COMPACTNESS OF \( p \)-ADIC INTEGRAL OPERATORS

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Abstract.
The \( p \)-adic counterparts of the classical integral operators are shown to be compact. This result is extended to integral operators \( C^n \to C^m \).

PRELIMINARIES
Throughout \( K \) is a non-archimedean valued complete field whose valuation \( | \cdot | \) is not trivial. For a compact topological space \( X \) the \( K \)-Banach space of all continuous functions \( f : X \to K \) with the norm \( f \mapsto \| f \|_\infty := \max \{|f(x)| : x \in X \} \) is denoted \( C(X \to K) \). The closed unit ball of a \( K \)-Banach space \( E \) is written \( B_E \). A (continuous) linear map \( T : E \to F \), where \( E, F \) are locally convex spaces over \( K \), is called compact if there is a neighbourhood \( U \) of \( 0 \) in \( E \) for which \( TU \) is a compactoid, where a subset \( Y \) of \( F \) is called a compactoid if for each neighbourhood \( V \) of \( 0 \) in \( F \) there exists a finitely generated absolutely convex set \( I \) such that \( Y \subseteq V + I \). The topological dual of \( E \) is \( E' \).

Further background on \( p \)-adic Functional Analysis can be found in [3].

1. INTEGRAL OPERATORS
Let \( X, Y \) be compact topological spaces, \( \mu \in C(Y \to K)' \), \( G \in C(X \times Y \to K) \). For each \( f \in C(Y \to K) \) and \( x \in X \) the function \( h_x : y \mapsto G(x,y)f(y) \) is continuous so the expression \( \mu(h_x) \) makes sense. Rather than \( \mu(h_x) \) we shall use the more convenient notation \( \int G(x,y)h(y)d\mu(y) \).

Theorem 1.1. With the above notations, the formula

\[
(Tf)(x) = \int G(x,y)f(y)d\mu(y)
\]

defines a compact operator \( T : C(Y \to K) \to C(X \to K) \).
Proof. We have seen already that \((Tf)(x)\) is well-defined. To prove continuity of \(Tf\), let \(a \in X\), \(\varepsilon > 0\). By compactness there is a neighbourhood \(U\) of \(a\) such that 
\[
|G(x, y) - G(a, y)| \leq \varepsilon \text{ for all } x \in U \text{ and } y \in Y.
\]
Then, for \(x \in U\)
\[
|(Tf)(x) - (Tf)(a)| = \left| \int (G(x, y) - G(a, y))f(y)d\mu(y) \right| \leq \varepsilon \|f\|_{\infty} \|\mu\|
\]
and the continuity of \(Tf\) follows; we even have equicontinuity of \(TB_{C(Y \to K)}\). From 
\[
\|Tf\| \leq \|G\|_{\infty} \|f\|_{\infty} \|\mu\|
\]
we obtain (uniform) boundedness of \(TB_{C(Y \to K)}\) implying compactness of \(T\) by the p-adic Ascoli Theorem [1] Theorem 1.8 and [2] Theorem 1.

In the next sections we shall interpret \(T\) as a map \(C^{n}(Y \to K) \to C^{n}(X \to K)\). To this end we need some preliminary definitions and results that will be treated in \$2\) and \$3.

2. \(C^{n}\)-FUNCTIONS OF ONE VARIABLE
We recall some definitions of [5], \$29. For a subset \(X\) of \(K\) and \(n \in \mathbb{N}\) we set
\[
\nabla^{n}X := \{(x_{1}, x_{2}, \ldots, x_{n}) \in X^{n} : \text{if } i \neq j \text{ then } x_{i} \neq x_{j}\}.
\]
The \(n\)th difference quotient \(\Phi_{n}f : \nabla^{n+1}X \to K\) of a function \(f : X \to K\) is inductively given by \(\Phi_{0}f := f\) and, for \(n \in \mathbb{N}\), by the formula
\[
(\Phi_{n}f)(x_{1}, x_{2}, \ldots, x_{n+1}) = \frac{(\Phi_{n-1}f)(x_{1}, x_{2}, \ldots, x_{n+1}) - (\Phi_{n-1}f)(x_{2}, x_{3}, \ldots, x_{n+1})}{x_{1} - x_{2}}.
\]
We say that \(f\) is a \(C^{n}\)-function (\(f \in C^{n}(X \to K)\), or shortly \(f \in C^{n}\)) if \(\Phi_{n}f\) can be extended to a continuous function \(X^{n+1} \to K\). If \(X\) has no isolated points the above extension is unique and denoted \(\overline{\Phi}_{n}f\). We set
\[
D_{n}f(x) := (\overline{\Phi}_{n}f)(x, x, \ldots, x) \quad (x \in X).
\]
Then ([5] Theorem 29.5) \(n!D_{n}f = f^{(n)}\) (so that \(D_{n}f = f^{(n)}/n!\) if the characteristic of \(K\) is zero). The set \(C^{n}(X \to K)\) is a \(K\)-algebra under pointwise operations.

Now assume that \(X\) is a compact subset of \(K\) without isolated points. Then for an \(f \in C^{n}(X \to K)\) the functions \(f, \Phi_{1}f, \ldots, \Phi_{n}f\) are all bounded so one may define
\[
\|f\|_{n} := \max(\|f\|_{\infty}, \|\Phi_{1}f\|_{\infty}, \ldots, \|\Phi_{n}f\|_{\infty})
\]
\[
(= \max(\|f\|_{\infty}, \|\overline{\Phi}_{1}f\|_{\infty}, \ldots, \|\overline{\Phi}_{n}f\|_{\infty}).
\]
It is shown in [4], Theorem 8.5 that $\|f\|_n$ is a norm on $C^n(X \to K)$ making it into a $K$-Banach space. Because also $\|fg\|_n \leq \|f\|_n \|g\|_n$ holds for $f, g \in C^n(X \to K)$ the space $C^n(X \to K)$ is even a $K$-Banach algebra.

Observe that, if $f \in C^n(X \to K)$ the functions $f, \Phi_1 f, \ldots, \Phi_n f$ are uniformly continuous.

3. $C^n$-FUNCTIONS OF TWO VARIABLES

Throughout §3 let $n, m \in \{0,1,2,\ldots\}$.

Let $X$ be a subset of $K$, let $Y$ be just a set and let $H : X \times Y \to K$. The $n$th difference quotient of $H$ with respect to the first variable is by definition the function

$$(x_1, \ldots, x_{n+1}, y) \mapsto (\Phi^{(1)}_n H)(x_1, \ldots, x_{n+1})$$

defined on $\nabla^{n+1}X \times Y$, where $h_y(x) := H(x, y) \ (x \in X, y \in Y)$.

Similarly, for a set $X$, a subset $Y$ of $K$ and a function $J : X \times Y \to K$ we define the $m$th difference quotient of $J$ with respect to the second variable to be the map

$$(x, y_1, y_2, \ldots, y_{m+1}) \mapsto (\Phi^{(2)}_m J)(y_1, \ldots, y_{m+1})$$

defined on $X \times \nabla^{m+1}Y$ where $j^y(x) := J(x, y) \ (x \in X, y \in Y)$.

We leave the proof of the following elementary lemma to the reader.

**Lemma 3.1.** Let $X, Y$ be subsets of $K$, let $G : X \times Y \to K$. Then

$$\Phi^{(2)}_m \Phi^{(1)}_n G = \Phi^{(1)}_n \Phi^{(2)}_m G.$$  

Now let $X, Y$ be subsets of $K$ without isolated points, and let $G : X \times Y \to K$. We say that $G \in C^{n,m}(X \times Y \to K)$ (or simply $G \in C^{n,m}$) if the function $\Phi^{(2)}_m \Phi^{(1)}_n G$ (or, equivalently, $\Phi^{(1)}_n \Phi^{(2)}_m G$) can be extended to a continuous function $X^{n+1} \times Y^{m+1} \to K$.

This extension is unique and denoted $\Phi^{(2)}_m \Phi^{(1)}_n G$ or $\Phi^{(1)}_n \Phi^{(2)}_m G$). Special cases are

$$D^{(1)}_n G(x, y) := (\Phi^{(2)}_n \Phi^{(1)}_n G)(x, x, \ldots, x, y)$$

and also expressions like $D^{(2)}_m \Phi^{(1)}_n G, \Phi^{(2)}_m D^{(1)}_n G$ make sense. The following facts are easily established. If $G \in C^{n,m}$ and $j, k \in \{0,1,\ldots\}, j \leq n, k \leq m$ then $G \in C^{j,k}$. If $G \in C^{n,m}$ then

$$n!m!D^{(1)}_n D^{(2)}_m = \frac{\partial^n}{\partial x^n} \frac{\partial^m}{\partial y^m} G.$$
and in particular we have equality of mixed partial derivatives: $\frac{\partial^n}{\partial y^n} \frac{\partial^m}{\partial x^m} G = \frac{\partial^m}{\partial y^m} \frac{\partial^n}{\partial x^n} G$. If $G \in C^{n,0}(X \times Y)$ then for each $y \in Y$ the function $x \mapsto G(x, y)$ is in $C^n(X \to K)$.

4. INTEGRAL OPERATORS ON $C^m$

Theorem 4.1. Let $X, Y$ be compact subsets of $K$ without isolated points. Let $G \in C^{n,m}(X \times Y \to K)$ and let $\mu \in C^m(Y \to K)'$. Then the formula

$$(Tf)(x) = \int G(x, y)f(y)d\mu(y)$$

defines a continuous linear map $T : C^m(Y \to K) \to C^n(X \to K)$. We have $\|T\| \leq \|\mu\|\|G\|_{n,m}$ where

$$\|G\|_{n,m} := \sup \{\|\Phi_k(\Phi_j(t)G\|_{\infty} : 0 \leq j \leq n, 0 \leq k \leq m\}. $$

Proof. Since $G \in C^{0,m}$ the function $y \mapsto G(x, y)$ is $C^m$ for every $x \in X$. So for $f \in C^m(Y \to K)$ the product $y \mapsto G(x, y)f(y)$ is a $C^m$-function on $Y$. It follows that $Tf$ is a $C^m$-function on $Y$ and $Tf$ is $C^n$ if and only if $T\mu$ is $C^n$. We even may conclude that \{Tf : f \in C^m(Y \to K), \|f\|_m \leq 1\} is equicontinuous. To estimate $\|T\|$, let $j \in \{0, 1, \ldots, n\}$ and $(x_1, \ldots, x_{j+1}) \in Y_{j+1}$. Then

$$\Delta := \|\Phi_nTf(x_1, \ldots, x_{n+1}) - (\Phi_nTf)(x'_1, \ldots, x'_{n+1})\|$$

$$= \|\int (\Phi_n G(x_1, \ldots, x_{n+1}, y) - \Phi_n G(x'_1, \ldots, x'_{n+1}, y))f(y)d\mu(y)\|$$

$$\leq \|h\|_m \|f\|_m \|\mu\|$$

where

$$h(y) = \Phi_n G(x_1, \ldots, x_{n+1}, y) - \Phi_n G(x'_1, \ldots, x'_{n+1}, y) \quad (y \in Y).$$

Now $\|h\|_m = \max\{\|\Phi_k h\|_\infty : 0 \leq k \leq m\}$ which is $< \varepsilon$ by (A). We see that $\Delta \leq \|f\|_m \|\mu\|\varepsilon$. It follows that $Tf$ is $C^n$; we even may conclude that \{Tf : f \in C^m(Y \to K), \|f\|_m \leq 1\} is equicontinuous. To estimate $\|T\|$, let $j \in \{0, 1, \ldots, n\}$ and $(x_1, \ldots, x_{j+1}) \in Y_{j+1}$. Then
\[|\Phi_j T f(x_1, \ldots, x_{j+1})| = \left| \int \Phi_j^{(1)} G(x_1, \ldots, x_{j+1}, y) f(y) d\mu(y) \right| \]
\[\leq \|\mu\| \|f\|_m \sup \{\|\Phi_k^{(2)} \Phi_j^{(1)} G\|_\infty : 0 \leq k \leq m\} \]
\[\leq \|\mu\| \|f\|_m \|G\|_{n,m}.\]

We see that \(\|T\| \leq \|\mu\| \|G\|_{n,m}.\)

**Corollary 4.2.** For all \(f \in C^n(Y \to K)\) and \(x \in X\)

\[
\frac{d^n(Tf)(x)}{dx^n} = \int \frac{\partial^n G(x, y)}{\partial x^n} f(y) d\mu(y).
\]

**Proof.** It is shown in the previous proof that

\[z \mapsto \Phi_n^{(1)} G(z, \cdot) \quad (z \in \nabla^{n+1} X)\]

has a continuous extension \(X^{n+1} \to C^m(Y \to K)\). Then, for \(x \in X\) we have (with \(z \in \nabla^{n+1} X\))

\[D_n(Tf)(x) = \Phi_n^{(1)}(Tf)(x, x, x, \ldots, x) = \lim_{z \to (x, x, \ldots)} \Phi_n^{(1)}(Tf)(z) \]
\[= \lim_{z \to (x, x, \ldots)} \int \Phi_n^{(1)} G(z, y) f(y) d\mu(y) = \]
\[= \int \lim_{z \to (x, x, \ldots)} \Phi_n^{(1)} G(z, y) f(y) d\mu(y) = \int D_n^1 G(x, y) f(y) d\mu(y).
\]

5. **COMPACTNESS OF INTEGRAL OPERATORS**

To prove compactness of the operator \(T : C^m(Y \to K) \to C^n(X \to K)\) of the previous section we shall combine the p-adic Ascoli Theorem with the following.

**Lemma 5.1.** Let \(X\) be a subset of \(K\), without isolated points. Let \(n \in \{0, 1, 2, \ldots\}\). The map

\[\tau_n : f \mapsto (f, \overline{\Phi}_1 f, \overline{\Phi}_2 f, \ldots, \overline{\Phi}_n f)\]

embeds \(C^n(X \to K)\) linearly and isometrically into \(C(X) \times C(X^2) \times \ldots \times C(X^{n+1})\).

**Proof.** Direct verification.

**Lemma 5.2.** Let \(Z_1, \ldots, Z_k (k \in \mathbb{N})\) be compact topological spaces and let, for each \(i \in \{1, \ldots, k\}\), \(\pi_i\) be the obvious projection \(\prod_{j=1}^k C(Z_j \to K) \to C(Z_i \to K)\). Then a
subset $S$ of $\prod_{j=1}^{k} C(Z_j \to K)$ is a compactoid if and only if for each $i \in \{1, \ldots, k\}$, $\pi_i(S)$ is bounded and equicontinuous in $C(Z_i \to K)$.

**Proof.** If $S$ is a compactoid and $i \in \{1, \ldots, k\}$ then, since $\pi_i$ is linear and continuous, $\pi_i(S)$ is a compactoid in $C(Z_i \to K)$, hence bounded and equicontinuous by the $p$-adic Ascoli Theorem. Conversely, if every $\pi_i(S)$ is bounded and equicontinuous then each $\pi_i(S)$ is a compactoid by the $p$-adic Ascoli Theorem, hence $\pi_1(S) \times \pi_2(S) \times \ldots \times \pi_k(S)$ is a compactoid. But then so is its subset $S$.

**Theorem 5.3.** The map $T$ of Theorem 4.1 is compact.

**Proof.** By Lemma 5.1 it is enough to prove that $\tau_n \circ T$ is compact; from Lemma 5.2 we see that it suffices to show that, for each $i \in \{0, 1, \ldots, n\}$ the set

$$\{ \Phi_i T f : f \in C^m(Y \to K), \|f\|_m \leq 1 \}$$

is (pointwise) bounded and equicontinuous in $C(X^{i+1} \to K)$.

But this was already observed in the proof of Theorem 4.1 for $i = n$; a similar argument works for each $i \in \{0, 1, \ldots, n\}$.

6. INTEGRAL OPERATORS ON $C^\infty$

Let $X, Y$ be compact subsets of $K$ without isolated points. The space

$$C^\infty(X \to K) := \bigcap_n C^n(X \to K)$$

has a natural locally convex topology induced by the norms $\| \cdot \|_n$ ($n \in \{0, 1, 2, \ldots\}$). If $G : X \times Y \to K$ is $C^\infty$ (i.e. $G \in C^{m,n}(X \times Y \to K)$ for each $n, m \in \{0, 1, \ldots\}$) and $\mu \in C^\infty(X \to K)'$ the formula

$$(*) \quad (Tf)(x) = \int G(x, y)f(y)d\mu(y)$$

defines a linear map $C^\infty(Y \to K) \to C^\infty(X \to K)$. By construction there is an $m \in \{0, 1, \ldots\}$ and a $C > 0$ such that $|\mu(f)| \leq C\|f\|_m$ for all $f \in C^\infty(Y \to K)$. For each $n \in \{0, 1, \ldots\}$ $(*)$ is the restriction of a compact integral operator $C^m(Y \to K) \to C^n(X \to K)$. It follows that $T$ maps $\{ f \in C^m(Y \to K) : \|f\|_m \leq 1 \}$ into a compactoid of $C^\infty(X \to K)$ i.e. that $T$ is a compact map $C^\infty(Y \to K) \to C^\infty(X \to K)$. 

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REFERENCES


