COMPACTIFICATION AND COMPACTOIDIFICATION

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Abstract. After discussing some of the many ways to get the Banaschewski compactification $\beta_0T$ of an arbitrary ultraregular space $T$, we develop another construction of $\beta_0T$ in Th. 2.1. Using those ideas, we develop an analog of $\beta_0T$—what we call a compactoidification $\kappa T$ of an ultraregular space $T$ in Sec. 3; $\kappa T$ is, in essence, a complete absolutely convex compactoid ‘superset’ of $T$ to which continuous maps of $T$ with precompact range into any complete absolutely convex compactoid subset may be ‘continuously extended.’

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1 The Many Faces

For any topological spaces $X$ and $Y$, $C(X,Y)$ and $C^*(X,Y)$ denote the spaces of continuous maps of $X$ into $Y$ and the continuous maps of $X$ into $Y$ with relatively compact range, respectively. To say that a topological space $X$ is ultraregular or ultranormal means, respectively, that the clopen sets are a basis or disjoint closed subsets of $X$ may be separated by clopen sets. A synonym for ultraregular is 0-dimensional. We have a slight preference for the former in order to avoid confusion with other notions of dimension. Throughout the discussion, $T$ denotes at least a Hausdorff space. For an ultraregular space $E$ containing at least two points and ultraregular $T$, B. Banaschewski [2] discovered a compactification $\beta_0T$ of $T$ in which every $x \in C^*(T,E)$ may be continuously extended to $\beta_0x \in C(\beta_0T, E)$. $\beta_0T$ is nowadays usually called the Banaschewski compactification of $T$. It functions as the natural analog of the Stone-Čech compactification ($\beta_0T$ is $\beta T$ for ultranormal $T$) in non-Archimedean analysis. Like the Stone-Čech compactification, the Banaschewski compactification is a protean entity, assuming many different guises. We discuss some of them in this section and then develop a new one in Sec. 2.

1.1 As a completion

Let $E$ be an ultraregular space containing at least two points and let $T$ be ultraregular. Let $C^*(T,E)$ denote the weakest uniform structure on $T$ making each $x \in C^*(T,E)$ uniformly continuous into the compact space $\text{cl } x(T)$ equipped with its unique compatible uniform
structure. By [1], pp. 92-93, since \( T \) is ultraregular, \( C^* (T, E) \) is compatible with the topology on \( T \) and \( C^* (T, E) \) is a precompact uniform structure on \( T \). Since \( C^* (T, E) \) is precompact, its completion \( \beta_0 T \) is compact and is called the Banaschewski compactification of \( T \). \( \beta_0 T \) is an ultraregular (\[2\], p. 131, Satz 2 or [1], p. 93, Theorem 1)—hence ultraregular—and, by the usual process of extension by continuity function from a dense subspace to the whole space, each \( x \in C^* (T, E) \) may be continuously extended to a unique continuous function \( \beta_0 x \in C^* (\beta_0 T, E) \). \( \beta_0 T \) is unique in a sense we discuss in the context of \( E \)-compactifications (Th. 1.6). At this point the reader may find the notation \( \beta_0 T \) curious. Why \( \beta_0 T \) and not \( \beta_E T \)?

As long as \( E \) is ultraregular and contains at least two points ([1], p. 93, [8], pp. 240-243), the uniformity \( C^* (T, E) \) does not depend on \( E \). A fundamental system of entourages for \( C^* (T, E) \), no matter what \( E \) is, is defined by the sets

\[ V_p = \bigcup \{ V \times V : V \in \mathcal{P} \} \]

where \( \mathcal{P} \) is any finite open (therefore clopen) cover of \( T \) by pairwise disjoint sets. The completion of \( T \) with respect to this uniformity is the way Banaschewski obtained \( \beta_0 T \). The definition of \( \beta_0 T \) as the completion of \( C^* (T, E) \), where \( E \) is the discrete space of integers was first given in [7], though the idea of treating compactifications as completions is due to Nachbin. The connection with the Stone-Čech compactification is the following.

**Definition 1.1** Let \( \mathcal{P} \) be a finite clopen cover of a topological space \( S \) by pairwise disjoint sets and let \( V \) denote the uniformity generated by \( \mathcal{V}_\mathcal{P} \). We say that \( S \) is strongly ultraregular if \( V = C^* (T, R) \).

**Theorem 1.2** ([8], pp. 251-2) (a) Every ultranormal \( T_1 \)-space \( S \) is strongly ultraregular.

(b) If a topological space \( S \) is strongly ultraregular then \( \beta_0 S = \beta S \).

**1.2 As an \( E \)-Compactification**

Tihonov proved that a completely regular space \( T \) may be characterized as one that is homeomorphic to a subspace of a product \([0,1]^m \) of unit intervals. Even though his name is not associated with it, he created the first version of the Stone-Čech compactification \( \beta T \) of \( T \) by then taking the closure of \( T \) in \([0,1]^m \). Engelking and Mrówka [5] developed analogous notions of \( E \)-completely regular space \( T \) and \( E \)-compactification \( \beta_E T \). Let \( S \) and \( E \) be two topological spaces. \( S \) is called \( E \)-completely regular if it is homeomorphic to a subspace of the \( m \)-fold topological product \( E^m \) for some cardinal \( m \). If \( E = R \) or \([0,1] \), this is the familiar notion of complete regularity. With \( 2 \) denoting the discrete space \([0,1] \), it happens that

**Theorem 1.3** ([16], p. 17) A topological space \( S \) is 2-completely regular if and only if it is an ultraregular \( T_0 \)-space.

An \( E \)-compact space is one which is homeomorphic to a closed subspace of a topological product \( E^m \) for some cardinal \( m \). The \( 2 \)-compact spaces are characterized as follows:

**Theorem 1.4** ([5], p.430, Example (iii)) A topological space \( S \) is 2-compact if and only if it is compact and ultraregular.
An $E$-compactification $\beta_E T$ of an $E$-completely regular space $T$ is

1. an $E$-compact space which contains $T$ as a dense subset and
2. ("the $E$-extension property") each $x \in C(T, E)$ may be extended to $\beta_E x \in C(\beta_E T, E)$.

The following analogs of properties of the Stone-Čech compactification obtain for $E$-compactifications.

**Theorem 1.5** ([5], p. 433, Theorem 4, [16], pp. 25-27, 4.3 and 4.4). An $E$-completely regular (Hausdorff) space $T$ has a Hausdorff $E$-compactification $\beta_E T$ with the following properties:

(a) If $S$ is an $E$-compact space then every continuous function $x : T \to S$ has a continuous extension $\tilde{x} : \beta_E T \to S$.

(b) The space $\beta_E T$ is unique in the sense that if $S$ is an $E$-compact space containing $T$ as a dense subset and such that every continuous $x : T \to E$ has a continuous extension to $S$, then $S$ is homeomorphic to $\beta_E T$ under a homeomorphism that is the identity on $T$.

(c) $T$ is $E$-compact if and only if $T = \beta_E T$.

How does this apply to $\beta_0 T$? Ultraregular spaces $T$ are 2-completely regular by Th. 1.3. Since $\beta_0 T$ is compact and ultranormal, it follows that $\beta_0 T$ is 2-compact by Th. 1.4. Therefore, by Th. 1.5(b) it follows that

**Theorem 1.6** UNIQUENESS OF $\beta_0 T$. $\beta_0 T$ is homeomorphic to $\beta_E T$ under a homeomorphism that is the identity on $T$, as would be any ultraregular compactification of an ultraregular $T$ with the $E$-extension property.

### 1.3 As a Space of Characters

Let $F$ be an ultraregular Hausdorff topological field so that $X = C^*(T, F)$ may be considered as an $F$-algebra. A character of $X$ is a nonzero algebra homomorphism from $X$ into $F$. Let the set $H$ of characters of $X$ be equipped with the weakest topology for which the maps $H \to F$, $h \mapsto h(x)$, are continuous for each $x \in C^*(T, F)$. For each $p \in \beta_0 T$, let $p^*$ denote the evaluation map at $p$, the map $C^*(T, F) \to F$, $x \mapsto \beta_0 x(p)$. It is trivial to verify that each $p^*$ is a character of $C^*(T, F)$. But more is true: You get all the characters of $C^*(T, F)$ this way. In fact, the map

$$A : \beta_0 T \to H$$

$$p \mapsto p^*$$

establishes a homeomorphism between $\beta_0 T$ and $H$. The details may be found in [1], Theorem 3 and [8], Theorem 8.15.
1.4 Characters Again

Once again $\beta_0 T$ is realized as a space of nonzero homomorphisms—ring homomorphisms this time—into the very simple (discrete) field $\mathbb{2}$ with 2 elements.

A commutative ring $X$ with identity in which each element is idempotent is called a Boolean ring. A subcollection $X$ of the set of subsets of a given set $T$ which is closed under union, intersection and set difference of any two of its members is called a ring of sets. Such a collection forms a ring in the usual algebraic sense if addition and multiplication are taken to be symmetric difference and intersection, respectively. If the sets in $X$ cover $T$ then $X$ is called a covering ring. Since $X$ must have a multiplicative identity (i.e., with respect to intersection) any covering ring must contain $T$ as an element. Any covering ring $X$ generates (in the sense that it is a subbase for) a ultraregular topology on $T$; the topology is ultraregular since the complement $T - A$ of any open set (member of $X$) must belong to $X$. In the converse direction, the class $\text{Cl}(T)$ of clopen subsets obviously constitutes a covering ring of any topological space $T$.

Let $X$ be a Boolean ring and endow $2^X$ with the product topology. The Stone space $S(X)$ of the Boolean ring $X$ is the subspace of $2^X$ of all nonzero ring homomorphisms of $X$ into $\mathbb{2}$. $S(X)$ is called the Stone space because of Stone's use of it in his remarkable characterization of compact ultraregular spaces.

**The Stone Representation Theorem** ([12], Theorem 4, [12], [4] p.227 or [6], pp. 77-80) If $T$ is a compact ultraregular space, then $T$ is homeomorphic to the Stone space of the Boolean ring $\text{Cl}(T)$ of clopen subsets of $T$. Conversely, the Stone space $S(X)$ of any Boolean ring $X$ is a compact ultraregular Hausdorff space and $X$ is ring-isomorphic to the Boolean ring $\text{Cl}(T)$ of clopen subsets of $S(X)$.

If $T$ is ultraregular then $\beta_0 T$ is the Stone space of $\text{Cl}(T)$. Indeed, the map $\beta : T \rightarrow S(\text{Cl}(T))$, $t \mapsto \beta t$, defined for $t \in T$ and $K \in \text{Cl}(T)$ by

\[(\beta t)(K) = \begin{cases} 1 \in 2 & t \in K \\ 0 \in 2 & t \notin K \end{cases}\]

is a homeomorphism of $T$ onto a dense subset of the compact ultraregular Hausdorff space $S(\text{Cl}(T))$.

1.5 As a Space of Measures

Let $T$ be ultraregular and let $\text{Cl}(T)$ be the ring (algebra, actually, since $T \in \text{Cl}(T)$) of clopen subsets of $T$, and let $F$ be an ultraregular Hausdorff topological field. A 0-1 measure on $T$ is a finitely additive set function $m : \text{Cl}(T) \rightarrow \{0, 1\} \subseteq F$ satisfying the condition:

\[m(U) = 0 \text{ and } U \supset V \in \text{Cl}(T) \implies m(V) = 0\]

in other words, that clopen subsets of sets of measure 0 also have measure 0. Measures $m_\ast$ "concentrated at points $t \in T$" (also called "purely atomic" or "the point mass at $t$") which
are 1 on a clopen set $U$ if $t \in U$ and 0 otherwise are 0-1 measures on $T$. The weak clopen topology for the collection $M$ of all 0-1 measures on $T$ has as a neighborhood base $m_0 \in M$

sets of the form

$$V(m_0; S_1, \ldots, S_n) = \{m \in M : m(S_j) = m_0(S_j), j = 1, \ldots, n\}$$

where the $S_j$ are clopen sets and $n \in \mathbb{N}$. It is trivial to verify that the map $t \mapsto m_t$ is a homeomorphism of $T$ into $M$. Using the techniques of [1] one can demonstrate that $M$ is a compact ultranormal Hausdorff space to which any $\gamma_0(T, F)$ may be continuously extended. It follows that $\beta_0 T = M$ in the sense of Th. 1.6.

Last, let us mention that $\beta_0 T$ may also be realized as a Wallman compactification utilizing the lattice of clopen subsets of $T$.

2 A New Approach

A construction of $\beta_0 T$ using the methods of non-Archimedean functional analysis is presented in Theorem 2.1. The proof hinges on the fact that, for a local field $F$, if $U$ is a neighborhood of 0 in a locally $F$-convex space $X$ then its polar $U^\circ$ is $\sigma(X', X)$-compact ([15], Th. 4.11). Note that $\sigma(X', X)$ is ultraregular since the seminorms $p_x(f) = |f(x)|$, $x \in X$, $f \in X'$, are non-Archimedean.

**Theorem 2.1** Let $F$ be a local field, let $T$ be ultraregular and let $C^*(T, F)$ denote the sup-normed space of all continuous $F$-valued functions on $T$ with relatively compact range. There is an ultranormal compactification $\beta_0 T$ of $T$ such that any $x \in C^*(T, F)$ may be continuously extended to a function $\beta_0 x \in C(\beta_0 T, F)$.

**Proof.** For $t \in T$, let $t^*$ denote the evaluation map $x \mapsto x(t)$ for any $x \in C^*(T, F)$. We note that each such $t^*$ is a continuous linear form (algebra homomorphism, actually) and is of norm one. Thus $T^* = \{t^* : t \in T\} \subset U$ where $U$ denotes the unit ball of the norm-dual $C^*(T, F)'$ of $C^*(T, F)$. Furthermore, the map $i : T \rightarrow C^*(T, F)'$, $t \mapsto t^*$, embeds $T$ homeomorphically in $C^*(T, F)'$ endowed with its weak-* topology by the following argument. The map $i$ is obviously injective. If a net $t_s \rightarrow t \in T$ then $x(t_s) \rightarrow x(t)$ for any $x \in C^*(T, F)$; hence $t_s^* \rightarrow t^*$ and therefore $i$ is continuous. To see that $i$ is a homeomorphism onto $i(K)$, let $K$ be a closed subset of $T$. Since $T$ is ultraregular, if $t \not\in K$ then there exists $x \in C^*(T, F)$ such that $x(t) = 0$ and $|x(K)| = r > 1$. Hence the polar $\{x\}^\circ$ of $\{x\}$ is a neighborhood of $t^*$ disjoint from $K^*$ and $K^*$ is a closed subset of $i(K)$. As $U$ is the polar of the unit ball of $C^*(T, F)$, it follows that $U$ is weak-*-compact ([15], Th. 4.11). Therefore the closure $cT$ in $U$ of (the homeomorphic image of) $T^*$ is compact in $C^*(T, F)'$ endowed with the weak-* topology. As to the continuous extendiblility of $x \in C^*(T, F)$, consider the canonical image $Jx$ of $x$ in the second algebraic dual of $C^*(T, F)$, i.e., for any $f \in C^*(T, F)'$, $Jx(f) = f(x)$. Clearly $Jx$ is weak-*-continuous on $C^*(T, F)'$; so, therefore, is its restriction $\beta_0 x = Jx |_{cT}$. Should this be called $cF T$ rather than $cT$? No topologically significant changes occur for different $F$'s: the compactness of the ultraregular space $cT$ and the fact that $T$ is $C^*$-embedded in $cT$ imply that $cT = \beta_0 T$ by Th. 1.6.
3 Compactoidification

In this section we construct a compactoidification $\kappa T$ of an ultraregular space $T$. $(F, |\cdot|)$ denotes a complete nontrivially ultravalued field throughout. As usual, we abbreviate ‘$F$-convex’ to ‘convex.’ A map $f$ defined on an absolutely convex subset $A$ of a vector space over $F$ with values in some absolutely convex set in a vector space over $F$ is called affine if $f(ax + by) = af(x) + bf(y)$ for all $x, y \in A$ and all $a, b \in F$ with $|a| < 1$ and $|b| < 1$.

**Definition 3.1** A compactoidification of an ultraregular space $T$ is a pair $(i, \kappa T)$ where $\kappa T$ is a complete absolutely convex compactoid subset of some Hausdorff locally convex space $E$ over $F$ and $i : T \to \kappa T$ is a continuous map with precompact range for which following extendibility property holds: For any complete absolutely convex compactoid subset $A$ of some Hausdorff locally convex space $E$ over $F$ and any continuous map $j : T \to A$ with precompact range, there exists a unique continuous affine map $J : \kappa T \to A$ such that $J \circ i = j$.

\[
\begin{array}{c}
\kappa T \\
i \uparrow J \\
T \to A \\
\end{array}
\]

**Theorem 3.2** A compactoidification is unique in the following natural sense: if $(i_1, \kappa_1 T)$ and $(i_2, \kappa_2 T)$ are compactoidifications of $T$ then there exists a unique affine homeomorphism $J_1 : \kappa_1 T \to \kappa_2 T$ such that $J_1 \circ i_1 = i_2$. Moreover, the map $i$ must be injective.

**Proof.** By definition, there exist unique continuous affine maps $J_1$ and $J_2$ such that $J_2 \circ i_1 = i_2$ and $J_1 \circ i_2 = i_1$. Thus, $J_1 \circ (J_2 \circ i_1) = J_1 \circ i_2 = i_1$.

\[
\begin{array}{c}
\kappa_1 T \\
i_1 \uparrow J_2 \\
T \to \kappa_2 T \\
\end{array}
\]

Since the identity map $I_1 : t \mapsto t$ of $\kappa_1 T$ onto $\kappa_1 T$ also satisfies $I_1 \circ i_1 = i_1$, it follows from the uniqueness that $I_1 = J_1 \circ J_2$. Similarly, $I_2 = J_2 \circ J_1$ where $I_2$ is the identity map of $\kappa_2 T$ onto $\kappa_2 T$. It follows that $J_1$ is a homeomorphism of $\kappa_1 T$ onto $\kappa_2 T$ and $J_2$ is its inverse. If $i_1(t_1) = i_2(t_2)$ then $i_2(t_1) = J_1 \circ i_1(t_1) = J_1 \circ i_1(t_2) = i_2(t_2)$ so if one of the maps $i$ is 1-1, all such $i$ must be. As shown in Theorem 3.3, there is an $i$ that is 1-1.

In the notation of Sec. 2:

**Theorem 3.3** Let $T$ be ultraregular and let the continuous dual $C^*(T, F)'$ of $C^*(T, F)$ carry the weak-* topology. Then

(a) the closed absolutely convex hull $\kappa T$ of $T^*$ is the unit ball $U$ of $C^*(T, F)'$ and

(b) the pair $(i, \kappa T)$ is a compactoidification of $T$.

**Proof.** Clearly the absolute convex hull $B$ of $T^*$ is contained in the unit ball $U$ of $C^*(T, F)'$. Since $U$ is a complete compactoid by the $p$-adic Alaoglu theorem ([9], Prop.
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3.1), so, therefore, is the closed absolutely convex hull $\kappa T$ of the compact set $\text{cl } T$. It follows from [10], Prop. 1.3 that $B$ is edged (i.e., if the valuation of $F$ is dense then $\text{cl } B = \bigcap \{a(\text{cl } B): a \in F, |a| > 1\}$) and therefore ([9], Th. 4.7) a polar set in $C^*(T, F)'$. If $\text{cl } B \neq U$ there must exist $g \in C^*(T, F)'$ such that $|g| \leq 1$ on $B$ and $|g(f)| > 1$ for some $f \in U \text{-- cl } B$. Since $g$ must be an evaluation map determined by some point $x \in C^*(T, F)$ by [9], Lemma 7.1, we have found an $x$ such that $|x(t)| = |t^*(x)| \leq 1$ for all $t \in T$ but $|f(x)| > 1$. This contradicts $\|f\| \leq 1$, the proof of (a) is complete.

(b) As in the proof of Th. 2.1, $i$ is a homeomorphism onto the precompact set $T$. To verify the extendibility requirement, let $A$ be a complete absolutely convex compactoid and let $j : T \to A$ be continuous with precompact range. We define the affine extension $J$ of $j$ on the absolutely convex hull $B$ of $T$ by taking $J\left(\sum_{i=1}^{n} a_i t_i\right) = \sum_{i=1}^{n} a_i j(t_i)$ for $a_i \in F, |a_i| \leq 1, i = 1, \ldots, n$. The definition makes sense because the $t_i$ are linearly independent for distinct $t_i$. Evidently $j = J \circ i$. To prove the continuity of $J$, let $s \to \mu_s = \sum_{i=1}^{n} a_i^t t_i$ be a net in $B$ convergent to 0 in the weak-* topology. Let $[A]$ denote the linear span of $A$ and note that for any $f \in [A]'$, the map $f \circ j \in C^*(T, F)$, since $j(T)$ is precompact. Thus,

$$f(J(\mu_s)) = f\left(\sum_{i=1}^{n} a_i^s j(t_i)\right) = \sum_{i=1}^{n} a_i^s f(j(t_i)) = \mu_s, (f \circ j) \to 0$$

and we conclude that $J(\mu_s) \to 0$ in the weak topology of $[A]$. As $A$ is of countable type, hence a polar space, the weak topology coincides with the initial one on the compactoid $A$ ([9], Th. 5.12) so $J(\mu_s) \to 0$ in $A$. By continuity and 'affinity,' $J$ extends uniquely to a continuous affine map of $\text{cl } B = \kappa T$ into $A$, since $A$ is complete.

References


