Non-reflexive and non-spherically complete subspaces of the $p$-adic space $l^\infty$

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ABSTRACT

By forming tensor products we construct natural examples of non-reflexive (Section 2) and non-spherically complete (Section 3) closed subspaces of the non-archimedean space $l^\infty$. Also, we study (Section 4) conditions under which two spherically complete Banach spaces are isomorphic; as an application we describe the spherical completion of the subspaces of $l^\infty$ constructed in the paper.

1. PRELIMINARIES

Throughout this paper $K$ is a non-archimedean valued field that is complete under the metric induced by the non-trivial valuation $|\cdot|$, and $E, F$ are pseudo-reflexive ([4], p. 60) non-archimedean Banach spaces over $K$. By $E \cong F$ we mean that $E$ and $F$ are isomorphic, i.e., there is a linear isometry from $E$ onto $F$. We will denote the completed tensor product of $E$ and $F$ in the sense of [4], p. 123 by $E \hat{\otimes} F$.

A Banach space is called spherically complete if every sequence of closed balls

$$B(a_1, r_1) \supset B(a_2, r_2) \supset \cdots$$

for which $r_1 > r_2 > \cdots$ has a non-empty intersection. By $E^\Sigma$ we will denote the spherical completion of a Banach space $E$ (see [4], p. 148).

A subset $B$ of $E$ is called compactoid if for every $r > 0$ there exists a finite set $S$
in $E$ such that $B \subset \text{co } S + B(0, r)$, where $\text{co } S$ denotes the absolutely convex hull of $S$.

$L(E, F)$ will denote the Banach space of all continuous linear maps from $E$ into $F$, endowed with the usual norm. The topological dual space of $E$ is $E' = L(E, K)$. $E$ is called reflexive if the canonical linear map $J_E$ from $E$ into $E''$ is a surjective isometry.

By $C(E, F)$ we will denote the closed subspace of $L(E, F)$ consisting of all compact linear maps from $E$ into $F$ (i.e., the maps $T \in L(E, F)$ for which the image of the closed unit ball of $E$ is a compactoid subset of $F$).

We say that $F$ is a strict quotient of $E$ if there exists a $T \in L(E, F)$ such that $\|T\| \leq 1$ and for every $y \in F$ there is $x \in E$ for which $Tx = y$ and $\|x\| = \|y\|$.

For unexplained terms and background we refer to [4].

2. NON-REFLEXIVE SUBSPACES OF $l^\infty$

It is well known that, if $K$ is spherically complete, no infinite-dimensional Banach space over $K$ is reflexive ([4], 4.16). However, non-spherically complete fields have a more satisfactory behaviour with regard to reflexivity. In fact, suppose that $K$ is not spherically complete. Then $c_0$ and $l^\infty$ are reflexive spaces (more generally, the spaces $c_0(N, s)$ and $l^\infty(N, s)$ are reflexive, for every function $s : N \to (0, \infty)$, [4], 4.22.ii). Also, every quotient and every closed subspace of $c_0(N, s)$ is reflexive ([2], 9.9). But quotients and closed subspaces of $l^\infty$ need not be reflexive. For quotients this is easily seen: $l^\infty/c_0$ is a non-reflexive quotient of $l^\infty$ since its dual is trivial ([4], 4.15). The construction of a non-reflexive closed subspace of $l^\infty$ is more laborious and was given in [4], 4.1.

In this section we are going to show that, by taking tensor products, we can construct in a simple way, natural examples of non-reflexive closed subspaces of $l^\infty$ when $K$ is not spherically complete. To do that we need some preliminary machinery.

Recall ([4], 4.34) that the maps

\[ U_{EF} : E' \hat{\otimes} F' \to (E \hat{\otimes} F)' \]
\[ V_{EF} : E \hat{\otimes} F \to (E' \hat{\otimes} F')' \]

given by

\[ U_{EF}(g \otimes h)(x \otimes y) = g(x)h(y) \]
\[ V_{EF}(x \otimes y)(g \otimes h) = g(x)h(y) \]
\[ (x \in E, g \in E', y \in F, h \in F') \]

are linear isometries. We need two facts.

**Lemma 2.1.** $U_{EF}$ is surjective iff $L(E, F') = C(E, F')$.

**Proof.** By 4.41 of [4]

\[ E' \hat{\otimes} F' \cong C(E, F') \]
and by 4.27 of [4]

\[(E \hat{\otimes} F)' \simeq L(E, F').\]

It is very easy to see that, under these identifications, the map \(U_{EF}\) converts into the canonical inclusion of \(C(E, F')\) into \(L(E, F')\). □

Lemma 2.2. The diagram

\[
\begin{array}{ccc}
(E \hat{\otimes} F)'' & \xrightarrow{U_{EF}'} & (E' \hat{\otimes} F')' \\
\downarrow J_{E \hat{\otimes} F} & & \downarrow V_{EF} \\
E \hat{\otimes} F & \xleftarrow{U'_{EF}} & E'' \hat{\otimes} F'' \\
\end{array}
\]

is commutative.

**Proof.** Direct verification. □

By using these lemmas we now can prove:

**Theorem 2.3.** Let \(K\) be not spherically complete, let \(G := l^\infty \hat{\otimes} l^\infty\). Then,

(i) \(G'\) is isomorphic to a closed subspace of \(l^\infty\).

(ii) \(G' \simeq c_0\).

(iii) \(G\) is not reflexive.

**Proof.** (i) Since \((c_0)' \simeq l^\infty\) ([4], 3.Q.ii), the map \(U_{c_0c_0}\) yields a linear isometry from \(l^\infty \hat{\otimes} l^\infty\) into a closed subspace of \((c_0 \hat{\otimes} c_0)'\). Now the conclusion follows from the fact that \(c_0 \hat{\otimes} c_0 \simeq c_0\) ([4], 4.R.ii).

(ii) We have \((l^\infty)' \simeq c_0\) ([4], 4.17) and \(L(l^\infty, c_0) = C(l^\infty, c_0)\) ([4], 5.19) so, by 2.1, \(U_{EF}\) is surjective when \(E := l^\infty, F := l^\infty\). Hence,

\[G' \simeq (l^\infty)' \hat{\otimes} (l^\infty)' \simeq c_0 \hat{\otimes} c_0 \simeq c_0.\]

(iii) Suppose \(l^\infty \hat{\otimes} l^\infty\) were reflexive; we derive a contradiction. In the diagram of 2.2 (again with \(E = F = l^\infty\)) the maps \(J_{E \hat{\otimes} F}\) and \(U_{EF}'\) are bijections hence so is \(V_{EF}\). Then \(U_{EF}'\) is surjective hence, by 2.1, \(L(E', F'') = C(E', F'')\), i.e. \(L(c_0, l^\infty) = C(c_0, l^\infty)\), a contradiction. □

**Remark 2.4.** (1) From 2.3(iii) we conclude that if \(K\) is not spherically complete, then \(l^\infty \hat{\otimes} l^\infty\) is not isomorphic to \(l^\infty\). (The same conclusion holds when \(K\) is spherically complete and the valuation on \(K\) is dense, see 3.2.)

(2) The following slight extension of 2.3 will be needed in 4.2. Suppose that \(K\) is not spherically complete. Let \(s : N \to \{s_1, s_2, \ldots\} \subset (0, \infty)\) be such that \(\{m \in N : s(m) = s_n\}\) is an infinite set for all \(n \in N\). Then, \(H := l^\infty \hat{\otimes} l^\infty(N, s)\) is isomorphic to a non-reflexive closed subspace of \(l^\infty(N, s)\) and \(H' \simeq c_0(N, 1/s)\).

Indeed, observe that if \(\{e_i : i \in N\}\) is the canonical orthogonal base of \(c_0\) and \(c_0(N, 1/s)\), then \(\{e_i \otimes e_j : i, j \in N\}\) is an orthogonal base of \(c_0 \hat{\otimes} c_0(N, 1/s)\) ([4], 4.30), and so
\[ c_0 \otimes c_0(N, 1/s) \simeq c_0(N, 1/s). \]

The rest follows, by 3.Q and 4.22 of [4], like in the proof of 2.3.

3. NON-SPHERICALLY COMPLETE SUBSPACES OF \( l^\infty \)

As it is well known ([4], 4.A), \( l^\infty \) is spherically complete if and only if \( K \) is spherically complete.

In this section we study when the space \( l^\infty \otimes l^\infty \) considered in 2.3 is spherically complete. To do that, recall that if \( E \) is a Banach space, then \( l^\infty \otimes E \) is isomorphic to the Banach space of all compactoid sequences on \( E \), endowed with the supremum norm ([4], 4.R.v).

**Theorem 3.1.** Let \( E \) be a Banach space over \( K \) containing an orthogonal sequence \( v_1, v_2, \ldots \) such that \( \|v_1\| > \|v_2\| > \cdots \) and \( \lim_n \|v_n\| = 1 \). Then, \( l^\infty \otimes E \) is not spherically complete. In particular, \( l^\infty \otimes l^\infty \) is not spherically complete if the valuation on \( K \) is dense.

**Proof.** Suppose \( l^\infty \otimes E \) is spherically complete. For each \( n = 1, 2, \ldots \) let \( f_n \) be a compactoid sequence on \( E \) defined by \( f_n(m) = v_m \) if \( m \leq n \) and \( f_n(m) = 0 \) if \( m > n \). Since

\[ B(f_n, \|v_{n+1}\|) \supset B(f_{n+1}, \|v_{n+2}\|) \]

for all \( n \), we derive the existence of a compactoid sequence \( f = (x_1, x_2, \ldots) \) in \( E \) such that \( \|f - f_n\|_u \leq \|v_{n+1}\| \) for all \( n \) (where \( \| \cdot \|_u \) denotes the supremum norm). Given \( i = 1, 2, \ldots, \) we have that

\[ \|x_i - v_i\| \leq \|f - f_n\|_u \leq \|v_{n+1}\| \]

for all \( n \geq i \)

and so \( \|x_i - v_i\| < \|v_i\| \). Hence, \( x_1, x_2, \ldots \) is an orthogonal sequence in \( E \) ([4], 5.B) and \( \|x_i\| = \|v_i\| > 1 \) for all \( i \), which implies that \( \{x_1, x_2, \ldots\} \) is not compactoid in \( E \) ([3], 2.2), a contradiction. \( \square \)

**Remark 3.2.** From 3.1 we conclude that if the valuation on \( K \) is dense and \( K \) is spherically complete, then \( l^\infty \otimes l^\infty \) is not isomorphic to \( l^\infty \) (compare 2.4.1).

For discretely valued fields the situation is completely different. In fact, we have:

**Proposition 3.3.** Suppose that the valuation on \( K \) is discrete. Then \( l^\infty \otimes l^\infty \) is isomorphic to \( l^\infty \) and is, in particular, spherically complete.

**Proof.** Observe that if \( K \) is discretely valued, \( l^\infty \) has an orthonormal base ([4], 5.16) and so \( l^\infty \simeq c_0(I) \) for some infinite set \( I \). From 4.R.ii of [4] we derive that \( l^\infty \otimes l^\infty \simeq l^\infty \). \( \square \)
With an eye on 3.1 and 3.3, the following question arises in a natural way. Describe the spherical completion of $l^\infty \otimes l^\infty$ when the valuation on $K$ is dense. The key to the answer (Corollary 4.4) is given by the next theorem.

**Theorem 4.1.** Let $E, F$ be spherically complete Banach spaces. Then the following are equivalent.

(i) $E \simeq F$.

(ii) There exist linear isometries from $E$ into $F$ and from $F$ into $E$.

(iii) $E$ is isomorphic to an orthocomplemented subspace of $F$ and $F$ is isomorphic to an orthocomplemented subspace of $E$.

(iv) There exist strict quotient maps from $E$ onto $F$ and from $F$ onto $E$.

**Proof.** Clearly (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv). Also, (ii) $\Rightarrow$ (iii) follows directly from [4], 4.7.

(iv) $\Rightarrow$ (i): let $\Gamma = \{|\lambda| : \lambda \in K, \lambda \neq 0\}$ be the value group of $K$ and let $H$ be a system of representatives of $(0, \infty) / \Gamma$. By the remark following 5.2 of [4] we know that for every $h \in H$, all the maximal orthogonal subsets of $\{x \in E : \|x\| \in h\Gamma\}$ (resp. $\{y \in F : \|y\| \in h\Gamma\}$) have the same cardinality. We denote by $N_h(E)$ (resp. $N_h(F)$) a set with this cardinality.

Let $X, Y$ be maximal orthogonal systems in $E - \{0\}$ and $F - \{0\}$ respectively, and let $[X]$ and $[Y]$ be the corresponding closed linear hulls. By 4.7 of [4] we have that $E = [X]^\vee$ and $F = [Y]^\vee$. So,

$$E \simeq \left( \bigoplus_{h \in H} c_0(N_h(E), h) \right)^\vee$$

and analogously

$$F \simeq \left( \bigoplus_{h \in H} c_0(N_h(F), h) \right)^\vee.$$  

From (iv) it follows that for each $h \in H$, the sets $N_h(E)$ and $N_h(F)$ have the same cardinality. Hence, $c_0(N_h(E), h) \simeq c_0(N_h(F), h)$ for each $h \in H$, which proves that $E \simeq F$. $\square$

**Remark 4.2.** Theorem 4.1 does not remain true when spherical completeness is dropped.

**Example.** Let $s_1, s_2, \ldots$ be a sequence in $(0, \infty)$ such that $s_1 > s_2 > \cdots$ and $\lim_n s_n = 1$. Make $s : N \to \{s_1, s_2, \ldots\} \subset (0, \infty)$ such that $\{m \in N : s(m) = s_n\}$ is an infinite set for all $n \in N$.

Take $E = l^\infty(N, s)$ and $F = l^\infty \otimes l^\infty(N, s)$. $E$ is, in a natural way, isometrically embedded in $F$. Also, there is a linear isometry from $F$ into $E$ (see 2.4.2). But $E$ is not isomorphic to $F$. (Apply 2.4.2 when $K$ is not spherically complete and 3.1, for $E = l^\infty(N, s)$, when $K$ is spherically complete.)
However, for arbitrary Banach spaces we do have the following.

**Corollary 4.3.** If there exists linear isometries from $E$ into $F$ and from $F$ into $E$, then $E^\vee \simeq F^\vee$.

**Proof.** It follows from 4.1(i) $\iff$ (ii) and [4], 4.42. \qed

This is enough material to prove our main result of this section.

**Corollary 4.4.** $(L^\infty \hat{\otimes} L^\infty)^\vee \simeq (L^\infty)^\vee$. In particular, if $K$ is spherically complete then $(L^\infty \hat{\otimes} L^\infty)^\vee \simeq L^\infty$.

**Proof.** We have an obvious embedding from $L^\infty$ into $L^\infty \hat{\otimes} L^\infty$ and an embedding from $L^\infty \hat{\otimes} L^\infty$ into $L^\infty$ (see 2.3(i)). Now apply 4.3. \qed

**Remark 4.5.**

1. It follows from 4.41 of [4] that $L^\infty \hat{\otimes} E$ is isomorphic to $C(c_0, E)$. Now suppose that $E$ is an infinite-dimensional spherically complete space (hence, $K$ is spherically complete ([4], 4.3) and $E$ contains an infinite orthogonal sequence ([4], 5.5)). Since $L(c_0, E)$ is also spherically complete ([4], 4.5), we can apply 4.2 of [4] and 3.1 to conclude that $C(c_0, E)$ is not orthocomplemented in $L(c_0, E)$ when the valuation on $K$ is dense (compare [1], 3.3).

2. Observe that $M := L^\infty \hat{\otimes} c_0$ is also isomorphic to a closed subspace of $L^\infty$. In contrast to 2.3 we have that if $K$ is not spherically complete, then $M$ is reflexive ([4], 4.R.1 and 4.22.ii). Also, $M$ is spherically complete if and only if the valuation on $K$ is discrete (see 3.1 and 3.3).

3. For $n \in \mathbb{N}$, $n \geq 2$, let $G_n = L^\infty \hat{\otimes} \cdot \cdot \cdot \hat{\otimes} L^\infty$. By induction, we can easily see that the results proved in this paper for $G_2 = L^\infty \hat{\otimes} L^\infty$ are also true for $G_n$. More concretely, we have:

   (i) $G_n$ is isomorphic to a closed subspace of $L^\infty$. If $K$ is not spherically complete, $G_n$ is a non-reflexive space for which $G_n \simeq c_0$ (see 2.3).

   (ii) The valuation on $K$ is discrete $\iff G_n \simeq L^\infty$ $\iff G_n$ is spherically complete (see 3.1 and 3.3).

   (iii) $(G_n)^\vee \simeq (L^\infty)^\vee$ (see 4.4).

These facts lead us to the following question.

**Problem.** Suppose that the valuation on $K$ is dense. Are $G_m$ and $G_n$ isomorphic for all (or some) $m \neq n$, $m, n > 1$?

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**REFERENCES**

