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COMPACTNESS OF p-ADIC INTEGRAL OPERATORS

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Abstract.
The p-adic counterparts of the classical integral operators are shown to be compact. This result is extended to integral operators $C^n \rightarrow C^m$.

PRELIMINARIES
Throughout $K$ is a non-archimedean valued complete field whose valuation $| |$ is not trivial. For a compact topological space $X$ the $K$-Banach space of all continuous functions $f : X \rightarrow K$ with the norm $f \mapsto \|f\|_\infty := \max \{|f(x)| : x \in X\}$ is denoted $C(X \rightarrow K)$. The closed unit ball of a $K$-Banach space $E$ is written $B_E$. A (continuous) linear map $T : E \rightarrow F$, where $E, F$ are locally convex spaces over $K$, is called compact if there is a neighbourhood $U$ of 0 in $E$ for which $TU$ is a compactoid, where a subset $Y$ of $F$ is called a compactoid if for each neighbourhood $V$ of 0 in $F$ there exists a finitely generated absolutely convex set $I$ such that $Y \subseteq V + I$. The topological dual of $E$ is $E'$.
Further background on p-adic Functional Analysis can be found in [3].

1. INTEGRAL OPERATORS
Let $X, Y$ be compact topological spaces, $\mu \in C(Y \rightarrow K)'$, $G \in C(X \times Y \rightarrow K)$. For each $f \in C(Y \rightarrow K)$ and $x \in X$ the function $h_x : y \mapsto G(x, y)f(y)$ is continuous so the expression $\mu(h_x)$ makes sense. Rather than $\mu(h_x)$ we shall use the more convenient notation $\int G(x, y)h(y)d\mu(y)$.

Theorem 1.1. With the above notations, the formula

$$(Tf)(x) = \int G(x, y)f(y)d\mu(y)$$

defines a compact operator $T : C(Y \rightarrow K) \rightarrow C(X \rightarrow K)$.
Proof. We have seen already that \((Tf)(x)\) is well-defined. To prove continuity of \(Tf\), let \(a \in X\), \(\epsilon > 0\). By compactness there is a neighbourhood \(U\) of \(a\) such that 
\[ |G(x,y) - G(a,y)| \leq \epsilon \text{ for all } x \in U \text{ and } y \in Y. \]
Then, for \(x \in U\)
\[ |(Tf)(x) - (Tf)(a)| = \left| \int (G(x,y) - G(a,y))f(y)d\mu(y) \right| \leq \epsilon \|f\|\|\mu\| \]
and the continuity of \(Tf\) follows; we even have equicontinuity of \(TB_{C(Y-K)}\). From 
\[ ||Tf|| \leq ||G||_{\infty}||f||_{\infty}||\mu|| \]

In the next sections we shall interpret \(T\) as a map \(C^n(Y \to K) \to C^n(X \to K)\). To this end we need some preliminary definitions and results that will be treated in §2 and §3.

2. \(C^n\)-FUNCTIONS OF ONE VARIABLE

We recall some definitions of [5], §29. For a subset \(X\) of \(K\) and \(n \in \mathbb{N}\) we set
\[ \nabla^n X := \{(x_1, x_2, \ldots, x_n) \in X^n : \text{if } i \neq j \text{ then } x_i \neq x_j\}. \]
The \(n\)th difference quotient \(\Phi_n f : \nabla^{n+1} X \to K\) of a function \(f : X \to K\) is inductively given by \(\Phi_0 f := f\) and, for \(n \in \mathbb{N}\), by the formula
\[ (\Phi_n f)(x_1, x_2, \ldots, x_{n+1}) = \frac{(\Phi_{n-1} f)(x_1, x_3, \ldots, x_{n+1}) - (\Phi_{n-1} f)(x_2, x_3, \ldots, x_{n+1})}{x_1 - x_2}. \]
We say that \(f\) is a \(C^n\)-function \((f \in C^n(X \to K)\), or shortly \(f \in C^n\)) if \(\Phi_n f\) can be extended to a continuous function \(X^{n+1} \to K\). If \(X\) has no isolated points the above extension is unique and denoted \(\overline{\Phi_n f}\). We set
\[ D_n f(x) := (\overline{\Phi_n f})(x, x, \ldots, x) \quad (x \in X). \]
Then ([5] Theorem 29.5) \(n!D_n f = f^{(n)}\) (so that \(D_n f = f^{(n)}/n!\) if the characteristic of \(K\) is zero). The set \(C^n(X \to K)\) is a \(K\)-algebra under pointwise operations.

Now assume that \(X\) is a compact subset of \(K\) without isolated points. Then for an \(f \in C^n(X \to K)\) the functions \(f, \Phi_1 f, \ldots, \Phi_n f\) are all bounded so one may define
\[ ||f||_n := \max(||f||_{\infty}, ||\Phi_1 f||_{\infty}, \ldots, ||\Phi_n f||_{\infty}) \]
\[ (= \max(||f||_{\infty}, ||\overline{\Phi_1 f}||_{\infty}, \ldots, ||\overline{\Phi_n f}||_{\infty}). \]
It is shown in [4], Theorem 8.5 that $\|f\|_n$ is a norm on $C^n(X \to K)$ making it into a $K$-Banach space. Because also $\|fg\|_n \leq \|f\|_n \|g\|_n$ holds for $f, g \in C^n(X \to K)$ the space $C^n(X \to K)$ is even a $K$-Banach algebra.

Observe that, if $f \in C^n(X \to K)$ the functions $f, \Phi_1 f, \ldots, \Phi_n f$ are uniformly continuous.

3. $C^n$-FUNCTIONS OF TWO VARIABLES

Throughout §3 let $n, m \in \{0, 1, 2, \ldots\}$.

Let $X$ be a subset of $K$, let $Y$ be just a set and let $H : X \times Y \to K$. The $n$th difference quotient of $H$ with respect to the first variable is by definition the function

$$(x_1, \ldots, x_{n+1}, y) \mapsto \frac{H(x_1, \ldots, x_{n+1}, y) - H(x_1, \ldots, x_{n}, y)}{x_{n+1} - x_n}$$

defined on $\nabla^{n+1}X \times Y$, where $h_y(x) := H(x, y)$ ($x \in X, y \in Y$).

Similarly, for a set $X$, a subset $Y$ of $K$ and a function $J : X \times Y \to K$ we define the $m$th difference quotient of $J$ with respect to the second variable to be the map

$$(x, y_1, y_2, \ldots, y_{m+1}) \mapsto \frac{J(x, y_1, y_2, \ldots, y_{m+1}) - J(x, y_1, \ldots, y_{m}, y)}{y_{m+1} - y_m}$$

defined on $X \times \nabla^{m+1}Y$ where $j^x(y) := J(x, y)$ ($x \in X, y \in Y$).

We leave the proof of the following elementary lemma to the reader.

**Lemma 3.1.** Let $X, Y$ be subsets of $K$, let $G : X \times Y \to K$. Then

$$\Phi^{(2)}_m \Phi^{(1)}_n G = \Phi^{(1)}_n \Phi^{(2)}_m G.$$ 

Now let $X, Y$ be subsets of $K$ without isolated points, and let $G : X \times Y \to K$. We say that $G \in C^{n,m}(X \times Y \to K)$ (or simply $G \in C^{n,m}$) if the function $\Phi^{(2)}_m \Phi^{(1)}_n G$ (or, equivalently, $\Phi^{(1)}_n \Phi^{(2)}_m G$) can be extended to a continuous function $X^{n+1} \times Y^{m+1} \to K$.

This extension is unique and denoted $\Phi^{(2)}_m \Phi^{(1)}_n G$ or $\Phi^{(1)}_n \Phi^{(2)}_m G$. Special cases are

$$D^{(1)}_n G(x, y) := (\Phi^{(2)}_0 \Phi^{(1)}_n G)(x, x, \ldots, x, y)$$

and also expressions like $D^{(2)}_m \Phi^{(1)}_n G$, $\Phi^{(2)}_m D^{(1)}_n G$ make sense. The following facts are easily established. If $G \in C^{n,m}$ and $j, k \in \{0, 1, \ldots\}$, $j \leq n$, $k \leq m$ then $G \in C^{j,k}$. If $G \in C^{n,m}$ then

$$n!m! D^{(1)}_n D^{(2)}_m = \frac{\partial^n}{\partial x^n} \frac{\partial^m}{\partial y^m} G.$$
and in particular we have equality of mixed partial derivatives: \( \frac{\partial}{\partial x^n} \frac{\partial}{\partial y^m} G = \frac{\partial}{\partial y^m} \frac{\partial}{\partial x^n} G. \)

If \( G \in C^{n,0}(X \times Y) \) then for each \( y \in Y \) the function \( x \mapsto G(x, y) \) is in \( C^n(X \to K) \).

4. INTEGRAL OPERATORS ON \( C^n \)

**Theorem 4.1.** Let \( X, Y \) be compact subsets of \( K \) without isolated points. Let \( G \in C^{n,m}(X \times Y \to K) \) and let \( \mu \in C^m(Y \to K)' \). Then the formula

\[
(Tf)(x) = \int G(x, y)f(y)\,d\mu(y)
\]

defines a continuous linear map \( T : C^m(Y \to K) \to C^n(X \to K) \). We have \( \|T\| \leq \|\mu\|\|G\|_{n,m} \) where

\[
\|G\|_{n,m} := \sup \{\|\Phi_k^{(2)} \Phi_j^{(1)} G\|_\infty : 0 \leq j \leq n, 0 \leq k \leq m\}.
\]

**Proof.** Since \( G \in C^{0,m} \) the function \( y \mapsto G(x, y) \) is \( C^m \) for every \( x \in X \). So for \( f \in C^m(Y \to K) \) the product \( y \mapsto G(x, y)f(y) \) is a \( C^m \)-function on \( Y \). It follows that (*) defines a \( K \)-linear map of \( C^m(Y \to K) \) into the space of all functions on \( X \). To check that \( Tf \) is a \( C^n \)-function let \( \varepsilon > 0 \). By uniform continuity there exists a \( \delta > 0 \) such that for all \( j \in \{0, 1, \ldots, n\}, k \in \{0, 1, \ldots, m\} \), all \( (x_1, \ldots, x_{j+1}), (x'_1, \ldots, x'_{j+1}) \in X^{j+1} \), all \( (y_1, \ldots, y_{k+1}) \in Y^{k+1} \),

\[
|\Phi_k^{(2)} \Phi_j^{(1)} G(x_1, \ldots, x_{j+1}, y_1, \ldots, y_{k+1}) - \Phi_k^{(2)} \Phi_j^{(1)} G(x'_1, \ldots, x'_{j+1}, y_1, \ldots, y_{k+1})| < \varepsilon
\]

whenever \( |x_1 - x'_1| < \delta, \ldots, |x_{j+1} - x'_{j+1}| < \delta \). Then for such \( (x_1, \ldots, x_{n+1}) \) and \( (x'_1, x'_2, \ldots, x'_{n+1}) \) in \( \nabla^{n+1} X \) we have

\[
\Delta := \|(\Phi_n Tf)(x_1, \ldots, x_{n+1}) - (\Phi_n Tf)(x'_1, \ldots, x'_{n+1})| = \left| \int (\Phi_n^{(1)} G(x_1, \ldots, x_{n+1}, y) - \Phi_n^{(1)} G(x'_1, \ldots, x'_{n+1}, y))f(y)\,d\mu(y) \right| \
\leq \|h\|_m \|f\|_m \|\mu\|
\]

where

\[
h(y) = \Phi_n^{(1)} G(x_1, \ldots, x_{n+1}, y) - \Phi_n^{(1)} G(x'_1, \ldots, x'_{n+1}, y) \quad (y \in Y).
\]

Now \( \|h\|_m = \max\{\|\Phi_k h\|_\infty : 0 \leq k \leq m\} \) which is \( < \varepsilon \) by (A). We see that \( \Delta \leq \|f\|_m \|\mu\| \varepsilon. \) It follows that \( Tf \) is \( C^n \); we even may conclude that \( \{\Phi_n Tf : f \in C^m(Y \to K), \|f\|_m \leq 1\} \) is equicontinuous. To estimate \( \|T\| \), let \( j \in \{0, 1, \ldots, n\} \) and \( (x_1, \ldots, x_{j+1}) \in \nabla^{j+1} X \). Then
\[|\Phi_j T f(x_1, \ldots, x_{j+1})| = \left| \int \Phi_j^{(1)} G(x_1, \ldots, x_{j+1}, y) f(y) d\mu(y) \right| \]
\[\leq \|\mu\| \|f\| m \sup \{ \|\Phi_k^{(2)} \Phi_j^{(1)} G\|_\infty : 0 \leq k \leq m \} \]
\[\leq \|\mu\| \|f\| m \|G\|_{n,m} .\]

We see that \(\|T\| \leq \|\mu\| \|G\|_{n,m} .\)

**Corollary 4.2.** For all \(f \in C^n(Y \to K)\) and \(x \in X\)
\[
\frac{d^n(Tf)(x)}{dx^n} = \int \frac{\partial^n G(x, y)}{\partial x^n} f(y) d\mu(y).
\]

**Proof.** It is shown in the previous proof that
\[z \mapsto \Phi_n^{(1)} G(z, \cdot) \quad (z \in \nabla^{n+1} X)\]
has a continuous extension \(X^{n+1} \to C^n(Y \to K)\). Then, for \(x \in X\) we have (with \(z \in \nabla^{n+1} X\))
\[
D_n(Tf)(x) = \Phi_n^{(1)} (Tf)(x, x, \ldots, x) = \lim_{z \to (x, \ldots, x)} \Phi_n^{(1)} (Tf)(z)
\]
\[= \lim_{z \to (x, \ldots, x)} \int \Phi_n^{(1)} G(z, y) f(y) d\mu(y) = \]
\[
= \int \lim_{z \to (x, \ldots, x)} \Phi_n^{(1)} G(z, y) f(y) d\mu(y) = \int D_n^1 G(x, y) f(y) d\mu(y).
\]

**5. COMPACTNESS OF INTEGRAL OPERATORS**

To prove compactness of the operator \(T : C^n(Y \to K) \to C^n(X \to K)\) of the previous section we shall combine the \(p\)-adic Ascoli Theorem with the following.

**Lemma 5.1.** Let \(X\) be a subset of \(K\), without isolated points. Let \(n \in \{0, 1, 2, \ldots\}\).

The map
\[\tau_n : f \mapsto (f, \Phi_1 f, \Phi_2 f, \ldots, \Phi_n f)\]
embeds \(C^n(X \to K)\) linearly and isometrically into \(C(X) \times C(X^2) \times \ldots \times C(X^{n+1})\).

**Proof.** Direct verification.

**Lemma 5.2.** Let \(Z_1, \ldots, Z_k (k \in \mathbb{N})\) be compact topological spaces and let, for each \(i \in \{1, \ldots, k\}\), \(\pi_i\) be the obvious projection \(\prod_{j=1}^k C(Z_j \to K) \to C(Z_i \to K)\). Then a
subset $S$ of $\prod_{j=1}^{k} C(Z_j \to K)$ is a compactoid if and only if for each $i \in \{1, \ldots, k\}$, $\pi_i(S)$ is bounded and equicontinuous in $C(Z_i \to K)$.

**Proof.** If $S$ is a compactoid and $i \in \{1, \ldots, k\}$ then, since $\pi_i$ is linear and continuous, $\pi_i(S)$ is a compactoid in $C(Z_i \to K)$, hence bounded and equicontinuous by the $p$-adic Ascoli Theorem. Conversely, if every $\pi_i(S)$ is bounded and equicontinuous then each $\pi_i(S)$ is a compactoid by the $p$-adic Ascoli Theorem, hence $\pi_1(S) \times \pi_2(S) \times \ldots \times \pi_k(S)$ is a compactoid. But then so is its subset $S$.

**Theorem 5.3.** The map $T$ of Theorem 4.1. is compact.

**Proof.** By Lemma 5.1 it is enough to prove that $\tau_n \circ T$ is compact; from Lemma 5.2 we see that it suffices to show that, for each $i \in \{0, 1, \ldots, n\}$ the set

$$\{ \tilde{\Phi}_i T f : f \in C^m(Y \to K), ||f||_m \leq 1 \}$$

is (pointwise) bounded and equicontinuous in $C(X^{i+1} \to K)$.

But this was already observed in the proof of Theorem 4.1 for $i = n$; a similar argument works for each $i \in \{0, 1, \ldots, n\}$.

6. **INTEGRAL OPERATORS ON $C^\infty$**

Let $X, Y$ be compact subsets of $K$ without isolated points. The space

$$C^\infty(X \to K) := \bigcap_n C^n(X \to K)$$

has a natural locally convex topology induced by the norms $|| \cdot ||_n (n \in \{0, 1, 2, \ldots\})$.

If $G : X \times Y \to K$ is $C^\infty$ (i.e. $G \in C^{n,m}(X \times Y \to K)$ for each $n, m \in \{0, 1, \ldots\}$) and $\mu \in C^\infty(X \to K)'$ the formula

$$(*) \quad (Tf)(x) = \int G(x,y) f(y) d\mu(y)$$

defines a linear map $C^{\infty}(Y \to K) \to C^{\infty}(X \to K)$. By construction there is an $m \in \{0, 1, \ldots\}$ and a $C > 0$ such that $|\mu(f)| \leq C ||f||_m$ for all $f \in C^{\infty}(Y \to K)$. For each $n \in \{0, 1, \ldots, \}$ (*) is the restriction of a compact integral operator $C^m(Y \to K) \to C^n(X \to K)$. It follows that $T$ maps $\{ f \in C^m(Y \to K) : ||f||_m \leq 1 \}$ into a compactoid of $C^\infty(X \to K)$ i.e. that $T$ is a compact map $C^\infty(Y \to K) \to C^\infty(X \to K)$. 

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REFERENCES


