Cardinality and Mackey Topologies of Non-Archimedean Banach and Fréchet Spaces (*)

by

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Summary. Let $K$ be a non-Archimedean complete non-trivially valued field. One obtains the cardinality of non-Archimedean Fréchet spaces of countable type over $K$. This enables us to get a new characterization of the spherical completeness of $K$ in terms of the Hahn-Banach theorem and the Mackey-Arens theorem.

1. Introduction. Throughout the paper $K$ is a non-Archimedean valued field which is complete with respect to the metric induced by the non-trivial valuation $|.|$. If $E = (E, \tau)$ is a locally convex space (lcs) over $K$ with topology $\tau$ we denote by $\mathcal{P}(E)$ or $\mathcal{P}$ the family of (non-Archimedean) $\tau$-continuous seminorms. By $E'$ and $E^*$ we denote the topological and algebraic dual of $E$, respectively. For the basic notions and properties about lcs we refer to [21] when $K$ is spherically complete and to [17] when $K$ is not spherically complete. We recall only the following. A non-empty subset $B$ of a vector space $E$ (over $K$) is called absolutely convex if $ax + \beta y \in B$ whenever $x, y \in B$ and $\alpha, \beta \in K$, $|\alpha| \leq 1$, $|\beta| \leq 1$. If $B$ is absolutely convex then the edged hull $B^e$ of $B$ is defined as follows. If $K$ has discrete valuation,
then $B^n = B$; if the valuation is dense then $B^n = \bigcap_{|\lambda| > 1} \lambda B$ (cf. [17]).

If $E$ is an lcs and $p \in \mathcal{P}(E)$ by $E_p$ we denote the quotient space $E / \ker p$ endowed with the natural norm. The space $E$ is said to be of countable type if, for every continuous seminorm $p$ on $E$ the normed space $E_p$ is of countable type. Recall that a normed space $E$ is of countable type if there exists a countable subset $X$ of $E$ such that the closure of the linear hull $[X]$ of $X$ in $E$ equals $E$; this means that either $\dim E < \infty$ or the completion of $E$ is isomorphic to the space $c_0$ of null $K$-sequences (cf. [16]). The weak topology $\sigma(E, E')$ on $E$ is of countable type (cf. [17]). A seminorm $p$ on a $K$-vector space $E$ is polar if $p = \sup\{|f| : f \in E^*, |f| \leq p\}$. An lcs $E$ is polar if there is a basis of continuous seminorms consisting of polar seminorms (cf. [17] also for examples). An lcs $E$ is polar iff the polar neighbourhoods of zero form a neighbourhood basis of zero for the topology of $E$ (cf. [17], Proposition 5.2). A metrizable and complete lcs will be called a Fréchet space. Two locally convex topologies on $E$ will be called compatible if they define the same continuous linear functionals on $E$.

It is natural to ask if $E$ admits the Mackey topology $\mu(E, E')$, i.e. the finest locally convex topology $E$ whose topological dual is still $E'$. In ([21], Theorem 4.17), van Tiel proved that the Mackey topology exists if $K$ is spherically complete; in this case $\mu(E, E')$ is determined by the seminorms $x \mapsto \sup\{|f(x)| : f \in E^*\}$, where $A$ runs through the collection of all subsets of $E'$ which are bounded and $c$-compact in the weak topology $\sigma(E', E)$. In [6] we proved that whenever $K$ is not spherically complete then the space $\ell^\infty$ of bounded $K$-sequences does not admit the Mackey topology; nevertheless, the weak topology $\sigma(\ell^\infty, c_0)$ is of countable type and $\ell^\infty$ admits the finest locally convex topology of countable type to be compatible with $\sigma(\ell^\infty, c_0)$, this follows from Theorem 2.1 of [18] (cf. also [5], Theorem 2) and our Corollary 3.2.

The following question is still open (cf. [17]). Let $E$ be a polar lcs over a non-spherically complete field. Does the polar Mackey topology exist on $E$ (i.e. the strongest polar topology on $E$ compatible with the original one)? We know only that the answer is positive for polarly barrelled and polarly bornological spaces (cf. [17], Corollary 7.9).

In this note (applying ideas of [7] and [8]) we extend results of [6] and [7] by showing the following.

**Theorem 1.1.** Let $E$ be a vector space over $K$ with $\dim E \geq K^\#$, where $K^\#$ denotes the cardinal number of $K$ and $\dim E$ denotes the algebraic dimension of $E$. Let $\Delta = \{(x, x) : x \in E\} \subset E \times E$. The following assertions are equivalent.

1. $K$ is spherically complete.
2. For every locally convex topology $\tau$ on $E$ the space $(E, \tau)$ admits the
Mackey topology.

(3) There exists a locally convex topology $\tau$ on $E$ with $(E, \tau)^{\prime} \neq E^*$ such that $(E, \tau)$ admits the Mackey topology.

(4) $(E, \tau)^{\prime} \neq \{0\}$ for every (complete) Hausdorff locally convex topology $\tau$ on $E$.

(5) For every $f \in E^*$ and seminorms $p, q$ on $E$ such that $|f| \leq \max\{p, q\}$, there exist $f_1, f_2 \in E^*$ such that $f = f_1 + f_2$ and $f_1$ is $p$-continuous and $f_2$ is $q$-continuous.

(6) The diagonal $\Delta$ has the following Hahn-Banach property. For every seminorm $p$ on $E \times E$ and every $f \in \Delta^*$ such that $|f| \leq p$, there exists a linear extension $h$ of $f$ to $E \times E$ such that $|h| \leq p$.

We will see, however (see Remark 3.4 (a)) that (6) implies (1) for any vector space $E$ with $\dim E \geq 1$.

Since every linear functional on a vector space $E$ is continuous with respect to the finest locally convex topology, Theorem 1.1 yields also the following.

**Corollary 1.2.** Let $E$ be a lcs with $\dim E \geq K\#$. Then $E$ admits the Mackey topology iff $K$ is spherically complete or every linear functional on $E$ is continuous.

We will see (Proposition 2.2) that $\dim E \geq K\#$ for any infinite-dimensional Fréchet space. This together with Theorem 1 yields the following.

**Corollary 1.3.** Let $K$ be not spherically complete. Then no infinite-dimensional Fréchet space admits the Mackey topology.

We recall that by ([17], Corollary 7.9), every polar Fréchet space admits the polar Mackey topology.

The problem seems to be much more complicated when we are looking for the polar Mackey topology for non-metrizable spaces. We prove the following.

**Theorem 1.4.** The following conditions on $K$ are equivalent.

1. For every (polar) lcs over $K$ the polar Mackey topology exists.

2. For every lcs $E$ over $K$ of finite type (i.e. each continuous seminorm on $E$ has finite-codimensional kernel) and bounded complete absolutely convex subsets $A$ and $B$ the closure $A + B$ of $A + B$ is complete.

3. For every index set $I$, every pair of closed, bounded, absolutely convex subsets $A$ and $B$ of $K^I$ we have $A + B \subset [A + B]$.

4. If $p, q$ are polar seminorms on some $K$-vector space $E$ and if $f \in E^*$ such that $|f| \leq \max\{p, q\}$, then $f = f_1 + f_2$, where $f_1, f_2 \in E^*$, $f_1$ is $p$-continuous, $f_2$ is $q$-continuous.
In Theorem 1.1 we used essentially the assumption “dim $E \geq K^\#$”. It is natural to ask whether there is a relation between the cardinality of $K$ and the linear dimension of any Fréchet space $E$ over $K$. We will see (Propositions 2.2 and 2.5) that dim $E \geq K^\#$ for any infinite-dimensional Fréchet space and that $E$ contains a closed subspace $F$ such that dim$(E/F) = K^\#$.

2. Dimension of Fréchet spaces. It is known that dim $E \geq 2^{\mathbb{N}_0}$ for any infinite-dimensional real or complex Fréchet space; dim $E = 2^{\mathbb{N}_0}$, if $E$ is additionally separable (cf. [12]). In [13] Popov showed that for every nonatomic measure space $(\Omega, \Sigma, \mu)$ with $\Sigma^\# > 2^{\mathbb{N}_0}$ there exists a subset $\Omega_1$ of $\Omega$ with $\mu(\Omega_1) > 0$ such that the real space $L^p(\Omega_1), 0 < p < 1$, has no Hausdorff $2^{\mathbb{N}_0}$-dimensional quotient; recall that $L^p(\Omega_1)$ has trivial topological dual.

We start with the following observation.

**Lemma 2.1.** dim $\ell^\infty = K^\#$.  

**Proof.** Since dim $K^\mathbb{N} = (K^\#)^{\mathbb{N}_0}$ (cf. [10], 5(3), p. 75) and $\ell^\infty \subset K^\mathbb{N}$, we have dim $\ell^\infty \leq (K^\#)^{\mathbb{N}_0}$. On the other hand, we can proceed as in the proof of 5(3) of ([10], p. 75) and obtain dim $\ell^\infty \geq K^\#$. Now the conclusion follows from $(K^\#)^{\mathbb{N}_0} = K^\#$ (cf. [15], Corollary 3.9). We give a direct proof of this fact: take any $x = (x_n) \in K^\mathbb{N}$. For every $n \in \mathbb{N}$ there exists a unique two-sided sequence $(a_{jn} : j \in \mathbb{Z})$ of a full set of representatives in $\{x \in K : |x| \leq 1\}$ modulo $\{x \in K : |x| \leq |\pi|, 0 < |\pi| < 1\}$ such that $a_{-jn} = 0$ for large $j \in \mathbb{N}$ and $x_n = \sum_{j=-\infty}^{\infty} a_{jn} \pi^j$ (cf. [19], Theorems 12.1 and 12.3). Let $\sigma : \mathbb{N} \to \mathbb{Z} \times \mathbb{Z}$ be a bijection. Then the map $T : ((x_n)) \mapsto \sum_{n \in \mathbb{N}} a_{\sigma(n)} \pi^n$ of $K^\mathbb{N}$ into $K$ is (as easily seen) injective.

**Proposition 2.2.** Let $E$ be a sequentially complete lcs over $K$ containing an infinite-dimensional bounded set $B$. Then there exists a continuous injective linear map $T : \ell^\infty \to E$. In particular, dim $E \geq K^\#$ for any infinite-dimensional Fréchet space $E$.

**Proof.** Since the linear span of the closed absolutely convex hull of $B$ endowed with the Minkowski functional norm topology is an infinite-dimensional Banach space, we may assume that $E$ is an infinite-dimensional Banach space. Hence $E$ contains a subspace isomorphic to $c_0$. Let $(e_n)$ be a Schauder basis of this subspace; we may assume that $e_n \to 0$. Then the map $T : \ell^\infty \to E, T((b_n)) = \sum_{n=1}^{\infty} b_n e_n$ is continuous and injective.

**Examples 2.3.** By Proposition 2.2 infinite-dimensional Fréchet spaces have the dimension at least $K^\#$. Proposition 2.2 applies also to get the same conclusion for the following spaces. (1) Any $(LF)$-space (i.e. the inductive limit space of an increasing sequence of infinite-dimensional Fréchet
spaces, see [12], for the definition). (2) Any topological product of infinite-dimensional Fréchet spaces. (3) Any perfect sequence space $\Lambda \neq \varphi$ endowed with its natural topology $\pi(\Lambda, \Lambda^*)$, where $\Lambda^*$ is the Köthe dual of $\Lambda$ and $\varphi$ is the space of all sequences in $K$ with only finitely many of non-zero coordinates. By ([1], Proposition 7), where the assumption that $K$ is spherically complete is not necessary, the space $\Lambda$ is complete. Moreover, if $(\alpha_n) \in \Lambda$, $\alpha_n \neq 0$, $n \in \mathbb{N}$, then $\{(\beta_n) : |\beta_n| \leq |\alpha_n|, n \in \mathbb{N}\}$ is an infinite-dimensional bounded set in $\Lambda$.

**Corollary 2.4.** If $E$ is an infinite-dimensional Fréchet space over $K$ of countable type, then $\dim E = K^\#$.

**Proof.** Obviously, $\dim E \geq K^\#$. The space $E$ is isomorphic to a subspace of a countable product of Banach spaces of countable type; hence $E$ is isomorphic to a subspace of the product $c_0^\mathbb{N}$. Now $\dim E \leq \dim c_0^\mathbb{N} \leq (K^\#)^{\mathbb{N}_0} = K^\#$ which completes the proof.

**Proposition 2.5.** Every infinite-dimensional Fréchet space $E = (E, \tau)$ contains a closed subspace $F$ such that $\dim(E/F) = K^\#$.

**Proof.** If $\tau = \sigma(E, E')$, then $E$ is isomorphic to the space $K^\mathbb{N}$ (cf. [20], Theorem 7). Now suppose that $\tau \neq \sigma(E, E')$. Then there exists a continuous seminorm $p$ on $E$ such that $\dim E_p = \infty$. Let $\|\cdot\|_p$ be the corresponding norm on $E_p$. Let, for some $t \in (0, 1)$, $(e_n)$ be a $t$-orthogonal sequence in $E_p$ with $a < \|e_n\|_p \leq 1$, $n \in \mathbb{N}$, $0 < a < 1$. Let $(f_n)$ be a similar sequence in $\ell^\infty/c_0$. Define a continuous linear map $T : [e_n : n \in \mathbb{N}] \rightarrow [f_n : n \in \mathbb{N}]$ by $T(\sum_{n \in \mathbb{N}} \lambda_n e_n) = \sum_{n \in \mathbb{N}} \lambda_n f_n$. Since $\ell^\infty/c_0$ is spherically complete ([16], Theorem 4.1), there exists by Ingleton's theorem (cf. [16], Theorem 4.8), a continuous linear extension $T_0 : E_p \rightarrow \ell^\infty/c_0$ of $T$. Let $F := \ker(T_0 \circ \pi_p)$, where $\pi_p : E \rightarrow E_p$ is the quotient map. Then $F$ is closed, so $E/F$ is a Fréchet space, infinite-dimensional, and $\dim E/F = \dim(\ker T_0) \leq \dim(\ell^\infty/c_0) \leq K^\#$. By Proposition 2.2, $\dim E/F \geq K^\#$.

**Remark 2.6.** (1) In ([4], Proposition 2.6) De Grande-De Kimpe proved that any non-normable Fréchet space $E$ has a subspace isomorphic to $K^\mathbb{N}$. Moreover, she showed that if $K$ is spherically complete, this subspace is complemented in $E$. The latter result is even true for non-spherically complete $K$ if the dual $E'$ separates the points of $E$ (cf. [20], Corollary 9.1 (iv)).

(2) Proposition 2.5 suggests also the following question (still open for the real or complex Banach spaces, cf. [11,14]). Which infinite-dimensional Fréchet spaces over $K$ admit infinite-dimensional quotients of countable type? We shall say that $E$ has a *Quotient* if $E$ has such a quotient. We have the following partial answer.
(a) A Banach space has a Quotient iff \( E \) contains a complemented copy of \( c_0 \) (cf. [3]).

(b) If the valuation of \( K \) is discrete, then any infinite dimensional Banach space has a Quotient. This follows from ([16], Corollary 4.14).

(c) The space \( \ell^\infty \) does not have a Quotient if \( K \) has a dense valuation. Indeed, by ([16], Corollary 5.19) every continuous linear map of \( \ell^\infty \) onto \( c_0 \) is compact.

(d) If \( E \) is spherically complete and \( K \) has a dense valuation, then \( E \) does not have a Quotient. This follows from ([16], Corollary 5.20).

3. Proof of Theorem 1.1. We start with the following Lemma.

**Lemma 3.1.** Let \((E, \tau)\) be a lcs over \( K \). Let \( \{\tau_\alpha\}_{\alpha \in A} \) be a family of locally convex topologies on \( E \) compatible with \( \tau \). Then the topology \( \xi = \sup \tau_\alpha \) is compatible with \( \tau \) provided \( K \) is spherically complete or every topology \( \tau_\alpha \) is of countable type.

**Proof.** If \( K \) is spherically complete we proceed as in the proof of Theorem 1 of [7]. Now assume that every \( \tau_\alpha \) is of countable type. Observe that \((E, \xi)\) is of countable type. In fact \((E, \xi)\) is isomorphic to the diagonal \( \Delta \) of the product \( \prod_{\alpha \in A}(E, \tau_\alpha) \). The last product and also \( \Delta \) (endowed with the relative topology) are spaces of countable type by ([17], Proposition 4.12). Since spaces of countable type are strongly polar, Theorem 4.2 of [17] combined with our Lemma 1 of [7] (and its proof) yields the following. Let \( f \in (E, \xi)' \), \( \varepsilon > 0 \). There exist seminorms \( p_{\alpha_1}, p_{\alpha_2}, \ldots, p_{\alpha_n} \) continuous in \( \tau_{\alpha_1}, \tau_{\alpha_2}, \ldots, \tau_{\alpha_n} \), respectively, such that \( |f| \leq \max\{p_{\alpha_1}, p_{\alpha_2}, \ldots, p_{\alpha_n}\} \).

Then there are \( f_1, f_2, \ldots, f_n \in E^* \) such that \( f = f_1 + f_2 + \ldots + f_n \) and \( |f_i| \leq (1 + \varepsilon)p_{\alpha_i}, 1 \leq i \leq n \). Then (by assumption) \( f \in (E, \tau)' \).

**Corollary 3.2** [5, 18]. Every lcs \((E, \tau)\) over \( K \) admits the finest locally convex topology \( \mu \) of countable type compatible with \( \tau \).

**Proof.** Let \( (\tau_\alpha)_{\alpha \in A} \) be the family of all locally convex topologies on \( E \) of countable type finer than \( \sigma(E, E') \) and compatible with \( \tau \); recall that \((E, \sigma(E, E'))\) is one of them. Then using Lemma 3.1 one gets that \( \mu = \sup_{\alpha \in A} \tau_\alpha \) is compatible with \( \tau \) and \((E, \mu)\) is of countable type.

**Remark 3.3.** It is known (cf. [9]) that when \( K \) is not spherically complete, the space \((\ell^\infty, \sigma(\ell^\infty, c_0))\) has a Schauder basis but \( \ell^\infty \) (with respect to the norm topology) is not of countable type. Using Lemma 3.1 (and its proof) one deduces also that if \((E, \tau)\) is an lcs such that \((E, \sigma(E, E'))\) has a Schauder basis \((e_n)\), then \( E \) admits the finest locally convex topology \( \xi \) such that \( \sigma(E, E') \leq \xi \leq \tau \) and \((e_n)\) is a Schauder basis for \((E, \xi)\). In fact, \( \xi \)
is the finest compatible topology of countable type between \( \sigma(E, E') \) and \( \tau \) (observe that \( \xi \) and \( \sigma(E, E') \) have the same convergent sequences by ([17], Proposition 4.11)).

Now we are ready to prove our Theorem 1.1.

**Proof of Theorem 1.1.** (1) \( \Rightarrow \) (2) follows from Lemma 3.1 and (2) \( \Rightarrow \) (3) from the following observation. Let \( (x_\alpha) \) be a Hamel basis for \( E \). Then the topology \( \tau \) defined by the norm \( ||x|| = \max_\alpha |t_\alpha| \), \( x = \sum_\alpha t_\alpha x_\alpha \), where \( t_\alpha \in K \), is as in (3). Now we prove (3) \( \Rightarrow \) (4). Assume that \( (E, \tau) \) is a lcs as in (3) and \( E \) admits a locally convex topology \( \varphi_1 \) such that \( (E, \varphi_1)' = \{0\} \).

Take \( f_0 \in E^* \setminus E' \) and \( x_0 \neq 0 \) in \( E \) such that \( f_0(x_0) = 2 \). Consider the map \( T : E \rightarrow E \) defined by \( T(x) = x - f_0(x)x_0 \), then \( T^2 = \text{id} \). Let \( \varphi_2 \) be the image by \( T \) of \( \varphi_1 \). Consider the topologies \( \psi_i = \sup \{\sigma(E, E'), \varphi_i\} \), \( i = 1, 2 \). Observe that both topologies \( \psi_i \) are compatible with the original one of \( E \): fix \( i \in \{1, 2\} \). Let \( f \in (E, \psi_i)' \). Then there exist \( f_1, f_2, \ldots, f_n \in E' \) and a \( \varphi_i \)-continuous seminorm \( p \) such that \( |f| \leq \max\{|f_1|, \ldots, |f_n|, p\} \).

Let \( H = \bigcap_{k=1}^n \{x \in E : f_k(x) = 0\} \). Then \( |f| \leq p \) on \( H \). Since \( H \) is a finite-codimensional subspace of \( E \), there exists a continuous linear extension of \( f \) to the space \( (E, \varphi_i) \). Since \( (E, \varphi_i)' = \{0\} \), we have \( f = 0 \) on \( H \). Hence \( f \) is a linear combination of \( f_1, \ldots, f_n \), so \( f \in E' \). On the other hand \( f_0 \) is continuous with respect to the topology \( \sup \{\varphi_1, \varphi_2\} \). In fact, if \( (x_\alpha) \) is a null net in \( E \) with respect to the topology \( \varphi = \sup \{\varphi_1, \varphi_2\} \), then \( T(x_\alpha) = x_\alpha - f_0(x_\alpha)x_0 \rightarrow 0 \) in \( \varphi_2 \) and \( x_\alpha \rightarrow 0 \) in \( \varphi_2 \). Now \( x_0 \neq 0 \), so \( f_0(x_\alpha) \rightarrow 0 \). By assumption (the Mackey topology \( \mu(E, E') \) exists) we have \( \sigma(E, E') \leq \sup \{\psi_1, \psi_2\} \leq \mu(E, E') \). This implies that \( f_0 \in E' \), a contradiction.

(4) \( \Rightarrow \) (1). Assume that \( K \) is not spherically complete. Since \( \dim E \geq K^\# \) and also \( \dim \ell^\infty/c_0 = K^\# \), there exists an index set \( A \) and a family of vector subspaces \( E_\alpha \) of \( E \), \( \alpha \in A \), such that \( E = \bigoplus_{\alpha \in A} E_\alpha \) (algebraically) with \( \dim E_\alpha = \dim \ell^\infty/c_0 \) for all \( \alpha \in A \). Endow every \( E_\alpha \) with the isomorphic copy of the original topology of \( \ell^\infty/c_0 \). Let \( \phi \) be the Hausdorff locally convex direct sum topology on \( E = \bigoplus_{\alpha \in A} E_\alpha \). Then \( (E, \phi) \) is a Hausdorff and complete lcs, cf. [21] (where the assumption that \( K \) is spherically complete is not necessary). From \( (\ell^\infty/c_0)' = \{0\} \) we obtain \( (E, \phi)' = \{0\} \).

(1) \( \Rightarrow \) (5). It is a direct consequence of our Lemma 1 of [7]. (5) \( \Rightarrow \) (2). If \( \mu(E, E') \) is the supremum topology of all locally convex topologies on \( E \) compatible with the original one, then \( \mu(E, E') \) is compatible with the original one; hence \( \mu(E, E') \) is the Mackey topology of \( E \). (1) \( \Rightarrow \) (6). This follows from Ingleton's Theorem, (cf. [21], Theorem 3.5). (6) \( \Rightarrow \) (5). We proceed similarly as in the proof of Lemma 1 of [7].

**Remark 3.4.** (1) Observe that the implication (6) \( \Rightarrow \) (1) is true for any
vector space \( E \) over \( K \) with \( \dim E \geq 1 \). In fact, assume that \( K \) is not spherically complete. First, suppose that \( \dim E = 1 \), i.e. the space \( E \) is algebraically isomorphic to \( K \). According to ([16], p. 68) there exists a norm \( \| \cdot \| \) on \( K^2 \), a one-dimensional space \( D \) and \( g \in D' \) with \( \| g(x) \| \leq \| x \| , x \in D \), that does not admit an extension \( g_0 \in (K^2, \| \cdot \|)' \) with \( \| g_0(x) \| \leq \| x \| , x \in K^2 \).

Now let \( T : K \times K \rightarrow E \times E \) be a linear bijection such that \( T(D) = \Delta \). Set \( f = g \circ T^{-1}\Delta \) and \( p(x) = \| T^{-1}(x) \| , x \in E \times E \). Then \( f \in \Delta^* , \| f(z) \| \leq p(z) \) for all \( z \in \Delta \). Suppose that there exists on \( E \times E \) a linear extension \( f_0 \) of \( f \) with \( \| f_0 \| \leq p \). Then \( f_0 \circ T \) is a linear extension of \( g \) satisfying \( \| f_0(Ty) \| \leq p(Ty) = \| y \| \) for all \( y \in K^2 \), which is a contradiction. Now suppose that \( \dim E > 1 \). Take a non-zero element \( a \in E \) and an algebraic complement \( S \) of \([a]\). Take on \( E \times E \) the seminorm \( (s, \lambda a) \times (s', \gamma a) \mapsto p(\lambda a, \gamma a) \), where \( p : [a] \times [a] \rightarrow \mathbb{R} \) is as above, and a linear functional \( h : A K \) defined by \( (s, \lambda a) \times (s, \gamma a) \mapsto f(\lambda a, \lambda a) \), where \( f \) is as above. Clearly \( |h| \leq p \) on \( \Delta \) but \( h \) cannot be extended to a linear functional \( h_0 \) on \( E \times E \) with \( \| h_0 \| \leq p \) by the first part.

(2) From the proof of Theorem 1.1, (3) \( \Rightarrow \) (4), one deduces also the following. Let \((E, \tau)\) be an lcs with trivial topological dual. Then there exists a family \((\tau_\alpha)_{\alpha \in A}\) of locally convex topologies on \( E \) such that every \((E, \tau_\alpha)\) is isomorphic to \((E, \tau)\) and \((E, \sup_{\alpha \in A} \tau_\alpha)' = E^* \). Therefore, if \( E \neq \{0\} \), then \((E, \tau)\) does not admit the Mackey topology.

(3) We do not know any example of a lcs \( E \) over a non-spherically complete \( K \) with \( \aleph_0 < \dim E < K^\# \) which admits the Mackey topology. Note that any lcs \((E, \tau)\) with \( \dim E = \aleph_0 \) admits the Mackey topology. Indeed, if \((\tau_\alpha)_{\alpha \in A}\) is the family of all locally convex topologies on \( E \) compatible with \( \tau \), then \((E, \sup_{\alpha \in A} \tau_\alpha)\) is of countable type ([17], Examples 4.5). Now Lemma 3.1 completes the proof.

(4) Using Corollary 1 we deduce also that the spaces considered in Examples 2.3 admit the Mackey topology iff \( K \) is spherically complete.

4. Proof of Theorem 1.4. Let \( E \) be a polar lcs. By a special covering of \( E' \) (cf. [17], Definition 7.3) we mean a covering \( G \) of \( E' \) such that

(a) each member of \( G \) is edged, \( \sigma(E', E) \)-bounded, \( \sigma(E', E) \)-complete;
(b) for each \( A, B \in G \) there is a \( C \in G \) such that \( A \cup B \subset C \);
(c) for each \( A \in G \) and \( \lambda \in K \) there is a \( B \in G \) with \( \lambda A \subset B \).

For a special covering \( G \) of \( E' \) the G-topology on \( E \) is the topology induced by the seminorms \( x \mapsto \sup_{f \in A} \| f(x) \| : f \in A \}, \) where \( A \) runs through \( G \).

In order to prove Theorem 2 we shall need the following.

**Lemma 4.1.** Let \( E \) be a polar lcs and suppose that the polar Mackey topology \( \mu(E, E') \) exists. Then the family \( G \) of all edged and absolutely convex
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complete compactoids in \((E', \sigma(E', E))\) is a special covering and \(\mu(E, E')\) equals the \(G\)-topology.

**Proof.** For each \(\mu(E, E')\)-continuous polar seminorm \(p\) on \(E\) set \(A_p = \{f \in E^* : |f| \leq p\}\). Then by ([17], Proposition 7.4 and its proof), the topology \(\mu(E, E')\) is the \(G\)-topology, where \(G = \{A_p : p\) is a \(\mu(E, E')\)-continuous seminorm on \(E\}\) is a special covering of \(E'\). Let \(A\) be an edged, absolutely convex complete compactoid in \((E, \sigma(E, E))\) is a special covering and \(\lambda(E, E)\) equals the \(Q\)-topology. Let \(A\) be an edged, absolutely convex complete compactoid in \((E', \sigma(E', E))\); we show that \(A \in G\). (This will show that \(Q\) is the largest possible special covering of \(E'\)).

Let \(Q_a = \{(\lambda A + F)^e : \lambda \in K, F \subset E', F\) be absolutely convex and \(\sigma(E', E)\)-bounded, \(\dim[F] < \infty\}\). It is easy to check that \(Q_a\) is a special covering of \(E'\); the completeness of \((\lambda A + F)^e\) follows from ([18], Theorem 4.1). By ([17], Proposition 7.4), the \(\lambda/\lambda\)-topology is compatible with the original one of \((E', \sigma(E', E))\). Hence \(A\) is in \(G\) and \(A \subset A_p\). Since by ([17], Proposition 4.10), the set \(A\) is a polar set, we get equality.

Now we are ready to prove Theorem 1.4.

**Proof** of Theorem 1.4. (1) \(\Rightarrow\) (2): Let \((F, r)\) be a lcs of finite type; then \(\sigma(F, F') = r\). The space \(E := (F', \sigma(F', F))\) admits the polar Mackey topology (by (1)) and by Lemma 4.1 it is the \(G\)-topology, where \(G\) equals the set of all edged, absolutely convex, complete compactoids in \((E', \sigma(E', E))\). By (b) above, for \(A, B \in G\) the set \(A + B\) is complete; (2) now follows because the natural map \((F, \sigma(F, F')) \rightarrow (E', \sigma(E', E))\) is an isomorphism.

(2) \(\Rightarrow\) (3): Note that \([A + B]\) is of finite type, being a subspace of \(K^I\). By (2) one gets that the closure of \(A + B\) in \([A + B]\) is complete; hence the closure of \(A + B\) in \(K^I\) lies in \([A + B]\).

(3) \(\Rightarrow\) (4): Let \((e_i : i \in I)\) be an algebraic base for \(E\) and endow \(E\) with the finest locally convex topology. Then \(E^* = E'\) and \((E', \sigma(E', E))\) is isomorphic to the product \(K^I\). Let \(p, q\) be two polar seminorms on \(E\), then \(A = \{f \in E' : |f| \leq p\}\) and \(B = \{f \in E' : |f| \leq q\}\) are edged, absolutely convex, bounded and complete. Hence we may apply (3) to conclude that \(A + B \subset [A + B]\). By polarity of \(p\) and \(q\) we have \((A + B)^e = (A + B)^{00} = (U \cap V)^0\), where \(U, V\) are the unit balls of \(p, q\), respectively, and so \(U \cap V\) is the unit ball in the normed topology defined by \(\max\{p, q\}\). Hence by (3) one gets \(\{f \in E' : |f| \leq \max\{p, q\}\} \subset (A + B)^e \subset [A + B] = [A] + [B]\). Thus, we can write every \(f \in E^*\) with \(|f| \leq \max\{p, q\}\) as \(g + h\), where \(g \in [A]\) and \(h \in [B]\); hence \(g\) is \(p\)-continuous and \(h\) is \(q\)-continuous.

(4) \(\Rightarrow\) (1): It suffices to show that if \(\tau_1\) and \(\tau_2\) are polar locally convex topologies on a lcs \((E, \tau)\ compatible with \(\tau\), then so is \(\sup\{\tau_1, \tau_2\}\). Let \(p\) be a \(\tau_1\)-continuous polar seminorm and \(q\) be a \(\tau_2\)-continuous polar seminorm on \(E\). Let \(f \in E^*\), \(|f| \leq \max\{p, q\}\). Then, by (4), \(f = g + h\), where \(g\) is
$\tau_1$-continuous, $h$ is $\tau_2$-continuous, i.e. $g, h \in (E, \tau)'$. Hence $f$ is $\tau$-continuous and the proof is complete.

REFERENCES


