COMPACTIFICATION AND COMPACTOIDIFICATION

E. Beckenstein, L. Narici, and W. Schikhof

Abstract. After discussing some of the many ways to get the Banaschewski compactification $\beta_0 T$ of an arbitrary ultraregular space $T$, we develop another construction of $\beta_0 T$ in Th. 2.1. Using those ideas, we develop an analog of $\beta_0 T$—what we call a compactoidification $\kappa T$ of an ultraregular space $T$ in Sec. 3; $\kappa T$ is, in essence, a complete absolutely convex compactoid 'superset' of $T$ to which continuous maps of $T$ with precompact range into any complete absolutely convex compactoid subset may be 'continuously extended.'

1991 Mathematics subject classification: 46S10, 54D35, 54C45

1 The Many Faces

For any topological spaces $X$ and $Y$, $C(X,Y)$ and $C^*(X,Y)$ denote the spaces of continuous maps of $X$ into $Y$ and the continuous maps of $X$ into $Y$ with relatively compact range, respectively. To say that a topological space $X$ is ultraregular or ultranormal means, respectively, that the clopen sets are a basis or disjoint closed subsets of $X$ may be separated by clopen sets. A synonym for ultraregular is 0-dimensional. We have a slight preference for the former in order to avoid confusion with other notions of dimension. Throughout the discussion, $T$ denotes at least a Hausdorff space. For an ultraregular space $E$ containing at least two points and ultraregular $T$, B. Banaschewski [2] discovered a compactification $\beta_0 T$ of $T$ in which every $x \in C^*(T,E)$ may be continuously extended to $\beta_0 x \in C(\beta_0 T, E)$. $\beta_0 T$ is nowadays usually called the Banaschewski compactification of $T$. It functions as the natural analog of the Stone-Čech compactification ($\beta T$ for ultranormal $T$) in non-Archimedean analysis. Like the Stone-Čech compactification, the Banaschewski compactification is a protean entity, assuming many different guises. We discuss some of them in this section and then develop a new one in Sec. 2.

1.1 As a completion

Let $E$ be an ultraregular space containing at least two points and let $T$ be ultraregular. Let $C^*(T,E)$ denote the weakest uniform structure on $T$ making each $x \in C^*(T,E)$ uniformly continuous into the compact space $\text{cl} \ x(T)$ equipped with its unique compatible uniform
structure. By [1], pp. 92-93, since \( T \) is ultraregular, \( C^* (T, E) \) is compatible with the topology on \( T \) and \( C^* (T, E) \) is a precompact uniform structure on \( T \). Since \( C^* (T, E) \) is precompact, its completion \( \beta_0 T \) is compact and is called the Banaschewski compactification of \( T \). \( \beta_0 T \) is ultranormal ([2], p. 131, Satz 2 or [1], p. 93, Theorem 1)—hence ultraregular—and, by the usual process of extension by continuity function from a dense subspace to the whole space, each \( x \in C^* (T, E) \) may be continuously extended to a unique continuous function \( \beta_0 x \in C^* (\beta_0 T, E) \). \( \beta_0 T \) is unique in a sense we discuss in the context of \( E \)-compactifications (Th. 1.6). At this point the reader may find the notation \( \beta_0 T \) curious. Why \( \beta_0 T \) and not \( \beta E T \)?

As long as \( E \) is ultraregular and contains at least two points ([1], p. 93, [8], pp. 240-243), the uniformity \( C^* (T, E) \) does not depend on \( E \). A fundamental system of entourages for \( C^* (T, E) \), no matter what \( E \) is, is defined by the sets

\[
V_\mathcal{P} = \bigcup \{ V \times V : V \in \mathcal{P} \}
\]

where \( \mathcal{P} \) is any finite open (therefore clopen) cover of \( T \) by pairwise disjoint sets. The completion of \( T \) with respect to this uniformity is the way Banaschewski obtained \( \beta_0 T \). The definition of \( \beta_0 T \) as the completion of \( C^* (T, E) \) where \( E \) is the discrete space of integers first given in [7], though the idea of treating compactifications as completions is due to Nachbin. The connection with the Stone-Cech compactification is the following.

**Definition 1.1** Let \( \mathcal{P} \) be a finite clopen cover of a topological space \( S \) by pairwise disjoint sets and let \( V \) denote the uniformity generated by \( V_\mathcal{P} \). We say that \( S \) is strongly ultraregular if \( V = C^* (T, \mathbb{R}) \).

**Theorem 1.2** ([8], pp. 251-2) (a) Every ultranormal \( T_1 \)-space \( S \) is strongly ultraregular.

(b) If a topological space \( S \) is strongly ultraregular then \( \beta_0 S = \beta S \).

### 1.2 As an \( E \)-Compactification

Tihonov proved that a completely regular space \( T \) may be characterized as one that is homeomorphic to a subspace of a product \([0,1]^m \) of unit intervals. Even though his name is not associated with it, he created the first version of the Stone-Čech compactification \( \beta T \) of \( T \) by then taking the closure of \( T \) in \([0,1]^m \). Engelking and Mrówka [5] developed analogous notions of \( E \)-completely regular space \( T \) and \( E \)-compactification \( \beta E T \). Let \( S \) and \( E \) be two topological spaces. \( S \) is called \( E \)-completely regular if it is homeomorphic to a subspace of the \( m \)-fold topological product \( E^m \) for some cardinal \( m \). If \( E = \mathbb{R} \) or \([0,1]\), this is the familiar notion of complete regularity. With \( 2 \) denoting the discrete space \([0,1]\), it happens that

**Theorem 1.3** ([16], p. 17) A topological space \( S \) is 2-completely regular if and only if it is an ultraregular \( T_0 \)-space.

An \( E \)-compact space is one which is homeomorphic to a closed subspace of a topological product \( E^m \) for some cardinal \( m \). The 2-compact spaces are characterized as follows:

**Theorem 1.4** ([5], p.430, Example (iii)) A topological space \( S \) is 2-compact if and only if it is compact and ultraregular.
An $E$-compactification $\beta ET$ of an $E$-completely regular space $T$ is

1. an $E$-compact space which contains $T$ as a dense subset and
2. ("the $E$-extension property") each $x \in C(T, E)$ may be extended to $\beta E x \in C(\beta ET, E)$.

The following analogs of properties of the Stone-Cech compactification obtain for $E$-compactifications.

**Theorem 1.5** ([5], p. 433, Theorem 4, [16], pp. 25-27, 4.3 and 4.4). An $E$-completely regular (Hausdorff) space $T$ has a Hausdorff $E$-compactification $\beta ET$ with the following properties:

(a) If $S$ is an $E$-compact space then every continuous function $x : T \to S$ has a continuous extension $\bar{x} : \beta ET \to S$.

(b) The space $\beta ET$ is unique in the sense that if $S$ is an $E$-compact space containing $T$ as a dense subset and such that every continuous $x : T \to E$ has a continuous extension to $S$, then $S$ is homeomorphic to $\beta ET$ under a homeomorphism that is the identity on $T$.

(c) $T$ is $E$-compact if and only if $T = \beta ET$.

How does this apply to $\beta_0 T$? Ultraregular spaces $T$ are $2$-completely regular by Th. 1.3. Since $\beta_0 T$ is compact and ultranormal, it follows that $\beta_0 T$ is $2$-compact by Th. 1.4. Therefore, by Th. 1.5(b) it follows that

**Theorem 1.6** UNIQUENESS OF $\beta_0 T$. $\beta_0 T$ is homeomorphic to $\beta ET$ under a homeomorphism that is the identity on $T$, as would be any ultraregular compactification of an ultraregular $T$ with the $E$-extension property.

### 1.3 As a Space of Characters

Let $F$ be an ultraregular Hausdorff topological field so that $X = C^*(T, F)$ may be considered as an $F$-algebra. A **character** of $X$ is a nonzero algebra homomorphism from $X$ into $F$. Let the set $H$ of characters of $X$ be equipped with the weakest topology for which the maps $H \to F$, $h \mapsto h(x)$, are continuous for each $x \in C^*(T, F)$. For each $p \in \beta_0 T$, let $p^*$ denote the evaluation map at $p$, the map $C^*(T, F) \to F$, $x \mapsto \beta_0 x(p)$. It is trivial to verify that each $p^*$ is a character of $C^*(T, F)$. But more is true: You get all the characters of $C^*(T, F)$ this way. In fact, the map

$$A : \beta_0 T \to H$$

$$p \mapsto p^*$$

establishes a homeomorphism between $\beta_0 T$ and $H$. The details may be found in [1], Theorem 3 and [8], Theorem 8.15.
1.4 Characters Again

Once again \( \beta_0 T \) is realized as a space of nonzero homomorphisms—ring homomorphisms this time—into the very simple (discrete) field \( \mathbb{2} \) with 2 elements.

A commutative ring \( X \) with identity in which each element is idempotent is called a Boolean ring. A subcollection \( X \) of the set of subsets of a given set \( T \) which is closed under union, intersection and set difference of any two of its members is called a ring of sets. Such a collection forms a ring in the usual algebraic sense if addition and multiplication are taken to be symmetric difference and intersection, respectively. If the sets in \( X \) cover \( T \) then \( X \) is called a covering ring. Since \( X \) must have a multiplicative identity (i.e., with respect to intersection) any covering ring must contain \( T \) as an element. Any covering ring \( X \) generates (in the sense that it is a subbase for) a ultraregular topology on \( T \); the topology is ultraregular since the complement \( T - A \) of any open set (member of \( X \)) must belong to \( X \). In the converse direction, the class \( \text{Cl}(T) \) of clopen subsets obviously constitutes a covering ring of any topological space \( T \).

Let \( X \) be a Boolean ring and endow \( 2^X \) with the product topology. The Stone space \( S(X) \) of the Boolean ring \( X \) is the subspace of \( 2^X \) of all nonzero ring homomorphisms of \( X \) into \( \mathbb{2} \). \( S(X) \) is called the Stone space because of Stone's use of it in his remarkable characterization of compact ultraregular spaces.

THE STONE REPRESENTATION THEOREM ([12], Theorem 4, [12], [4] p.227 or [6], pp. 77-80) If \( T \) is a compact ultraregular space, then \( T \) is homeomorphic to the Stone space of the Boolean ring \( \text{Cl}(T) \) of clopen subsets of \( T \). Conversely, the Stone space \( S(X) \) of any Boolean ring \( X \) is a compact ultraregular Hausdorff space and \( X \) is ring-isomorphic to the Boolean ring \( \text{Cl}(T) \) of clopen subsets of \( S(X) \).

If \( T \) is ultraregular then \( \beta_0 T \) is the Stone space of \( \text{Cl}(T) \). Indeed, the map \( \beta : T \to S(\text{Cl}(T)), t \mapsto \beta t \), defined for \( t \in T \) and \( K \in \text{Cl}(T) \) by

\[
(\beta t)(K) = \begin{cases} 
1 & t \in K \\
0 & t \notin K 
\end{cases} 
\]

is a homeomorphism of \( T \) onto a dense subset of the compact ultraregular Hausdorff space \( S(\text{Cl}(T)) \).

1.5 As a Space of Measures

Let \( T \) be ultraregular and let \( \text{Cl}(T) \) be the ring (algebra, actually, since \( T \in \text{Cl}(T) \)) of clopen subsets of \( T \), and let \( F \) be an ultraregular Hausdorff topological field. A 0-1 measure on \( T \) is a finitely additive set function \( m : \text{Cl}(T) \to \{0,1\} \subset F \) satisfying the condition:

\[
m(U) = 0 \quad \text{and} \quad U \supset V \in \text{Cl}(T) \implies m(V) = 0
\]

in other words, that clopen subsets of sets of measure 0 also have measure 0. Measures \( m_t \) 'concentrated at points \( t \in T \)' (also called 'purely atomic' or 'the point mass at \( t \)') which
are 1 on a clopen set \( U \) if \( t \in U \) and 0 otherwise are 0-1 measures on \( T \). The weak clopen topology for the collection \( M \) of all 0-1 measures on \( T \) has as a neighborhood base \( m_0 \in M \) sets of the form

\[
V(m_0; S_1, \ldots, S_n) = \{m \in M : m(S_j) = m_0(S_j), j = 1, \ldots, n\}
\]

where the \( S_j \) are clopen sets and \( n \in \mathbb{N} \). It is trivial to verify that the map \( t \rightarrow m_t \) is a homeomorphism of \( T \) into \( M \). Using the techniques of [1] one can demonstrate that \( M \) is a compact ultranormal Hausdorff space to which any \( \mu \in \mathcal{M}(T, F) \) may be continuously extended. It follows that \( \beta_0 T = M \) in the sense of Th. 1.6.

Last, let us mention that \( \beta_0 T \) may also be realized as a Wallman compactification utilizing the lattice of clopen subsets of \( T \).

2 A New Approach

A construction of \( \beta_0 T \) using the methods of non-Archimedean functional analysis is presented in Theorem 2.1. The proof hinges on the fact that, for a local field \( F \), if \( U \) is a neighborhood of 0 in a locally \( F \)-convex space \( X \) then its polar \( U^\circ \) is \( \sigma(X', X) \)-compact ([15], Th. 4.11). Note that \( \sigma(X', X) \) is ultraregular since the seminorms \( p_x(f) = |f(x)|, x \in X, f \in X' \), are non-Archimedean.

**Theorem 2.1** Let \( F \) be a local field, let \( T \) be ultraregular and let \( C^*(T, F) \) denote the sup-normed space of all continuous \( F \)-valued functions on \( T \) with relatively compact range. There is an ultranormal compactification \( \beta_0 T \) of \( T \) such that any \( x \in C^*(T, F) \) may be continuously extended to a function \( \beta_0 x \in C(\beta_0 T, F) \).

**Proof.** For \( t \in T \), let \( t^* \) denote the evaluation map \( x \mapsto x(t) \) for any \( x \in C^*(T, F) \). We note that each such \( t^* \) is a continuous linear form (algebra homomorphism, actually) and is of norm one. Thus \( T^* = \{t^* : t \in T\} \subset U \) where \( U \) denotes the unit ball of the norm-dual \( C^*(T, F)' \) of \( C^*(T, F) \). Furthermore, the map \( i : T \rightarrow C^*(T, F)', t \mapsto t^* \), embeds \( T \) homeomorphically in \( C^*(T, F)' \) endowed with its weak-* topology by the following argument. The map \( i \) is obviously injective. If a net \( t_s \rightarrow t \in T \) then \( x(t_s) \rightarrow x(t) \) for any \( x \in C^*(T, F) \); hence \( t_s^* \rightarrow t^* \) and therefore \( i \) is continuous. To see that \( i \) is a homeomorphism onto \( i(K) \), let \( K \) be a closed subset of \( T \). Since \( T \) is ultraregular, if \( t \notin K \) then there exists \( x \in C^*(T, F) \) such that \( x(t) = 0 \) and \( |x(K)| = r > 1 \). Hence the polar \( \{x\}^\circ \) of \( \{x\} \) is a neighborhood of \( t^* \) disjoint from \( K^* \) and \( K^* \) is a closed subset of \( i(K) \). As \( U \) is the polar of the unit ball of \( C^*(T, F) \), it follows that \( U \) is weak-*-compact ([15], Th. 4.11). Therefore the closure \( cT \) in \( U \) of \( (\text{the homeomorphic image of } T^*) \) is compact in \( C^*(T, F)' \) endowed with the weak-* topology. As to the continuous extendibility of \( x \in C^*(T, F) \), consider the canonical image \( Jx \) of \( x \) in the second algebraic dual of \( C^*(T, F) \), i.e., for any \( f \in C^*(T, F)' \), \( Jx(f) = f(x) \). Clearly \( Jx \) is weak-continuous on \( C^*(T, F)' \); so, therefore, is its restriction \( \beta_0 x = Jx|_{cT} \). Should this be called \( c_T^r \) rather than \( cT \)? No topologically significant changes occur for different \( F's \): the compactness of the ultraregular space \( cT \) and the fact that \( T \) is \( C^* \)-embedded in \( cT \) imply that \( cT = \beta_0 T \) by Th. 1.6.
3 Compactoidification

In this section we construct a compactoidification $\kappa T$ of an ultraregular space $T$. $(F, |\cdot|)$ denotes a complete nontrivially ultravalued field throughout. As usual, we abbreviate ‘$F$-convex’ to ‘convex.’ A map $f$ defined on an absolutely convex subset $A$ of a vector space over $F$ with values in some absolutely convex set in a vector space over $F$ is called affine if $f(ax + by) = af(x) + bf(y)$ for all $x, y \in A$ and all $a, b \in F$ with $|a| \leq 1$ and $|b| \leq 1$.

Definition 3.1 A compactoidification of an ultraregular space $T$ is a pair $(i, \kappa T)$ where $\kappa T$ is a complete absolutely convex compactoid subset of some Hausdorff locally convex space $E$ over $F$ and $i : T \to \kappa T$ is a continuous map with precompact range for which following extendibility property holds: For any complete absolutely convex compactoid subset $A$ of some Hausdorff locally convex space $E$ over $F$ and any continuous map $j : T \to A$ with precompact range, there exists a unique continuous affine map $J : \kappa T \to A$ such that $J \circ i = j$.

\[ \kappa T \]
\[ i \uparrow \]
\[ T \quad \xrightarrow{J} \quad A \]

Theorem 3.2 A compactoidification is unique in the following natural sense: if $(i_1, \kappa_1 T)$ and $(i_2, \kappa_2 T)$ are compactoidifications of $T$ then there exists a unique affine homeomorphism $J_1 : \kappa_1 T \to \kappa_2 T$ such that $J_1 \circ i_1 = i_2$. Moreover, the map $i$ must be injective.

Proof. By definition, there exist unique continuous affine maps $J_1$ and $J_2$ such that $J_2 \circ i_1 = i_2$ and $J_1 \circ i_2 = i_1$. Thus, $J_1 \circ (J_2 \circ i_1) = J_1 \circ i_2 = i_1$.

\[ \kappa_1 T \]
\[ i_1 \uparrow \]
\[ T \quad \xrightarrow{J_1} \quad \kappa_2 T \]

Since the identity map $I_1 : t \mapsto t$ of $\kappa_1 T$ onto $\kappa_1 T$ also satisfies $I_1 \circ i_1 = i_1$, it follows from the uniqueness that $I_1 = J_1 \circ J_2$. Similarly, $I_2 = J_2 \circ J_1$ where $I_2$ is the identity map of $\kappa_2 T$ onto $\kappa_2 T$. It follows that $J_1$ is a homeomorphism of $\kappa_1 T$ onto $\kappa_2 T$ and $J_2$ is its inverse. If $i_1(t_1) = i_1(t_2)$ then $i_2(t_1) = J_1 \circ i_1(t_1) = J_1 \circ i_1(t_2) = i_1(t_2)$ so if one of the maps $i$ is 1-1, all such $i$ must be. As shown in Theorem 3.3, there is an $i$ that is 1-1.

In the notation of Sec. 2:

Theorem 3.3 Let $T$ be ultraregular and let the continuous dual $C^*(T, F)'$ of $C^*(T, F)$ carry the weak-* topology. Then

(a) the closed absolutely convex hull $\kappa T$ of $T^*$ is the unit ball $U$ of $C^*(T, F)'$ and

(b) the pair $(i, \kappa T)$ is a compactoidification of $T$.

Proof. Clearly the absolute convex hull $B$ of $T^*$ is contained in the unit ball $U$ of $C^*(T, F)'$. Since $U$ is a complete compactoid by the $p$-adic Alaoglu theorem ([9], Prop.
Compactification and compactoidification

3.1), so, therefore, is the closed absolutely convex hull \( \kappa T \) of the compact set \( \text{cl } T^* \).

It follows from \([10], \text{Prop. 1.3}\) that if \( B \) is edged (i.e., if the valuation of \( F \) is dense then \( \text{cl } B = \cap \{ a( \text{cl } B) : a \in F , |a| > 1 \} \) and therefore \([9], \text{Th. 4.7}\) a polar set in \( C^*(T,F)' \).

If \( \text{cl } B \neq U \) there must exist \( g \in C^*(T,F)' \) such that \( |g| \leq 1 \) on \( B \) and \( |g(f)| > 1 \) for some \( f \in U - \text{cl } B \). Since \( g \) must be an evaluation map determined by some point \( x \in C^*(T,F) \) by \([9], \text{Lemma 7.1}\), we have found an \( x \) such that \( |x(t)| = |t^*(x)| \leq 1 \) for all \( t \in T \) but \( |f(x)| > 1 \). As this contradicts \( \|f\| \leq 1 \), the proof of (a) is complete.

(b) As in the proof of Th. 2.1, \( i \) is a homeomorphism onto the precompact set \( T^* \). To verify the extendibility requirement, let \( A \) be a complete absolutely convex compactoid and \( j : T \to A \) be continuous with precompact range. We define the affine extension \( J \) of \( j \) on the absolutely convex hull \( B \) of \( T^* \) by taking \( J(\sum_{i=1}^{n} a_i t_i^*) = \sum_{i=1}^{n} a_i j(t_i) \) for \( a_i \in F , |a_i| \leq 1 , i = 1, \ldots, n \). The definition makes sense because the \( t_i^* \) are linearly independent for distinct \( t_i \). Evidently \( j = J \circ i \). To prove the continuity of \( J \), let \( s \to \mu_s = \sum_{i=1}^{n} \alpha_i^* t_i^* \) be a net in \( B \) convergent to 0 in the weak-* topology. Let \( [A] \) denote the linear span of \( A \) and note that for any \( f \in [A]' \), the map \( f \circ j \in C^*(T,F)' \), since \( j(T) \) is precompact. Thus,

\[
J(\mu_s) = J \left( \sum_{i=1}^{n} \alpha_i^* j(t_i^*) \right) = \sum_{i=1}^{n} \alpha_i^* f(j(t_i^*)) = \mu_s , (f \circ j) \to 0
\]

and we conclude that \( J(\mu_s) \to 0 \) in the weak topology of \( [A] \). As \( A \) is of countable type, hence a polar space, the weak topology coincides with the initial one on the compactoid \( A \) \([9], \text{Th. 5.12}\) so \( J(\mu_s) \to 0 \) in \( A \). By continuity and ‘affinity,’ \( J \) extends uniquely to a continuous affine map of \( \text{cl } B = \kappa T \) into \( A \), since \( A \) is complete.

References


St. John's University
Staten Island, NY 10301 USA
e-mail: beckenst at sjuvm.stjohns.edu

St. John's University
Jamaica, NY 11439 USA
e-mail: naricil at sjuvm.stjohns.edu
fax: 718-380-0353

Matematisch Instituut
K. U. Nijmegen
Toernooiveld
6525 ED Nijmegen, The Netherlands
e-mail: schikhof at sci.kun.nl