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TENSOR PRODUCT AND $p$-ADIC VECTOR VALUED CONTINUOUS FUNCTIONS

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Abstract. The stability of some classes of non-archimedean locally convex spaces by forming topological tensor products is considered in this paper. As a result we study several properties of different spaces of continuous functions, which we describe in terms of topological tensor products. The theory also leads to quite natural examples of non-reflexive spaces (such as $l^\infty \hat{\otimes} l^\infty$ over a non-spherically complete field, see Corollary 3.6).

1. Preliminaries

Throughout this paper $K$ is a non-archimedean valued field that is complete under the metric induced by the non-trivial valuation $|\cdot|$, and $E,F$ are Hausdorff locally convex spaces over $K$.

A subset $B$ of $E$ is called compactoid if for each zero-neighbourhood $V$ in $E$ there exists a finite set $S$ in $E$ such that $B \subset \text{co}(S) + V$, where $\text{co}(S)$ denotes the absolutely convex hull of $S$. A linear map $T: E \longrightarrow F$ is called compact if there exists a zero-neighbourhood $U$ in $E$ such that $T(U)$ is compactoid in $F$. We denote by $C(E,F)$ the vector space of the compact maps from $E$ to $F$. Then, obviously $C(E,F) \subset L(E,F)$ where $L(E,F)$ stands for the vector space of the continuous linear maps from $E$ to $F$.

If $p$ is a (non-archimedean) continuous seminorm on $E$ we denote by $E_p$ the vector space $E/\text{Ker} p$ and by $\pi_p : E \longrightarrow E_p$ the canonical surjection. The space $E_p$ is normed by $\| \pi_p(x) \| = p(x)$ ($x \in E$). If $p,q$ are (non-archimedean) continuous seminorms on $E,F$ respectively, we denote by $p \otimes q$ the seminorm

$$z \rightarrow \inf \{ \max_{1 \leq i \leq n} \| x_i \| \cdot \| y_i \| : n \in \mathbb{N}, \quad z = \sum_{i=1}^{n} x_i \otimes y_i, \quad x_i \in E, \quad y_i \in F \}$$

on the tensor product $E \otimes F$, on which we always consider the topology generated by the seminorms $p \otimes q$.

$E$ is called nuclear if for every continuous seminorm $p$ on $E$ there exists a continuous seminorm $q$ on $E$ with $p \leq q$ such that $\Phi_{pq} : E_q \longrightarrow E_p$ is compact, where $\Phi_{pq}$ is

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the unique map from \( E_q \) to \( E_p \) such that \( \Phi_{pq} \circ \pi_q = \pi_p \). E is called of countable type if for every continuous seminorm \( p \) on \( E \), the normed space \( E_p \) is of countable type. E is called an Orlicz-Pettis space ((O.P)-space) if every weakly convergent sequence in E is convergent. Observe that

\[ E \text{ nuclear } \Rightarrow E \text{ of countable type } \Rightarrow E \text{ (O.P)} \]

([8], Proposition 1.3 and [7], Theorem 1.3), but the converses are not true in general (if \( K \) is spherically complete every \( E \) over \( K \) is (O.P); also \( c_0 \) is an space of countable type which is not nuclear).

Now, suppose that \( E \) is a normed space over \( K \). For \( x \in E \) and \( r > 0 \), \( B(x, r) \) will denote the set \( \{ y \in E : \| y - x \| \leq r \} \). We say that a collection \( (e_i)_{i \in I} \) in \( E \) is a base of \( E \) if for each \( x \in E \) there exists a unique \( (\lambda_i) \subset K^I \) such that \( \{ i \in I : \lambda_i \geq \varepsilon \} \) is finite for each \( \varepsilon > 0 \), and \( x = \sum_{i \in I} \lambda_i e_i \). If \( E \) is complete this definition coincides with the usual one ([8] Lemma 3.1). Observe that every normed space with a base is an (O.P)-space ([7], Theorem 1.3). The topological dual space of \( E \) is \( E' = L(E, K) \) and the topological bidual of \( E \) is \( E'' = (E')' \) (where we assume \( E' \) endowed with the usual norm). \( E \) is called pseudoreflexive if the canonical map \( J_E : E \longrightarrow E'', x \longrightarrow J_E(x), \quad J_E(x)(x') = x'(x) \quad (x \in E, \ x' \in E') \), is a linear isometry from \( E \) onto \( J_E(E) \). If in addition \( J_E \) is a surjective map, \( E \) is called reflexive.

For unexplained terms and background we refer to [8] (locally convex spaces) and [11] (normed spaces).

2. Tensor Product and Spaces of Continuous Functions

For a Hausdorff zero-dimensional topological space \( X \) and a locally convex space \( E \) over \( K \) we define

\( C(X, E) \): The space of all continuous functions \( X \longrightarrow E \), endowed with the topology \( \tau_c \) of compact convergence.

\( PC(X, E) \): The space of all continuous functions \( f : X \longrightarrow E \) for which \( f(X) \) is precompact, endowed with the topology \( \tau_u \) of uniform convergence.

\( P(X, E) \): The space of all continuous functions \( f : X \longrightarrow E \) for which \( f(X) \) is compactoid, endowed with the topology \( \tau_u \) of uniform convergence.

When \( E = K \), we will write \( C(X) \), \( PC(X) \) and \( P(X) \) respectively.

The aim of this section is to describe when the spaces of continuous functions defined above are nuclear, (O.P), or of countable type and when, if \( E \) is a normed space, \( PC(X, E) \) and \( P(X, E) \) have a base. To do that we first prove three structure theorems.

**Theorem 2.1.** i) \( E \otimes F \) is a nuclear space (resp. an (O.P)-space, a space of countable type) if and only if \( E \) and \( F \) are nuclear spaces (resp. (O.P)-spaces, spaces of countable type).
ii) If $E$ and $F$ are normed spaces, then $E \otimes F$ has a base if and only if $E$ and $F$ have a base.

Proof.

i) For nuclear spaces the property was proved in [3], Theorem 2.10.

For the case of the (O.P)-property see [7], Theorem 2.1.

Now, suppose that $E \otimes F$ is of countable type. Since $E$ and $F$ can be identified with subspaces of $E \otimes F$, we conclude that $E$ and $F$ are also of countable type.

Conversely, suppose that $E$ and $F$ are spaces of countable type. Let $p$ be a continuous seminorm on $E$ and $q$ a continuous seminorm on $F$. Then, the normed spaces $E_p \otimes E_q$ and $(E \otimes F)_{p \otimes q}$ are isometrically isomorphic ([4], Proposition 15.4.4). By [11], Exercise 4.R.ii), $(E \otimes F)_{p \otimes q}$ is a normed space of countable type. Hence, $E \otimes F$ is of countable type.

ii) It is a direct consequence of Corollary 3.8 and Exercise 4.R.ii) of [11]. □

Theorem 2.2. Let $D$ be a dense subspace of $E$. Then,

i) $E$ is nuclear (resp. of countable type) if and only if $D$ is nuclear (resp. of countable type).

ii) If $E$ is metrizable, $E$ is an (O.P)-space if and only if $D$ is an (O.P)-space.

iii) If $E$ is a normed space, $E$ has a base if and only if $D$ has a base.

Proof.

i) For nuclear spaces the property was proved in [3], Corollary 2.7. For the case of spaces of countable type see [8], Proposition 4.12.

ii) It was proved in [7], Theorem 2.2.

iii) See [9], Lemma 3.1. □

Theorem 2.3. $C(X) \otimes E$ (resp. $PC(X) \otimes E$, $P(X) \otimes E$) is linearly homeomorphic to a dense subspace of $C(X,E)$ (resp. $PC(X,E)$, $P(X,E)$).

Proof. For the fact that $C(X) \otimes E$ (resp. $PC(X) \otimes E$) is linearly homeomorphic to a dense subspace of $C(X,E)$ (resp. $PC(X,E)$), see the proof of Theorem 4.2 of [3] (resp. the proof of Theorem 2.4 of [7]).

Now, consider the linear map $T: P(X) \otimes E \rightarrow P(X,E)$, where for

$$z = \sum_{m=1}^{n} f_m \otimes a_m \in P(X) \otimes E,$$

$T(z)$ is defined by

$$T(z)(x) = \sum_{m=1}^{n} f_m(x)a_m \quad (x \in X).$$

It is straightforward to verify that $T$ is a linear homeomorphism from $P(X) \otimes E$ onto $\text{Im}T$. To see that $\text{Im}T$ is dense in $P(X,E)$, take $f \in P(X,E)$ and $p$ a
continuous seminorm on \( E \). Since \( \pi_p \circ f \in P(X, E_p) \), we can apply Exercise 4.R.v) of [11] to derive the existence of a sequence \( \alpha_1, \alpha_2, \ldots \) in \( P(X) \) with \( \| \alpha_n \| \leq 1 \) for all \( n \), and a \( t \)-orthogonal sequence \( (0 < t < 1) \pi_p(a_1), \pi_p(a_2), \ldots \) in \( E_p \) such that

\[
\limsup_{n \to \infty} \sup_{x \in X} \| \pi_p f(x) - \sum_{m=1}^{n} \alpha_m(x) \pi_p(a_m) \|_p = \limsup_{n \to \infty} \sup_{x \in X} p(f(x) - \sum_{m=1}^{n} \alpha_m(x)a_m) = 0,
\]

and we are done. □

Then, with regard to the nuclearity of the spaces of continuous functions considered in this section, we have

**Theorem 2.4.** ([3], Theorems 3.3 and 4.2). The following are equivalent:

i) \( C(X, E) \) is nuclear.

ii) \( C(X) \) and \( E \) are nuclear.

iii) \( E \) is nuclear and every compact subset of \( X \) is finite.

**Proposition 2.5.** The following are equivalent:

i) \( P(X, E) \) is nuclear.

ii) \( P(X) \) and \( E \) are nuclear.

iii) \( PC(X, E) \) is nuclear.

iv) \( PC(X) \) and \( E \) are nuclear.

v) \( X \) is finite and \( E \) is nuclear.

**Proof.** The implications i) \( \Rightarrow \) iii) and v) \( \Rightarrow \) ii) are obvious. Also, the equivalences i) \( \Leftrightarrow \) ii) and iii) \( \Leftrightarrow \) iv) follow from 2.1, 2.2 and 2.3. iv) \( \Rightarrow \) v) If \( PC(X) \) is nuclear, then it is finite-dimensional and so \( X \) is finite. □

The \((O.P)\)-property is described in the following results.

**Theorem 2.6.** ([7], Theorem 2.4 and Corollary 2.5). Let \( E \) be a metrizable space. Then, the following are equivalent:

i) \( C(X, E) \) is an \((O.P)\)-space.

ii) \( PC(X, E) \) is an \((O.P)\)-space.

iii) \( E \) is an \((O.P)\)-space.

**Proposition 2.7.** Let \( E \) be a metrizable space. Then, the following are equivalent:

i) \( P(X, E) \) is an \((O.P)\)-space.

ii) \( P(X) \) and \( E \) are \((O.P)\)-spaces.

If in addition \( K \) is not spherically complete, properties i) and ii) are equivalent to

iii) \( X \) is pseudocompact (i.e. every continuous map \( f : X \to \mathbb{R} \) is bounded) and \( E \) is an \((O.P)\)-space.
Proof. The equivalence \(i) \Leftrightarrow ii)\) follows from 2.1, 2.2 and 2.3.

For non-spherically complete fields, the equivalence \(ii) \Leftrightarrow iii)\) is a direct consequence of Corollary 2.7 of [7]. □

If \(X\) is compact and \(E\) is metrizable, in [6], Theorem 4.1 it was proved that \(C(X,E)\) is separable if and only if \(E\) is separable and \(X\) is ultrametrizable. Propositions 2.8 - 2.10 constitute an extension of this result.

**Proposition 2.8.** The following are equivalent:

i) \(C(X,E)\) is of countable type.

ii) \(C(X)\) and \(E\) are of countable type.

iii) \(E\) is of countable type and every compact subset of \(X\) is ultrametrizable.

Proof. The equivalence \(i) \Leftrightarrow ii)\) follows from 2.1, 2.2 and 2.3. \(ii) \Leftrightarrow iii)\) Take a compact subset \(A\) of \(X\) and consider the corresponding seminorm \(p_A\) on \(C(X)\), given by \(p_A(f) = \sup_{x \in A} |f(x)|\). From [11], Theorem 5.24 we know that the associated normed space \(C(X)/\text{Ker}p_A\) is isometrically isomorphic to the space \(C(A)\) endowed with the supremum norm. Now the conclusion follows from [11], Exercise 3.T. □

**Proposition 2.9.** The following are equivalent:

i) \(PC(X,E)\) is of countable type.

ii) \(PC(X)\) and \(E\) are of countable type.

iii) \(X^\xi\) is ultrametrizable and \(E\) is of countable type (where \(X^\xi\) denotes the Banaschewski compactification of \(X\)).

Proof. The equivalence \(i) \Leftrightarrow ii)\) follows from 2.1, 2.2 and 2.3.

The equivalence \(ii) \Leftrightarrow iii)\) is a consequence of [11], Exercise 3.T and the fact that \(PC(X)\) is isometrically isomorphic to \(C(X^\xi)\) (see [11], proof of Corollary 5.23). □

**Proposition 2.10.** The following are equivalent:

i) \(P(X,E)\) is of countable type.

ii) \(P(X)\) and \(E\) are of countable type.

iii) \(X\) is pseudocompact and ultrametrizable and \(E\) is of countable type.

iv) \(X\) is compact and ultrametrizable and \(E\) is of countable type.

Proof. The equivalence \(i) \Leftrightarrow ii)\) follows from 2.1, 2.2 and 2.3. 

\(ii) \Rightarrow iii)\) If \(P(X)\) is of countable type, then it has a base ([11], Theorem 3.16), and so \(X\) is pseudocompact ([11], Corollary 5.25). Hence, \(P(X) = PC(X)\). Now the implication follows from Proposition 2.9.

\(iii) \Leftrightarrow iv)\) Recall that every pseudocompact ultrametrizable space is compact ([1], Theorem 1.5).

\(iv) \Rightarrow i)\) One verifies that if \(X\) is compact \(P(X,E) = PC(X,E)\). Now apply Proposition 2.9. □
Similarly, we can prove

**Proposition 2.11.** Let $E$ be a normed space.

i) $PC(X,E)$ has a base if and only if $E$ has a base.

ii) $P(X,E)$ has a base if and only if $X$ is pseudocompact and $E$ has a base.

3. **Tensor product, reflexivity and spherical completeness.**

In the sequel $E,F$ will denote Banach spaces over $K$. By $E \simeq F$ we mean that $E$ and $F$ are isometrically isomorphic, i.e., there is a linear isometry from $E$ onto $F$. Also, $E \hat{\otimes} F$ will denote the completion of $E \otimes F$. By Corollary 4.21 of [11], it follows that, for each $w \in E \hat{\otimes} F$, its norm is given by

$$
\| w \| = \inf \left\{ \max_{i \in \mathbb{N}} \| a_i \| \cdot \| b_i \| : a_1, a_2, \ldots \in E; b_1, b_2, \ldots \in F; w = \sum a_i \otimes b_i \right\}
$$

Observe that if in the proof of Theorem 2.3 we take $X = \mathbb{N}$ endowed with the discrete topology we have that $P(X) = l^\infty$ and so the map $T$ constructed in that proof gives us a linear isometry from $l^\infty \hat{\otimes} E$ onto the Banach space $P(\mathbb{N}, E)$ of all the compactoid sequences in $E$, endowed with the supremum norm $\| . \|_u$ (Compare with [11], Exercise 4.R.v)). This kind of spaces will be very useful in this section to obtain examples of non-reflexive and non-spherically complete Banach spaces (see Corollary 3.6 and Theorem 3.7). To do that we need some preliminary machinery.

Recall ([11], Corollary 4.34) that there exists a linear isometry $U$ of $E' \otimes F'$ into $(E \otimes F)'$ given by

$$
U(g \otimes h)(x \otimes y) = g(x)h(y) \quad (x \in E, \ g \in E', \ y \in F, \ h \in F')
$$

Also, by Theorem 4.41 of [11]

(1) \quad $E' \hat{\otimes} F' \simeq C(E, F')$

and by Theorem 4.27 of [11]

(2) \quad $(E \hat{\otimes} F)' \simeq L(E, F')$

It is very easy to see that, under these identifications, the map $U$ converts into the canonical inclusion of $C(E, F')$ in $L(E, F')$. So,

**Theorem 3.1.** $U$ is surjective if and only if $L(E, F') = C(E, F')$.

**Corollary 3.2.** 1) Suppose that

1.a) $K$ is spherically complete and $E,F$ are infinite-dimensional spaces or

1.b) $E$ contains a complemented subspace linearly homeomorphic to $c_0$ and $F'$ is infinite-dimensional.

Then $U$ is not surjective.

2) Suppose that $K$ is not spherically complete and let $F$ be a Banach space whose dual is isometrically isomorphic to $c_0$ (e.g. $F = l^\infty$, [11] Theorem 4.17). Then, the following properties are equivalent:
2.i) $U$ is surjective.
2.ii) $L(E, c_0) = C(E, c_0)$.
2.iii) $E$ does not contain a complemented subspace linearly homeomorphic to $c_0$.

Proof. 1) We only prove 1.a). The proof of 1.b) is similar.

Since $E, F$ are infinite-dimensional, there are closed subspaces $D_1 \subset E, D_2 \subset F'$ which are linearly homeomorphic to $c_0$. Let $T : D_1 \rightarrow D_2$ be a linear homeomorphism. Since $F'$ is spherically complete, $T$ can be extended to a $\overline{T} \in L(E, F')$ which, by Theorem 4.40 of [11] is not compact. Now apply Theorem 3.1.

2) It follows from Theorem 4.8 of [2] and Theorem 3.1. □

Although $U$ is not in general a surjective map, we have

Proposition 3.3. $\text{Im} U$ is weakly*-dense in $(E \hat{\otimes} F)'$.

Proof. It is straightforward to verify that, under the identification given in (2), the weak*-topology on $(E \hat{\otimes} F)'$ gives on $L(E, F')$ the locally convex topology $\tau$ defined by the seminorms $p_{x,y} (x \in E, y \in F)$ given by $p_{x,y}(T) = (T(x))(y)$ $(T \in L(E, F'))$. So, it is enough to prove that $C(E, F')$ is $\tau$-dense in $L(E, F')$. To see that, take $x \in E, y \in F$ and $T \in L(E, F')$. We can assume that $T(x) \neq 0$. Since the dual of $F'$ separates the points of $F'$, there exists a continuous linear projection $P$ from $F'$ onto the linear hull of $\{T(x)\}$. Thus, $S = P \circ T \in L(E, F')$ is a continuous linear map of finite rank (and hence compact) for which $p_{x,y}(T - S) = 0$. □

Now, assume that $E$ and $F$ are pseudoreflexive spaces. Then ([11], Corollary 4.34), $E \hat{\otimes} F$ is pseudoreflexive and there is a linear isometry $V$ of $E \hat{\otimes} F$ into $(E' \hat{\otimes} F')'$ for which

$$(V(x \otimes y))(g \otimes h) = g(x)h(y) \quad (x \in E, y \in F, g \in E', h \in F')$$

Lemma 3.4. Let $E, F$ be pseudoreflexive Banach spaces. If $V$ is surjective, then $L(E', F) = C(E', F)$. The converse is also true if $E$ and $F$ are reflexive spaces.

Proof. By considering $E'$ and $F'$ instead of $E$ and $F$ in the construction of the map $U$, we obtain a linear isometry $W$ of $E' \hat{\otimes} F'$ into $(E' \hat{\otimes} F')'$ such that $(W(x'' \otimes y''))(g \otimes h) = x''(g)y''(h)$ $(x'' \in E'', y'' \in F'', g \in E', h \in F')$ and $W \circ (J_E \otimes J_F) = V$. Hence, the surjectivity of $V$ implies (or is equivalent to, if $E, F$ are reflexive) the surjectivity of $W$. Now, apply Theorem 3.1. □

Theorem 3.5. Let $E, F$ be pseudoreflexive Banach spaces such that $L(E, F') = C(E, F')$.
If $E \hat{\otimes} F$ is reflexive, then $L(E', F) = C(E', F)$. The converse is also true if $E$ and $F$ are reflexive spaces.

Proof. Since $L(E, F') = C(E, F')$ we derive from Theorem 3.1 that the map $U$ (and hence its adjoint $U'$) is a surjective isometry. On the other hand, since $V = U' \circ J_{E \hat{\otimes} F}$ we can apply Lemma 3.4 to obtain the desired conclusions. □
Remark 1. The assumption \( L(E, F') = C(E, F') \) cannot be dropped in Theorem 3.5.

Example 1. Suppose that \( K \) is not spherically complete and take \( E = l^\infty, F = c_0 \). Clearly \( L(E, F') \neq C(E, F') \). Also, \( E\hat{\otimes} F \) is reflexive ([11, Exercise 4.R.i) and Theorem 4.22.ii)). However, \( L(E', F) \neq C(E', F) \).

For non-spherically complete fields, in [11], 4.J an example was constructed of a non-reflexive closed subspace \( D \) of \( l^\infty \), whose dual \( D' \) is isometrically isomorphic to \( c_0 \). Now, as a consequence of the above results we can construct, in a less laborial way, natural examples of non-reflexive subspaces of \( l^\infty \).

Corollary 3.6. Suppose that \( K \) is not spherically complete. For each \( n > 1 \) let \( G_n = l^\infty \hat{\otimes} \ldots \hat{\otimes} l^\infty \). Then,

i) \( G_n \) is isometrically isomorphic to a weakly dense subspace of \( l^\infty \).

ii) \( G'_n \simeq c_0 \).

iii) \( G_n \) is a non-reflexive space.

Proof.

i) By taking \( E = F = c_0 \), the map \( U \) yields a linear isometry from \( l^\infty \hat{\otimes} l^\infty \) into a weakly*-dense subspace of \( (c_0 \hat{\otimes} c_0)' \) (Proposition 3.3). Also, by Exercise 4.R.ii) of [11] we have that \( c_0 \hat{\otimes} c_0 \simeq c_0 \) and so \( (c_0 \hat{\otimes} c_0)' \simeq l^\infty \). We conclude that property i) is true for \( n = 2 \) (recall that \( l^\infty \) is reflexive when \( K \) is not spherically complete, [11 Theorem 4.17]). By induction we can prove that i) is true for all \( n > 1 \). Indeed, suppose that \( G_n \) satisfies i). Since the map \( U \) is surjective when \( E = F = l^\infty \) (Corollary 3.2.2)), we have that \( G_{n+1} = G_n \hat{\otimes} l^\infty \) is isometrically isomorphic to a weakly dense subspace of \( l^\infty \hat{\otimes} l^\infty \), which implies that \( G_{n+1} \) has also property i).

ii) This follows by induction and applying Corollary 3.2.2).

iii) It is a direct consequence of Theorem 3.5 (with \( E = l^\infty \) and \( F = G_{n-1} \)) and ii) (observe that \( E \) and \( F \) are pseudoreflexive spaces).

From Corollary 3.6.iii) we conclude that if \( K \) is not spherically complete then, for each \( n > 1 \), \( l^\infty \hat{\otimes} \ldots \hat{\otimes} l^\infty \) is not isometrically isomorphic to \( l^\infty \). The same fact occurs when \( K \) is spherically complete and the valuation on \( K \) is dense. Indeed, one just has to combine the spherical completeness of \( l^\infty \) ([11, 4.A) with the following.

Theorem 3.7. Suppose that the valuation on \( K \) is dense. Let \( (E, \| \cdot \|) \) be a Banach space over \( K \) containing an infinite orthogonal sequence in \( E - \{0\} \). Then, \( l^\infty \hat{\otimes} E \) is not spherically complete. In particular, \( l^\infty \hat{\otimes} \ldots \hat{\otimes} l^\infty \) is not spherically complete.

Proof. We shall prove that \( P(N, E) \) is not spherically complete (recall that \( P(N, E) \simeq l^\infty \hat{\otimes} E \)).

By assumption, there exists an orthogonal sequence \( (u_1, u_2, \ldots) \) in \( E \) such that \( \| u_1 \| > \| u_2 \| > \ldots \) and \( \lim_n \| u_n \| = 1 \).
For each \( n = 1, 2, \ldots \) let \( f_n \in P(N, E) \) defined by \( f_n(m) = u_m \) if \( m \leq n \) and \( f_n(m) = 0 \) if \( m > n \).

Suppose that \( P(N, E) \) is spherically complete. Since \( B(f_n, \| u_{n+1} \|) \supset B(f_{n+1}, \| u_{n+2} \|) \) for all \( n \), we derive the existence of a compactoid sequence \( f = (x_1, x_2, \ldots) \) in \( E \) such that \( \| f - f_n \|_u \leq \| u_{n+1} \| \) for all \( n \). Given \( i = 1, 2, \ldots \) we have that \( \| x_i - u_i \| \leq \| f - f_n \|_u \leq \| u_{n+1} \| \) for all \( n \geq i \) and so \( \| x_i - u_i \| \leq 1 \leq \| u_i \| \). Hence, \( (x_1, x_2, \ldots) \) is an orthogonal sequence in \( E \) such that \( \| x_i \| = \| u_i \| > 1 \) for all \( i \), which implies that \( \{x_1, x_2, \ldots\} \) is not compactoid in \( E \) ([10], Theorem 2.2) : a contradiction. \( \square \)

For discretely valued fields the situation is completely different. In fact, we have:

**Proposition 3.8.** Suppose that the valuation on \( K \) is discrete. Then, for each \( n > 1 \), \( G_n = l^\infty \hat{\otimes} \ldots \hat{\otimes} l^\infty \) isometrically isomorphic to \( l^\infty \) and so \( G_n \) is spherically complete.

**Proof.** It is enough to prove that \( G_2 \simeq l^\infty \). To see that, observe that, if \( K \) is discretely valued, \( l^\infty \) has an orthonormal basis ([11], Theorem 5.16) and so \( l^\infty \simeq c_0(I) \) for some infinite set \( I \). By Exercise 4.R.ii) of [11] we have that \( G_2 \simeq c_0(I \times I) \). Since \( I \times I \) has the same cardinality as \( I \) (see [5]), we conclude that \( c_0(I \times I) \simeq c_0(I) \) and we are done. \( \square \)

**Remarks.**
1. Corollary 3.6 and Theorem 3.7 show us that, in general, the classes of reflexive and spherically complete Banach spaces are not closed for forming of tensor products (Compare with Theorem 2.1).
2. From Corollary 3.6, the following question arises in a natural way:
   **Problem.** Is the space \( D \) constructed in [11], 4.J isometrically isomorphic to some \( G_n \) ?
3. In this section we have obtained that:
   "The valuation on \( K \) is discrete if and only if \( G_n \simeq l^\infty \) for all \( n > 1 \) if and only if there exists \( n > 1 \) such that \( G_n \simeq l^\infty \)

   This fact leads us to the following question, which is closely related to the above one:
   **Problem.** Suppose that the valuation on \( K \) is dense. Are \( G_m \) and \( G_n \) isometrically isomorphic for all (or some) \( m \neq n \), \( m, n > 1 \) ?

**References**


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