COMPACTNESS OF p-ADIC INTEGRAL OPERATORS

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Abstract.
The p-adic counterparts of the classical integral operators are shown to be compact. This result is extended to integral operators $C^n \to C^m$.

PRELIMINARIES
Throughout $K$ is a non-archimedean valued complete field whose valuation $|\cdot|$ is not trivial. For a compact topological space $X$ the $K$-Banach space of all continuous functions $f : X \to K$ with the norm $f \mapsto \|f\|_\infty := \max\{|f(x)| : x \in X\}$ is denoted $C(X \to K)$. The closed unit ball of a $K$-Banach space $E$ is written $B_E$. A (continuous) linear map $T : E \to F$, where $E, F$ are locally convex spaces over $K$, is called compact if there is a neighbourhood $U$ of $0$ in $E$ for which $TU$ is a compactoid, where a subset $Y$ of $F$ is called a compactoid if for each neighbourhood $V$ of $0$ in $F$ there exists a finitely generated absolutely convex set $I$ such that $Y \subseteq V + I$. The topological dual of $E$ is $E'$.

Further background on p-adic Functional Analysis can be found in [3].

1. INTEGRAL OPERATORS
Let $X, Y$ be compact topological spaces, $\mu \in C(Y \to K)'$, $G \in C(X \times Y \to K)$. For each $f \in C(Y \to K)$  and $x \in X$ the function $h_x : y \mapsto G(x, y)f(y)$ is continuous so the expression $\mu(h_x)$ makes sense. Rather than $\mu(h_x)$ we shall use the more convenient notation $\int G(x, y)h(y)d\mu(y)$.

Theorem 1.1. With the above notations, the formula

$$(Tf)(x) = \int G(x, y)f(y)d\mu(y)$$

defines a compact operator $T : C(Y \to K) \to C(X \to K)$. 27
Proof. We have seen already that $(Tf)(x)$ is well-defined. To prove continuity of $Tf$, let $a \in X$, $\epsilon > 0$. By compactness there is a neighbourhood $U$ of $a$ such that $|G(x,y) - G(a,y)| \leq \epsilon$ for all $x \in U$ and $y \in Y$. Then, for $x \in U$

$$|(Tf)(x) - (Tf)(a)| = \left| \int (G(x,y) - G(a,y))f(y)\,d\mu(y) \right| \leq \epsilon \|f\|_\infty \|\mu\|$$

and the continuity of $Tf$ follows; we even have equicontinuity of $TB_{C(Y-K)}$. From $\|Tf\| \leq \|G\|_\infty \|f\|_\infty \|\mu\|$ we obtain (uniform) boundedness of $TB_{C(Y-K)}$ implying compactness of $T$ by the p-adic Ascoli Theorem [1] Theorem 1.8 and [2] Theorem 1.

In the next sections we shall interpret $T$ as a map $C^n(Y \to K) \to C^n(X \to K)$. To this end we need some preliminary definitions and results that will be treated in §2 and §3.

2. $C^n$-FUNCTIONS OF ONE VARIABLE

We recall some definitions of [5], §29. For a subset $X$ of $K$ and $n \in \mathbb{N}$ we set

$$\nabla^n X := \{(x_1, x_2, \ldots, x_n) \in X^n : \text{if } i \neq j \text{ then } x_i \neq x_j \}.$$ 

The $n$th difference quotient $\Phi_n f : \nabla^{n+1} X \to K$ of a function $f : X \to K$ is inductively given by $\Phi_0 f := f$ and, for $n \in \mathbb{N}$, by the formula

$$(\Phi_n f)(x_1, x_2, \ldots, x_{n+1}) = \frac{(\Phi_{n-1} f)(x_1, x_3, \ldots, x_{n+1}) - (\Phi_{n-1} f)(x_2, x_3, \ldots, x_{n+1})}{x_1 - x_2}.$$ 

We say that $f$ is a $C^n$-function ($f \in C^n(X \to K)$, or shortly $f \in C^n$) if $\Phi_n f$ can be extended to a continuous function $X^{n+1} \to K$. If $X$ has no isolated points the above extension is unique and denoted $\overline{\Phi}_n f$. We set

$$D_n f(x) := (\overline{\Phi}_n f)(x, x, \ldots, x) \quad (x \in X).$$

Then ([5] Theorem 29.5) $n! D_n f = f^{(n)}$ (so that $D_n f = f^{(n)}/n!$ if the characteristic of $K$ is zero). The set $C^n(X \to K)$ is a $K$-algebra under pointwise operations.

Now assume that $X$ is a compact subset of $K$ without isolated points. Then for an $f \in C^n(X \to K)$ the functions $f, \Phi_1 f, \ldots, \Phi_n f$ are all bounded so one may define

$$\|f\|_n := \max(\|f\|_\infty, \|\Phi_1 f\|_\infty, \ldots, \|\Phi_n f\|_\infty)$$

$$= \max(\|f\|_\infty, \|\overline{\Phi}_1 f\|_\infty, \ldots, \|\overline{\Phi}_n f\|_\infty).$$

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It is shown in [4], Theorem 8.5 that $f \mapsto ||f||_n$ is a norm on $C^n(X \to K)$ making it into a $K$-Banach space. Because also $||fg||_n \leq ||f||_n||g||_n$ holds for $f, g \in C^n(X \to K)$ the space $C^n(X \to K)$ is even a $K$-Banach algebra.

Observe that, if $f \in C^n(X \to K)$ the functions $f, \Phi_1 f, \ldots, \Phi_n f$ are uniformly continuous.

3. $C^n$-FUNCTIONS OF TWO VARIABLES

Throughout §3 let $n, m \in \{0,1,2,\ldots\}$.

Let $X$ be a subset of $K$, let $Y$ be just a set and let $H : X \times Y \to K$. The $n$th difference quotient of $H$ with respect to the first variable is by definition the function

$$(x_1, \ldots, x_{n+1}, y) \mapsto (\Phi^{(1)}_n H)(x_1, \ldots, x_{n+1}, y)$$

defined on $\nabla^{n+1} X \times Y$, where $h_y(x) := H(x,y)$ ($x \in X, y \in Y$).

Similarly, for a set $X$, a subset $Y$ of $K$ and a function $J : X \times Y \to K$ we define the $m$th difference quotient of $J$ with respect to the second variable to be the map

$$(x, y_1, y_2, \ldots, y_{m+1}) \mapsto (\Phi^{(2)}_m J)(y_1, \ldots, y_{m+1})$$

defined on $X \times \nabla^{m+1} Y$ where $j^x(y) := J(x,y)$ ($x \in X, y \in Y$).

We leave the proof of the following elementary lemma to the reader.

**Lemma 3.1.** Let $X, Y$ be subsets of $K$, let $G : X \times Y \to K$. Then

$$\Phi^{(2)}_m \Phi^{(1)}_n G = \Phi^{(1)}_n \Phi^{(2)}_m G.$$ 

Now let $X, Y$ be subsets of $K$ without isolated points, and let $G : X \times Y \to K$. We say that $G \in C^{n,m}(X \times Y \to K)$ (or simply $G \in C^{n,m}$) if the function $\Phi^{(2)}_m \Phi^{(1)}_n G$ (or, equivalently, $\Phi^{(1)}_n \Phi^{(2)}_m G$) can be extended to a continuous function $X^{n+1} \times Y^{m+1} \to K$.

This extension is unique and denoted $\overline{\Phi^{(2)}_m} \Phi^{(1)}_n G$ or $\Phi^{(1)}_n \overline{\Phi^{(2)}_m} G$. Special cases are

$$D^{(1)}_n G(x, y) := (\Phi^{(2)}_n \Phi^{(1)}_n G)(x, x, \ldots, x, y)$$

$$D^{(2)}_m D^{(1)}_n(x, y) := (\Phi^{(2)}_m \Phi^{(1)}_n G)(x, x, \ldots, x, y, y, \ldots, y)$$

and also expressions like $D^{(2)}_m \Phi^{(1)}_n G$, $\overline{\Phi^{(2)}_m} D^{(1)}_n G$ make sense. The following facts are easily established. If $G \in C^{n,m}$ and $j, k \in \{0,1,\ldots\}$, $j \leq n$, $k \leq m$ then $G \in C^{j,k}$. If $G \in C^{n,m}$ then

$$n!m! D^{(1)}_n D^{(2)}_m = \frac{\partial^n}{\partial x^n} \frac{\partial^m}{\partial y^m} G$$

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and in particular we have equality of mixed partial derivatives: \( \frac{\partial^m}{\partial x^n \partial y^m} G = \frac{\partial^m}{\partial y^n \partial x^m} G \).

If \( G \in C^{n,0}(X \times Y) \) then for each \( y \in Y \) the function \( x \mapsto G(x, y) \) is in \( C^n(X \to K) \).

4. INTEGRAL OPERATORS ON \( C^n \)

**Theorem 4.1.** Let \( X, Y \) be compact subsets of \( K \) without isolated points. Let \( G \in C^{n,m}(X \times Y \to K) \) and let \( \mu \in C^m(Y \to K)' \). Then the formula

\[
(Tf)(x) = \int G(x, y)f(y)d\mu(y)
\]

defines a continuous linear map \( T : C^m(Y \to K) \to C^n(X \to K) \). We have \( \|T\| \leq \|\mu\||G||_{n,m} \) where

\[
\|G\|_{n,m} := \sup\{\|\Phi_k^{(2)}\Phi_j^{(1)}G\|_\infty : 0 \leq j \leq n, 0 \leq k \leq m\}.
\]

**Proof.** Since \( G \in C^{0,m} \) the function \( y \mapsto G(x, y) \) is \( C^m \) for every \( x \in X \). So for \( f \in C^m(Y \to K) \) the product \( y \mapsto G(x, y)f(y) \) is a \( C^m \)-function on \( Y \). It follows that (*) defines a \( K \)-linear map of \( C^m(Y \to K) \) into the space of all functions on \( X \). To check that \( Tf \) is a \( C^n \)-function let \( \varepsilon > 0 \). By uniform continuity there exists a \( \delta > 0 \) such that for all \( j \in \{0, 1, \ldots, n\}, k \in \{0, 1, \ldots, m\} \), all \( (x_1, \ldots, x_{j+1}), (x'_1, \ldots, x'_{j+1}) \in X^{j+1} \), all \( (y_1, \ldots, y_{k+1}) \in Y^{k+1} \),

\[
(\Delta) \|\Phi_k^{(2)}\Phi_j^{(1)}G(x_1, \ldots, x_{j+1}, y_1, \ldots, y_{k+1}) - \Phi_k^{(2)}\Phi_j^{(1)}G(x'_1, \ldots, x'_{j+1}, y_1, \ldots, y_{k+1})\| < \varepsilon
\]

whenever \( |x_1 - x'_1| < \delta, \ldots, |x_{j+1} - x'_{j+1}| < \delta \). Then for such \((x_1, \ldots, x_{n+1})\) and \((x'_1, x'_2, \ldots, x'_{n+1})\) in \( \nabla^{n+1} X \) we have

\[
\Delta := \|\Phi_n Tf)(x_1, \ldots, x_{n+1}) - (\Phi_n Tf)(x'_1, \ldots, x'_{n+1})\|
\]

\[
= \big| \int (\Phi_n^{(1)}G(x_1, \ldots, x_{n+1}, y) - \Phi_n^{(1)}G(x'_1, \ldots, x'_{n+1}, y))f(y)d\mu(y)\big|
\]

\[
\leq \|h\|_m\|f\|_m\|\mu\|
\]

where

\[
h(y) = \Phi_n^{(1)}G(x_1, \ldots, x_{n+1}, y) - \Phi_n^{(1)}G(x'_1, \ldots, x'_{n+1}, y) \quad (y \in Y).
\]

Now \( \|h\|_m = \max\{\|\Phi_k h\|_\infty : 0 \leq k \leq m\} \) which is \( < \varepsilon \) by (\(\Delta\)). We see that \( \Delta \leq \|f\|_m\|\mu\|\varepsilon \). It follows that \( Tf \) is \( C^n \); we even may conclude that \( \{\Phi_n Tf : f \in C^m(Y \to K), \|f\|_m \leq 1\} \) is equicontinuous. To estimate \( \|T\| \), let \( j \in \{0, 1, \ldots, n\} \) and \((x_1, \ldots, x_{j+1}) \in \nabla^{j+1} X \). Then
\[ |\Phi_j T f(x_1, \ldots, x_{j+1})| = \left| \int \Phi_j^{(1)} G(x_1, \ldots, x_{j+1}, y) f(y) d\mu(y) \right| \]
\[ \leq \|\mu\| \|f\|_{m} \sup \{ \|\Phi_k^{(2)} \Phi_j^{(1)} G\|_{\infty} : 0 \leq k \leq m \} \]
\[ \leq \|\mu\| \|f\|_{m} \|G\|_{n,m}. \]

We see that \(\|T\| \leq \|\mu\| \|G\|_{n,m}.\)

**Corollary 4.2.** For all \(f \in C^n(Y \to K)\) and \(x \in X\)

\[ \frac{d^n(Tf)(x)}{dx^n} = \int \frac{\partial^n G(x, y)}{\partial x^n} f(y) d\mu(y). \]

**Proof.** It is shown in the previous proof that

\[ z \mapsto \Phi_n^{(1)} G(z, \cdot) \quad (z \in \nabla^{n+1} X) \]

has a continuous extension \(X^{n+1} \to C^n(Y \to K)\). Then, for \(x \in X\) we have (with \(z \in \nabla^{n+1} X\))

\[ D_n(Tf)(x) = \Phi_n^{(1)}(Tf)(x, x, x, \ldots, x) = \lim_{z \to (x, x, \ldots)} \Phi_n^{(1)}(Tf)(z) \]
\[ = \lim_{z \to (x, \ldots, x)} \int \Phi_n^{(1)} G(z, y) f(y) d\mu(y) = \]
\[ = \int \lim_{z \to (x, \ldots, x)} \Phi_n^{(1)} G(z, y) f(y) d\mu(y) = \int D_n^1 G(x, y) f(y) d\mu(y). \]

**5. COMPACTNESS OF INTEGRAL OPERATORS**

To prove compactness of the operator \(T : C^n(Y \to K) \to C^n(X \to K)\) of the previous section we shall combine the p-adic Ascoli Theorem with the following.

**Lemma 5.1.** Let \(X\) be a subset of \(K\), without isolated points. Let \(n \in \{0, 1, 2, \ldots\}\). The map

\[ \tau_n : f \mapsto (f, \overline{\Phi}_1 f, \overline{\Phi}_2 f, \ldots, \overline{\Phi}_n f) \]

embeds \(C^n(X \to K)\) linearly and isometrically into \(C(X) \times C(X^2) \times \ldots \times C(X^{n+1})\).

**Proof.** Direct verification.

**Lemma 5.2.** Let \(Z_1, \ldots, Z_k \ (k \in \mathbb{N})\) be compact topological spaces and let, for each \(i \in \{1, \ldots, k\}\), \(\pi_i\) be the obvious projection \(\prod_{j=1}^k C(Z_j \to K) \to C(Z_i \to K)\). Then a
subset $S$ of $\prod_{j=1}^{k} C(Z_j \to K)$ is a compactoid if and only if for each $i \in \{1, \ldots, k\}$, $\pi_i(S)$ is bounded and equicontinuous in $C(Z_i \to K)$. 

**Proof.** If $S$ is a compactoid and $i \in \{1, \ldots, k\}$ then, since $\pi_i$ is linear and continuous, $\pi_i(S)$ is a compactoid in $C(Z_i \to K)$, hence bounded and equicontinuous by the $p$-adic Ascoli Theorem. Conversely, if every $\pi_i(S)$ is bounded and equicontinuous then each $\pi_i(S)$ is a compactoid by the $p$-adic Ascoli Theorem, hence $\pi_1(S) \times \pi_2(S) \times \ldots \times \pi_k(S)$ is a compactoid. But then so is its subset $S$.

**Theorem 5.3.** The map $T$ of Theorem 4.1. is compact.

**Proof.** By Lemma 5.1 it is enough to prove that $\tau_n \circ T$ is compact; from Lemma 5.2 we see that it suffices to show that, for each $i \in \{0,1,\ldots,n\}$ the set 

$$\{\tilde{\mathbf{T}}_iTf : f \in C^m(Y \to K), \|f\|_m \leq 1\}$$

is (pointwise) bounded and equicontinuous in $C(X^{i+1} \to K)$. But this was already observed in the proof of Theorem 4.1 for $i = n$; a similar argument works for each $i \in \{0,1,\ldots,n\}$.

### 6. INTEGRAL OPERATORS ON $C^\infty$

Let $X, Y$ be compact subsets of $K$ without isolated points. The space

$$C^\infty(X \to K) := \bigcap_n C^n(X \to K)$$

has a natural locally convex topology induced by the norms $\|f\|_n$ ($n \in \{0,1,2,\ldots\}$). If $G : X \times Y \to K$ is $C^\infty$ (i.e. $G \in C^{n,m}(X \times Y \to K)$ for each $n, m \in \{0,1,\ldots\}$) and $\mu \in C^\infty(X \to K)'$ the formula

$$\text{(\ast)} \quad (Tf)(x) = \int G(x,y)f(y)d\mu(y)$$

defines a linear map $C^\infty(Y \to K) \to C^\infty(X \to K)$. By construction there is an $m \in \{0,1,\ldots\}$ and a $C > 0$ such that $|\mu(f)| \leq C\|f\|_m$ for all $f \in C^\infty(Y \to K)$. For each $n \in \{0,1,\ldots\}$ (\ast) is the restriction of a compact integral operator $C^n(Y \to K) \to C^n(X \to K)$. It follows that $T$ maps $\{f \in C^m(Y \to K) : \|f\|_m \leq 1\}$ into a compactoid of $C^\infty(X \to K)$ i.e. that $T$ is a compact map $C^\infty(Y \to K) \to C^\infty(X \to K)$. 

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REFERENCES


