A SCALAR FIELD FOR WHICH $C_0$ HAS NO HAHN–BANACH PROPERTY

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Abstract. Let $K$ be a field with a valuation $| |$ of infinite rank for which $(K, | |)$ is complete, separable and such that the value group is countable. We show that the Banach space $c_0$ consisting of all null sequences in $K$ admits a closed subspace $S$ and a continuous linear function $S \rightarrow K$ that cannot be extended to a continuous linear function $c_0 \rightarrow K$.

This surprising result is in contrast not only to Keller's example [2] of a separable inner product Banach space over $K$ resembling classical Hilbert space, but also to the Hahn–Banach property of $c_0$ over a field with a rank 1 valuation [4, 3.16].


Preliminaries. (For details on valuations see e.g. [5]).

Let $G$ be a multiplicatively written totally ordered abelian group. We require $G \neq \{1\}$ implying that $G$ is torsion free. We add an element $0$ such that $0 < g$ for all $g \in G$.

Let $K$ be a field. A valuation (with value group $G$) is a surjection $| | : K \rightarrow G \cup \{0\}$ such that (i) $|x| = 0$ if and only if $x = 0$ (ii) $|xy| = |x| \cdot |y|$ (iii) $|x + y| \leq \max(|x|, |y|)$ for all $x, y \in K$.

Remark. Usually $G$ is written additively (with reversed ordering and an element $+\infty$ adjoined) but here we prefer the multiplicative notation because of its analogy with the absolute value.

For $a \in K$, $\varepsilon \in G$ we write $B(a, \varepsilon) := \{x \in K : |x - a| \leq \varepsilon\}$. The topology induced naturally by the valuation makes $K$ into a topological field. A subgroup $H$ of $G$ is called convex if $g, k \in H$, $g < h$ implies $\{t \in G : g \leq t \leq h\} \subset H$.

Henceforth we shall assume that

(i) $(K, | |)$ is complete (i.e. each Cauchy net converges) and separable,
(ii) \( G \) is countable.
Then it follows that
(iii) \( K \) is metrizable,
(iv) for every convex subgroup \( H \) of \( G \), \( H \neq \{1\}, H \neq G \) there exists a strictly decreasing sequence \( g_1, g_2, \ldots \) in \( G \) such that \( H = \bigcap \{g \in G : g_{n+1}^{-1} \leq g \leq g_n\} \).
We further assume that the valuation has infinite rank which in our case means that
(v) \( G \) is the union of convex subgroups \( H_1, H_2, \ldots \) for which
\[
\{1\} \subsetneq H_1 \subsetneq H_2 \subsetneq \ldots
\]
We indicate an example of a field satisfying (i) – (v) ([3], 2.1). Let, for each \( i \in \mathbb{N} \), \( G_i \) be the ordered group \( \mathbb{Z} \), written multiplicatively, with positive generator \( a_i \). Let \( G := \bigoplus_i G_i \), ordered antilexicographically. The following formulas define a valuation on \( \mathbb{Q}(X_1, X_2, \ldots) \), the field of rational functions in countably many variables over \( \mathbb{Q} \).
\[
|q| := \begin{cases} 
0 & \text{if } q \in \mathbb{Q}, q = 0 \\
(1,1,\ldots) \in G & \text{if } q \in \mathbb{Q}, q \neq 0 
\end{cases}
\]
\[
|X_i| := (b_{i1}, b_{i2}, \ldots) \in G 
\]
where \( b_{ij} = a_i \) if \( j = i \), \( b_{ij} = 1 \) if \( j \neq i \).
It is clear that the completion \( K \) of \( \mathbb{Q}(X_1, X_2, \ldots) \) with respect to the valuation \( | \cdot | \) satisfies (i), (ii) above.
Also (v) is true if we take \( H_n = \bigoplus_{i=1}^n G_i = \{(b_1, b_2, \ldots) \in G : b_i = 1 \text{ for } i > n\} \).
We finally note that this \( K \) also served as the scalar field of the Hilbert space constructed by Keller [2].

In this note a norm on a \( K \)-vector space \( E \) is a map \( \| \cdot \| : E \to G \cup \{0\} \) (we do not need the more usual notion of norm with extended range) satisfying (i) \( \|x\| = 0 \) if and only if \( x = 0 \) (ii) \( \|\lambda x\| = |\lambda| \|x\| \) (iii) \( \|x + y\| \leq \max(\|x\|, \|y\|) \) for all \( x, y \in E, \lambda \in K \). The dual space \( E' \) is the space of all continuous linear functions \( E \to K \). \( E = (E, \| \cdot \|) \) is called a \( K \)-Banach space if \( E \) is complete.
Let \( E_1, E_2, \ldots \) be \( K \)-Banach spaces. By \( \bigoplus E_n \) we mean the space consisting of all sequences \( x = (x_n)_{n \in \mathbb{N}} \), where \( x_n \in E_n \) for each \( n \), for which \( \lim_{n \to \infty} x_n = 0 \) normed by \( x \mapsto \max_n \|x_n\| \).
As usual, we denote \( \bigoplus K \) by \( c_0 \). It follows easily that \( \bigoplus E_n \) is a \( K \)-Banach space. If each \( E_n \) is separable then so is \( \bigoplus E_n \).
Let \( E, F \) be \( K \)-Banach spaces. We say that a surjective linear continuous map \( \pi : E \to F \) is a quotient map (and \( F \) is a quotient of \( E \)) if \( \|y\| = \inf \{\|x\| : x \in E, \pi(x) = y\} \) for each \( y \in F \).
The following Lemma is a non-archimedean version of the 'second half of the Open Mapping Theorem'. Although the proof is not materially different from the classical one we include it to convince the reader of this very fact.

Denote the 'closed unit ball' \( \{ x \in E : \|x\| \leq 1 \} \) of a \( K \)-Banach space \( E \) by \( B_E \), and the closure of a set \( X \subset E \) by \( \overline{X} \).

**Lemma.** Let \( E, F \) be \( K \)-Banach spaces, let \( \pi : E \to F \) be a continuous linear map. Suppose \( \pi(B_E) \) is a zero neighbourhood in \( F \). Then \( \pi(B_E) = \overline{\pi(B_E)} \) (and in particular \( \pi(B_E) \) is a zero neighbourhood).

**Proof.** There exist \( \lambda_0, \lambda_1, \ldots \in K \) such that \( \lambda_0 = 1, |\lambda_0| > |\lambda_1| > \cdots, \lim_{n \to \infty} \lambda_n = 0 \) and \( \lambda_1 B_F \subset \overline{\pi(B_E)} \). Let \( y \in \overline{\pi(B_E)} \); we construct an \( x \in B_E \) with \( \pi(x) = y \). There exists an \( x_0 \in B_E \) with \( \|y - \pi(x_0)\| \leq |\lambda_1|^2 \). Then

\[
y = \pi(\lambda_0 x_0) + \lambda_1 y_1 \text{ where } y_1 \in \lambda_1 B_F \subset \overline{\pi(B_E)}.
\]

Next, we can take an \( x_1 \in B_E \) with \( \|y_1 - \pi(x_1)\| \leq |\lambda_2| \) so

\[
y = \pi(\lambda_0 x_0) + \pi(\lambda_1 x_1) + \lambda_2 y_2 \text{ where } y_2 \in \lambda_1 B_F \subset \overline{\pi(B_E)}.
\]

Similarly we can find an \( x_2 \in B_E \) with \( \|y_2 - \pi(x_2)\| \leq |\lambda_2| \). Then

\[
y = \pi(\lambda_0 x_0) + \pi(\lambda_1 x_1) + \pi(\lambda_2 x_2) + \lambda_3 y_3 \text{ where } y_3 \in \lambda_2 B_F \subset \overline{\pi(B_E)}
\]

etc. Inductively we obtain \( x_1, x_2, \ldots \in B_E \) such that \( y = \sum_{n=0}^{\infty} \pi(\lambda_n x_n) \). By completeness \( x := \sum_{n=0}^{\infty} \lambda_n x_n \) exists in \( E \), \( \|x\| = \max_n \|\lambda_n x_n\| \leq 1 \) and \( \pi(x) = y \).

**§1. A SEPARABLE BANACH SPACE WITHOUT THE HAHN-BANACH PROPERTY**

As a stepping stone we construct a separable \( K \)-Banach space \( E \) having no Hahn-Banach property (Theorem 1.5). Let us first observe that continuous maps between \( K \)-Banach spaces are bounded. This is a simple consequence of our assumption that the range of the norm is \( G \cup \{0\} \), enabling us to normalize each nonzero vector by a suitable scalar multiplication. (If one allows norms to have a more general range, continuous linear maps are not always bounded [1]). For our purpose we just need the following Lemma.

**Lemma 1.1.** For a \( K \)-Banach space \( E \), every \( f \in E' \) is bounded i.e. there exists a \( c \in G \) such that \( |f(x)| \leq c\|x\| \) for all \( x \in E \).

The next Lemma tells us that \( K \) is not spherically complete.
**Lemma 1.2.** Let $\lambda_1, \lambda_2, \ldots \in K$ be such that $|\lambda_1| > |\lambda_2| > \cdots$ is bounded below in $G$. Then there exist $a_1, a_2, \ldots \in K$ such that $B(a_1, |\lambda_1|), B(a_2, |\lambda_2|), \ldots$ form a nest with empty intersection.

**Proof.** Let $\{b_1, b_2, \ldots\}$ be dense in $K$. The equivalence relation $|x - y| \leq |\lambda_1|$ divides $K$ into balls, so there is one, say $B(a_1, |\lambda_1|)$, that does not contain $b_1$. Similarly, the ball $B(a_1, |\lambda_1|)$ is divided into (at least two) balls of radius $|\lambda_2|$ so that we can find one, say $B_2(a_2, |\lambda_2|)$, that does not contain $b_2$ etc. Inductively we arrive at a nest $B(a_1, |\lambda_1|) \supset B_2(a_2, |\lambda_2|) \supset \cdots$ where $b_n \not\in B(a_n, |\lambda_n|)$ for each $n$. Then $B := \bigcap_n B(a_n, |\lambda_n|)$ contains none of the $b_n$, so it has empty intersection. On the other hand, if $B \neq \emptyset$ it would contain a ball with radius $|\lambda|$ where $\lambda \in K$, $0 < |\lambda| \leq |\lambda_n|$ for each $n$. We conclude that $B = \emptyset$.

**Lemma 1.3.** Let $\lambda_1, \lambda_2, \ldots \in K$ be such that $H := \{s \in G : |\lambda_n|^{-1} \leq s \leq |\lambda_n| \text{ for all } n\}$ is a convex subgroup, $H \neq \{1\}$. If $a_1, a_2, \ldots \in K$ are such that $B(a_1, |\lambda_1|), B(a_2, |\lambda_2|), \ldots$ is a nest with empty intersection then so is for each $c \in H$, $c \geq 1$, $B(a_1, c|\lambda_1|), B(a_2, c|\lambda_2|), \ldots$.

**Proof.** For each $n$, $a_{n+1} \in B(a_n, |\lambda_n|) \subseteq B(a_n, c|\lambda_n|)$ so the balls $B(a_n, c|\lambda_n|)$ ($n \in \mathbb{N}$) form a nest. To complete the proof we show that any $x \in \bigcap_n B(a_n, c|\lambda_n|)$ is also in $\bigcap_n B(a_n, |\lambda_n|)$.

To this end it suffices to show that $x \in B(a_1, |\lambda_1|)$. Suppose $|x - a_1| > |\lambda_1|$. Since $|a_m - a_1| \leq |\lambda_1|$ we have, for all $m$, $|x - a_m| = |x - a_1|$ so there is an $\alpha \in K$ such that

$$|\alpha| = |x - a_m| \leq c|\lambda_m| \quad (m \in \mathbb{N}).$$

Also $|\lambda_m|^{-1} \leq 1 \leq c^{-1}|\lambda_1| < c^{-1}|\alpha| \leq |\lambda_m|^{-1} \leq c^{-1}|\alpha| \leq |\lambda_m|^{-1}$ for each $m$. Hence $|\alpha| \in H$ conflicting $|\alpha| > |\lambda_1|$.

**Lemma 1.4.** (Compare [4], p. 68) For every convex subgroup $H$ of $G$, $H \neq \{1\}$, $H \neq G$ there exist $\lambda_1, \lambda_2, \ldots, a_1, a_2, \ldots \in K$ as in Lemma 1.3.

The formula

$$\|(\lambda, \mu)\| = \lim_{m \to \infty} |\lambda - \mu a_m|$$

defines a norm on $K^2$. The function $f : (\lambda, 0) \mapsto \lambda$ (\(\lambda \in K\)) satisfies $|f(x)| \leq \|x\| \quad (x \in K \times \{0\})$ but for any extension $\tilde{f} \in (K^2)'$ we have $|\tilde{f}(x)| \leq c\|x\| \quad (x \in K^2)$ for no $c \in H$.

**Proof.** Choose, for each $n$, $\lambda_n \in K$ such that $|\lambda_n| = g_n$ where $g_n$ is as in (iv) of the Preliminaries. For the existence of $a_1, a_2, \ldots$ combine Lemmas 1.2 and 1.3. After observing that for each $\xi \in K$ the sequence $m \mapsto |\xi - a_m|$ is eventually constant the proof that $\|\cdot\|$ is a norm becomes straightforward. Obviously $\|(\lambda, 0)\| = |\lambda| = |f(\lambda, 0)|$ for each $\lambda \in K$.

Now let $\tilde{f} \in (K^2)'$ be an extension of $f$ and suppose

\[ (*) \quad |\tilde{f}(x)| \leq c\|x\| \quad (x \in K^2) \]
for some \( c \in H \). Then \( c \geq 1 \). Writing \( \alpha := -\tilde{f}(0,1) \) formula (*) becomes

\[
|\lambda - \mu \alpha| \leq c \lim_{m \to \infty} |\lambda - \mu a_m| \quad (\lambda, \mu \in K),
\]

implying

\[
|\lambda - \alpha| \leq c \lim_{m \to \infty} |\lambda - a_m| \quad (\lambda \in K),
\]

whence

\[
|a_n - \alpha| \leq c \lim_{m \to \infty} |a_n - a_m| \quad (n \in \mathbb{N})
\]

But \( |a_n - a_m| \leq |\lambda_n| \) for \( m \geq n \) so that \( |a_n - \alpha| \leq c|\lambda_n| \) for all \( n \in \mathbb{N} \), in other words \( \alpha \in \bigcap B(a_n, c|\lambda_n|) = \emptyset \), a contradiction.

**Theorem 1.5.** There exist a separable \( K \)-Banach space \( E \), a subspace \( D \) of \( E \) and an element \( f \in D \) that cannot be extended to an element of \( E' \).

**Proof.** Let \( G \) be the union of the convex subgroups \( \{1\} \subseteq H_1 \subseteq H_2 \subseteq \ldots \). By applying Lemma 1.4 to \( H := H_n \) we can find for each \( n \in \mathbb{N} \) a norm \( || \cdot ||_n \) on \( K^2 \) such that for any linear extension \( \tilde{f}_n \) of \( f_n : (\lambda,0) \mapsto \lambda \) \( (\lambda \in K) \) we have

\[
(*) \quad |\tilde{f}_n(x)| \leq c||x||_n \quad (x \in K^2) \quad \text{for no } c \in H_n.
\]

Now set \( E := \bigoplus_n (K^2, || \cdot ||_n) \) and denote its norm by \( || \cdot || \). \( E \) is separable. Let \( D \) be the subspace of all \( (\lambda_n, \mu_n)_{n \in \mathbb{N}} \) for which \( \mu_n = 0 \) for all \( n \) and define \( f : D \to K \) by the formula

\[
f((\lambda_n,0)_{n \in \mathbb{N}}) = \sum_{n=1}^{\infty} f_n((\lambda_n,0)) = \sum_{n=1}^{\infty} \lambda_n.
\]

Then \( |f(x)| \leq ||x|| \quad (x \in D) \). However, if \( \tilde{f} \in E' \) is an extension then by Lemma 1.1 there exists a \( c \in G \) for which \( |\tilde{f}(x)| \leq c||x|| \quad (x \in E) \). Then \( c \in H_n \) for some \( n \) and the restriction of \( \tilde{f} \) to \( (K^2, || \cdot ||_n) \) is an extension of \( f_n \) for which \( |\tilde{f}(x)| \leq c||x||_n \quad (x \in (K^2, || \cdot ||_n)) \) conflicting (*). It follows that \( F \) has no continuous linear extension \( E \to K \).

**§2. The Space \( c_0 \) Does Not Have the Hahn-Banach Property**

**Theorem 2.1.** Every separable \( K \)-Banach space \( E \) is a quotient of \( c_0 \).

**Proof.** The set \( \{y \in E : ||y|| = 1\} \) is separable; let \( \{y_1, y_1, \ldots\} \) be a dense subset. Define \( \pi : c_0 \to E \) by the formula

\[
\pi((\xi_1, \xi_2, \ldots)) = \sum_{i=1}^{\infty} \xi_i y_i.
\]
Then clearly \( \|\pi(x)\| \leq \|x\| \) for all \( x \in c_0 \). For each \( \lambda \in K, 0 < |\lambda| \leq 1 \) the set \( \{\lambda y_1, \lambda y_2, \ldots\} \) is dense in \( \{x \in E : \|x\| = |\lambda|\} \). It follows that \( \pi(B_{c_0}) \) is dense in \( B_E \). By the Lemma of the Preliminaries we may conclude that \( \pi(B_{c_0}) = B_E \) and also \( \pi(\lambda B_{c_0}) = \lambda B_E \) for each \( \lambda \in K \). Now let \( y \in E, \|y\| = |\lambda| \). Then there exists an \( x \in c_0 \) with \( \pi(x) = y \) and \( \|x\| \leq |\lambda| \). But also \( |\lambda| = \|y\| = \|\pi(x)\| \leq \|x\| \), so \( \|x\| = |\lambda| \). It follows that \( \pi \) is a quotient map.

**Remark.** We even proved that \( E \) is a strict quotient of \( c_0 \) i.e., \( \|y\| = \min\{\|x\| : x \in c_0, \pi(x) = y\} \) for each \( y \in E \).

**Theorem 2.2.** There exist a (closed) subspace \( S \) of \( c_0 \) and a \( g \in S' \) that cannot be extended to an element of \( c_0' \).

**Proof.** Let \( E, D, f \) be as in Theorem 1.5 and let \( \pi : c_0 \rightarrow E \) be the strict quotient map of Theorem 2.1. Set \( S := \pi^{-1}(D) \). The function \( g := f \circ \pi \) is in \( S' \). If \( \bar{g} \in c_0' \) were an extension of \( g \) then the unique map \( \bar{f} \) making the diagram

\[
\begin{array}{ccc}
 c_0 & \xrightarrow{\pi} & E \\
 \bar{g} \downarrow & & \downarrow \bar{f} \\
 K & & \\
\end{array}
\]

commute, is a linear extension of \( f \). But \( \bar{f} \) is also continuous (by Lemma 1.1 there is a \( c \in G \) such that \( |\bar{g}(z)| \leq c\|z\| \) for all \( z \in c_0 \). Let \( y \in E \), choose an \( x \in c_0 \) with \( \pi(x) = y \), \( \|x\| = \|y\| \). Then \( |\bar{f}(y)| = |\bar{f}(\pi(x))| = |\bar{g}(x)| \leq c\|x\| = c\|y\| \), a contradiction by Lemma 1.5.

Some further questions (recall that we assumed throughout that norm values are in \( G \cup \{0\} \) and that \( (K, |\ |) \) satisfies (i)-(v) of the Preliminaries):

1. Does there exist an infinite-dimensional \( K \)-Banach space having the Hahn-Banach property?
2. Does there exist a separable \( K \)-Banach space whose dual does not separate points (resp. is trivial)?
3. Does there exist a separable \( K \)-Banach space without a Schauder base? In particular, does the space \( E \) of Theorem 1.5 have a Schauder base?
4. It is not very hard to see that any Banach space over a spherically complete valued field \( (L, |\ |) \) has the Hahn-Banach property. Now let \( (L, |\ |) \) be a field with a complete valuation of infinite rank that is not spherically complete. Does it follow that \( c_0 \) does not have the Hahn-Banach property?

**References**

A scalar field for which $C$-zero has...


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