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COMPACTNESS OF p-ADIC INTEGRAL OPERATORS

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Abstract.
The p-adic counterparts of the classical integral operators are shown to be compact. This result is extended to integral operators \( C^n \rightarrow C^m \).

PRELIMINARIES

Throughout \( K \) is a non-archimedean valued complete field whose valuation \( | \cdot | \) is not trivial. For a compact topological space \( X \) the \( K \)-Banach space of all continuous functions \( f : X \rightarrow K \) with the norm \( f \mapsto \|f\|_\infty := \max \{|f(x)| : x \in X\} \) is denoted \( C(X \rightarrow K) \). The closed unit ball of a \( K \)-Banach space \( E \) is written \( B_E \). A (continuous) linear map \( T : E \rightarrow F \), where \( E, F \) are locally convex spaces over \( K \), is called compact if there is a neighbourhood \( U \) of 0 in \( E \) for which \( TU \) is a compactoid, where a subset \( Y \) of \( F \) is called a compactoid if for each neighbourhood \( V \) of 0 in \( F \) there exists a finitely generated absolutely convex set \( I \) such that \( Y \subseteq V + I \). The topological dual of \( E \) is \( E' \).

Further background on p-adic Functional Analysis can be found in [3].

1. INTEGRAL OPERATORS

Let \( X, Y \) be compact topological spaces, \( \mu \in C(Y \rightarrow K)' \), \( G \in C(X \times Y \rightarrow K) \). For each \( f \in C(Y \to K) \) and \( x \in X \) the function \( h_x : y \mapsto G(x,y)f(y) \) is continuous so the expression \( \mu(h_x) \) makes sense. Rather than \( \mu(h_x) \) we shall use the more convenient notation \( \int G(x,y)h(y)d\mu(y) \).

Theorem 1.1. With the above notations, the formula

\[
(Tf)(x) = \int G(x,y)f(y)d\mu(y)
\]

defines a compact operator \( T : C(Y \rightarrow K) \rightarrow C(X \rightarrow K) \).
Proof. We have seen already that $(Tf)(x)$ is well-defined. To prove continuity of $Tf$, let $a \in X$, $\varepsilon > 0$. By compactness there is a neighbourhood $U$ of $a$ such that $|G(x,y) - G(a,y)| \leq \varepsilon$ for all $x \in U$ and $y \in Y$. Then, for $x \in U$

$$|(Tf)(x) - (Tf)(a)| = \left| \int (G(x,y) - G(a,y))f(y)d\mu(y) \right| \leq \varepsilon \|f\|_\infty \|\mu\|$$

and the continuity of $Tf$ follows; we even have equicontinuity of $TB_{C(Y-K)}$. From $||Tf|| \leq ||G||_\infty ||f||_\infty ||\mu||$ we obtain (uniform) boundedness of $TB_{C(Y-K)}$ implying compactness of $T$ by the p-adic Ascoli Theorem [1] Theorem 1.8 and [2] Theorem 1.

In the next sections we shall interpret $T$ as a map $C^m(Y \to K) \to C^n(X \to K)$. To this end we need some preliminary definitions and results that will be treated in §2 and §3.

2. $C^n$-FUNCTIONS OF ONE VARIABLE

We recall some definitions of [5], §29. For a subset $X$ of $K$ and $n \in \mathbb{N}$ we set

$$\nabla^n X := \{(x_1, x_2, \ldots, x_n) \in X^n : \text{if } i \neq j \text{ then } x_i \neq x_j\}.$$ 

The $n$th difference quotient $\Phi_n f : \nabla^{n+1} X \to K$ of a function $f : X \to K$ is inductively given by $\Phi_0 f := f$ and, for $n \in \mathbb{N}$, by the formula

$$(\Phi_n f)(x_1, x_2, \ldots, x_{n+1}) = \frac{(\Phi_{n-1} f)(x_1, x_3, \ldots, x_{n+1}) - (\Phi_{n-1} f)(x_2, x_3, \ldots, x_{n+1})}{x_1 - x_2}.$$ 

We say that $f$ is a $C^n$-function ($f \in C^n(X \to K)$, or shortly $f \in C^n$) if $\Phi_n f$ can be extended to a continuous function $X^{n+1} \to K$. If $X$ has no isolated points the above extension is unique and denoted $\overline{\Phi_n f}$. We set

$$D_n f(x) := (\overline{\Phi_n f})(x, x, \ldots, x) \quad (x \in X).$$

Then ([5] Theorem 29.5) $n!D_n f = f^{(n)}$ (so that $D_n f = f^{(n)}/n!$ if the characteristic of $K$ is zero). The set $C^n(X \to K)$ is a $K$-algebra under pointwise operations.

Now assume that $X$ is a compact subset of $K$ without isolated points. Then for an $f \in C^n(X \to K)$ the functions $f, \Phi_1 f, \ldots, \Phi_n f$ are all bounded so one may define

$$||f||_n := \max(||f||_\infty, ||\Phi_1 f||_\infty, \ldots, ||\Phi_n f||_\infty)$$

$$= \max(||f||_\infty, ||\overline{\Phi_1 f}||_\infty, \ldots, ||\overline{\Phi_n f}||_\infty).$$
It is shown in [4], Theorem 8.5 that \( \|f\|_n \) is a norm on \( C^n(X \to K) \) making it into a \( K \)-Banach space. Because also \( \|fg\|_n \leq \|f\|_n \|g\|_n \) holds for \( f, g \in C^n(X \to K) \) the space \( C^n(X \to K) \) is even a \( K \)-Banach algebra.

Observe that, if \( f \in C^n(X \to K) \) the functions \( f, \Phi_1 f, \ldots, \Phi_n f \) are uniformly continuous.

3. \( C^n \)-FUNCTIONS OF TWO VARIABLES

Throughout §3 let \( n, m \in \{0, 1, 2, \ldots\} \).

Let \( X \) be a subset of \( K \), let \( Y \) be just a set and let \( H : X \times Y \to K \). The \( n \)th difference quotient of \( H \) with respect to the first variable is by definition the function

\[
(x_1, \ldots, x_{n+1}, y) \mapsto (\Phi_{n+1} H)(x_1, \ldots, x_{n+1})
\]

defined on \( \nabla^{n+1} X \times Y \), where \( h_y(x) := H(x, y) \) \((x \in X, y \in Y)\).

Similarly, for a set \( X \), a subset \( Y \) of \( K \) and a function \( J : X \times Y \to K \) we define the \( m \)th difference quotient of \( J \) with respect to the second variable to be the map

\[
(x, y_1, y_2, \ldots, y_{m+1}) \mapsto (\Phi_{m+1} J)(y_1, \ldots, y_{m+1})
\]

defined on \( X \times \nabla^{m+1} Y \) where \( j^x(y) := J(x, y) \) \((x \in X, y \in Y)\).

We leave the proof of the following elementary lemma to the reader.

**Lemma 3.1.** Let \( X, Y \) be subsets of \( K \), let \( G : X \times Y \to K \). Then

\[
\Phi_{m}^{(2)} \Phi_{n}^{(1)} G = \Phi_{n}^{(1)} \Phi_{m}^{(2)} G.
\]

Now let \( X, Y \) be subsets of \( K \) without isolated points, and let \( G : X \times Y \to K \). We say that \( G \in C^{n,m}(X \times Y \to K) \) (or simply \( G \in C^{n,m} \)) if the function \( \Phi_{m}^{(2)} \Phi_{n}^{(1)} G \) (or, equivalently, \( \Phi_{n}^{(1)} \Phi_{m}^{(2)} G \)) can be extended to a continuous function \( X^{n+1} \times Y^{m+1} \to K \).

This extension is unique and denoted \( \Phi_{m}^{(2)} \Phi_{n}^{(1)} \) or \( \Phi_{n}^{(1)} \Phi_{m}^{(2)} G \). Special cases are

\[
D_{n}^{(1)} G(x, y) := (\Phi_{n}^{(1)} G)(x, x, \ldots, x, y)
\]

and also expressions like \( D_{n}^{(2)} \Phi_{n}^{(1)} G, \Phi_{m}^{(2)} D_{n}^{(1)} G \) make sense. The following facts are easily established. If \( G \in C^{n,m} \) and \( j, k \in \{0, 1, \ldots\}, j \leq n, k \leq m \) then \( G \in C^{j,k} \). If \( G \in C^{n,m} \) then

\[
n! m! D_{n}^{(1)} D_{m}^{(2)} = \frac{\partial^n}{\partial x^n} \frac{\partial^m}{\partial y^m} G
\]
and in particular we have equality of mixed partial derivatives: \( \frac{\partial^n}{\partial x^m \partial y^n} G = \frac{\partial^n}{\partial y^m \partial x^n} G \).

If \( G \in C^{n,0}(X \times Y) \) then for each \( y \in Y \) the function \( x \mapsto G(x, y) \) is in \( C^n(X \to K) \).

4. INTEGRAL OPERATORS ON \( C^n \)

**Theorem 4.1.** Let \( X, Y \) be compact subsets of \( K \) without isolated points. Let \( G \in C^{n,m}(X \times Y \to K) \) and let \( \mu \in C^m(Y \to K)' \). Then the formula

\[
(Tf)(x) = \int G(x, y)f(y)d\mu(y)
\]

defines a continuous linear map \( T : C^m(Y \to K) \to C^n(X \to K) \). We have \( \|T\| \leq \|\mu\|\|G\|_{n,m} \) where

\[
\|G\|_{n,m} := \sup\{\|\Phi^{(j)}_k G\|_\infty : 0 \leq j \leq n, 0 \leq k \leq m\}.
\]

**Proof.** Since \( G \in C^{0,m} \) the function \( y \mapsto G(x, y) \) is \( C^n \) for every \( x \in X \). So for \( f \in C^m(Y \to K) \) the product \( y \mapsto G(x, y)f(y) \) is a \( C^n \)-function on \( Y \). It follows that \( (\ast) \) defines a \( K \)-linear map of \( C^m(Y \to K) \) into the space of all functions on \( X \). To check that \( Tf \) is a \( C^n \)-function let \( \varepsilon > 0 \). By uniform continuity there exists a \( \delta > 0 \) such that for all \( j \in \{0, 1, \ldots, n\} \), \( k \in \{0, 1, \ldots, m\} \), all \( (x_1, \ldots, x_{j+1}), (x'_1, \ldots, x'_{j+1}) \in X^{j+1} \), all \( (y_1, \ldots, y_{k+1}) \in Y^{k+1} \),

\[
(\Phi^{(j)}_k G(x_1, \ldots, x_{j+1}, y_1, \ldots, y_{k+1}) - \Phi^{(j)}_k G(x'_1, \ldots, x'_{j+1}, y_1, \ldots, y_{k+1})) < \varepsilon
\]

whenever \( |x_1 - x'_1| < \delta, \ldots, |x_{j+1} - x'_{j+1}| < \delta \). Then for such \( (x_1, \ldots, x_{n+1}) \) and \( (x'_1, x'_2, \ldots, x'_{n+1}) \) in \( \nabla^{n+1} X \) we have

\[
\Delta := \|(\Phi_n Tf)(x_1, \ldots, x_{n+1}) - (\Phi_n Tf)(x'_1, \ldots, x'_{n+1})\|
\]
\[
= \left| \int (\Phi^{(1)}_n G(x_1, \ldots, x_{n+1}, y) - \Phi^{(1)}_n G(x'_1, \ldots, x'_{n+1}, y))f(y)d\mu(y) \right|
\]
\[
\leq \|h\|_m \|f\|_m \|\mu\|
\]

where

\[
h(y) = \Phi^{(1)}_n G(x_1, \ldots, x_{n+1}, y) - \Phi^{(1)}_n G(x'_1, \ldots, x'_{n+1}, y) \quad (y \in Y).
\]

Now \( \|h\|_m = \max\{\|\Phi_k h\|_\infty : 0 \leq k \leq m\} \) which is \( < \varepsilon \) by \( (\Lambda) \). We see that \( \Delta \leq \|f\|_m \|\mu\| \varepsilon \). It follows that \( Tf \) is \( C^n \); we even may conclude that \( \{\Phi_n Tf : f \in C^m(Y \to K), \|f\|_m \leq 1\} \) is equicontinuous. To estimate \( \|T\| \), let \( j \in \{0, 1, \ldots, n\} \) and \( (x_1, \ldots, x_{j+1}) \in \nabla^{j+1} X \). Then
\[ |\Phi_jTf(x_1, \ldots, x_{j+1})| = \left| \int \Phi_j^{(1)} G(x_1, \ldots, x_{j+1}, y) f(y) d\mu(y) \right| \]
\[ \leq \|\mu\| \|f\|_{m} \sup \{ \|\Phi_k^{(2)} \Phi_j^{(1)} G\|_{\infty} : 0 \leq k \leq m \} \]
\[ \leq \|\mu\| \|f\|_{m} \|G\|_{n,m}. \]

We see that \(\|T\| \leq \|\mu\| \|G\|_{n,m}.\)

**Corollary 4.2.** For all \(f \in C^n(Y \to K)\) and \(x \in X\)
\[ \frac{d^n(Tf)(x)}{dx^n} = \int \frac{\partial^n G(x, y)}{\partial x^n} f(y) d\mu(y). \]

**Proof.** It is shown in the previous proof that
\[ z \mapsto \Phi_n^{(1)} G(z, \cdot) \quad (z \in \nabla^{n+1} X) \]
has a continuous extension \(X^{n+1} \to C^n(Y \to K)\). Then, for \(x \in X\) we have (with \(z \in \nabla^{n+1} X\))
\[ D_n(Tf)(x) = \Phi_n^{(1)} (Tf)(x, x, x, \ldots, x) = \lim_{z \to (x, x, \ldots)} \Phi_n^{(1)} (Tf)(z) \]
\[ = \lim_{z \to (x, \ldots, x)} \int \Phi_n^{(1)} G(z, y) f(y) d\mu(y) = \]
\[ = \int \lim_{z \to (x, \ldots, x)} \Phi_n^{(1)} G(z, y) f(y) d\mu(y) = \int D_n^1 G(x, y) f(y) d\mu(y). \]

### 5. COMPACTNESS OF INTEGRAL OPERATORS

To prove compactness of the operator \(T : C^n(Y \to K) \to C^n(X \to K)\) of the previous section we shall combine the \(p\)-adic Ascoli Theorem with the following.

**Lemma 5.1.** Let \(X\) be a subset of \(K\), without isolated points. Let \(n \in \{0, 1, 2, \ldots\}\). The map
\[ \tau_n : f \mapsto (f, \overline{\Phi}_1 f, \overline{\Phi}_2 f, \ldots, \overline{\Phi}_n f) \]
embeds \(C^n(X \to K)\) linearly and isometrically into \(C(X) \times C(X^2) \times \ldots \times C(X^{n+1})\).

**Proof.** Direct verification.

**Lemma 5.2.** Let \(Z_1, \ldots, Z_k (k \in \mathbb{N})\) be compact topological spaces and let, for each \(i \in \{1, \ldots, k\}\), \(\pi_i\) be the obvious projection \(\prod_{j=1}^k C(Z_j \to K) \to C(Z_i \to K)\). Then a
subset $S$ of $\prod_{j=1}^{k} C(Z_j \to K)$ is a compactoid if and only if for each $i \in \{1, \ldots, k\}$, $\pi_i(S)$ is bounded and equicontinuous in $C(Z_i \to K)$.

**Proof.** If $S$ is a compactoid and $i \in \{1, \ldots, k\}$ then, since $\pi_i$ is linear and continuous, $\pi_i(S)$ is a compactoid in $C(Z_i \to K)$, hence bounded and equicontinuous by the $p$-adic Ascoli Theorem. Conversely, if every $\pi_i(S)$ is bounded and equicontinuous then each $\pi_i(S)$ is a compactoid by the $p$-adic Ascoli Theorem, hence $\pi_1(S) \times \pi_2(S) \times \ldots \times \pi_k(S)$ is a compactoid. But then so is its subset $S$.

**Theorem 5.3.** The map $T$ of Theorem 4.1. is compact.

**Proof.** By Lemma 5.1 it is enough to prove that $\tau_n \circ T$ is compact; from Lemma 5.2 we see that it suffices to show that, for each $i \in \{0, 1, \ldots, n\}$ the set

$$\{ \Phi_i T f : f \in C^m(Y \to K), \|f\|_m \leq 1 \}$$

is (pointwise) bounded and equicontinuous in $C(X^{i+1} \to K)$.

But this was already observed in the proof of Theorem 4.1 for $i = n$; a similar argument works for each $i \in \{0, 1, \ldots, n\}$.

### 6. INTEGRAL OPERATORS ON $C^\infty$

Let $X, Y$ be compact subsets of $K$ without isolated points. The space

$$C^\infty(X \to K) := \bigcap_n C^n(X \to K)$$

has a natural locally convex topology induced by the norms $\| \cdot \|_n (n \in \{0, 1, 2, \ldots\})$. If $G : X \times Y \to K$ is $C^\infty$ (i.e. $G \in C^{n,m}(X \times Y \to K)$ for each $n, m \in \{0, 1, \ldots\}$) and $\mu \in C^\infty(X \to K)'$ the formula

$$(*) \quad (T f)(x) = \int G(x, y) f(y) d\mu(y)$$

defines a linear map $C^\infty(Y \to K) \to C^\infty(X \to K)$. By construction there is an $m \in \{0, 1, \ldots\}$ and a $C > 0$ such that $|\mu(f)| \leq C\|f\|_m$ for all $f \in C^\infty(Y \to K)$. For each $n \in \{0, 1, \ldots\}$ $(*)$ is the restriction of a compact integral operator $C^m(Y \to K) \to C^n(X \to K)$. It follows that $T$ maps $\{ f \in C^m(Y \to K) : \|f\|_m \leq 1 \}$ into a compactoid of $C^\infty(X \to K)$ i.e. that $T$ is a compact map $C^\infty(Y \to K) \to C^\infty(X \to K)$. 

REFERENCES


