COMPACTIFICATION AND COMPACTOIDIFICATION

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Abstract. After discussing some of the many ways to get the Banaschewski compactification \( \beta_0 T \) of an arbitrary ultraregular space \( T \), we develop another construction of \( \beta_0 T \) in Th. 2.1. Using those ideas, we develop an analog of \( \beta_0 T \)—what we call a compactoidification \( \kappa T \) of an ultraregular space \( T \) in Sec. 3; \( \kappa T \) is, in essence, a complete absolutely convex compactoid 'superset' of \( T \) to which continuous maps of \( T \) with precompact range into any complete absolutely convex compactoid subset may be 'continuously extended.'

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1 The Many Faces

For any topological spaces \( X \) and \( Y \), \( C(X, Y) \) and \( C^*(X, Y) \) denote the spaces of continuous maps of \( X \) into \( Y \) and the continuous maps of \( X \) into \( Y \) with relatively compact range, respectively. To say that a topological space \( X \) is ultraregular or ultranormal means, respectively, that the clopen sets are a basis or disjoint closed subsets of \( X \) may be separated by clopen sets. A synonym for ultraregular is \( 0 \)-dimensional. We have a slight preference for the former in order to avoid confusion with other notions of dimension. Throughout the discussion, \( T \) denotes at least a Hausdorff space. For an ultraregular space \( E \) containing at least two points and ultraregular \( T \), B. Banaschewski [2] discovered a compactification \( \beta_0 T \) of \( T \) in which every \( x \in C^*(T, E) \) may be continuously extended to \( \beta_0 x \in C(\beta_0 T, E) \). \( \beta_0 T \) is nowadays usually called the Banaschewski compactification of \( T \). It functions as the natural analog of the Stone-Čech compactification (\( \beta T \) is \( \beta T \) for ultranormal \( T \)) in non-Archimedean analysis. Like the Stone-Čech compactification, the Banaschewski compactification is a protean entity, assuming many different guises. We discuss some of them in this section and then develop a new one in Sec. 2.

1.1 As a completion

Let \( E \) be an ultraregular space containing at least two points and let \( T \) be ultraregular. Let \( C^*(T, E) \) denote the weakest uniform structure on \( T \) making each \( x \in C^*(T, E) \) uniformly continuous into the compact space \( \text{cl} \ x(T) \) equipped with its unique compatible uniform
structure. By \cite{1}, pp. 92-93, since $T$ is ultra-regular, $C^* (T, E)$ is compatible with the topology on $T$ and $C^* (T, E)$ is a precompact uniform structure on $T$. Since $C^* (T, E)$ is precompact, its completion $\beta_0 T$ is compact and is called the Banaschewski compactification of $T$. $\beta_0 T$ is ultranormal (\cite{2}, p. 131, Satz 2 or \cite{1}, p. 93, Theorem 1)—hence ultra-regular—and, by the usual process of extension by continuity function from a dense subspace to the whole space, each $x \in C^* (T, E)$ may be continuously extended to a unique continuous function $\beta_0 x \in C^* (\beta_0 T, E)$. $\beta_0 T$ is unique in a sense we discuss in the context of $E$-compactifications (Th. 1.6). At this point the reader may find the notation $\beta_0 T$ curious. Why $\beta_0 T$ and not $\beta_0 E T$? As long as $E$ is ultra-regular and contains at least two points (\cite{1}, p. 93, \cite{8}, pp. 240-243), the uniformity $C^* (T, E)$ does not depend on $E$! A fundamental system of entourages for $C^* (T, E)$, no matter what $E$ is, is defined by the sets
\[ V_P = \bigcup \{ V \times V : V \in P \} \]
where $P$ is any finite open (therefore clopen) cover of $T$ by pairwise disjoint sets. The completion of $T$ with respect to this uniformity is the way Banaschewski obtained $\beta_0 T$. The definition of $\beta_0 T$ as the completion of $C^* (T, E)$ where $E$ is the discrete space of integers was first given in \cite{7}, though the idea of treating compactifications as completions is due to Nachbin. The connection with the Stone-Cech compactification is the following.

**Definition 1.1** Let $P$ be a finite clopen cover of a topological space $S$ by pairwise disjoint sets and let $V$ denote the uniformity generated by $V_P$. We say that $S$ is strongly ultra-regular if $V = C^* (T, R)$.

**Theorem 1.2** (\cite{8}, pp. 251-2) (a) Every ultra-regular $T_1$-space $S$ is strongly ultra-regular.

(b) If a topological space $S$ is strongly ultra-regular then $\beta_0 S = \beta S$.

### 1.2 As an $E$-Compactification

Tihonov proved that a completely regular space $T$ may be characterized as one that is homeomorphic to a subspace of a product $[0,1]^m$ of unit intervals. Even though his name is not associated with it, he created the first version of the Stone-Cech compactification $\beta T$ of $T$ by then taking the closure of $T$ in $[0,1]^m$. Engelking and Mrówka \cite{5} developed analogous notions of $E$-completely regular space $T$ and $E$-compactification $\beta E T$. Let $S$ and $E$ be two topological spaces. $S$ is called $E$-completely regular if it is homeomorphic to a subspace of the $m$-fold topological product $E^m$ for some cardinal $m$. If $E = R$ or $[0,1]$, this is the familiar notion of complete regularity. With $2$ denoting the discrete space $\{0,1\}$, it happens that

**Theorem 1.3** (\cite{16}, p. 17) A topological space $S$ is 2-completely regular if and only if it is an ultra-regular $T_0$-space.

An $E$-compact space is one which is homeomorphic to a closed subspace of a topological product $E^m$ for some cardinal $m$. The 2-compact spaces are characterized as follows:

**Theorem 1.4** (\cite{5}, p.430, Example (iii)) A topological space $S$ is 2-compact if and only if it is compact and ultra-regular.
An \(E\)-compactification \(\beta_E T\) of an \(E\)-completely regular space \(T\) is

1. an \(E\)-compact space which contains \(T\) as a dense subset and
2. ('the \(E\)-extension property') each \(x \in C(T, E)\) may be extended to \(\beta_E x \in C(\beta_E T, E)\).

The following analogs of properties of the Stone-ŻCech compactification obtain for \(E\)-compactifications.

Theorem 1.5 ([5], p. 433, Theorem 4, [16], pp. 25-27, 4.3 and 4.4). An \(E\)-completely regular (Hausdorff) space \(T\) has a Hausdorff \(E\)-compactification \(\beta_E T\) with the following properties:

(a) If \(S\) is an \(E\)-compact space then every continuous function \(x : T \to S\) has a continuous extension \(\tilde{x} : \beta_E T \to S\).

(b) The space \(\beta_E T\) is unique in the sense that if \(S\) is an \(E\)-compact space containing \(T\) as a dense subset and such that every continuous \(x : T \to E\) has a continuous extension to \(S\), then \(S\) is homeomorphic to \(\beta_E T\) under a homeomorphism that is the identity on \(T\).

(c) \(T\) is \(E\)-compact if and only if \(T = \beta_E T\).

How does this apply to \(\beta_0 T\)? Ultraregular spaces \(T\) are 2-completely regular by Th. 1.3. Since \(\beta_0 T\) is compact and ultranormal, it follows that \(\beta_0 T\) is 2-compact by Th. 1.4. Therefore, by Th. 1.5(b) it follows that

Theorem 1.6 UNIQUENESS OF \(\beta_0 T\). \(\beta_0 T\) is homeomorphic to \(\beta_2 T\) under a homeomorphism that is the identity on \(T\), as would be any ultraregular compactification of an ultraregular \(T\) with the \(E\)-extension property.

1.3 As a Space of Characters

Let \(F\) be an ultraregular Hausdorff topological field so that \(X = C^* (T, F)\) may be considered as an \(F\)-algebra. A character of \(X\) is a nonzero algebra homomorphism from \(X\) into \(F\). Let the set \(H\) of characters of \(X\) be equipped with the weakest topology for which the maps \(H \to F, h \mapsto h(x)\), are continuous for each \(x \in C^* (T, F)\). For each \(p \in \beta_0 T\), let \(p^*\) denote the evaluation map at \(p\), the map \(C^* (T, F) \to F, x \mapsto \beta_0 x (p)\). It is trivial to verify that each \(p^*\) is a character of \(C^* (T, F)\). But more is true: You get all the characters of \(C^* (T, F)\) this way. In fact, the map

\[
A : \beta_0 T \to H, \quad p \mapsto p^*
\]

establishes a homeomorphism between \(\beta_0 T\) and \(H\). The details may be found in [1], Theorem 3 and [8], Theorem 8.15.
1.4 Characters Again

Once again $\beta_0 T$ is realized as a space of nonzero homomorphisms—ring homomorphisms this time—into the very simple (discrete) field 2 with 2 elements.

A commutative ring $X$ with identity in which each element is idempotent is called a Boolean ring. A subcollection $X$ of the set of subsets of a given set $T$ which is closed under union, intersection and set difference of any two of its members is called a ring of sets. Such a collection forms a ring in the usual algebraic sense if addition and multiplication are taken to be symmetric difference and intersection, respectively. If the sets in $X$ cover $T$ then $X$ is called a covering ring. Since $X$ must have a multiplicative identity (i.e., with respect to intersection) any covering ring must contain $T$ as an element. Any covering ring $X$ generates (in the sense that it is a subbase for) a ultrafilter topology on $T$; the topology is ultrafilter since the complement $T - A$ of any open set (member of $X$) must belong to $X$. In the converse direction, the class $\text{Cl}(T)$ of clopen subsets obviously constitutes a covering ring of any topological space $T$.

Let $X$ be a Boolean ring and endow $2^X$ with the product topology. The Stone space $S(X)$ of the Boolean ring $X$ is the subspace of $2^X$ of all nonzero ring homomorphisms of $X$ into 2. $S(X)$ is called the Stone space because of Stone's use of it in his remarkable characterization of compact ultraregular spaces.

**The Stone Representation Theorem** ([12], Theorem 4, [12], [4] p.227 or [6], pp. 77-80) If $T$ is a compact ultraregular space, then $T$ is homeomorphic to the Stone space of the Boolean ring $\text{Cl}(T)$ of clopen subsets of $T$. Conversely, the Stone space $S(X)$ of any Boolean ring $X$ is a compact ultraregular Hausdorff space and $X$ is ring-isomorphic to the Boolean ring $\text{Cl}(T)$ of clopen subsets of $S(X)$.

If $T$ is ultrafilter regular then $\beta_0 T$ is the Stone space of $\text{Cl}(T)$. Indeed, the map $\beta : T \rightarrow S(\text{Cl}(T)), t \mapsto \beta t$, defined for $t \in T$ and $K \in \text{Cl}(T)$ by

$$ (\beta t)(K) = \begin{cases} 1 & t \in K \\ 0 & t \notin K \end{cases} $$

is a homeomorphism of $T$ onto a dense subset of the compact ultraregular Hausdorff space $S(\text{Cl}(T))$.

1.5 As a Space of Measures

Let $T$ be ultrafilter regular and let $\text{Cl}(T)$ be the ring (algebra, actually, since $T \in \text{Cl}(T)$) of clopen subsets of $T$, and let $F$ be an ultrafilter regular Hausdorff topological field. A 0-1 measure on $T$ is a finitely additive set function $m : \text{Cl}(T) \rightarrow \{0, 1\} \subset F$ satisfying the condition:

$$ m(U) = 0 \quad \text{and} \quad U \supset V \in \text{Cl}(T) \quad \Rightarrow \quad m(V) = 0 $$

in other words, that clopen subsets of sets of measure 0 also have measure 0. Measures $m_t$, 'concentrated at points $t \in T$' (also called 'purely atomic' or 'the point mass at $t$') which
are 1 on a clopen set \( U \) if \( t \in U \) and 0 otherwise are 0-1 measures on \( T \). The weak clopen topology for the collection \( M \) of all 0-1 measures on \( T \) has as a neighborhood base \( m_0 \in M \) sets of the form

\[
V(m_0; S_1, \ldots, S_n) = \{ m \in M : m(S_j) = m_0(S_j), j = 1, \ldots n \}
\]

where the \( S_j \) are clopen sets and \( n \in \mathbb{N} \). It is trivial to verify that the map \( t \rightarrow m_t \) is a homeomorphism of \( T \) into \( M \). Using the techniques of [1] one can demonstrate that \( M \) is a compact ultranormal Hausdorff space to which any \( \beta_{0T} (T, \{0\}) \) may be continuously extended. It follows that \( \beta_0 T = M \) in the sense of Th. 1.6.

Last, let us mention that \( \beta_0 T \) may also be realized as a Wallman compactification utilizing the lattice of clopen subsets of \( T \).

2 A New Approach

A construction of \( \beta_0 T \) using the methods of non-Archimedean functional analysis is presented in Theorem 2.1. The proof hinges on the fact that, for a local field \( F \), if \( U \) is a neighborhood of 0 in a locally \( F \)-convex space \( X \) then its polar \( U^\circ \) is \( (X', X) \)-compact ([15], Th. 4.11). Note that \( (X', X) \) is ultraregular since the seminorms \( p_x(f) = |f(x)|, x \in X, f \in X' \), are non-Archimedean.

**Theorem 2.1** Let \( F \) be a local field, let \( T \) be ultraregular and let \( C^*(T, F) \) denote the sup-normed space of all continuous \( F \)-valued functions on \( T \) with relatively compact range. There is an ultranormal compactification \( \beta_0 T \) of \( T \) such that any \( x \in C^*(T, F) \) may be continuously extended to a function \( \beta_0 x \in C(\beta_0 T, F) \).

**Proof.** For \( t \in T \), let \( t^* \) denote the evaluation map \( x \mapsto x(t) \) for any \( x \in C^*(T, F) \). We note that each such \( t^* \) is a continuous linear form (algebra homomorphism, actually) and is of norm one. Thus \( T^* = \{ t^* : t \in T \} \subset U \) where \( U \) denotes the unit ball of the norm-dual \( C^*(T, F)' \) of \( C^*(T, F) \). Furthermore, the map \( i : T \rightarrow C^*(T, F)', t \mapsto t^* \), embeds \( T \) homeomorphically in \( C^*(T, F)' \) endowed with its weak-* topology by the following argument. The map \( i \) is obviously injective. If a net \( t_s \rightarrow t \in T \) then \( x(t_s) \rightarrow x(t) \) for any \( x \in C^*(T, F) \); hence \( t_s^* \rightarrow t^* \) and therefore \( i \) is continuous. To see that \( i \) is a homeomorphism onto \( i(K) \), let \( K \) be a closed subset of \( T \). Since \( T \) is ultraregular, if \( t \notin K \) then there exists \( x \in C^*(T, F) \) such that \( x(t) = 0 \) and \( |x(K)| = r > 1 \). Hence the polar \( \{ x \}^\circ \) of \( \{ x \} \) is a neighborhood of \( t^* \) disjoint from \( K^* \) and \( K^* \) is a closed subset of \( i(K) \). As \( U \) is the polar of the unit ball of \( C^*(T, F) \), it follows that \( U \) is weak-*compact ([15], Th. 4.11). Therefore the closure \( cT \) in \( U \) of (the homeomorphic image of ) \( T^* \) is compact in \( C^*(T, F)' \) endowed with the weak-* topology. As to the continuous extendibility of \( x \in C^*(T, F) \), consider the canonical image \( Jx \) of \( x \) in the second algebraic dual of \( C^*(T, F) \), i.e., for any \( f \in C^*(T, F)' \), \( Jx(f) = f(x) \). Clearly \( Jx \) is weak-*continuous on \( C^*(T, F)' \); so, therefore, is its restriction \( \beta_0 x = Jx |_{cT} \). Should this be called \( c_T \) rather than \( cT \)? No topologically significant changes occur for different \( F \)'s: the compactness of the ultraregular space \( cT \) and the fact that \( T \) is \( C^* \)-embedded in \( cT \) imply that \( cT = \beta_0 T \) by Th. 1.6.
3 Compactoidification

In this section we construct a compactoidification $\kappa T$ of an ultraregular space $T$. $(F, |\cdot|)$ denotes a complete nontrivially ultravalued field throughout. As usual, we abbreviate 'F-convex' to 'convex.' A map $f$ defined on an absolutely convex subset $A$ of a vector space over $F$ with values in some absolutely convex set in a vector space over $F$ is called affine if $f(ax + by) = af(x) + bf(y)$ for all $x, y \in A$ and all $a, b \in F$ with $|a| < 1$ and $|b| < 1$.

**Definition 3.1** A compactoidification of an ultraregular space $T$ is a pair $(i, \kappa T)$ where $\kappa T$ is a complete absolutely convex compactoid subset of some Hausdorff locally convex space $E$ over $F$ and $i : T \to \kappa T$ is a continuous map with precompact range for which following extendibility property holds: For any complete absolutely convex compactoid subset $A$ of some Hausdorff locally convex space $E$ over $F$ and any continuous map $j : T \to A$ with precompact range, there exists a unique continuous affine map $J : \kappa T \to A$ such that $J \circ i = j$.

**Theorem 3.2** A compactoidification is unique in the following natural sense: if $(i_1, \kappa_1 T)$ and $(i_2, \kappa_2 T)$ are compactoidifications of $T$ then there exists a unique continuous affine homeomorphism $J_1 : \kappa_1 T \to \kappa_2 T$ such that $J_1 \circ i_1 = i_2$. Moreover, the map $i$ must be injective.

**Proof.** By definition, there exist unique continuous affine maps $J_1$ and $J_2$ such that $J_2 \circ i_1 = i_2$ and $J_1 \circ i_2 = i_1$. Thus, $J_1 \circ (J_2 \circ i_1) = J_1 \circ i_2 = i_1$.

Since the identity map $I_1 : t \mapsto t$ of $\kappa_1 T$ onto $\kappa_1 T$ also satisfies $I_1 \circ i_1 = i_1$, it follows from the uniqueness that $I_1 = J_1 \circ J_2$. Similarly, $I_2 = J_2 \circ J_1$ where $I_2$ is the identity map of $\kappa_2 T$ onto $\kappa_2 T$. It follows that $J_1$ is a homeomorphism of $\kappa_1 T$ onto $\kappa_2 T$ and $J_2$ is its inverse. If $i_1(t_1) = i_2(t_2)$ then $i_2(t_1) = J_1 \circ i_1(t_1) = J_1 \circ i_1(t_2) = i_2(t_2)$ so if one of the maps $i$ is 1-1, all such $i$ must be. As shown in Theorem 3.3, there is an $i$ that is 1-1.

In the notation of Sec. 2:

**Theorem 3.3** Let $T$ be ultraregular and let the continuous dual $C^\ast(T,F)'$ of $C^\ast(T,F)$ carry the weak-* topology. Then

(a) the closed absolutely convex hull $\kappa T$ of $T^\ast$ is the unit ball $U$ of $C^\ast(T,F)'$

(b) the pair $(i, \kappa T)$ is a compactoidification of $T$.

**Proof.** Clearly the absolute convex hull $B$ of $T^\ast$ is contained in the unit ball $U$ of $C^\ast(T,F)'$. Since $U$ is a complete compactoid by the $p$-adic Alaoglu theorem ([9], Prop.
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3.1), so, therefore, is the closed absolutely convex hull \( \kappa T \) of the compact set \( \text{cl} T \).

It follows from [10], Prop. 1.3 that \( B \) is edged (i.e., if the valuation of \( F \) is dense then \( \text{cl} B = \cap \{ a( \text{cl} B) : a \in F, |a| > 1 \} \) and therefore ([9], Th. 4.7) a polar set in \( C^*(T, F)' \).

If \( \text{cl} B \neq U \) there must exist \( g \in C^*(T, F)' \) such that \( |g| \leq 1 \) on \( B \) and \( |g(f)| > 1 \) for some \( f \in U - \text{cl} B \). Since \( g \) must be an evaluation map determined by some point \( x \in C^*(T, F) \) by [9], Lemma 7.1, we have found an \( x \) such that \( |x(t)| = |x(\cdot)(x)| \leq 1 \) for all \( t \in T \) but \( |f(x)| > 1 \). As this contradicts \( \|f\| \leq 1 \), the proof of (a) is complete.

(b) As in the proof of Th. 2.1, \( i \) is a homeomorphism onto the precompact set \( T \). To verify the extendibility requirement, let \( A \) be a complete absolutely convex compactoid and let \( j : T \to A \) be continuous with precompact range. We define the affine extension \( J \) of \( j \) on the absolutely convex hull \( B \) of \( T \) by taking \( J \left( \sum_{i=1}^{n} a_i t_i \right) = \sum_{i=1}^{n} a_i j(t_i) \) for \( a_i \in F, |a_i| \leq 1, i = 1, \ldots, n \). The definition makes sense because the \( t_i \) are linearly independent for distinct \( t_i \). Evidently \( j = J \circ i \). To prove the continuity of \( J \), let \( s \to \mu_s = \sum_{i=1}^{n} a_i^s t_i \) be a net in \( B \) convergent to 0 in the weak-* topology. Let \([A]\) denote the linear span of \( A \) and note that for any \( f \in [A]' \), the map \( f \circ j \in C^*(T, F) \), since \( j(T) \) is precompact. Thus,

\[
f(J(\mu_s)) = f \left( \sum_{i=1}^{n} a_i^s j(t_i) \right) = \sum_{i=1}^{n} a_i^s f(j(t_i)) = \mu_s(f \circ j) \to 0
\]

and we conclude that \( J(\mu_s) \to 0 \) in the weak topology of \([A]\). As \( A \) is of countable type, hence a polar space, the weak topology coincides with the initial one on the compactoid \( A \) ([9], Th. 5.12) so \( J(\mu_s) \to 0 \) in \( A \). By continuity and ‘affinity,’ \( J \) extends uniquely to a continuous affine map of \( \text{cl} B = \kappa T \) into \( A \), since \( A \) is complete.

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