such that $f = f_1 + f_2 + \ldots + f_n$ and $|f_i| \leq (1 + \varepsilon)p_a$, $1 \leq i \leq n$. Then (by assumption) $f \in (E, \tau)'$.

**Corollary 3.2** [5, 18]. Every lcs $(E, \tau)$ over $K$ admits the finest locally convex topology $\mu$ of countable type compatible with $\tau$.

**Proof.** Let $(\tau_\alpha)_{\alpha \in A}$ be the family of all locally convex topologies on $E$ of countable type finer than $\sigma(E, E')$ and compatible with $\tau$; recall that $(E, \sigma(E, E'))$ is one of them. Then using Lemma 3.1 one gets that $\mu = \sup_{\alpha \in A} \tau_\alpha$ is compatible with $\tau$ and $(E, \mu)$ is of countable type.

**Remark 3.3.** It is known (cf. [9]) that when $K$ is not spherically complete, the space $(\ell^\infty, \sigma(\ell^\infty, c_0))$ has a Schauder basis but $\ell^\infty$ (with respect to the norm topology) is not of countable type. Using Lemma 3.1 (and its proof) one deduces also that if $(E, \tau)$ is an lcs such that $(E, \sigma(E, E'))$ has a Schauder basis $(e_n)$, then $E$ admits the finest locally convex topology $\xi$ such that $\sigma(E, E') \leq \xi \leq \tau$ and $(e_n)$ is a Schauder basis for $(E, \xi)$. In fact, $\xi$
is the finest compatible topology of countable type between $\sigma(E,E')$ and $\tau$ (observe that $\xi$ and $\sigma(E,E')$ have the same convergent sequences by ([17], Proposition 4.11)).

Now we are ready to prove our Theorem 1.1.

Proof of Theorem 1.1. (1) $\Rightarrow$ (2) follows from Lemma 3.1 and (2) $\Rightarrow$ (3) from the following observation. Let $(x_\alpha)$ be a Hamel basis for $E$. Then the topology $\tau$ defined by the norm $\|x\| = \max_\alpha |t_\alpha|$, $x = \sum_\alpha t_\alpha x_\alpha$, where $t_\alpha \in K$, is as in (3). Now we prove (3) $\Rightarrow$ (4). Assume that $(E,\tau)$ is a lcs as in (3) and $E$ admits a locally convex topology $\varphi_1$ such that $(E,\varphi_1)' = \{0\}$. Take $f_0 \in E^* \setminus E'$ and $x_0 \neq 0$ in $E$ such that $f_0(x_0) = 2$. Consider the map $T : E \to E$ defined by $T(x) = x - f_0(x)x_0$, then $T^2 = \text{id}$. Let $\varphi_2$ be the image by $T$ of $\varphi_1$. Consider the topologies $\psi_i = \sup\{\sigma(E,E')$, $\varphi_i\}$, $i = 1,2$. Observe that both topologies $\psi_i$ are compatible with the original one of $E$: fix $i \in \{1,2\}$. Let $f \in (E,\psi_i)'$. Then there exist $f_1,f_2,\ldots,f_n \in E'$ and a $\varphi_i$-continuous seminorm $p$ such that $|f| \leq \max\{|f_1|,\ldots,|f_n|,p\}$. Let $H = \bigcap_{k=1}^n \{x \in E : f_k(x) = 0\}$. Then $|f| \leq p$ on $H$. Since $H$ is a finite-codimensional subspace of $E$, there exists a continuous linear extension of $f$ to the space $(E,\psi_i)$. Since $(E,\psi_i)' = \{0\}$, we have $f = 0$ on $H$. Hence $f$ is a linear combination of $f_1,\ldots,f_n$, so $f \in E'$. On the other hand $f_0$ is continuous with respect to the topology $\sup\{\varphi_1,\varphi_2\}$. In fact, if $(x_\alpha)$ is a null net in $E$ with respect to the topology $\varphi = \sup\{\varphi_1,\varphi_2\}$, then $T(x_\alpha) = x_\alpha - f_0(x_\alpha)x_0 \to 0$ in $\varphi_2$ and $x_\alpha \to 0$ in $\varphi_2$. Now $x_0 \neq 0$, so $f_0(x_\alpha) \to 0$. By assumption (the Mackey topology $\mu(E,E')$ exists) we have $\sigma(E,E') \leq \sup\{\psi_1,\psi_2\} \leq \mu(E,E')$. This implies that $f_0 \in E'$, a contradiction.

(4) $\Rightarrow$ (1). Assume that $K$ is not spherically complete. Since $\dim E \geq K^\#$ and also $\dim \ell^\infty/c_0 = K^\#$, there exists an index set $A$ and a family of vector subspaces $E_\alpha$ of $E$, $\alpha \in A$, such that $E = \bigoplus_{\alpha \in A} E_\alpha$ (algebraically) with $\dim E_\alpha = \dim \ell^\infty/c_0$ for all $\alpha \in A$. Endow every $E_\alpha$ with the isomorphic copy of the original topology of $\ell^\infty/c_0$. Let $\phi$ be the Hausdorff locally convex direct sum topology on $E = \bigoplus_{\alpha \in A} E_\alpha$. Then $(E,\phi)$ is a Hausdorff and complete lcs, cf. [21] (where the assumption that $K$ is spherically complete is not necessary). From $(\ell^\infty/c_0)' = \{0\}$ we obtain $(E,\phi)' = \{0\}$.

(1) $\Rightarrow$ (5). It is a direct consequence of our Lemma 1 of [7]. (5) $\Rightarrow$ (2). If $\mu(E,E')$ is the supremum topology of all locally convex topologies on $E$ compatible with the original one, then $\mu(E,E')$ is compatible with the original one; hence $\mu(E,E')$ is the Mackey topology of $E$. (1) $\Rightarrow$ (6). This follows from Ingleton’s Theorem, (cf. [21], Theorem 3.5). (6) $\Rightarrow$ (5). We proceed similarly as in the proof of Lemma 1 of [7].

Remark 3.4. (1) Observe that the implication (6) $\Rightarrow$ (1) is true for any
vector space $E$ over $K$ with $\dim E \geq 1$. In fact, assume that $K$ is not spherically complete. First, suppose that $\dim E = 1$, i.e. the space $E$ is algebraically isomorphic to $K$. According to ([16], p. 68) there exists a norm $\| \cdot \|$ on $K^2$, a one-dimensional space $D$ and $g \in D'$ with $|g(x)| \leq \|x\|$, $x \in D$, that does not admit an extension $g_0 \in (K^2, \| \cdot \|)'$ with $|g_0(x)| \leq \|x\|$, $x \in K^2$. Now let $T : K \times K \to E \times E$ be a linear bijection such that $T(D) = \Delta$. Set $f = g \circ T^{-1}|\Delta$ and $p(x) = \|T^{-1}(x)\|$, $x \in E \times E$. Then $f \in \Delta^*$, $|f(z)| \leq p(z)$ for all $z \in \Delta$. Suppose that there exists on $E \times E$ a linear extension $f_0$ of $f$ with $|f_0| \leq p$. Then $f_0 \circ T$ is a linear extension of $g$ satisfying $|f_0(Ty)| \leq p(Ty) = \|y\|$ for all $y \in K^2$, which is a contradiction. Now suppose that $\dim E > 1$. Take a non-zero element $a \in E$ and an algebraic complement $S$ of $[a]$. Take on $E \times E$ the seminorm $(s, \lambda a) \times (s', \gamma a) \mapsto p(\lambda a, \gamma a)$, where $p : [a] \times [a] \to \mathbb{R}$ is as above, and a linear functional $h : A K$ defined by $(s, \lambda a) \times (s, \lambda a) \mapsto f(\lambda a, \lambda a)$, where $f$ is as above. Clearly $|h| \leq p$ on $\Delta$ but $h$ cannot be extended to a linear functional $h_0$ on $E \times E$ with $|h_0| \leq p$ by the first part.

(2) From the proof of Theorem 1.1, (3) $\Rightarrow$ (4), one deduces also the following. Let $(E, \tau)$ be an lcs with trivial topological dual. Then there exists a family $(\tau_\alpha)_{\alpha \in A}$ of locally convex topologies on $E$ such that every $(E, \tau_\alpha)$ is isomorphic to $(E, \tau)$ and $(E, \sup_{\alpha \in A} \tau_\alpha)' = E^*$. Therefore, if $E \neq \{0\}$, then $(E, \tau)$ does not admit the Mackey topology.

(3) We do not know any example of a lcs $E$ over a non-spherically complete $K$ with $K < \dim E < K^#$ which admits the Mackey topology. Note that any lcs $(E, \tau)$ with $\dim E = K_0$ admits the Mackey topology. Indeed, if $(\tau_\alpha)_{\alpha \in A}$ is the family of all locally convex topologies on $E$ compatible with $\tau$, then $(E, \sup_{\alpha \in A} \tau_\alpha)$ is of countable type ([17], Examples 4.5). Now Lemma 3.1 completes the proof.

(4) Using Corollary 1 we deduce also that the spaces considered in Examples 2.3 admit the Mackey topology iff $K$ is spherically complete.

4. Proof of Theorem 1.4. Let $E$ be a polar lcs. By a special covering of $E'$ (cf. [17], Definition 7.3) we mean a covering $\mathcal{G}$ of $E'$ such that

(a) each member of $\mathcal{G}$ is edged, $\sigma(E', E)$-bounded, $\sigma(E', E)$-complete;
(b) for each $A, B \in \mathcal{G}$ there is a $C \in \mathcal{G}$ such that $A \cup B \subset C$;
(c) for each $A \in \mathcal{G}$ and $\lambda \in K$ there is a $B \in \mathcal{G}$ with $\lambda A \subset B$.

For a special covering $\mathcal{G}$ of $E'$ the $\mathcal{G}$-topology on $E$ is the topology induced by the seminorms $x \mapsto \sup\{|f(x)| : f \in A\}$, where $A$ runs through $\mathcal{G}$.

In order to prove Theorem 2 we shall need the following.

Lemma 4.1. Let $E$ be a polar lcs and suppose that the polar Mackey topology $\mu(E, E')$ exists. Then the family $\mathcal{G}$ of all edged and absolutely convex
complete compactoids in \((E', \sigma(E', E))\) is a special covering and \(\mu(E, E')\) equals the \(G\)-topology.

Proof. For each \(\mu(E, E')\)-continuous polar seminorm \(p\) on \(E\) set \(A_p = \{f \in E^*: |f| \leq p\}\). Then by ([17], Proposition 7.4 and its proof), the topology \(\mu(E, E')\) is the \(G\)-topology, where \(G = \{A_p: p\) is a \(\mu(E, E')\)-continuous seminorm on \(E\}\) is a special covering of \(E'\). Let \(A\) be an edged, absolutely convex complete compactoid in \((E\setminus \sigma(E, E))\) is a special covering and \(\sigma(E, E)\) equals the \(Q\)-topology. Let \(A\) be an edged, absolutely convex complete compactoid in \((E', \sigma(E', E))\); we show that \(A \in G\). (This will show that \(G\) is the largest possible special covering of \(E'\)). Let \(A = \{(\lambda A + F)^e: \lambda \in K, F \subset E', F\) be absolutely convex and \(\sigma(E', E)\)-bounded, \(\dim[F] < \infty\}\). It is easy to check that \(G_A\) is a special covering of \(E'\); the completeness of \((\lambda A + F)^e\) follows from ([18], Theorem 4.1). By ([17], Proposition 7.4), the \(\sigma/\sigma\)-topology is compatible with the original one of \(E\). Hence \(A = A\) is in \(G\) and \(A \subset A_p\). Since by ([17], Proposition 4.10), the set \(A\) is a polar set, we get equality.

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. \((1) \Rightarrow (2)\): Let \((F, \tau)\) be a lcs of finite type; then \(\sigma(F, F') = \tau\). The space \(E := (F', \sigma(F', F))\) admits the polar Mackey topology (by \((1)\)) and by Lemma 4.1 it is the \(G\)-topology, where \(G\) equals the set of all edged, absolutely convex, complete compactoids in \((E', \sigma(E', E))\). By \((b)\) above, for \(A, B \in G\) the set \(\overline{A+B}\) is complete; \((2)\) now follows because the natural map \((F, \sigma(F, F')) \rightarrow (E', \sigma(E', E))\) is an isomorphism.

\((2) \Rightarrow (3)\): Note that \([A+B]\) is of finite type, being a subspace of \(K^I\). By \((2)\) one gets that the closure of \(A+B\) in \([A+B]\) is complete; hence the closure of \(A+B\) in \(K^I\) lies in \([A+B]\).

\((3) \Rightarrow (4)\): Let \((e_i: i \in I)\) be an algebraic base for \(E\) and endow \(E\) with the finest locally convex topology. Then \(E^* = E'\) and \((E', \sigma(E', E))\) is isomorphic to the product \(K^I\). Let \(p, q\) be two polar seminorms on \(E\), then \(A = \{f \in E': |f| \leq p\}\) and \(B = \{f \in E': |f| \leq q\}\) are edged, absolutely convex, bounded and complete. Hence we may apply \((3)\) to conclude that \(\overline{A+B} \subset [A+B]\). By polarity of \(p\) and \(q\) we have \((\overline{A+B})^e = (A+B)^00 = (U \cap V)^0\), where \(U, V\) are the unit balls of \(p, q\), respectively, and so \(U \cap V\) is the unit ball in the normed topology defined by \(\max\{p, q\}\). Hence by \((3)\) one gets \(\{f \in E': |f| \leq \max\{p, q\}\} \subset (\overline{A+B})^e \subset [A+B] = [A] + [B]\). Thus, we can write every \(f \in E^*\) with \(|f| \leq \max\{p, q\}\) as \(g+h\), where \(g \in [A]\) and \(h \in [B]\); hence \(g\) is \(p\)-continuous and \(h\) is \(q\)-continuous.

\((4) \Rightarrow (1)\): It suffices to show that if \(\tau_1\) and \(\tau_2\) are polar locally convex topologies on a lcs \((E, \tau)\) compatible with \(\tau\), then so is \(\sup\{\tau_1, \tau_2\}\). Let \(p\) be a \(\tau_1\)-continuous polar seminorm and \(q\) be a \(\tau_2\)-continuous polar seminorm on \(E\). Let \(f \in E^*, |f| \leq \max\{p, q\}\). Then, by \((4)\), \(f = g+h\), where \(g\) is
$\tau_1$-continuous, $h$ is $\tau_2$-continuous, i.e. $g,h \in (E,\tau)'$. Hence $f$ is $\tau$-continuous and the proof is complete.

REFERENCES


