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COMPACTNESS OF p-ADIC INTEGRAL OPERATORS

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Abstract.
The p-adic counterparts of the classical integral operators are shown to be compact. This result is extended to integral operators \( C^n \to C^m \).

PRELIMINARIES
Throughout \( K \) is a non-archimedean valued complete field whose valuation \(|\cdot|\) is not trivial. For a compact topological space \( X \) the \( K \)-Banach space of all continuous functions \( f : X \to K \) with the norm \( f \mapsto \|f\|_\infty := \max\{|f(x)| : x \in X\} \) is denoted \( C(X \to K) \). The closed unit ball of a \( K \)-Banach space \( E \) is written \( B_E \). A (continuous) linear map \( T : E \to F \), where \( E, F \) are locally convex spaces over \( K \), is called compact if there is a neighbourhood \( U \) of 0 in \( E \) for which \( TU \) is a compactoid, where a subset \( Y \) of \( F \) is called a compactoid if for each neighbourhood \( V \) of 0 in \( F \) there exists a finitely generated absolutely convex set \( I \) such that \( Y \subseteq V + I \). The topological dual of \( E \) is \( E' \).

Further background on p-adic Functional Analysis can be found in [3].

1. INTEGRAL OPERATORS
Let \( X, Y \) be compact topological spaces, \( \mu \in C(Y \to K)' \), \( G \in C(X \times Y \to K) \). For each \( f \in C(Y \to K) \) and \( x \in X \) the function \( h_x : y \mapsto G(x, y)f(y) \) is continuous so the expression \( \mu(h_x) \) makes sense. Rather than \( \mu(h_x) \) we shall use the more convenient notation \( \int G(x, y)h(y)d\mu(y) \).

Theorem 1.1. With the above notations, the formula

\[
(Tf)(x) = \int G(x, y)f(y)d\mu(y)
\]

defines a compact operator \( T : C(Y \to K) \to C(X \to K) \).
Proof. We have seen already that \((Tf)(x)\) is well-defined. To prove continuity of \(Tf\), let \(a \in X\), \(\varepsilon > 0\). By compactness there is a neighbourhood \(U\) of \(a\) such that \(|G(x,y) - G(a,y)| \leq \varepsilon\) for all \(x \in U\) and \(y \in Y\). Then, for \(x \in U\)

\[
|(Tf)(x) - (Tf)(a)| = \left| \int (G(x,y) - G(a,y))f(y)d\mu(y) \right| \leq \varepsilon \|f\|_\infty \|\mu\|
\]

and the continuity of \(Tf\) follows; we even have equicontinuity of \(TB_{C(Y \to K)}\). From \(|\|Tf\|\| \leq \|G\|_\infty \|f\|_\infty \|\mu\|\) we obtain (uniform) boundedness of \(TB_{C(Y \to K)}\) implying compactness of \(T\) by the p-adic Ascoli Theorem [1] Theorem 1.8 and [2] Theorem 1.

In the next sections we shall interpret \(T\) as a map \(C^n(Y \to K) \to C^n(X \to K)\). To this end we need some preliminary definitions and results that will be treated in §2 and §3.

2. \(C^n\)-FUNCTIONS OF ONE VARIABLE

We recall some definitions of [5], §29. For a subset \(X\) of \(K\) and \(n \in \mathbb{N}\) we set

\[
\nabla^n X := \{(x_1, x_2, \ldots, x_n) \in X^n : \text{if } i \neq j \text{ then } x_i \neq x_j\}.
\]

The \(n\)th difference quotient \(\Phi_n f : \nabla^{n+1} X \to K\) of a function \(f : X \to K\) is inductively given by \(\Phi_0 f := f\) and, for \(n \in \mathbb{N}\), by the formula

\[
(\Phi_n f)(x_1, x_2, \ldots, x_{n+1}) = \frac{(\Phi_{n-1} f)(x_1, x_3, \ldots, x_{n+1}) - (\Phi_{n-1} f)(x_2, x_3, \ldots, x_{n+1})}{x_1 - x_2}.
\]

We say that \(f\) is a \(C^n\)-function (\(f \in C^n(X \to K)\), or shortly \(f \in C^n\)) if \(\Phi_n f\) can be extended to a continuous function \(X^{n+1} \to K\). If \(X\) has no isolated points the above extension is unique and denoted \(\overline{\Phi}_n f\). We set

\[
D_n f(x) := (\overline{\Phi}_n f)(x, x, \ldots, x) \quad (x \in X).
\]

Then ([5] Theorem 29.5) \(n! D_n f = f^{(n)}\) (so that \(D_n f = f^{(n)}/n!\) if the characteristic of \(K\) is zero). The set \(C^n(X \to K)\) is a \(K\)-algebra under pointwise operations.

Now assume that \(X\) is a compact subset of \(K\) without isolated points. Then for an \(f \in C^n(X \to K)\) the functions \(f, \Phi_1 f, \ldots, \Phi_n f\) are all bounded so one may define

\[
\|f\|_n := \max(\|f\|_\infty, \|\Phi_1 f\|_\infty, \ldots, \|\Phi_n f\|_\infty)
\]

\[
(= \max(\|f\|_\infty, \|\overline{\Phi}_1 f\|_\infty, \ldots, \|\overline{\Phi}_n f\|_\infty)).
\]

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It is shown in [4], Theorem 8.5 that $\|f\|_n$ is a norm on $C^n(X \to K)$ making it into a $K$-Banach space. Because also $\|fg\|_n \leq \|f\|_n \|g\|_n$ holds for $f, g \in C^n(X \to K)$ the space $C^n(X \to K)$ is even a $K$-Banach algebra.

Observe that, if $f \in C^n(X \to K)$ the functions $f, \Phi_1 f, \ldots, \Phi_n f$ are uniformly continuous.

3. $C^n$-FUNCTIONS OF TWO VARIABLES

Throughout §3 let $n, m \in \{0,1,2,\ldots\}$.

Let $X$ be a subset of $K$, let $Y$ be just a set and let $H : X \times Y \to K$. The $n$th difference quotient of $H$ with respect to the first variable is by definition the function

$$(x_1, \ldots, x_{n+1}, y) \xmapsto \frac{\Phi_{n+1}^{(1)} H}{h_y} (x_1, \ldots, x_{n+1})$$

defined on $\nabla^{n+1} X \times Y$, where $h_y(x) := H(x, y) \ (x \in X, y \in Y)$.

Similarly, for a set $X$, a subset $Y$ of $K$ and a function $J : X \times Y \to K$ we define the $m$th difference quotient of $J$ with respect to the second variable to be the map

$$(x, y_1, y_2, \ldots, y_{m+1}) \xmapsto \frac{\Phi_{m+1}^{(2)} J}{j_x^m} (y_1, \ldots, y_{m+1})$$

defined on $X \times \nabla^{m+1} Y$ where $j_x^m(y) := J(x, y) \ (x \in X, y \in Y)$.

We leave the proof of the following elementary lemma to the reader.

**Lemma 3.1.** Let $X, Y$ be subsets of $K$, let $G : X \times Y \to K$. Then

$$\Phi_n^{(2)} \Phi_n^{(1)} G = \Phi_n^{(1)} \Phi_n^{(2)} G.$$  

Now let $X, Y$ be subsets of $K$ without isolated points, and let $G : X \times Y \to K$. We say that $G \in C_n^{m,:}(X \times Y \to K)$ (or simply $G \in C_n^{m,:}$) if the function $\Phi_n^{(2)} \Phi_n^{(1)} G$ (or, equivalently, $\Phi_n^{(1)} \Phi_n^{(2)} G$) can be extended to a continuous function $X^{n+1} \times Y^{m+1} \to K$.

This extension is unique and denoted $\Phi_n^{(2)} \Phi_n^{(1)} G$ or $\Phi_n^{(1)} \Phi_n^{(2)} G$). Special cases are

$$D_n^{(1)} G(x, y) := (\Phi_n^{(2)} \Phi_n^{(1)} G)(x, x, \ldots, x, y)$$

$$D_n^{(2)} D_n^{(1)} (x, y) := (\Phi_n^{(2)} \Phi_n^{(1)} G)(x, x, \ldots, x, y, y, \ldots, y)$$

and also expressions like $D_n^{(2)} \Phi_n^{(1)} G$, $\Phi_n^{(2)} D_n^{(1)} G$ make sense. The following facts are easily established. If $G \in C_n^{m,:}$ and $j, k \in \{0,1,\ldots\}$, $j \leq n$, $k \leq m$ then $G \in C^{j,k}$. If $G \in C_n^{m,:}$ then

$$n!m! D_n^{(1)} D_m^{(2)} = \frac{\partial^n}{\partial x^n} \frac{\partial^m}{\partial y^m} G.$$  

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and in particular we have equality of mixed partial derivatives: \( \frac{\partial}{\partial x^n} \frac{\partial}{\partial y^m} G = \frac{\partial}{\partial y^m} \frac{\partial}{\partial x^n} G \). If \( G \in C^{n,0}(X \times Y) \) then for each \( y \in Y \) the function \( x \mapsto G(x,y) \) is in \( C^n(X \to K) \).

4. **INTEGRAL OPERATORS ON \( C^n \)**

**Theorem 4.1.** Let \( X, Y \) be compact subsets of \( K \) without isolated points. Let \( G \in C^{n,m}(X \times Y \to K) \) and let \( \mu \in C^m(Y \to K)' \). Then the formula

\[
(Tf)(x) = \int G(x,y)f(y)d\mu(y)
\]

defines a continuous linear map \( T : C^m(Y \to K) \to C^n(X \to K) \). We have \( \|T\| \leq \|\mu\| \|G\|_{n,m} \) where

\[
\|G\|_{n,m} := \sup \{ \|\Phi_k^{(2)} \Phi_j^{(1)} G\|_{\infty} : 0 \leq j \leq n, 0 \leq k \leq m \}.
\]

**Proof.** Since \( G \in C^{0,m} \) the function \( y \mapsto G(x,y) \) is \( C^m \) for every \( x \in X \). So for \( f \in C^m(Y \to K) \) the product \( y \mapsto G(x,y)f(y) \) is a \( C^m \)-function on \( Y \). It follows that \((*)\) defines a \( K \)-linear map of \( C^m(Y \to K) \) into the space of all functions on \( X \). To check that \( Tf \) is a \( C^n \)-function let \( \varepsilon > 0 \). By uniform continuity there exists a \( \delta > 0 \) such that for all \( j \in \{0,1,\ldots,n\} \), \( k \in \{0,1,\ldots,m\} \), all \( (x_1,\ldots,x_{j+1}),(x'_1,\ldots,x'_{j+1}) \in X^{j+1} \), all \( (y_1,\ldots,y_{k+1}) \in Y^{k+1} \),

\[
(\forall) \quad |\Phi_k^{(2)} \Phi_j^{(1)} G(x_1,\ldots,x_{j+1},y_1,\ldots,y_{k+1}) - \Phi_k^{(2)} \Phi_j^{(1)} G(x'_1,\ldots,x'_{j+1},y_1,\ldots,y_{k+1})| < \varepsilon
\]

whenever \( |x_j-x'_j| < \delta, \ldots, |x_{j+1}-x'_{j+1}| < \delta \). Then for such \( (x_1,\ldots,x_{n+1}) \) and \( (x'_1,\ldots,x'_{n+1}) \) in \( \nabla^{n+1}X \) we have

\[
\Delta := |(\Phi_n Tf)(x_1,\ldots,x_{n+1}) - (\Phi_n Tf)(x'_1,\ldots,x'_{n+1})| \leq \|h\|_m \|f\|_m \|\mu\|
\]

where

\[
h(y) = \Phi_n^{(1)} G(x_1,\ldots,x_{n+1},y) - \Phi_n^{(1)} G(x'_1,\ldots,x'_{n+1},y) \quad (y \in Y).
\]

Now \( \|h\|_m = \max\{ \|\Phi_k h\|_{\infty} : 0 \leq k \leq m \} \) which is \( < \varepsilon \) by \((\forall)\). We see that \( \Delta \leq \|f\|_m \|\mu\|\varepsilon \). It follows that \( Tf \) is \( C^n \); we even may conclude that \( \{\Phi_n Tf : f \in C^m(Y \to K), \|f\|_m \leq 1\} \) is equicontinuous. To estimate \( \|T\| \), let \( j \in \{0,1,\ldots,n\} \) and \( (x_1,\ldots,x_{j+1}) \in \nabla^{j+1}X \). Then
We see that \( \|T\| \leq \|\mu\| \|G\|_{n,m} \).

**Corollary 4.2.** For all \( f \in C^n(Y \to K) \) and \( x \in X \)

\[
\frac{d^n(Tf)(x)}{dx^n} = \int \frac{\partial^n G(x,y)}{\partial x^n} f(y)d\mu(y).
\]

**Proof.** It is shown in the previous proof that

\[
(z \mapsto \Phi_n^{(1)}G(z,\cdot) \quad (z \in \nabla^{n+1}X)\]

has a continuous extension \( X^{n+1} \to C^n(Y \to K) \). Then, for \( x \in X \) we have (with \( z \in \nabla^{n+1}X \))

\[
D_n(Tf)(x) = \Phi_n^{(1)}(Tf)(x,x,x,\ldots,x) = \lim_{z \to (x,x,\ldots,x)} \Phi_n^{(1)}(Tf)(z)
= \lim_{z \to (x,x,\ldots,x)} \int \Phi_n^{(1)}G(z,y)f(y)d\mu(y) = \int \lim_{z \to (x,x,\ldots,x)} \Phi_n^{(1)}G(z,y)f(y)d\mu(y) = \int D^n_1 G(x,y)f(y)d\mu(y).
\]

### 5. COMPACTNESS OF INTEGRAL OPERATORS

To prove compactness of the operator \( T : C^n(Y \to K) \to C^n(X \to K) \) of the previous section we shall combine the \( p \)-adic Ascoli Theorem with the following.

**Lemma 5.1.** Let \( X \) be a subset of \( K \), without isolated points. Let \( n \in \{0,1,2,\ldots\} \). The map

\[
\tau_n : f \mapsto (f, \overline{\Phi}_1 f, \overline{\Phi}_2 f, \ldots, \overline{\Phi}_n f)
\]

embeds \( C^n(X \to K) \) linearly and isometrically into \( C(X) \times C(X^2) \times \ldots \times C(X^{n+1}) \).

**Proof.** Direct verification.

**Lemma 5.2.** Let \( Z_1, \ldots, Z_k (k \in \mathbb{N}) \) be compact topological spaces and let, for each \( i \in \{1, \ldots, k\} \), \( \pi_i \) be the obvious projection \( \prod_{j=1}^k C(Z_j \to K) \to C(Z_i \to K) \). Then a
subset $S$ of $\prod_{j=1}^{k} C(Z_j \to K)$ is a compactoid if and only if for each $i \in \{1, \ldots, k\}$, $\pi_i(S)$
is bounded and equicontinuous in $C(Z_i \to K)$.

**Proof.** If $S$ is a compactoid and $i \in \{1, \ldots, k\}$ then, since $\pi_i$ is linear and continuous,$\pi_i(S)$ is a compactoid in $C(Z_i \to K)$, hence bounded and equicontinuous by the $p$-adic
Ascoli Theorem. Conversely, if every $\pi_i(S)$ is bounded and equicontinuous then each
$\pi_i(S)$ is a compactoid by the $p$-adic Ascoli Theorem, hence $\pi_1(S) \times \pi_2(S) \times \ldots \times \pi_k(S)$
is a compactoid. But then so is its subset $S$.

**Theorem 5.3.** The map $T$ of Theorem 4.1. is compact.

**Proof.** By Lemma 5.1 it is enough to prove that $\tau_n \circ T$ is compact; from Lemma 5.2
we see that it suffices to show that, for each $i \in \{0, 1, \ldots, n\}$ the set

$$\{ \overline{T}_iTf : f \in C^m(Y \to K), \|f\|_m \leq 1 \}$$

is (pointwise) bounded and equicontinuous in $C(X^{i+1} \to K)$.

But this was already observed in the proof of Theorem 4.1 for $i = n$; a similar argument
works for each $i \in \{0, 1, \ldots, n\}$.

## 6. INTEGRAL OPERATORS ON $C^\infty$

Let $X, Y$ be compact subsets of $K$ without isolated points. The space

$$C^\infty(X \to K) := \bigcap_n C^n(X \to K)$$

has a natural locally convex topology induced by the norms $\| \cdot \|_n$ ($n \in \{0, 1, 2, \ldots\}$).

If $G : X \times Y \to K$ is $C^\infty$ (i.e. $G \in C^{n,m}(X \times Y \to K)$ for each $n, m \in \{0, 1, \ldots\}$) and
$\mu \in C^\infty(X \to K)'$ the formula

$$(\ast) \quad (Tf)(x) = \int G(x, y)f(y)d\mu(y)$$

defines a linear map $C^\infty(Y \to K) \to C^\infty(X \to K)$. By construction there is an
$m \in \{0, 1, \ldots\}$ and a $C > 0$ such that $|\mu(f)| \leq C\|f\|_m$ for all $f \in C^\infty(Y \to K)$. For
each $n \in \{0, 1, \ldots\}$ $(\ast)$ is the restriction of a compact integral operator

$C^m(Y \to K) \to C^n(X \to K)$. It follows that $T$ maps $\{ f \in C^m(Y \to K) : \|f\|_m \leq 1 \}$ into a compactoid
of $C^\infty(X \to K)$ i.e. that $T$ is a compact map $C^\infty(Y \to K) \to C^\infty(X \to K)$.  

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REFERENCES


