COMPACTNESS OF $p$-ADIC INTEGRAL OPERATORS

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Abstract.
The $p$-adic counterparts of the classical integral operators are shown to be compact. This result is extended to integral operators $C^n \to C^m$.

PRELIMINARIES
Throughout $K$ is a non-archimedean valued complete field whose valuation $|\cdot|$ is not trivial. For a compact topological space $X$ the $K$-Banach space of all continuous functions $f : X \to K$ with the norm $f \mapsto \|f\|_\infty := \max\{|f(x)| : x \in X\}$ is denoted $C(X \to K)$. The closed unit ball of a $K$-Banach space $E$ is written $B_E$. A (continuous) linear map $T : E \to F$, where $E, F$ are locally convex spaces over $K$, is called compact if there is a neighbourhood $U$ of 0 in $E$ for which $TU$ is a compactoid, where a subset $Y$ of $F$ is called a compactoid if for each neighbourhood $V$ of 0 in $F$ there exists a finitely generated absolutely convex set $I$ such that $Y \subseteq V + I$. The topological dual of $E$ is $E'$.

Further background on $p$-adic Functional Analysis can be found in [3].

1. INTEGRAL OPERATORS
Let $X, Y$ be compact topological spaces, $\mu \in C(Y \to K)'$, $G \in C(X \times Y \to K)$. For each $f \in C(Y \to K)$ and $x \in X$ the function $h_x : y \mapsto G(x,y)f(y)$ is continuous so the expression $\mu(h_x)$ makes sense. Rather than $\mu(h_x)$ we shall use the more convenient notation $\int G(x,y)h(y)d\mu(y)$.

Theorem 1.1. With the above notations, the formula

$$(Tf)(x) = \int G(x,y)f(y)d\mu(y)$$

defines a compact operator $T : C(Y \to K) \to C(X \to K)$. 

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Proof. We have seen already that \((Tf)(x)\) is well-defined. To prove continuity of \(Tf\), let \(a \in X\), \(\varepsilon > 0\). By compactness there is a neighbourhood \(U\) of \(a\) such that 
\[ |G(x, y) - G(a, y)| \leq \varepsilon \text{ for all } x \in U \text{ and } y \in Y. \]
Then, for \(x \in U\)
\[ |(Tf)(x) - (Tf)(a)| = \left| \int (G(x, y) - G(a, y)) f(y) d\mu(y) \right| \leq \varepsilon \|f\|_\infty \|\mu\| \]
and the continuity of \(Tf\) follows; we even have equicontinuity of \(Tbc(Y - K')\). From 
\[ \|Tf\| \leq \|G\|_\infty \|f\|_\infty \|\mu\| \]

In the next sections we shall interpret \(T\) as a map \(C^n(Y \to K') \to C^n(X \to K')\). To this end we need some preliminary definitions and results that will be treated in §2 and §3.

2. \(C^n\)-FUNCTIONS OF ONE VARIABLE

We recall some definitions of [5], §29. For a subset \(X\) of \(K\) and \(n \in \mathbb{N}\) we set

\[ \nabla^n X := \{(x_1, x_2, \ldots, x_n) \in X^n : \text{if } i \neq j \text{ then } x_i \neq x_j\}. \]

The \(n\)th difference quotient \(\Phi_n f : \nabla^{n+1} X \to K\) of a function \(f : X \to K\) is inductively given by \(\Phi_0 f := f\) and, for \(n \in \mathbb{N}\), by the formula

\[ (\Phi_n f)(x_1, x_2, \ldots, x_{n+1}) = \frac{(\Phi_{n-1} f)(x_1, x_3, \ldots, x_{n+1}) - (\Phi_{n-1} f)(x_2, x_3, \ldots, x_{n+1})}{x_1 - x_2}. \]

We say that \(f\) is a \(C^n\)-function (\(f \in C^n(X \to K)\), or shortly \(f \in C^n\)) if \(\Phi_n f\) can be extended to a continuous function \(X^{n+1} \to K\). If \(X\) has no isolated points the above extension is unique and denoted \(\overline{\Phi}_n f\). We set

\[ D_n f(x) := (\overline{\Phi}_n f)(x, x, \ldots, x) \quad (x \in X). \]

Then ([5] Theorem 29.5) \(n!D_n f = f^{(n)}\) (so that \(D_n f = f^{(n)}/n!\) if the characteristic of \(K\) is zero). The set \(C^n(X \to K)\) is a \(K\)-algebra under pointwise operations.

Now assume that \(X\) is a compact subset of \(K\) without isolated points. Then for an \(f \in C^n(X \to K)\) the functions \(f, \Phi_1 f, \ldots, \Phi_n f\) are all bounded so one may define

\[ \|f\|_n := \max(\|f\|_\infty, \|\Phi_1 f\|_\infty, \ldots, \|\Phi_n f\|_\infty) \]

\[ (= \max(\|f\|_\infty, \|\overline{\Phi}_1 f\|_\infty, \ldots, \|\overline{\Phi}_n f\|_\infty). \]
It is shown in [4], Theorem 8.5 that \( f \| n \) is a norm on \( C^n(X \to K) \) making it into a \( K \)-Banach space. Because also \( \| fg \|_n \leq \| f \|_n \| g \|_n \) holds for \( f, g \in C^n(X \to K) \) the space \( C^n(X \to K) \) is even a \( K \)-Banach algebra. Observe that, if \( f \in C^n(X \to K) \) the functions \( f, \Phi_1 f, \ldots, \Phi_n f \) are uniformly continuous.

3. \( C^n \)-FUNCTIONS OF TWO VARIABLES

Throughout §3 let \( n, m \in \{0, 1, 2, \ldots\} \).

Let \( X \) be a subset of \( K \), let \( Y \) be just a set and let \( H : X \times Y \to K \). The \( n \)th difference quotient of \( H \) with respect to the first variable is by definition the function

\[
(x_1, \ldots, x_{n+1}, y) \xrightarrow{\Phi^{(1)}_n H} (\Phi_n h_y)(x_1, \ldots, x_{n+1})
\]
defined on \( \nabla^{n+1}X \times Y \), where \( h_y(x) := H(x, y) \) \((x \in X, y \in Y)\).

Similarly, for a set \( X \), a subset \( Y \) of \( K \) and a function \( J : X \times Y \to K \) we define the \( m \)th difference quotient of \( J \) with respect to the second variable to be the map

\[
(x, y_1, y_2, \ldots, y_{m+1}) \xrightarrow{\Phi^{(2)}_m J} (\Phi_m j^x)(y_1, \ldots, y_{m+1})
\]
defined on \( X \times \nabla^{m+1}Y \) where \( j^x(y) := J(x, y) \) \((x \in X, y \in Y)\).

We leave the proof of the following elementary lemma to the reader.

**Lemma 3.1.** Let \( X, Y \) be subsets of \( K \), let \( G : X \times Y \to K \). Then

\[
\Phi^{(2)}_m \Phi^{(1)}_n G = \Phi^{(1)}_n \Phi^{(2)}_m G.
\]

Now let \( X, Y \) be subsets of \( K \) without isolated points, and let \( G : X \times Y \to K \). We say that \( G \in C^{n,m}(X \times Y \to K) \) (or simply \( G \in C^{n,m} \)) if the function \( \Phi^{(2)}_m \Phi^{(1)}_n G \) (or, equivalently, \( \Phi^{(1)}_n \Phi^{(2)}_m G \)) can be extended to a continuous function \( X^{n+1} \times Y^{m+1} \to K \). This extension is unique and denoted \( \Phi^{(2)}_m \Phi^{(1)}_n G \) or \( \Phi^{(1)}_n \Phi^{(2)}_m G \). Special cases are

\[
D^{(1)}_n G(x, y) := (\Phi^{(2)}_n \Phi^{(1)}_n G)(x, x, \ldots, x, y)
\]
\[
D^{(2)}_m D^{(1)}_n G(x, y) := (\Phi^{(2)}_m \Phi^{(1)}_n G)(x, x, \ldots, x, y, y, \ldots, y)
\]

and also expressions like \( D^{(2)}_m \Phi^{(1)}_n G, \Phi^{(2)}_m D^{(1)}_n G \) make sense. The following facts are easily established. If \( G \in C^{n,m} \) and \( j, k \in \{0, 1, \ldots\}, j \leq n, k \leq m \) then \( G \in C^{j,k} \). If \( G \in C^{n,m} \) then

\[
n! m! D^{(1)}_n D^{(2)}_m = \frac{\partial^n}{\partial x^n} \frac{\partial^m}{\partial y^m} G
\]
and in particular we have equality of mixed partial derivatives: \( \frac{\partial^2}{\partial x^m \partial y^n} G = \frac{\partial^2}{\partial y^m \partial x^n} G \). If \( G \in C^{\mu,0}(X \times Y) \) then for each \( y \in Y \) the function \( x \mapsto G(x,y) \) is in \( C^n(X \to K) \).

4. INTEGRAL OPERATORS ON \( C^n \)

**Theorem 4.1.** Let \( X, Y \) be compact subsets of \( K \) without isolated points. Let \( G \in C^{n,m}(X \times Y \to K) \) and let \( \mu \in C^m(Y \to K)' \). Then the formula

\[
(Tf)(x) = \int G(x,y)f(y)d\mu(y)
\]

defines a continuous linear map \( T : C^m(Y \to K) \to C^n(X \to K) \). We have \( \|T\| \leq \|\mu\|\|G\|_{n,m} \) where

\[
\|G\|_{n,m} := \sup\{\|\Phi^{(2)}_k \Phi^{(1)}_j G\|_{\infty} : 0 \leq j \leq n, 0 \leq k \leq m\}.
\]

**Proof.** Since \( G \in C^{0,m} \) the function \( y \mapsto G(x,y) \) is \( C^m \) for every \( x \in X \). So for \( f \in C^m(Y \to K) \) the product \( y \mapsto G(x,y)f(y) \) is a \( C^m \)-function on \( Y \). It follows that \( (*) \) defines a \( K \)-linear map of \( C^m(Y \to K) \) into the space of all functions on \( X \). To check that \( Tf \) is a \( C^n \)-function let \( \varepsilon > 0 \). By uniform continuity there exists a \( \delta > 0 \) such that for all \( j \in \{0,1,\ldots,n\}, k \in \{0,1,\ldots,m\} \), all \((x_1,\ldots,x_{j+1}),(x'_1,\ldots,x'_{j+1}) \in X^{j+1} \), all \((y_1,\ldots,y_k+1) \in Y^{k+1} \),

\[
(\forall) \ |\Phi^{(2)}_k \Phi^{(1)}_j G(x_1,\ldots,x_{j+1},y_1,\ldots,y_{k+1}) - \Phi^{(2)}_k \Phi^{(1)}_j G(x'_1,\ldots,x'_{j+1},y_1,\ldots,y_{k+1})| < \varepsilon
\]

whenever \(|x_1-x'_1| < \delta, \ldots, |x_{j+1}-x'_{j+1}| < \delta \). Then for such \((x_1,\ldots,x_{n+1})\) and \((x'_1,\ldots,x'_{n+1})\) in \( Y^{n+1} \) we have

\[
\Delta := |(\Phi_nTf)(x_1,\ldots,x_{n+1}) - (\Phi_nTf)(x'_1,\ldots,x'_{n+1})|
\]

\[
= \left| \int (\Phi^{(1)}_n G(x_1,\ldots,x_{n+1},y) - \Phi^{(1)}_n G(x'_1,\ldots,x'_{n+1},y))f(y)d\mu(y) \right|
\]

\[
\leq \|h\|_m \|f\|_m \|\mu\|
\]

where

\[
h(y) = \Phi^{(1)}_n G(x_1,\ldots,x_{n+1},y) - \Phi^{(1)}_n G(x'_1,\ldots,x'_{n+1},y) \quad (y \in Y).
\]

Now \( \|h\|_m = \max\{\|\Phi_k h\|_{\infty} : 0 \leq k \leq m\} \) which is \( \varepsilon \) by \((\forall)\). We see that \( \Delta \leq \|f\|_m \|\mu\| \varepsilon \). It follows that \( Tf \) is \( C^n \); we even may conclude that \( \{\Phi_nTf : f \in C^m(Y \to K), \|f\|_m \leq 1\} \) is equicontinuous. To estimate \( \|T\| \), let \( j \in \{0,1,\ldots,n\} \) and \((x_1,\ldots,x_{j+1}) \in Y^{j+1} \). Then...
\[ |\Phi_j T f(x_1, \ldots, x_{j+1})| = \left| \int \Phi_j^{(1)} G(x_1, \ldots, x_{j+1}, y) f(y) d\mu(y) \right| \]
\[ \leq \|\mu\| \|f\|_m \sup \{\|\Phi_k^{(2)} \Phi_j^{(1)} G\|_{\infty} : 0 \leq k \leq m\} \]
\[ \leq \|\mu\| \|f\|_m \|G\|_{n,m}. \]

We see that \( \|T\| \leq \|\mu\| \|G\|_{n,m} \).

**Corollary 4.2.** For all \( f \in C^n(Y \to K) \) and \( x \in X \)

\[ \frac{d^n(Tf)(x)}{dx^n} = \int \frac{\partial^n G(x, y)}{\partial x^n} f(y) d\mu(y). \]

**Proof.** It is shown in the previous proof that

\[ z \mapsto \Phi_n^{(1)} G(z, \cdot) \quad (z \in \nabla^{n+1} X) \]

has a continuous extension \( X^{n+1} \to C^m(Y \to K) \). Then, for \( x \in X \) we have (with \( z \in \nabla^{n+1} X \))

\[ D_n(Tf)(x) = \Phi_n^{(1)}(Tf)(x, x, \ldots, x) = \lim_{z \to (x, x, \ldots)} \Phi_n^{(1)}(Tf)(z) \]
\[ = \lim_{z \to (x, x, \ldots)} \int \Phi_n^{(1)} G(z, y) f(y) d\mu(y) = \]
\[ = \int \lim_{z \to (x, x, \ldots)} \Phi_n^{(1)} G(z, y) f(y) d\mu(y) = \int D_n^1 G(x, y) f(y) d\mu(y). \]

5. **COMPACTNESS OF INTEGRAL OPERATORS**

To prove compactness of the operator \( T : C^m(Y \to K) \to C^n(X \to K) \) of the previous section we shall combine the \( p \)-adic Ascoli Theorem with the following.

**Lemma 5.1.** Let \( X \) be a subset of \( K \), without isolated points. Let \( n \in \{0, 1, 2, \ldots\} \).

The map

\[ \tau_n : f \mapsto (\overline{f}, \overline{f}_1 f, \overline{f}_2 f, \ldots, \overline{f}_n f) \]

embeds \( C^n(X \to K) \) linearly and isometrically into \( C(X) \times C(X^2) \times \ldots \times C(X^{n+1}) \).

**Proof.** Direct verification.

**Lemma 5.2.** Let \( Z_1, \ldots, Z_k \ (k \in \mathbb{N}) \) be compact topological spaces and let, for each \( i \in \{1, \ldots, k\} \), \( \pi_i \) be the obvious projection \( \prod_{j=1}^k C(Z_j \to K) \to C(Z_i \to K) \). Then a
subset \( S \) of \( \prod_{j=1}^{k} C(Z_j \to K) \) is a compactoid if and only if for each \( i \in \{1, \ldots, k\} \), \( \pi_i(S) \) is bounded and equicontinuous in \( C(Z_i \to K) \).

**Proof.** If \( S \) is a compactoid and \( i \in \{1, \ldots, k\} \) then, since \( \pi_i \) is linear and continuous, \( \pi_i(S) \) is a compactoid in \( C(Z_i \to K) \), hence bounded and equicontinuous by the \( p \)-adic Ascoli Theorem. Conversely, if every \( \pi_i(S) \) is bounded and equicontinuous then each \( \pi_i(S) \) is a compactoid by the \( p \)-adic Ascoli Theorem, hence \( \pi_1(S) \times \pi_2(S) \times \ldots \times \pi_k(S) \) is a compactoid. But then so is its subset \( S \).

**Theorem 5.3.** The map \( T \) of Theorem 4.1. is compact.

**Proof.** By Lemma 5.1 it is enough to prove that \( \tau_n \circ T \) is compact; from Lemma 5.2 we see that it suffices to show that, for each \( i \in \{0, 1, \ldots, n\} \) the set

\[
\{ \Phi_i T f : f \in C^m(Y \to K), \|f\|_m \leq 1 \}
\]

is (pointwise) bounded and equicontinuous in \( C(X^{i+1} \to K) \).

But this was already observed in the proof of Theorem 4.1 for \( i = n \); a similar argument works for each \( i \in \{0, 1, \ldots, n\} \).

**6. INTEGRAL OPERATORS ON \( C^\infty \)**

Let \( X, Y \) be compact subsets of \( K \) without isolated points. The space

\[
C^\infty(X \to K) := \bigcap_n C^n(X \to K)
\]

has a natural locally convex topology induced by the norms \( \|f\|_n (n \in \{0, 1, 2, \ldots\}) \).

If \( G : X \times Y \to K \) is \( C^\infty \) (i.e. \( G \in C^{m,m}(X \times Y \to K) \) for each \( n, m \in \{0, 1, \ldots\} \)) and \( \mu \in C^\infty(X \to K)' \) the formula

\[
(Tf)(x) = \int G(x, y)f(y)d\mu(y)
\]

defines a linear map \( C^\infty(Y \to K) \to C^\infty(X \to K) \). By construction there is an \( m \in \{0, 1, \ldots\} \) and a \( C > 0 \) such that \( |\mu(f)| \leq C\|f\|_m \) for all \( f \in C^\infty(Y \to K) \). For each \( n \in \{0, 1, \ldots, \} \) \( (*) \) is the restriction of a compact integral operator \( C^m(Y \to K) \to C^n(X \to K) \). It follows that \( T \) maps \( \{ f \in C^m(Y \to K) : \|f\|_m \leq 1 \} \) into a compactoid of \( C^\infty(X \to K) \) i.e. that \( T \) is a compact map \( C^\infty(Y \to K) \to C^\infty(X \to K) \).
REFERENCES


