Non-reflexive and non-spherically complete subspaces of the \( p \)-adic space \( l^\infty \)

by C. Perez-Garcia* and W.H. Schikhof

Departamento de Matematicas, Universidad de Cantabria, Avda. de los Castros s/n,
39071 Santander, Spain

Department of Mathematics, Catholic University, Toernooiveld, 6525 ED Nijmegen, the Netherlands

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ABSTRACT

By forming tensor products we construct natural examples of non-reflexive (Section 2) and non-spherically complete (Section 3) closed subspaces of the non-archimedean space \( l^\infty \). Also, we study (Section 4) conditions under which two spherically complete Banach spaces are isomorphic; as an application we describe the spherical completion of the subspaces of \( l^\infty \) constructed in the paper.

1. PRELIMINARIES

Throughout this paper \( K \) is a non-archimedean valued field that is complete under the metric induced by the non-trivial valuation \( | \cdot | \), and \( E, F \) are pseudo-reflexive ([4], p. 60) non-archimedean Banach spaces over \( K \). By \( E \cong F \) we mean that \( E \) and \( F \) are isomorphic, i.e., there is a linear isometry from \( E \) onto \( F \). We will denote the completed tensor product of \( E \) and \( F \) in the sense of [4], p. 123 by \( E \hat{\otimes} F \).

A Banach space is called \textit{spherically complete} if every sequence of closed balls

\[ B(a_1, r_1) \supset B(a_2, r_2) \supset \cdots \]

for which \( r_1 > r_2 > \cdots \) has a non-empty intersection. By \( E^\vee \) we will denote the spherical completion of a Banach space \( E \) (see [4], p. 148).

A subset \( B \) of \( E \) is called \textit{compactoid} if for every \( r > 0 \) there exists a finite set \( S\)
in $E$ such that $B \subset \text{co} \ S + B(0, r)$, where $\text{co} \ S$ denotes the absolutely convex hull of $S$.

$L(E, F)$ will denote the Banach space of all continuous linear maps from $E$ into $F$, endowed with the usual norm. The topological dual space of $E$ is $E' = L(E, K)$. $E$ is called reflexive if the canonical linear map $J_E$ from $E$ into $E''$ is a surjective isometry.

By $C(E, F)$ we will denote the closed subspace of $L(E, F)$ consisting of all compact linear maps from $E$ into $F$ (i.e., the maps $T \in L(E, F)$ for which the image of the closed unit ball of $E$ is a compactoid subset of $F$).

We say that $F$ is a strict quotient of $E$ if there exists a $T \in L(E, F)$ such that $\|T\| \leq 1$ and for every $y \in F$ there is $x \in E$ for which $Tx = y$ and $\|x\| = \|y\|$.

For unexplained terms and background we refer to [4].

2. NON-REFLEXIVE SUBSPACES OF $l^\infty$

It is well known that, if $K$ is spherically complete, no infinite-dimensional Banach space over $K$ is reflexive ([4], 4.16). However, non-spherically complete fields have a more satisfactory behaviour with regard to reflexivity. In fact, suppose that $K$ is not spherically complete. Then $c_0$ and $l^\infty$ are reflexive spaces (more generally, the spaces $c_0(N, s)$ and $l^\infty(N, s)$ are reflexive, for every function $s : N \to (0, \infty)$, [4], 4.22.ii). Also, every quotient and every closed subspace of $c_0(N, s)$ is reflexive ([2], 9.9). But quotients and closed subspaces of $l^\infty$ need not be reflexive. For quotients this is easily seen: $l^\infty/c_0$ is a non-reflexive quotient of $l^\infty$ since its dual is trivial ([4], 4.15). The construction of a non-reflexive closed subspace of $l^\infty$ is more laborious and was given in [4], 4.1.

In this section we are going to show that, by taking tensor products, we can construct in a simple way, natural examples of non-reflexive closed subspaces of $l^\infty$ when $K$ is not spherically complete. To do that we need some preliminary machinery.

Recall ([4], 4.34) that the maps

$U_{EF} : E' \hat{\otimes} F' \to (E \hat{\otimes} F)'$

$V_{EF} : E \hat{\otimes} F \to (E' \hat{\otimes} F')'$

given by

$U_{EF}(g \otimes h)(x \otimes y) = g(x)h(y)$

$V_{EF}(x \otimes y)(g \otimes h) = g(x)h(y)$

$(x \in E, \ g \in E', \ y \in F, \ h \in F')$

are linear isometries. We need two facts.

**Lemma 2.1.** $U_{EF}$ is surjective iff $L(E, F') = C(E, F')$.

**Proof.** By 4.41 of [4]

$E' \hat{\otimes} F' \simeq C(E, F')$

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and by 4.27 of [4]

$$(E \otimes F)' \simeq L(E, F').$$

It is very easy to see that, under these identifications, the map $U_{EF}$ converts into the canonical inclusion of $C(E, F')$ into $L(E, F')$. □

**Lemma 2.2.** The diagram

$$
\begin{array}{ccc}
(E \otimes F)'' & \xrightarrow{U_{EF}'} & (E' \otimes F')' \\
\downarrow J_{E \otimes F} & & \downarrow J_{E' \otimes F'} \\
E \otimes F & \xrightarrow{V_{EF}} & E'' \otimes F''
\end{array}
$$

is commutative.

**Proof.** Direct verification. □

By using these lemmas we now can prove:

**Theorem 2.3.** Let $K$ be not spherically complete, let $G := l^\infty \otimes l^\infty$. Then,

(i) $G$ is isomorphic to a closed subspace of $l^\infty$.

(ii) $G' \simeq c_0$.

(iii) $G$ is not reflexive.

**Proof.** (i) Since $(c_0)' \simeq l^\infty$ ([4], 3.Q.ii), the map $U_{c_0c_0}$ yields a linear isometry from $l^\infty \otimes l^\infty$ into a closed subspace of $(c_0 \otimes c_0)'$. Now the conclusion follows from the fact that $c_0 \otimes c_0 \simeq c_0$ ([4], 4.R.ii).

(ii) We have $(l^\infty)' \simeq c_0$ ([4], 4.17) and $L(l^\infty, c_0) = C(l^\infty, c_0)$ ([4], 5.19) so, by 2.1, $U_{EF}$ is surjective when $E := l^\infty, F := l^\infty$. Hence,

$$G' \simeq (l^\infty)' \otimes (l^\infty)' \simeq c_0 \otimes c_0 \simeq c_0.$$

(iii) Suppose $l^\infty \otimes l^\infty$ were reflexive; we derive a contradiction. In the diagram of 2.2 (again with $E = F = l^\infty$) the maps $J_{E \otimes F}$ and $U_{EF}'$ are bijections hence so is $V_{EF}$. Then $U_{EF}'$ is surjective hence, by 2.1, $L(E', F'') = C(E', F'')$, i.e. $L(c_0, l^\infty) = C(c_0, l^\infty)$, a contradiction. □

**Remark 2.4.** (1) From 2.3(iii) we conclude that if $K$ is not spherically complete, then $l^\infty \otimes l^\infty$ is not isomorphic to $l^\infty$. (The same conclusion holds when $K$ is spherically complete and the valuation on $K$ is dense, see 3.2.)

(2) The following slight extension of 2.3 will be needed in 4.2. Suppose that $K$ is not spherically complete. Let $s : N \to \{s_1, s_2, \ldots\} \subset (0, \infty)$ be such that $\{m \in N : s(m) = s_n\}$ is an infinite set for all $n \in N$. Then, $H := l^\infty \otimes l^\infty(N, s)$ is isomorphic to a non-reflexive closed subspace of $l^\infty(N, s)$ and $H' \simeq c_0(N, 1/s)$.

Indeed, observe that if $\{e_i : i \in N\}$ is the canonical orthogonal base of $c_0$ and $c_0(N, 1/s)$, then $\{e_i \otimes e_j : i, j \in N\}$ is an orthogonal base of $c_0 \otimes c_0(N, 1/s)$ ([4], 4.30), and so
$c_0 \otimes c_0(N, 1/s) \simeq c_0(N, 1/s)$.

The rest follows, by 3.9 and 4.22 of [4], like in the proof of 2.3.

3. NON-SPHERICALLY COMPLETE SUBSPACES OF $l^\infty$

As it is well known ([4], 4.A), $l^\infty$ is spherically complete if and only if $K$ is spherically complete.

In this section we study when the space $l^\infty \otimes l^\infty$ considered in 2.3 is spherically complete. To do that, recall that if $E$ is a Banach space, then $l^\infty \otimes E$ is isomorphic to the Banach space of all compactoid sequences on $E$, endowed with the supremum norm ([4], 4.R.v).

**Theorem 3.1.** Let $E$ be a Banach space over $K$ containing an orthogonal sequence $v_1, v_2, \ldots$ such that $\|v_1\| > \|v_2\| > \cdots$ and $\lim_n \|v_n\| = 1$. Then, $l^\infty \otimes E$ is not spherically complete. In particular, $l^\infty \otimes l^\infty$ is not spherically complete if the valuation on $K$ is dense.

**Proof.** Suppose $l^\infty \otimes E$ is spherically complete. For each $n = 1, 2, \ldots$ let $f_n$ be a compactoid sequence on $E$ defined by $f_n(m) = v_m$ if $m \leq n$ and $f_n(m) = 0$ if $m > n$. Since

$$B(f_n, \|v_{n+1}\|) \supseteq B(f_{n+1}, \|v_{n+2}\|) \quad \text{for all } n,$$

we derive the existence of a compactoid sequence $f = (x_1, x_2, \ldots)$ in $E$ such that $\|f - f_n\|_u \leq \|v_{n+1}\|$ for all $n$ (where $\|\cdot\|_u$ denotes the supremum norm). Given $i = 1, 2, \ldots$, we have that

$$\|x_i - v_i\| \leq \|f - f_n\|_u \leq \|v_{n+1}\| \quad \text{for all } n \geq i$$

and so $\|x_i - v_i\| < \|v_i\|$. Hence, $x_1, x_2, \ldots$ is an orthogonal sequence in $E$ ([4], 5.B) and $\|x_i\| = \|v_i\| > 1$ for all $i$, which implies that $\{x_1, x_2, \ldots\}$ is not compactoid in $E$ ([3], 2.2), a contradiction. □

**Remark 3.2.** From 3.1 we conclude that if the valuation on $K$ is dense and $K$ is spherically complete, then $l^\infty \otimes l^\infty$ is not isomorphic to $l^\infty$ (compare 2.4.1).

For discretely valued fields the situation is completely different. In fact, we have:

**Proposition 3.3.** Suppose that the valuation on $K$ is discrete. Then $l^\infty \otimes l^\infty$ is isomorphic to $l^\infty$ and is, in particular, spherically complete.

**Proof.** Observe that if $K$ is discretely valued, $l^\infty$ has an orthonormal base ([4], 5.16) and so $l^\infty \simeq c_0(I)$ for some infinite set $I$. From 4.R.ii of [4] we derive that $l^\infty \otimes l^\infty \simeq l^\infty$. □
With an eye on 3.1 and 3.3, the following question arises in a natural way. Describe the spherical completion of \( l^\infty \hat{\otimes} l^\infty \) when the valuation on \( K \) is dense. The key to the answer (Corollary 4.4) is given by the next theorem.

**Theorem 4.1.** Let \( E, F \) be spherically complete Banach spaces. Then the following are equivalent.

(i) \( E \simeq F \).

(ii) There exist linear isometries from \( E \) into \( F \) and from \( F \) into \( E \).

(iii) \( E \) is isomorphic to an orthocomplemented subspace of \( F \) and \( F \) is isomorphic to an orthocomplemented subspace of \( E \).

(iv) There exist strict quotient maps from \( E \) onto \( F \) and from \( F \) onto \( E \).

**Proof.** Clearly (i) \( \Rightarrow \) (ii) and (iii) \( \Rightarrow \) (iv). Also, (ii) \( \Rightarrow \) (iii) follows directly from [4], 4.7.

(iv) \( \Rightarrow \) (i): let \( \Gamma = \{ |\lambda| : \lambda \in K, \ \lambda \neq 0 \} \) be the value group of \( K \) and let \( H \) be a system of representatives of \( (0, \infty)/\Gamma \). By the remark following 5.2 of [4] we know that for every \( h \in H \), all the maximal orthogonal subsets of \( \{ x \in E : \| x \| \in h\Gamma \} \) (resp. of \( \{ y \in F : \| y \| \in h\Gamma \} \) ) have the same cardinality. We denote by \( N_h(E) \) (resp. \( N_h(F) \) ) a set with this cardinality.

Let \( X, Y \) be maximal orthogonal systems in \( E - \{ 0 \} \) and \( F - \{ 0 \} \) respectively, and let \( [X] \) and \( [Y] \) be the corresponding closed linear hulls. By 4.7 of [4] we have that \( E = [X]^\vee \) and \( F = [Y]^\vee \). So,

\[
E \simeq \left( \bigoplus_{h \in H} c_0(N_h(E), h) \right)^\vee
\]

and analogously

\[
F \simeq \left( \bigoplus_{h \in H} c_0(N_h(F), h) \right)^\vee.
\]

From (iv) it follows that for each \( h \in H \), the sets \( N_h(E) \) and \( N_h(F) \) have the same cardinality. Hence, \( c_0(N_h(E), h) \simeq c_0(N_h(F), h) \) for each \( h \in H \), which proves that \( E \simeq F \). \( \square \)

**Remark 4.2.** Theorem 4.1 does not remain true when spherical completeness is dropped.

**Example.** Let \( s_1, s_2, \ldots \) be a sequence in \( (0, \infty) \) such that \( s_1 > s_2 > \cdots \) and \( \lim_n s_n = 1 \). Make \( s : N \to \{ s_1, s_2, \ldots \} \subset (0, \infty) \) such that \( \{ m \in N : s(m) = s_n \} \) is an infinite set for all \( n \in N \).

Take \( E = l^\infty(N, s) \) and \( F = l^\infty \hat{\otimes} l^\infty(N, s) \). \( E \) is, in a natural way, isometrically embedded in \( F \). Also, there is a linear isometry from \( F \) into \( E \) (see 2.4.2). But \( E \) is not isomorphic to \( F \). (Apply 2.4.2 when \( K \) is not spherically complete and 3.1, for \( E = l^\infty(N, s) \), when \( K \) is spherically complete.)
However, for arbitrary Banach spaces we do have the following.

**Corollary 4.3.** If there exists linear isometries from $E$ into $F$ and from $F$ into $E$, then $E^\vee \simeq F^\vee$.

**Proof.** It follows from 4.1(i) ⇔ (ii) and [4], 4.42. □

This is enough material to prove our main result of this section.

**Corollary 4.4.** $(\ell^\infty \hat{\otimes} \ell^\infty)^\vee \simeq (\ell^\infty)^\vee$. In particular, if $K$ is spherically complete then $(\ell^\infty \hat{\otimes} \ell^\infty)^\vee \simeq \ell^\infty$.

**Proof.** We have an obvious embedding from $\ell^\infty$ into $\ell^\infty \hat{\otimes} \ell^\infty$ and an embedding from $\ell^\infty \hat{\otimes} \ell^\infty$ into $\ell^\infty$ (see 2.3(i)). Now apply 4.3. □

**Remark 4.5.** (1) It follows from 4.41 of [4] that $\ell^\infty \hat{\otimes} E$ is isomorphic to $C(c_0, E)$. Now suppose that $E$ is an infinite-dimensional spherically complete space (hence, $K$ is spherically complete ([4], 4.3) and $E$ contains an infinite orthogonal sequence ([4], 5.5)). Since $L(c_0, E)$ is also spherically complete ([4], 4.5), we can apply 4.2 of [4] and 3.1 to conclude that $C(c_0, E)$ is not ortho-complemented in $L(c_0, E)$ when the valuation on $K$ is dense (compare [1], 3.3).

(2) Observe that $M := \ell^\infty \hat{\otimes} c_0$ is also isomorphic to a closed subspace of $\ell^\infty$.

In contrast to 2.3 we have that if $K$ is not spherically complete, then $M$ is reflexive ([4], 4.R.i and 4.22.ii). Also, $M$ is spherically complete if and only if the valuation on $K$ is discrete (see 3.1 and 3.3).

(3) For $n \in \mathbb{N}$, $n \geq 2$, let $G_n = \ell^\infty \hat{\otimes} \cdots \hat{\otimes} \ell^\infty$. By induction, we can easily see that the results proved in this paper for $G_2 = \ell^\infty \hat{\otimes} \ell^\infty$ are also true for $G_n$. More concretely, we have:

(i) $G_n$ is isomorphic to a closed subspace of $\ell^\infty$. If $K$ is not spherically complete, $G_n$ is a non-reflexive space for which $G_n' \simeq c_0$ (see 2.3).

(ii) The valuation on $K$ is discrete $\Leftrightarrow G_n \simeq \ell^\infty \Leftrightarrow G_n$ is spherically complete (see 3.1 and 3.3).

(iii) $(G_n)^\vee \simeq (\ell^\infty)^\vee$ (see 4.4).

These facts lead us to the following question.

**Problem.** Suppose that the valuation on $K$ is dense. Are $G_m$ and $G_n$ isomorphic for all (or some) $m \neq n$, $m, n > 1$?

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**REFERENCES**

