ORTHOCOMPLEMENTATION IN $p$-ADIC BANACH SPACES

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Abstract.
This paper deals with classes of Banach spaces $E$ over a non-archimedean valued field for which every one-dimensional subspace satisfies some orthocomplementation property. They are described in terms of polarity for the balls of $E$ (section 1) and in terms of compactoidity for the balls of the dual space $E'$ (section 2). This study yields (Lemma 1.6) the solution of an open problem raised by the second author in [8]. Finally, the stability properties of these spaces are discussed in section 3.

0. PRELIMINARIES

Throughout $K$ is a non-archimedean valued field that is complete with respect to the metric induced by the non-trivial valuation $|\cdot|$. By $|K|$ we will denote the set $\{|\lambda| : \lambda \in K\}$.

For fundamentals on Banach spaces and locally convex spaces over $K$ we refer to [9] and [4] respectively.

Let $E$ be a $K$-vector space. A subset $A$ of $E$ is absolutely convex if it is a module over the ring $\{\lambda \in K : |\lambda| \leq 1\}$. For a set $B$ in $E$ we denote by $[B]$ the $K$-vector space generated by $B$, and by $coB$ the smallest absolutely convex set of $E$ containing $B$. For an absolutely convex set $A$ in $E$ we set $A^c := \cap \{\lambda A : |\lambda| > 1\}$ if the valuation of $K$ is dense, $A^c := A$ otherwise, and $A^* := \cup \{\lambda A : |\lambda| < 1\}$. For a (non-archimedean) seminorm $p$ on $E$, $Ker p$ will be the set $\{x \in E : p(x) = 0\}$. Recall that $p$ is called polar if $p = \sup\{|f| : f \in E^*, |f| \leq p\}$, where $E^*$ is the algebraic dual space of $E$.

Let $E$ be a locally convex space over $K$. For a set $B$ in $E$ we denote by $\overline{[B]}$ the closed linear hull of $B$. An absolutely convex set $A$ in $E$ is called:

a) compactoid: if for each neighbourhood $U$ of 0 there exist $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in E$ such that $A \subseteq U + co \{x_1, \ldots, x_n\}$,

b) $c'$-compact: if in the above we may choose $x_1, \ldots, x_n \in A$.

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c) compactoid of finite type: if for each neighbourhood \( U \) of 0 there exists a bounded finite-dimensional set \( S \subseteq A \) such that \( A \subseteq U + S \).

Clearly \( A \) \( c' \)-compact \( \Rightarrow \) \( A \) compactoid of finite type \( \Rightarrow \) \( A \) compactoid. The converses are not true in general (see [5] and [7]).

Let \( (E, \| \cdot \|) \) be a Banach space over \( K \). For each \( r > 0 \), \( B_E(r^-) \) (resp. \( B_E(r) \)) will mean \( \{ x \in E : \|x\| < r \} \) (resp. \( \{ x \in E : \|x\| \leq r \} \)). By \( E' \) we denote the topological dual space of \( E \). Recall that \( E' \) is again a Banach space endowed with the norm

\[
\|f\| = \sup \left\{ \frac{|f(x)|}{\|x\|} : x \in E, x \neq 0 \right\} \quad (f \in E').
\]

Let \( t \in (0, 1] \) and let \( S, T \) be sets in \( E \). We say that \( S \) is \( t \)-orthogonal to \( T \) (and write \( S \perp t T \)) if for all \( x \in S, \ y \in T \):

\[
\|\lambda x + \mu y\| \geq t \max(\|\lambda x\|, \|\mu y\|) \quad (\lambda, \mu \in K).
\]

If \( I \) is an index set, a family \( \{e_i : i \in I\} \) of elements of \( E \) is called a base of \( E \) if for each \( x \in E \) there is a unique \( (\lambda_i)_{i \in I} \in K^I \) such that \( x = \sum_{i \in I} \lambda_i x_i \). If in addition \( \|\sum_{i \in J} \alpha_i x_i\| \geq t \max_{i \in J} \|\alpha_i x_i\| \) for all \( J \subseteq I, J \) finite and all \( \alpha_i \in K \) \( (i \in J) \), we say that \( \{e_i : i \in I\} \) is a \( t \)-orthogonal base of \( E \). For \( t = 1 \), we write orthogonal instead of \( 1 \)-orthogonal and \( \perp \) instead of \( \perp 1 \).

Now, let \( D \) be a finite-dimensional subspace of \( (E, \| \cdot \|) \). We say that \( D \) has an orthogonal almost complement if there exists a closed subspace \( H \) of finite codimension in \( E \) such that \( D \perp H \). Such an \( H \) is called an orthogonal almost complement of \( D \). (If, in addition, \( D + H = E \) it is customary to drop the word 'almost' in the above.) Also, \( D \) is called almost orthocomplemented if for each \( t \in (0, 1) \) there exists a closed subspace \( H_t \) of finite codimension in \( E \) such that \( D \perp H_t \) and \( D + H_t = E \). Such an \( H_t \) is called a \( t \)-orthogonal complement of \( D \).

Recall that \( (E, \| \cdot \|) \) is called norm-polar if \( \| \cdot \| \) is a polar norm on \( E \) (or equivalently, every one-dimensional subspace of \( E \) is almost orthocomplemented, Theorem 1.2). We also consider in this paper the following related classes of Banach spaces: \( E \) is called Hilbertian (resp. almost Hilbertian) if every one-dimensional subspace of \( E \) is orthocomplemented (resp. has an orthogonal almost complement). One verifies (for the second implication see Proposition 3.5(ii))

\[
E \text{ Hilbertian} \Rightarrow E \text{ almost Hilbertian} \Rightarrow E \text{ norm-polar}.
\]

Also, if \( K \) is spherically complete, then every Banach space over \( K \) is Hilbertian ([9], Lemma 4.35). Hence,

**FROM NOW ON IN THIS PAPER E WILL BE A BANACH SPACE OVER A NON-SPHERICALLY COMPLETE FIELD K.**
1. HILBERTIAN SPACES AND POLAR SETS

For an absolutely convex set $A \subseteq E$ we set (see [4])

$$A^\circ = \{ f \in E' : |f(a)| \leq 1 \text{ for all } a \in A \}$$

$$A^{*\circ} = \{ x \in E : |f(x)| \leq 1 \text{ for all } f \in A^\circ \}$$

$A$ is called a polar set if $A = A^{*\circ}$.

In the same spirit we define (see [10])

$$A^\square = \{ f \in E' : |f(a)| < 1 \text{ for all } a \in A \}$$

$$A^{*\square} = \{ x \in E : |f(x)| < 1 \text{ for all } f \in A^\square \}$$

$A$ is called a pseudopolar set if $A = A^{*\square}$.

One verifies that $A^{*\circ} = (A^\square)^\circ$ ([4], Prop. 4.10.) and that

$$A \subseteq \overline{A}^\sigma \subseteq A^{*\square} \subseteq A^{*\circ}$$

for every absolutely set $A \subseteq E$ (where $\overline{A}^\sigma$ denotes the closure of $A$ with respect to the weak topology $\sigma(E, E')$ on $E$). We construct a set $A$ for which all the inclusions appearing in $(I)$ are strict.

**Example 1.1.** Let $E = K \oplus K_\nu^2 \oplus F$, where $K_\nu^2$ is the two-dimensional Banach space constructed in [9], p. 68, and where $F$ is the vector space $c_0$ endowed with a norm $N$ which is equivalent to the supremum norm on $c_0$, but with the property that if $x, y \in F$ are such that $x$ is $N$-orthogonal to $y$, then $x = y = 0$ (to see that such a norm exists, consider in the spherical completion $\hat{K}$ of $K$, a sequence $a_1, a_2, \ldots$ consisting of $K$-linearly independent elements. By taking on $[a_1, a_2, \ldots]$ the norm induced by the valuation of $\hat{K}$, we obtain an infinite-dimensional Banach space of countable type over $K$, for which there are no non-zero mutually orthogonal elements, see [9], Example 5.E).

Let $A \subseteq E$ be given by

$$A = R \oplus T \oplus S$$

where $R$, $T$ and $S$ are the open unit balls of $K$, $K_\nu^2$ and $F$ respectively. It is easy to see that

$$\overline{A}^\sigma = R \oplus T \oplus S^\circ$$

$$A^{*\square} = R \oplus T^\circ \oplus S^\circ$$

$$A^{*\circ} = R^\circ \oplus T^\circ \oplus S^\circ$$

and so $A \not\subseteq \overline{A}^\sigma \not\subseteq A^{*\square} \not\subseteq A^{*\circ}$.

It is well-known (see [4]) that the norm-polar Banach spaces are precisely those Banach spaces $E$ for which $B_E(1)$ is polar. Norm-polar spaces can also be described in terms of a complementation property or a Hahn-Banach property as follows.
Theorem 1.2. The following are equivalent.

i) $E$ is norm-polar.

ii) For each one-(finite-)dimensional subspace $D$, for each $\varepsilon > 0$, and for each $f \in D'$, there exists an extension $\overline{f} \in E'$ such that $\|\overline{f}\| \leq (1 + \varepsilon)\|f\|$.

iii) Every one-(finite-)dimensional subspace of $E$ is almost orthocomplemented.

iv) $B_E(1)$ is polar.

Proof. With a simple adaption of the proof of Lemma 4.35.iii) of [9] we can derive that if the one-dimensional version of iii) holds then so does the finite-dimensional one. So, it is enough to prove the theorem for the case of one-dimensional subspaces.

The equivalences i) $\iff$ ii) and i) $\iff$ iv) were proved in [1], Theorem 2.1 and [4], Proposition 3.4 respectively.

ii) $\Rightarrow$ iii): Let $D = [x]$ ($x \in E - \{0\}$) be a one-dimensional subspace of $E$, let $t \in (0,1)$. The linear map $[x] \rightarrow K : \lambda x \rightarrow \lambda$ has norm $\|x\|^{-1}$. By ii), there exists an extension $\overline{f} \in E'$ such that $\|\overline{f}\| \leq t^{-1}\|x\|^{-1}$. Then, $\ker \overline{f}$ is a $t$-orthogonal complement of $D$.

iii) $\Rightarrow$ ii): Let $D$ be a one-dimensional subspace of $E$, let $\varepsilon > 0$ and let $f \in D'$. By iii), there exists a continuous linear projection $P : E \rightarrow D$ with $\|P\| \leq 1 + \varepsilon$. Then, $\overline{f} := f \circ P \in E'$ extends $f$ and $\|\overline{f}\| \leq (1 + \varepsilon)\|f\|$.

Remark 1.3. By using Propositions 3.4 and 4.10 of [4], it is not hard to see that iv) of above is also equivalent to each one of the following statements.

v) $B_E(r)$ is polar for each $r > 0$.

vi) $B_E(1)$ is pseudopolar.

vii) $B_E(r)$ is pseudopolar for each $r > 0$.

viii) $B_E(1)$ is weakly closed.

ix) $B_E(r)$ is weakly closed for each $r > 0$.

Now we are going to describe the Banach spaces $E$ for which $B_E(r^-)$ is pseudopolar (weakly closed) for each $r > 0$. In the same vein as Theorem 1.2 we can prove:

Theorem 1.4. The following are equivalent.

i) $E$ is Hilbertian.

ii) For each one-(finite-)dimensional subspace $D$, for each $f \in D'$, there exists an extension $\overline{f} \in E'$ such that $\|\overline{f}\| = \|f\|$.

iii) Every finite-dimensional subspace of $E$ is orthocomplemented.

iv) $B_E(r^-)$ is pseudopolar for each $r > 0$.

Proof. The equivalence i) $\iff$ iii) was proved in [9], Lemma 4.35. Also, i) $\iff$ ii) follows in a similar way as ii) $\iff$ iii) in Theorem 1.2.
ii) \( \Rightarrow \) iv): Let \( r \in (0, \infty) \). Take \( x \in E, \|x\| \geq r \). By ii), there exists an \( \overline{f} \in E' \) such that \( |\overline{f}(x)| = 1 \) and \( \|\overline{f}\| = \|x\|^{-1} \) (and hence \( \overline{f} \in (B_E(r^{-1}))' \)). So, \(-x \notin (B_E(r^{-1}))'\).

iv) \( \Rightarrow \) i): Let \( x \in E - \{0\} \) and let \( r := \|x\| \). By iv), there exists a \( g \in E' \) such that \( \|g\| \leq r^{-1} \) and \( |g(x)| \geq 1 \). Hence, \( |g(x)| = \|g\| \|x\| \), which implies that \( \ker g \) is an orthogonal complement of \( [x] \) in \( E \).

We shall describe almost Hilbertian spaces in a similar way as we did in Theorems 1.2 and 1.4 for norm-polar and Hilbertian spaces respectively. To this end we need the following lemmas.

**Lemma 1.5.** Let \( D_1, D_2, W_1, W_2 \) be subspaces of \( E \). If \( D_1 \perp D_2, W_1 \perp W_2 \) and \( W_2 \subset D_2 \), then \( D_1 + W_2 \perp D_2 \cap W_1 \).

**Proof.** Let \( a \in D_1, b \in W_2, c \in D_2 \cap W_1 \). Then \( b + c \in D_2 \) and so \( \|a + b + c\| \geq \|b + c\| \). By Lemma 3.2 of [9] we obtain that \( \|a + b + c\| \geq \|a\| \). But also \( b \perp c \) so that \( \|b + c\| \geq \max(\|b\|, \|c\|) \). Hence,

\[
\|a + b + c\| \geq \max(\|a\|, \|b\|, \|c\|) \geq \max(\|a + b\|, \|c\|)
\]

and the result follows.

**Lemma 1.6.** (This gives an affirmative answer to the problem raised in [8], §5.) If \( E \) is almost Hilbertian, then every finite-dimensional subspace of \( E \) has an orthogonal almost complement.

**Proof.** It suffices to prove: if \( D \) is a finite-dimensional subspace of \( E \) having an orthogonal almost complement and if \( a \in E - D \), then \( D_1 = D + [a] \) also has an orthogonal almost complement. In fact, there is a closed subspace \( H \) of \( E \) of finite codimension such that \( D \perp H \). Now we distinguish two cases.

a) \( D \) is not orthocomplemented in \( D_1 \). We prove that \( D_1 \perp H \). Let \( x \in D_1, h \in H \). To see that \( x \perp h \) we may assume \( x \notin D \). Then \( x \) is not orthogonal to \( D \) so there is a \( d \in D \) with \( \|x - d\| < \|d\| = \|x\| \). Then \( \|x - h\| = \|x - d + d - h\| = \max(\|x - d\|, \|d - h\|) \geq \|d\| = \|x\| \). It follows easily that \( \|x - h\| \geq \max(\|x\|, \|h\|) \) (Lemma 3.2 of [9]).

b) \( D \) has an orthogonal complement in \( D_1 \). We may assume that \( a \perp D \). Also, we may assume that \( [a] \) and \( D + H \) are not orthogonal (if they are then \( H \perp D_1 \)), so there exists a \( v \in D + H \) for which \( \|a - v\| < \|a\| \). Then \( a \perp D \) implies \( v \perp D \). Write \( v = d + h \) \((d \in D, h \in H)\). Since \( E \) is almost Hilbertian, there is a closed subspace \( S \) of \( E \) of finite codimension such that \([h] \perp S \). By Lemma 1.5, \( D + K_v = D + K_h \perp H \cap S \). We finish the proof by showing that \( D_1 \perp H \cap S \) (observe that \( H \cap S \) has finite codimension in \( E \)). For that, let \( x \in D, \lambda \in K - \{0\}, c \in H \cap S \). Since

\[
\|x + \lambda v + c\| \geq \|\lambda v\| > |\lambda| \|a - v\|,
\]

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we deduce that
\[ \|x + \lambda a + c\| = \|x + \lambda v + c\| \geq \max(\|x\|, \|c\|, \|\lambda v\|) \]
\[ = \max(\|x\|, \|c\|, \|\lambda a\|) \geq \]
\[ \geq \|x + \lambda a\| \]

and we are done.

**Theorem 1.7.** The following are equivalent.

i) \( E \) is almost Hilbertian.

ii) For each one-(finite-)dimensional subspace \( D \) there exists a closed finite-codimensional subspace \( H \) of \( E \) such that \( D \subset H \) and every \( f \in D' \) admits an extension to an element of \( H' \) with the same norm.

iii) Every finite-dimensional subspace of \( E \) has an orthogonal almost complement.

iv) \( B_E(r^-) \) is weakly closed for each \( r > 0 \).

**Proof.** The equivalence i) \( \iff \) iii) follows from Lemma 1.6. Also, i) \( \iff \) ii) can be proved as ii) \( \iff \) iii) in Theorem 1.2.

i) \( \Rightarrow \) iv): Let \( r \in (0, \infty) \). Let \( x \in E \), \( \|x\| \geq r \). By i), there is a closed subspace \( H \) of \( E \) with finite codimension in \( E \) such that \( [x] \perp H \). Let \( \pi : E \to E/H \) be the canonical surjection and let \( q \) be the norm on \( E/H \) associated to \( \| \cdot \| \). Then, \( p = q \circ \pi \) is a weakly continuous seminorm on \( E \) for which \( p(x) = \|x\| \geq r \) and \( p(B_E(r^-)) \subset [0, r) \). So, \( \{ y \in E : p(x) < r \} \) is a weakly open set which contains \( B_E(r^-) \) and which does not contain \( x \). Hence, \( B_E(r^-) \) is weakly closed.

iv) \( \Rightarrow \) i): Let \( x \in E - \{0\} \) and let \( r := \|x\| \). By iv), there is a weak neighbourhood \( U \) of 0 such that \( (x + U) \cap B_E(r^-) = \emptyset \), i.e., \( \|x + u\| \geq \|x\| \) for all \( u \in U \). Now \( U \) contains a closed subspace \( H \) of finite codimension and since \( \|x + h\| \geq \|x\| \) for all \( h \in H \) we have \( [x] \perp H \).

**Remarks 1.8.**

1) In contrast to the statements appearing in Remark 1.3, there are Banach spaces \( E \) for which property iv) of Theorem 1.4 (resp. Theorem 1.7) is not equivalent to \( "B_E(1^-) \) is pseudopolar" (resp. "\( B_E(1^-) \) is weakly closed").

**Example.** Suppose that \( |K| \neq [0, \infty) \) and let \( r > 0 \) such that \( r \not\in |K| \). Let \( N \) be the norm on \( c_0 \) considered in Example 1.1 and let \( E := (c_0, s) \) where

\[ s(x) = rN(x) \quad (x \in E). \]

Clearly,

\[ \{ x \in E : s(x) < 1 \} = \{ x \in E : s(x) \leq 1 \} \]

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(recall that \( N(x) \in \mathbb{K} \) for all \( x \in c_0 \)). Hence, \( B_E(1^-) \) is polar (and hence pseudopolar and weakly closed). But there are not non-trivial mutually \( s \)-orthogonal elements in \( E \). So, \( E \) is not almost Hilbertian (and hence \( E \) is not Hilbertian).

2) Looking at properties iii) of Theorems 1.2, 1.4 and 1.7, it seems natural to consider the class of Banach spaces \( E \) satisfying the following property.

"For each one-(finite-)dimensional subspace \( D \), there exists a closed subspace \( H \) of finite codimension such that \( D \) is almost orthocomplemented in \( H'' \)."

Since every closed subspace of finite codimension of a Banach space \( E \) is almost orthocomplemented in \( E \) ([1], proof of Theorem 4.7.i) \( \Rightarrow \) ii)), we conclude that the above property is nothing but norm-polarity of \( E \).

3) Also, looking at properties iv) of Theorems 1.2, 1.4 and 1.7 one might think of the following property for a Banach space \( E \).

"\( B_E(r^-) \) is polar for each \( r > 0 \)."

But one can easily see that if \( E \) has this property then \( B_E(r^-) = B_E(r) \) for each \( r > 0 \), and so \( E = \{0\} \).

2. HILBERTIAN SPACES AND COMPACTOID SETS

In this section we give several new descriptions of Hilbertian and almost Hilbertian spaces in terms of compactoidity properties of the balls in the dual space.

The following lemma will be crucial for our purpose.

**Lemma 2.1.** Let \( D \) be a finite-dimensional subspace of \( E \). Let \( n \in \mathbb{N} \) \((n \geq 1)\). Consider the following statements.

i) There exists a closed subspace \( H \) of \( E \) with \( \dim E/H = n \) such that \( D \perp H \).

ii) For each \( r > 0 \) there exists \( S_r \subset B_{E'}(r^-) \) with \( \dim[S_r] = n \) such that \( B_{E'}(r^-) \subset D^0 + S_r \).

iii) There exists \( S \subset B_{E'}(1^-) \) with \( \dim[S] = n \) such that \( B_{E'}(1^-) \subset D^0 + S \).

Then we have i) \( \Rightarrow \) ii) \( \Rightarrow \) iii).

If in addition \( E \) is norm-polar, then i) - iii) are equivalent.

**Proof.** i) \( \Rightarrow \) ii): The formula \( q(x) = \text{dist}(x, H) = \inf \{ ||x - h|| : h \in H \} \) defines a continuous seminorm on \( E \) with \( \dim(E/\text{Ker}q) = n \) such that \( q \leq || \cdot || \) and \( q = || \cdot || \) on \( D \).

Now, let \( r > 0 \) be given. Let

\[
T_r = \{ f \in E' : |f| \leq rq \}.
\]
We see that $\dim[T_r] = n$. Also, since $q \leq \| \cdot \|$ we have that $T_r \subset B_{E'}(r)$. We now shall prove that

$$B_{E'}(r^-) \subset D^0 + S_r$$

where $S_r = (T_r)'$.

In fact, let $f \in B_{E'}(r^-)$. Then there is a $\lambda \in K$, $0 < |\lambda| < 1$ with $|f| \leq |\lambda|r \cdot \| \cdot \|$. Choose $\lambda' \in K$ with $|\lambda| < |\lambda'| < 1$. Since $q = \| \cdot \|$ on $D$ we have $|f| \leq |\lambda'|rq$ on $D$, so we can extend $f$ to a $g \in E'$ with $|g| \leq |\lambda'|rq$. (This is because $\dim(E/\text{Ker}q) < \infty$ so $q$ is polar.) Now write

$$f = (f - g) + g.$$

Since $f = g$ on $D$ we have $f - g \in D^0$. Also, $|g| \leq |\lambda'|rq$ and so $(\lambda')^{-1}g \in T_r$, which implies that $g \in (T_r)'$.

Clearly ii) $\Rightarrow$ iii).

Now, assume that $E$ is norm-polar.

iii) $\Rightarrow$ i): There exists $S \subset B_{E'}(1^-)$ with $\dim[S] = n$ such that

$$B_{E'}(1^-) = (D^0 \cap B_{E'}(1^-)) + S.$$

Set $q(x) = \sup_{h \in S} |h(x)|$ $(x \in E)$. Then $q$ is a continuous seminorm on $E$ for which $\dim(E/\text{Ker}q) = n$. Also, since $E$ is norm-polar, for each $x \in E$ we have

$$q(x) \leq \sup_{\|h\| < 1} |h(x)| = \sup_{\|h\| \leq 1} |h(x)| = \|x\|$$

and so $q \leq \| \cdot \|$. Further, if $x \in D$ then

$$\|x\| = \sup_{\|f\| < 1} |f(x)| = \sup_{\|h\| < 1} \{|h(x) + t(x)| : h \in D^0 \cap B_{E'}(1^-), t \in S\} = \sup_{t \in S} |t(x)| = q(x)$$

and hence $q = \| \cdot \|$ on $D$.

Finally, for $d \in D$ and $y \in \text{Ker}q$ we have:

$$\|d + y\| \geq q(d + y) = q(d) = \|d\|$$

which means that $D \perp \text{Ker}q$. We conclude that $H = \text{Ker}q$ satisfies the conditions required in i).

**Remark 2.2.** Properties i) - iii) of Lemma 2.1 are not equivalent in general.

**Example.** Take $E = (\ell^\infty, \nu)$ where

$$\nu(x) = \max(\|x\|_\infty, 2\text{dist}(x, c_0)) \quad (x \in \ell^\infty).$$
Set \( n = 1 \) and \( D = [e] \) where \( e = (1, 1, 1, \ldots) \). Since \( D \) is orthocomplemented in \( (\ell^\infty, \| \cdot \|_{\infty}) \) and the identity map \( E' \to (\ell^\infty, \| \cdot \|_{\infty})' \) is an isometry ([9], Ex.4) we deduce that iii) is true for \( E \).

Now, we prove that i) is not true for \( E \). If it were true then \( D \) is orthocomplemented in \( E \) and so there would be an \( f \in E' - \{0\} \) such that \( |f(e)| = \|f\|\|e\| \). Set \( (a_1, a_2, \ldots) \in c_0 \) such that \( f(x) = \sum_{n=1}^{\infty} a_n x_n \) for all \( x = (x_1, x_2, \ldots) \in E \). Then, \( |\sum_{n=1}^{\infty} a_n| = 2\max_{n=1}^{\infty} |a_n| \), which is a contradiction.

Next, we shall apply Lemma 2.1 to characterize Hilbertian spaces by means the following variant of compactoidity:

**Definition 2.3.** Let \( F \) be a locally convex space over \( K \) and let \( A \) be an absolutely convex subset of \( F \). \( A \) is called nearly \( c' \)-compact if for every zero-neighbourhood \( U \) in \( F \) there exist \( n \in \mathbb{N} \) and bounded sets \( S_1, \ldots, S_n \) contained in \( A \) with \( \dim[X_i] = 1 \) for all \( i = 1, \ldots, n \) and such that \( A \subset U + S_1 + \ldots + S_n \).

Observe that

\[
A \text{ nearly } c' \text{-compact} \Rightarrow A \text{ nearly } c' \text{-compact} \Rightarrow \Rightarrow A \text{ compactoid of finite type.}
\]

But the converses are not true in general (see Theorems 2.5 and 2.6).

**Lemma 2.4.** Let \( F \) and \( A \) be as in Definition 2.3. Suppose that the topology of \( F \) is generated by a family \( \mathcal{P} \) of non-archimedean seminorms such that \( \dim(E/\ker p) < \infty \) for all \( p \in \mathcal{P} \). Then the following properties are equivalent.

i) \( A \) is nearly \( c' \)-compact (resp. \( A \) is of finite type).

ii) For every closed subspace \( H \) of finite codimension there are bounded sets \( S_1, \ldots, S_n \) contained in \( A \) with \( \dim[S_i] = 1 \) for all \( i = 1, \ldots, n \) (resp. there exists a finite-dimensional bounded set \( S \subset A \)) such that \( A \subset H + S_1 + \ldots + S_n \) (resp. \( A \subset H + S \)).

**Proof.** We prove the result for nearly \( c' \)-compact sets. For the case of sets of finite type the proof is similar.

i) \( \Rightarrow \) ii) (Observe that this implication holds for any locally convex space \( F \)): We may assume that \([A] = F\).

\( H \) has the form \( H = D^0 \) where \( D \) is a finite-dimensional subspace of \( F' \). Let \( f_1, \ldots, f_m \) be a base of \( D \). There exist \( x_1, \ldots, x_m \in F \) with \( f_i(x_j) = \delta_{ij} \) \( (i, j = 1, \ldots, m) \). Since \([A] = F\), there exists a \( \lambda \in K \) \( \lambda \neq 0 \) such that \( \lambda x_i \in A \) for each \( i \in \{1, \ldots, m\} \). Set

\[
U = \bigcap_{i=1}^{m} \{x \in F : |f_i(x)| \leq |\lambda|\}.
\]

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Then \( U \) is a zero-neighbourhood in \( F \). Since \( A \) is nearly \( c' \)-compact there are bounded subsets \( S_1, \ldots, S_r \) of \( A \) with \( \dim[\mathcal{S}_h] = 1 \) for all \( h = 1, \ldots, r \) such that \( A \subset U + S_1 + \ldots + S_r \). Let \( x \in U \). Write \( x = y + z \) where

\[
y = x - \sum_{i=1}^{m} f_i(x)x_i \]

\[
z = \sum_{i=1}^{m} f_i(x)x_i.
\]

Now, since \( x \in U \), we have that \( z \in T_1 + \ldots + T_m \) where \( T_i = \text{co} \{\lambda x_i\} \) (\( i = 1, \ldots, m \)) (observe that \( T_i \subset A \) for all \( i \)). Also, for \( j \in \{1, \ldots, m\} \), \( f_j(y) = 0 \) and so \( y \in D^0 \). Hence, we conclude that

\[
A \subset H + T_1 + \ldots + T_m + S_1 + \ldots + S_r
\]

and we are done.

ii) \( \Rightarrow \) i): Observe that every zero-neighbourhood in \( E \) contains a closed subspace of finite codimension.

Putting Lemmas 2.1 and 2.4 together we can now prove:

**Theorem 2.5.** The following statements i), ii) and iii) are equivalent.

- i) \( E \) is Hilbertian.
- ii) \( E \) is norm-polar and \( B_{E'}(1) \) is nearly \( c' \)-compact in \( \sigma(E', E) \).
- iii) \( E \) is norm-polar and \( B_{E'}(1^-) \) is nearly \( c' \)-compact in \( \sigma(E', E) \).

Also, the following statements i') and ii') are equivalent.

- i') \( E \) is Hilbertian and \( \|x\| \in |K| \) for all \( x \in E \).
- ii') \( E \) is norm-polar and \( B_{E'}(1) \) is \( c' \)-compact in \( \sigma(E', E) \).

**Proof.**

i) \( \Rightarrow \) ii): Clearly \( E \) is norm-polar (see Proposition 3.5).

Now, let \( H \) be a \( \sigma(E', E) \)-closed subspace of \( E' \) with \( \dim(E'/H) = n < \infty \) \( (n \in \mathbb{N}) \).

There are \( x_1, \ldots, x_n \in E \) such that

\[
H = \{f \in E' : f(x_i) = 0 \text{ for all } i = 1, \ldots, n\}.
\]

For each \( m = 1, \ldots, n \), set

\[
H_m = \{f \in E' : f(x_i) = 0 \text{ for all } i = 1, \ldots, m\}.
\]

Since \( E \) is Hilbertian, it follows from Lemma 2.1 that there exist a \( g \in B_{E'}(1^-) \) and an \( s > 0 \) such that

\[
B_{E'}(1^-) \subset H_1 + S_1
\]
where $S_1 = B_K(s) \cdot g$. Clearly $g \not\in H_1$ and so $H_1 \cap S_1 = \{0\}$, which implies that
\[
B_{E'}(1) = (B_{E'}(1^-))^c \subset H_1 + (S_1)^c.
\]
Hence, $T_1 = (S_1)^c$ is an absolutely convex subset of $B_{E'}(1)$ with $\dim[T_1] = 1$ such that
\[
B_{E'}(1) \subset H_1 + T_1 \quad (II)
\]
or, equivalently,
\[
B_{E'}(1) = B_{H_1}(1) + T_1 \quad (III)
\]
On the other hand, if $M_1$ is an orthogonal complement of $[x_1]$, we have that $H_1$ is isometrically isomorphic to $M_1'$ via the isomorphism $f \in H_1 \rightarrow f|_{M_1} \in M_1'$. This isometry maps $H_2$ onto a $\sigma(M_1', M_1)$-closed subspace of $M_1'$. Since $M_1$ is again Hilbertian, we can apply (II) to $M_1$ instead of $E$ to find a set $T_2 \subset B_{H_1}(1)$ with $\dim[T_2] = 1$ such that
\[
B_{H_1}(1) \subset H_2 + T_2
\]
and by (III) it follows that
\[
B_{E'}(1) \subset H_2 + T_1 + T_2.
\]
Inductively, we can prove that there exist subsets $T_1, \ldots, T_n$ of $B_{E'}(1)$ with $\dim[T_i] = 1$ for all $i = 1, \ldots, n$ such that $B_{E'}(1) \subset H + T_1 + \ldots + T_n$. Now, the nearly $c'$-compactness of $B_{E'}(1)$ follows from Lemma 2.4.

ii) $\Rightarrow$ iii): One can easily prove that if $A$ is a nearly $c'$-compact subset of a locally convex space, then $A^i$ is also nearly $c'$-compact.

iii) $\Rightarrow$ i): Let $D$ be a one-dimensional subspace of $E$. By iii) and Lemma 2.4 there are absolutely convex sets $S_1, \ldots, S_n$ in $B_{E'}(1^-)$ with $\dim[S_i] = 1$ for all $i = 1, \ldots, n$ such that $B_{E'}(1^-) \subset D^0 + S_1 + \ldots + S_n$. Also, since $\dim(E'/D^0) = 1$ there exists $m \in \{1, \ldots, n\}$ such that $\pi(S_1) + \ldots + \pi(S_n) = \pi(S_m)$ (where $\pi : E' \rightarrow E'/D^0$ is the canonical surjection) and so $D^0 + S_1 + \ldots + S_n = D^0 + S_m$. Hence, $B_{E'}(1^-) \subset D^0 + S_m$ which implies that $D$ is orthocomplemented (Lemma 2.1).

i') $\Rightarrow$ ii'): By [6], Theorem 3.2 it suffices to prove that $\max\{|f(x)| : \|f\| \leq 1\}$ exists for each $x \in E$. Since $\|x\| \in [K]$ we may assume that $\|x\| = 1$. For such $x$ we must prove
\[
\max\{|f(x)| : \|f\| \leq 1\} = 1.
\]
Since $E$ is Hilbertian, $[x]$ has an orthogonal complement $H$. For the function $f : \lambda x + h \rightarrow \lambda \ (\lambda \in K, h \in H)$ we have $|f(x)| = 1$. Also, for $\lambda \in K, h \in H$,
\[
|f(\lambda x + h)| = |\lambda| = \|\lambda x\| \leq \max(\|\lambda x\|, \|h\|) = \|\lambda x + h\|
\]
so that \( \|f\| \leq 1 \).

ii') \( \Rightarrow \) i'): It is straightforward to verify that if \( A \) is a \( c' \)-compact set of a locally convex space, then \( A^i \) is nearly \( c' \)-compact. Hence, ii') implies iii), which is i). Further, by norm-polarity and \( c' \)-compactness, for each \( x \in E \) we have \( \|x\| = \sup\{|f(x)| : f \in B_{E'}(1)\} = \max\{|f(x)| : f \in B_{E'}(1)\} \in |K| \).

Also, as a direct consequence of Lemmas 2.1 and 2.4 we derive:

**Theorem 2.6.** The following are equivalent.

i) \( E \) is almost Hilbertian.

ii) \( E \) is norm-polar and \( B_{E'}(1^-) \) is of finite type in \( \sigma(E', E) \).

**Remarks 2.7.**

1) Observe that with the same proof as in Theorem 2.5 we can see that \( E \) is Hilbertian if and only if \( E \) satisfies one of the two following equivalent conditions:

ii') \( E \) is norm-polar and \( B_{E'}(r) \) is nearly \( c' \)-compact in \( \sigma(E', E) \) for each \( r > 0 \).

iii') \( E \) is norm-polar and \( B_{E'}(r^-) \) is nearly \( c' \)-compact in \( \sigma(E', E) \) for each \( r > 0 \).

Analogously, one verifies that \( E \) is almost Hilbertian if and only if \( E \) satisfies

ii') \( E \) is norm-polar and \( B_{E'}(r^-) \) is of finite type in \( \sigma(E', E) \) for each \( r > 0 \).

2) However, property ii') of Theorem 2.5 does not imply in general that \( B_{E'}(r) \) is \( c' \)-compact in \( \sigma(E', E) \) for each \( r > 0 \).

Indeed, observe that if \( E \) has property ii') of Theorem 2.5, then \( B_{E'}(r) \) is \( c' \)-compact if and only if \( r \in |K| \) (recall that for each \( x \in E \), there is \( f \in B_{E'}(r) \) such that \( |f(x)| = \sup\|g\| \leq r |g(x)| = r \|x\| \)).

3) One could think of considering also the property "\( B_{E'}(1^-) \) is \( c' \)-compact in \( \sigma(E', E) \)." But this possibility is not interesting at all because, if \( E \neq \{0\} \), \( B_{E'}(1^-) = B_{E'}(1)^i \) is never \( c' \)-compact in \( \sigma(E', E) \).

4) The last part of Remark 2.7.1 yields the following natural question.

**Problem.** Let \( E \) be an almost Hilbertian space. Does it imply that \( B_{E'}(r) \) is of finite type for each \( r > 0 \)? (Observe that if \( E \) is norm-polar the converse is true. Indeed, if \( B_{E'}(r) \) is of finite type then \( B_{E'}(r^-) = B_{E'}(r)^i \) is of finite type. Now, apply Remark 2.7.1).

### 3. Stability Properties and Examples

Following [9], if \( I \) is an index set and \( \{E_i\}_{i \in I} \) is a family of Banach spaces over \( K \), by \( \times_{i \in I} E \) we will denote the set of all elements \( a \) of the cartesian product \( \prod_{i \in I} E_i \), for which the set \( \{\|a_i\| : i \in I\} \) is bounded. This \( \times_{i \in I} E_i \) is a Banach space endowed with
the norm \( \|a\| = \sup_{i \in I} \|a_i\| \). The elements \( a \) of \( \prod_{i \in I} E_i \) for which, for each \( \varepsilon > 0 \), the set \( \{ i \in I : \|a_i\| \geq \varepsilon \} \) is finite form a closed subspace of \( \times_{i \in I} E_i \), denoted by \( \oplus_{i \in I} E_i \).

Then, we have

**Theorem 3.1.**

i) A subspace of a norm-polar (resp. a Hilbertian, an almost Hilbertian) subspace is a norm-polar (resp. a Hilbertian, an almost Hilbertian) space.

ii) If \( \{ E_i \}_{i \in I} \) is a family of norm-polar (resp. Hilbertian, almost Hilbertian) spaces, then \( \oplus_{i \in I} E_i \) is again a norm-polar (resp. a Hilbertian, an almost Hilbertian) space.

iii) If \( \{ E_i \}_{i \in I} \) is a family of norm-polar spaces, then \( \times_{i \in I} E_i \) is again a norm-polar space. If in addition \( I \) is finite and every \( E_i \) \((i \in I)\) is a Hilbertian (resp. an almost Hilbertian) space, then \( \prod_{i \in I} E_i \) is again a Hilbertian (resp. an almost Hilbertian) space.

iv) If \( E \) is a norm-polar (resp. a Hilbertian, an almost Hilbertian) space and \( D \) is a finite-dimensional subspace of \( E \), then \( E/D \) is again a norm-polar (resp. a Hilbertian, an almost Hilbertian) space.

**Proof.** We prove i), ii) and iv) for norm-polar spaces. Similar proofs work for Hilbertian and almost Hilbertian spaces.

i) Let \( E \) be a norm-polar space and let \( M \) be a subspace. For each \( x \in M - \{0\} \), \( [x] \) is almost orthocomplemented in \( E \) and hence in \( M \). By Theorem 1.2, \( M \) is norm-polar.

ii) Let \( x = (x_i)_{i \in I} \in \oplus_{i \in I} E_i \), \( x \neq 0 \) and let \( t \in (0,1) \) be given. There is a \( j \in I \) such that \( \|x_j\| = \|x\| \). Also, since \( E_j \) is norm-polar, \( [x_j] \) has a \( t \)-orthogonal complement \( S_j \) in \( E_j \). Take \( S = \oplus_{i \in I} S_i \) where \( S_i = E_i \) if \( i \neq j \). Then, for each \( s = (s_i)_{i \in I} \in S \),

\[
\|x + s\| = \max_i \|x_i + s_i\| \geq \|x_j + s_j\| \geq t\|x_j\| = t\|x\|
\]

and so, \( S \) is a \( t \)-orthogonal complement of \( [x] \) in \( \oplus_{i \in I} E_i \). Now, apply Theorem 1.2.

iii) Let \( G = \times_{i \in I} E_i \) and let \( x = (x_i)_{i \in I} \in G \), \( x \neq 0 \) and let \( \varepsilon > 0 \) be given. We have to show that there exists an \( f \in G' \) with \( \|f\| \leq 1 \) such that \( \|x\| - \varepsilon \leq \|f(x)\| \). For that, let \( j \in I \) be such that \( \|x_j\| - \varepsilon/2 \leq \|x_j\| \). Since \( E_j \) is norm-polar, there is \( f_j \in E_j' \) with \( \|f_j\| \leq 1 \) such that \( \|x_j\| - \varepsilon/2 \leq |f_j(x_j)| \). Then \( f : G \to K \), \( y = (y_i)_{i \in I} \to f_j(y_j) \) satisfies the required conditions.

Now, assume that \( I \) is finite. Then, the conclusion follows directly from ii).

iv) Let \( x \in E/D \), \( x \neq 0 \) and let \( t \in (0,1) \) be given. There is \( y \in E \) such that \( \pi(y) = x \) (where \( \pi : E \to E/D \) is the canonical surjection). Since \( E \) is norm-polar, \( D + [y] \) has a \( t \)-orthogonal complement \( H \) in \( E \) (Theorem 1.2). Then, for each \( h \in H \),

\[
\|\pi(h) - x\| = \inf_{d \in D} \|h - y - d\| \geq t \inf_{d \in D} \|y - d\| = t\|\pi(y)\|
\]

and so \( \pi(H) \) is a \( t \)-orthogonal complement of \( Kx \). Now, the norm-polarity of \( E/D \) again follows by Theorem 1.2.
Remarks 3.2.

1) The product $\times_{i\in I} E_i$ of a family of Hilbertian (almost Hilbertian) spaces is not always a Hilbertian (almost Hilbertian) space.

Example: Clearly $K$ is a Hilbertian (and hence almost Hilbertian) space. However, $\ell_\infty = \times_{n\in \mathbb{N}} K$ is not almost Hilbertian (its 'open' unit ball is not weakly closed, see [3]).

2) The class of norm-polar (resp. Hilbertian, almost Hilbertian) spaces is not closed for forming of quotients.

Indeed, for every Banach space $E$ one can make a quotient map $c_0(I) \to E$ if $I$ has adequate cardinal. Now, the conclusion follows by [9], Lemma 4.35(ii).

It is well-known that for norm-polar Banach spaces $E$ and $F$, their tensor product $E \hat{\otimes} F$ is also a norm-polar Banach space ([9], Corollary 4.34).

To study the stability of the Hilbertian and almost Hilbertian property under the forming of tensor products we need the following preliminary result.

Lemma 3.3. Let $E, F$ be Banach spaces over $K$, let $D, S$ be closed subspaces of $E$ and let $G, T$ be closed subspaces of $F$. Suppose that $D \perp S$ and $G \perp T$. Then

$$D \hat{\otimes} F + E \hat{\otimes} G \perp S \hat{\otimes} T.$$ 

Proof. By Lemma 4.30.ii) of [9] we may assume that $E, F$ are of countable type. Let $x \in D \hat{\otimes} F + E \hat{\otimes} G$, $y \in S \hat{\otimes} T$ and $t \in (0, 1)$. We shall prove that $\|x - y\| \geq t\|y\|$.

$E$ has a $t$-orthogonal base $\{e_i : i \in \Lambda_E\}$ where $\Lambda_E \subset \mathbb{N}$, such that $\{E_i : i \in \Lambda_D\}$ is a base for $D$ for some $\Lambda_D \subset \Lambda_E$, and such that $\{e_i : i \in \Lambda_S\}$ is a base for $S$ for some $\Lambda_S \subset \Lambda_E$, where $\Lambda_S \cap \Lambda_D = \emptyset$ ([9], Theorem 3.16).

Similarly $F$ has a $t$-orthogonal base $\{f_i : i \in \Lambda_F\}$ where $\Lambda_F \subset \mathbb{N}$, such that $\{f_i : i \in \Lambda_G\}$ is a base for $G$ for some $\Lambda_G \subset \Lambda_F$, and such that $\{f_i : i \in \Lambda_T\}$ is a base for $T$ for some $\Lambda_T \subset \Lambda_F$, where $\Lambda_T \cap \Lambda_G = \emptyset$.

Then, $\{e_i \otimes e_j : (i, j) \in \Lambda_E \times \Lambda_F\}$ is a $t$-orthogonal base for $E \hat{\otimes} F$. Also, we can expand the elements $x$ and $y$ as follows.

$$x = \sum_{i \in \Lambda_D} \lambda_{ij} e_i \otimes e_j + \sum_{i \in \Lambda_E} \mu_{ij} e_i \otimes e_j$$

$$y = \sum_{i \in \Lambda_S} \zeta_{ij} e_i \otimes e_j$$

($\lambda_{ij}, \mu_{ij}, \zeta_{ij} \in K$ for all $i, j$). Since

$$(\Lambda_S \times \Lambda_T) \cap ((\Lambda_D \times \Lambda_E) \cup (\Lambda_E \times \Lambda_G)) = \emptyset,$$
we have
\[ \|x - y\| \geq t \max_{i \in A} \max_{j \in A_T} \|\zeta_{ij} e_i \otimes c_j\| \geq t\|y\|. \]

**Theorem 3.4.** Let \( E, F \) be non-zero Banach spaces over \( K \). Then, \( E \hat{\otimes} F \) is a Hilbertian (resp. an almost Hilbertian space) if and only if \( E \) and \( F \) are Hilbertian (resp. almost Hilbertian) spaces.

**Proof.**

We prove the result for Hilbertian spaces. For the case of almost Hilbertian spaces the proof is similar. First, suppose that \( E, F \) are Hilbertian spaces.

Let \( x \in E \hat{\otimes} F, x \neq 0 \). We can write
\[ x = \sum_{n=1}^{\infty} e_n \otimes f_n \]
where \( e_n \in E \) and \( f_n \in F \) for all \( n \in \mathbb{N} \) ([9], Lemma 4.30). There is an \( m \in \mathbb{N} \) such that
\[ \|x - \sum_{n=1}^{m} e_n \otimes f_n\| < \|x\|. \quad (IV) \]

Set \( \{e_1, \ldots, e_N\} \) and \( T = \{f_1, \ldots, f_N\} \). Since \( S, T \) are Hilbertian spaces (Theorem 3.1) we have that \( S, T \) have orthogonal bases (Proposition 3.5.iv) and so \( S \hat{\otimes} T \) has also an orthogonal base ([9], Exercise 4.R.i). Hence, \( \sum_{n=1}^{m} e_n \otimes f_n \) is orthocomplemented in \( S \hat{\otimes} T \). By (IV) it is enough to prove that \( S \hat{\otimes} T \) is orthocomplemented in \( E \hat{\otimes} F \). To see that, let \( D \) be an orthogonal complement of \( S \) in \( E \) and let \( G \) be an orthogonal complement of \( T \) in \( F \). Then, by the previous lemma \( D \hat{\otimes} F + E \hat{\otimes} G \) is an orthogonal complement of \( S \hat{\otimes} T \).

Now, suppose that \( E \hat{\otimes} F \) is a Hilbertian space. Since \( E \) and \( F \) can be isometrically identified with subspaces of \( E \hat{\otimes} F \), we conclude that \( E \) and \( F \) are Hilbertian spaces (Theorem 3.1.i)).

In [4] and [9] we can find several examples of spaces which are (and which are not) norm-polar. The next result gives us some examples of Hilbertian and almost Hilbertian spaces and their relation to norm-polar spaces.

**Proposition 3.5.**

i) Every Hilbertian space is almost Hilbertian.

ii) Every almost Hilbertian space is a norm-polar space.

iii) Every Banach space with an orthogonal base is a Hilbertian space.
iv) If $E$ is a Banach space of countable type, then $E$ is Hilbertian if and only if $E$ has an orthogonal base.

v) Every finite-dimensional space is an almost Hilbertian space.

vi) If $\{D_i\}_{i \in I}$ is a family of finite-dimensional spaces then $\oplus_{i \in I} D_i$ is an almost Hilbertian space.

vii) If $E$ is an infinite-dimensional Banach space for which there are no non-zero orthogonal elements, then $E$ is not an almost Hilbertian space.

viii) There are almost Hilbertian spaces which are not Hilbertian.

ix) There are norm-polar spaces which are not almost Hilbertian.

Proof.

i), v) and vii) are obvious.

ii) Let $D$ be a one-dimensional subspace of $E$ and let $f \in D'$ with $|f| \leq \| \cdot \|$ on $D$. Since $E$ is almost Hilbertian, there exists a closed subspace $M$ of $E$ of finite codimension such that $D$ is orthocomplemented in $M$. Also, since $M$ is almost orthocomplemented (([1], proof of Theorem 4.7.i) ⇒ ii)), we conclude that for every $\varepsilon > 0$ there exists $\overline{f} \in E'$ extending $f$ with $|\overline{f}| \leq (1 + \varepsilon)\| \cdot \|$. Hence, $E$ is norm-polar.

For iii) and iv) see [9].

vi) It is a direct consequence of v) and Theorem 3.1.

viii) If $E$ is a Hilbertian space (e.g. $c_0$) and $K_2^2$ is the two-dimensional space appearing in Example 1.1, it follows by the above properties and by Theorem 3.1 that $E \oplus K_2^2$ is an almost Hilbertian space which is not Hilbertian.

ix) Let $F = (c_0, N)$ be the Banach space of countable type considered in Example 1.1. By Theorem 3.16 of [9] $F$ is norm-polar. By vii), $F$ is not almost Hilbertian.

Remark 3.6. From considering the properties iii) and iv) of Proposition 3.5, the following question arises in a natural way.

Problem. Does every Hilbertian space have an orthogonal base?

Now, we are going to apply the above results to give some examples of norm-polar (resp. Hilbertian, almost Hilbertian) spaces consisting of some spaces of vector-valued continuous functions.

For a Hausdorff zerodimensional topological space $X \neq \emptyset$ and a Banach space $E$ we define

$C_b(X, E)$: The space of all bounded continuous functions $X \to E$, endowed with the supremum norm,

$PC(X, E)$ (resp. $P(X, E)$): The space of all continuous functions $f : X \to E$ for which $f(X)$ is precompact (resp. compactoid), endowed with the supremum norm.
When \( E = K \) we will write \( C_b(X), PC(X) \) and \( P(X) \) instead of \( C_b(X, K), PC(X, K) \) and \( P(X, K) \). Observe that \( C_b(X) = P(X) \).

It is straightforward to verify that \( C_b(X, E) \) (resp. \( PC(X, E), P(X, E) \)) is a norm-polar space if and only if \( E \) is polar.

Also, as in exercise 4.1 of [9] and Theorem 1.3 of [2] one can easily prove that \( PC(X) \otimes E \) (resp. \( P(X \otimes E) \)) is isometrically isomorphic to \( PC(X, E) \) (resp. \( P(X, E) \)).

On the other hand, since \( PC(X) \) has an orthogonal base ([9], Corollary 5.23) we have that \( PC(X) \) is a Hilbertian (and hence almost Hilbertian) space (Proposition 3.5.iii)). So, as a direct consequence of Theorem 3.4 we conclude:

**Proposition 3.7.** The following are equivalent.

i) \( PC(X, E) \) is a Hilbertian (resp. an almost Hilbertian) space.

ii) \( E \) is a Hilbertian (resp. an almost Hilbertian) space.

The picture changes when we consider \( C_b(X, E) \) and \( P(X, E) \):

**Proposition 3.8.** The following are equivalent.

i) \( C_b(X, E) \) is a Hilbertian (resp. an almost Hilbertian) space.

ii) \( P(X, E) \) is a Hilbertian (resp. an almost Hilbertian) space.

iii) \( X \) is pseudocompact and \( E \) is a Hilbertian (resp. an almost Hilbertian) space.

**Proof.**

i) \( \Rightarrow \) ii) It follows from Theorem 3.1.i).

ii) \( \Rightarrow \) iii) If \( X \) is not pseudocompact we can find a countable infinite clopen partition \( X = \bigcup_n X_n \). Choose \( e \in E \setminus \{0\} \) and define \( T: \ell^\infty \to P(X, E) \) by the formula

\[
T(\alpha_1, \alpha_2, \ldots)(x) = \alpha_n e \text{ if } n \in \mathbb{N}, \ x \in X_n.
\]

We see that \( \|T\alpha\| = \|\alpha\|_\infty \|e\| \) for \( \alpha = (\alpha_1, \alpha_2, \ldots) \in \ell^\infty \) so \( \ell^\infty \) is a Hilbertian (resp. almost Hilbertian) space, which is a contradiction (see [3]).

iii) \( \Rightarrow \) i) One verifies that, if \( X \) is pseudocompact, then \( C_b(X, E) = PC(X, E) \). Now apply Proposition 3.7.

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