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THE WEIERSTRASS-STONE APPROXIMATION THEOREM
FOR p-ADIC $C^n$-FUNCTIONS

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Abstract.
Let $K$ be a non-Archimedean valued field. Then, on compact subsets of $K$, every $K$-valued $C^n$-function can be approximated in the $C^n$-topology by polynomial functions (Theorem 1.4). This result is extended to a Weierstrass-Stone type theorem (Theorem 2.10).

INTRODUCTION

The non-archimedean version of the classical Weierstrass Approximation Theorem - the case $n = 0$ of the Abstract - is well known and named after Kaplansky ([1], 5.28). To investigate the case $n = 1$ first let us return to the Archimedean case and consider a real-valued $C^1$-function $f$ on the unit interval. To find a polynomial function $P$ such that both $|f - P|$ and $|f' - P'|$ are smaller or equal than a prescribed $\varepsilon > 0$ one simply can apply the standard Weierstrass Theorem to $f'$ obtaining a polynomial function $Q$ for which $|f' - Q| \leq \varepsilon$. Then $x \mapsto P(x) := f(0) + \int_0^x Q(t)dt$ solves the problem.

Now let $f : X \to K$ be a $C^1$-function where $K$ is a non-archimedean valued field and $X \subseteq K$ is compact. Lacking an indefinite integral the above method no longer works. There do exist continuous linear antiderivations ([3], §64) but they do not map polynomials into polynomials ([3], Ex. 30.C). A further complicating factor is that the natural norm for $C^1$-functions on $X$ is given by

$$f \mapsto \max\{|f(x)| : x \in X\} \lor \max\left\{\left|\frac{f(x) - f(y)}{x - y}\right| : x, y \in X, x \neq y\right\}$$

rather than the more classical formula

$$f \mapsto \max\{|f(x)| : x \in X\} \lor \max\{|f'(x)| : x \in X\}.$$ 

(Observe that in the real case both formulas lead to the same norm thanks to the Mean Value Theorem, see [3], §§26,27 for further discussions.)
Thus, to obtain non-archimedean $C^n$-Weierstrass-Stone Theorems for $n \in \{1, 2, \ldots\}$ our methods will necessarily deviate from the 'classical' ones.

0. PRELIMINARIES

1. Throughout $K$ is a non-archimedean complete valued field whose valuation $|\cdot|$ is not trivial. For $a \in K$, $r > 0$ we write $B(a, r) := \{x \in K : |x-a| \leq r\}$, the 'closed' ball about $a$ with radius $r$. 'Clopen' is an abbreviation for 'closed and open'. The function $x \mapsto x$ ($x \in K$) is denoted $\xi$. The $K$-valued characteristic function of a subset $Y$ of $K$ is written $\chi_Y$. For a set $Z$, a function $f : Z \to K$ and a set $W \subset Z$ we define $\|f\|_W := \sup\{|f(z)| : z \in W\}$ (allowing the value $\infty$). The cardinality of a set $\Gamma$ is $\#\Gamma$. $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$, $\mathbb{N} := \{1, 2, 3, \ldots\}$.

We now recall some facts from [2], [3] on $C^n$-theory.

2. For a set $Y \subset K$, $n \in \mathbb{N}$ we set $\nabla^n Y := \{(y_1, y_2, \ldots, y_n) \in Y^n : i \neq j \Rightarrow y_i \neq y_j\}$. For $f : Y \to K$, $n \in \mathbb{N}_0$ we define its $n$th difference quotient $\Phi_n f : \nabla^{n+1} Y \to K$ inductively by $\Phi_0 f := f$ and the formula

$$\Phi_n f(y_1, \ldots, y_{n+1}) = \frac{\Phi_{n-1} f(y_1, y_3, \ldots, y_{n+1}) - \Phi_{n-1} f(y_2, y_3, \ldots, y_{n+1})}{y_1 - y_2}$$

$f$ is called a $C^n$-function if $\Phi_n f$ can be extended to a continuous function on $Y^{n+1}$.

The set of all $C^n$-functions $Y \to K$ is denoted $C^n(Y \to K)$. The function $f : Y \to K$ is a $C^\infty$-function if it is in $C^\infty(Y \to K) := \bigcap_{n=0}^{\infty} C^n(Y \to K)$. The space $C^0(Y \to K)$, consisting of all continuous functions $Y \to K$ is sometimes written as $C(Y \to K)$.

FROM NOW ON IN THIS PAPER $X$ IS A NONEMPTY COMPACT SUBSET OF $K$ WITHOUT ISOLATED POINTS.

3. Since $X$ has no isolated points we have for an $f \in C^n(X \to K)$ that the continuous extension of $\Phi_n f$ to $X^n$ is unique; we denote this extension by $\overline{\Phi}_n f$. Also we write

$$D_n f(a) := \overline{\Phi}_n f(a, a, \ldots, a) \quad (a \in X)$$

The following facts are proved in [2] and [3].

Proposition 0.3.

(i) For each $n \in \mathbb{N}_0$ the space $C^n(X \to K)$ is a $K$-algebra under pointwise operations.

(ii) $C^0(X \to K) \supset C^1(X \to K) \supset \ldots$
(iii) If \( f \in C^n(X \to K) \) then \( f \) is \( n \) times differentiable and \( j!D_jf = f^{(j)} \) for each \( j \in \{0, 1, \ldots, n\} \). More generally, if \( i, j \in \{0, 1, \ldots, n\}, i+j \leq n \) then \( \binom{i+j}{i}D_iD_jf = D_{i+j}f \).

(iv) If \( f \in C^n(X \to K) \) then for \( x, y \in X \) we have Taylor's formula

\[
f(x) = f(y) + (x-y)D_if(y) + \cdots + (x-y)^{n-1}D_{n-1}f(y) + (x-y)^n\rho_1f(x, y),
\]

where \( \rho_1f(x, y) = \Phi_nf(x, y, y, \ldots, y) \).

4. Since \( X \) is compact the difference quotients \( \Phi_i f \) \( (0 \leq i \leq n) \) are bounded if \( f \in C^n(X \to K) \). We set

\[
\|f\|_{n,X} := \max\{\|\Phi_i f\|_{n+1,X} : 0 \leq i \leq n\}.
\]

Then \( \|f\|_{0,X} = \|f\|_X \). We quote the following from [2] and [3].

**Proposition 0.4.** Let \( n \in \mathbb{N}_0 \).

(i) The function \( \| \|_{n,X} \) is a norm on \( C^n(X \to K) \) making it into a \( K \)-Banach algebra.

(ii) The local polynomials form a dense subset of \( C^n(X \to K) \).

(iii) The function

\[
f \mapsto \|f\|_{n,X} := \max_{0 \leq i \leq n-1} \|D_if\|_X \vee \|\rho_1f\|_X
\]

(see Proposition 0.3 (iv)) also is a norm on \( C^n(X \to K) \). We have

\[
\|f\|_{n,X} = \max\{\|D_if\|_{n-i,X} : 0 \leq i \leq n\} \quad (f \in C^n(X \to K)).
\]

**Remarks**

1. Proposition 0.4 (ii) will also follow from Proposition 2.8.

2. In general \( \| \|_{n,X} \) is not equivalent to \( \| \|_{n,X} \) for \( n \geq 3 \) (see [3], Example 83.2).
1 THE WEIERSTRASS THEOREM FOR $C^n$-FUNCTIONS

The following product rule for difference quotients is easily proved by induction with respect to \( j \).

Let \( f, g : X \to K \), let \( j \in \mathbb{N}_0 \). Then for all \( (x_1, \ldots, x_{j+1}) \in \nabla^{j+1}X \) we have

\[
\Phi_j(fg)(x_1, \ldots, x_{j+1}) = \sum_{k=0}^j \Phi_k f(x_1, \ldots, x_{k+1}) \Phi_{j-k} g(x_{k+1}, \ldots, x_{j+1}).
\]

Or, less precise,

\[
\Phi_j(fg)(x_1, \ldots, x_{j+1}) = \sum_{k=0}^j \Phi_k f(z_k) \Phi_{j-k} g(u_{j-k})
\]

for certain \( z_k \in \nabla^{k+1}X \), \( u_{j-k} \in \nabla^{j-k+1}X \).

In the sequel we need an extension of this formula to finite products of functions. The proof is straightforward by induction with respect to \( N \).

**Lemma 1.1. (Product Rule)** Let \( h_1, \ldots, h_N : X \to K \), let \( j \in \mathbb{N}_0 \). Then for all \( (x_1, \ldots, x_{j+1}) \in \nabla^{j+1}X \) we have

\[
\Phi_j\left(\prod_{s=1}^N h_s\right)(x_1, \ldots, x_{j+1}) = \sum_{\sigma} \prod_{s=1}^N \Phi_{j_s}(z_{\sigma,s})
\]

where the sum is taken over all \( \sigma := (j_1, \ldots, j_N) \in \mathbb{N}_0^N \) for which \( j_1 + \cdots + j_N = j \) and where \( z_{\sigma,s} \in \nabla^{j_s+1}X \) for each \( s \in \{1, \ldots, N\} \). (In fact, \( z_{\sigma,1} = (x_1, \ldots, x_{j_1+1}), z_{\sigma,2} = (x_{j_1+1}, \ldots, x_{j_1+j_2+1}), \ldots, z_{\sigma,N} = (x_{j_1+\cdots+j_{N-1}+1}, \ldots, x_{j+1}) \).

The following key lemma grew out of [1], 5.28.

**Lemma 1.2.** Let \( 0 < \delta < 1, 0 < \varepsilon < 1 \), let \( B = B_0 \cup B_1 \cup \cdots \cup B_m \) where \( B_0, \ldots, B_m \) are pairwise disjoint 'closed' balls in \( K \) of radius \( \delta \). Then, for each \( n \in \{0, 1, \ldots\} \) there exists a polynomial function \( P : K \to K \) such that \( \|P - \xi B_0\|_{n,B} \leq \varepsilon \).

**Proof.** We may assume \( 0 \in B_0 \). Choose \( c_1 \in B_1, \ldots, c_m \in B_m \); we may assume that \( |c_1| \leq |c_2| \leq \cdots \leq |c_m| \). Then \( \delta < |c_1| \). We shall prove the following statement by induction with respect to \( n \).

Let \( k \in \mathbb{N} \) be such that \( (\delta/|c_1|)^k \leq \varepsilon n^k, k > n \). Let \( t_1, t_2, \ldots, t_m \in \mathbb{N} \) be such that for all \( \ell \in \{1, \ldots, m\} \)

\[
\left| \frac{c_{t_{\ell}}}{c_{t_{\ell-1}}} \right|^{t_{\ell}} \leq \varepsilon n^k
\]
defines a polynomial function \( P : K \to K \) for which
\[
\| P - \xi_{B_0} \|_{n, B} \leq \varepsilon.
\]
The case \( n = 0 \) is proved in [1], 5.28. To prove the step \( n - 1 \to n \) we first observe that from the induction hypothesis (with \( \varepsilon \) replaced by \( \varepsilon \delta \)) it follows that
\[
\| P - \xi_{B_0} \|_{n-1, B} \leq \varepsilon \delta
\]
So it remains to be shown that
\[
|\Phi_n(P - \xi_{B_0})(x_1, \ldots, x_{n+1})| \leq \varepsilon
\]
for all \( (x_1, \ldots, x_{n+1}) \in \nabla^{n+1} B \). Now, if \( |x_i - x_j| > \delta \) for some \( i, j \in \{1, \ldots, n + 1\} \) we have, using (2),
\[
|\Phi_n(P - \xi_{B_0})(x_1, \ldots, x_{n+1})| = |x_i - x_j|^{-1} |\Phi_{n-1}(P - \xi_{B_0})(x_1, \ldots, x_{i-1}, x_j, x_{i+1}, \ldots, x_{n+1}) - \Phi_{n-1}(P - \xi_{B_0})(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1})| \leq \delta^{-1} \varepsilon \delta = \varepsilon.
\]
So this reduces the proof of (3) to the case where \( |x_i - x_j| \leq \delta \) for all \( i, j \in \{1, \ldots, n + 1\} \); in other words we may assume that \( x_1, \ldots, x_{n+1} \) are all in the same \( B_\ell \) for some \( \ell \in \{0,1,\ldots,m\} \). But then, after observing that \( n \geq 1 \), we have \( \Phi_n \xi_{B_0}(x_1, \ldots, x_{n+1}) = 0 \) so it suffices to prove the following.
If \( \ell \in \{0,1,\ldots,m\} \) and \( x_1, \ldots, x_{n+1} \in B_\ell \) are pairwise distinct then
\[
|\Phi_n P(x_1, \ldots, x_{n+1})| \leq \varepsilon
\]
To prove it we introduce, with \( \ell \in \{1,\ldots,m\} \) fixed, the constants \( M_i (i \in \{1,\ldots,n\}) \) by
\[
M_i := \begin{cases} 1 & \text{if } i > \ell \\ \delta/|c_1| & \text{if } i = \ell \\ |c_\ell/c_i|^k & \text{if } i < \ell \end{cases}
\]
and use the following three steps.

**Step 1.** For each \( \ell \in \{0,1,\ldots,n\} \), \( i \in \{1,\ldots,n\} \) we have
\[
|\Phi_j(1 - (x_i/c_i)^k)| \leq \begin{cases} 1 & \text{if } \ell = 0, j = 0 \\ \delta^{-i}(\delta/|c_1|)^k & \text{if } \ell = 0, j > 0 \\ \delta^{-i} M_i & \text{if } \ell > 0. \end{cases}
\]
Proof.

a. The case $j = 0$. Then for $x \in B_t$ we have

- if $i > \ell$ then $|1 - (\frac{x}{c_i})^k| = 1$
- if $i = \ell$ then $|1 - (\frac{x}{c_i})^k| = \frac{\delta}{|c_i|} \leq \delta$
- if $i < \ell$ then $|1 - (\frac{x}{c_i})^k| = |\frac{x}{c_i}|^k = \frac{\delta}{|c_i|^k}$

and the statement follows.

b. The case $j > 0$. Then $\Phi_j(1) = 0$ so that

$$\Phi_j(1 - (\frac{X}{c_i})^k) = \frac{1}{c_i^k} \Phi_j(\lambda^k)$$

Let $(x_1, \ldots, x_{j+1}) \in \nabla^{j+1} B_t$. By the Product Rule 1.1, $\Phi_j(\lambda^k)(x_1, \ldots, x_{j+1})$ is a sum of terms of the form $\prod_{s=1}^{k} (\Phi_j, \lambda^k)(z_s)$. Such a term is 0 if one of the $j_s$ is $> 1$, so we only have to deal with $j_s = 0$ (then $\Phi_j, \lambda^k = \lambda^k$) or $j_s = 1$ (then $\Phi_j, \lambda = 1$). The latter case occurs $j$ times (as $\sum_{s=1}^{k} j_s = j$) and it follows that

$$\prod_{s=1}^{k} (\Phi_j, \lambda^k)(z_s)$$

is a product of $k-j$ distinct terms taken from $\{x_1, \ldots, x_{j+1}\}$ (observe that, indeed, $j < k$ since $j \leq n < k$), so its absolute value is $\leq |c_{\ell}|^{k-j}$. It follows that $\|\Phi_j(1 - (\frac{X}{c_i})^k)\|_{\nu^{j+1} B_t} \leq |c_{\ell}|^{k-j}/|c_i|^k$ from which we conclude

- if $\ell = 0$: $|c_{\ell}|^{k-j}/|c_i|^k \leq \delta^{k-j}/|c_i|^k = \delta^{j}(\delta/|c_i|)^k$,
- if $\ell > 0$: $|c_{\ell}|^{k-j}/|c_i|^k \leq |c_{\ell}^{-j}| < \delta^{-j} = \delta^{-j}M_i$
- if $i = \ell > 0$: $|c_{\ell}|^{k-j}/|c_i|^k \leq |c_{\ell}^{-j}| \leq c_{\ell}^{-j} = \delta^{-j}(\frac{\delta}{|c_i|})^j \leq \delta^{-j}M_i$
- if $i < \ell$: $|c_{\ell}|^{k-j}/|c_i|^k \leq |c_{\ell}^{-j}| |c_i|^k \leq \delta^{-j}M_i$

and step 1 is proved.

Step 2. For each $j \in \{0, 1, \ldots, n\}$, $i \in \{1, \ldots, n\}$ we have

$$\|\Phi_j(1 - (\frac{X}{c_i})^k)^{i_{\ell}}\|_{\nu^{j+1} B_t} \leq \begin{cases} 1 & \text{if } \ell = 0, j = 0 \\ \delta^{-j}(|\frac{\delta}{|c_i|}|)^k & \text{if } \ell = 0, j > 0 \\ \delta^{-j}M_i^{i_{\ell}} & \text{if } \ell > 0 \end{cases}$$

Proof. The case $j = 0$ follows directly from Step 1, part a, so assume $j > 0$. By the Product Rule 1.1 applied to $h_s = 1 - (\frac{X}{c_i})^k$ for all $s \in \{1, \ldots, s\}$ we have for $(x_1, \ldots, x_{j+1}) \in \nabla^{j+1} B_t$ that $\Phi_j(1 - (\frac{X}{c_i})^k)^{i_{\ell}}(x_1, \ldots, x_{j+1})$ is a sum of terms of the form

$$\prod_{s=1}^{s_i} \Phi_j, 1 - (\frac{X}{c_i})^k)(z_s)$$
where \( j_1 + \cdots + j_s = j \). If \( \ell = 0 \) it follows from Step 1 that the value of (5) is
\[
\leq \prod \delta^{-j_i}(\frac{\delta}{|c_i|})^{t_i}
\]
where the product is taken over all \( s \) in the nonempty set \( \Gamma := \{s \in \{1, \ldots, t_i\} : j_s > 0\} \), so the product is \( \leq \delta^{-j}(\frac{\delta}{|c_i|})^{t_i} \). If \( \ell > 0 \) it follows from Step 1 that the value of (5) is \( \leq \prod_{s=1}^{t_i} \delta^{-j_s} M_i = \delta^{-j} M_i^{t_i} \).

The statement of Step 1 follows.

**Step 3.** Proof of (4). Again, the Product Rule 1.1, now applied to \( h_i = (1 - (\frac{X}{c_i})^{k_i}) \) for \( i \in \{1, \ldots, m\} \) tells us that for \( (x_1, \ldots, x_{n+1}) \in \nabla^{n+1} B_\ell \) the expression \( \Phi_n P(x_1, \ldots, x_{n+1}) \) is a sum of terms of the form
\[
\prod_{i=1}^{m} \Phi_{n_i}(1 - (\frac{X}{c_i})^{k_i})(z_s)
\]
where \( n_1 + \cdots + n_m = n \). If \( \ell = 0 \) we have by Step 2 that the value of (6) is \( \leq \prod \delta^{-n_i}(\frac{\delta}{|c_i|})^{t_i} \) where the product is taken over \( i \) in the nonempty set \( \Gamma := \{i : n_i \neq 0\} \), so the product is \( \leq \delta^{-n}(\frac{\delta}{|c_i|})^{k} \leq \delta^{-n} \cdot \varepsilon \delta^n = \varepsilon \), where we used the assumption \( (\delta/|c_i|)^k \leq \varepsilon \delta^n \). We see that \( |\Phi_n P(x_1, \ldots, x_{n+1})| \leq \varepsilon \) if \( (x_1, \ldots, x_n) \in B_0 \).

Now let \( \ell > 0 \). By Step 2 we have that the absolute value of (6) is \( \leq \prod_{i=1}^{m} \delta^{-n_i} M_i^{t_i} = \delta^{-n} M_1^{t_1} \cdots M_m^{t_m} = \delta^{-n} \cdot |\frac{c_i}{c_1}|^{k_1} \cdots |\frac{c_i}{c_{i-1}}|^{k_{t_i}|k_i|} \frac{k}{k_i} \) which is \( \leq \delta^{-n} \varepsilon \delta^n \) by (1). This proves (4) and the Lemma.

**Corollary 1.3.** For every locally constant \( f : X \to K \), for every \( n \in \mathbb{N}_0 \) and \( \varepsilon > 0 \) there exists a polynomial function \( P : K \to K \) such that \( ||f - P||_{n,X} \leq \varepsilon \).

**Proof.** There exist a \( \delta \in (0,1) \), pairwise disjoint 'closed' balls \( B_1, \ldots, B_m \) of radius \( \delta \) covering \( X \) and \( \lambda_1, \ldots, \lambda_m \in K \) such that
\[
f(x) = \sum_{i=1}^{m} \lambda_i \xi_B_i(x) \quad (x \in X)
\]
By Lemma 1.2 there exist polynomials \( P_1, \ldots, P_m \) such that \( ||\xi_B_i - P_i||_{n,X} \leq ||\xi_B_i - P_i||_{n,\cup B_i} \leq \varepsilon (|\lambda_i| + 1)^{-1} \) for each \( i \in \{1, \ldots, m\} \). Then \( P := \sum_{i=1}^{m} \lambda_i P_i \) is a polynomial function and \( ||f - P||_{n,X} \leq \max_i ||\lambda_i(\xi_B_i - P_i)||_{n,X} \leq \max_i |\lambda_i| \varepsilon (|\lambda_i| + 1)^{-1} \leq \varepsilon \).

**Theorem 1.4.** (\( C^n \)-Weierstrass Theorem) For each \( n \in \mathbb{N}_0 \), \( f \in C^n(X \to K) \) and \( \varepsilon > 0 \) there exists a polynomial function \( P : K \to K \) such that \( ||f - P||_{n,X} \leq \varepsilon \).

**Proof.** There is by Proposition 0.4 a local polynomial \( g : K \to K \) with \( ||f - g||_{n,X} \leq \varepsilon \). This \( g \) has the form \( g = \sum_{i=1}^{m} Q_i h_i \) where \( Q_1, \ldots, Q_m \) are polynomials and \( h_1, \ldots, h_m \)
are locally constant. By Corollary 1.3 we can find polynomials $P_1, \ldots, P_m$ for which
\[ \|h_i - P_i\|_{n,X} \leq \varepsilon(\|Q_i\|_{n,X} + 1) \] for each $i$. Then $P := \sum_{i=1}^{m} Q_i P_i$ is a polynomial and
\[ \|g - P\|_{n,X} \leq \varepsilon. \] It follows that $\|f - P\|_{n,X} \leq \max(\|f - g\|_{n,X}, \|g - P\|_{n,X}) \leq \varepsilon$.

Remarks.
1. In the case where $X = \mathbb{Z}_p$, $K \supseteq \mathbb{Q}_p$ the above Theorem 1.4 is not new: The Mahler base $e_0, e_1, \ldots$ of $C(\mathbb{Z}_p \rightarrow K)$ defined by $e_m(x) = \left( \binom{x}{m} \right)$ is proved in [3], §54 to be a Schauder base for $C^n(\mathbb{Z}_p \rightarrow K)$, for each $n$.

2. It follows directly from Theorem 1.4 that the polynomial functions $X \rightarrow K$ form a dense subset of $C^\infty(X \rightarrow K)$.

2. A WEIERSTRASS-STONE THEOREM FOR $C^n$-FUNCTIONS

For this Theorem (2.10) we will need the continuity of $g \mapsto g \circ f$ in the $C^n$-topologies (Proposition 2.5). To prove it we need some technical lemmas that are in the spirit of [3], §77.

Let $n \in \mathbb{N}$. For a function $h : \nabla^n X \rightarrow K$ we define $\Delta h : \nabla^{n+1} X \rightarrow K$ by the formula
\[
\Delta h(x_1, x_2, \ldots, x_{n+1}) = \frac{h(x_1, x_3, x_4, \ldots, x_{n+1}) - h(x_2, x_3, \ldots, x_{n+1})}{x_1 - x_2}
\]
We have the following product rule.

Lemma 2.1. (Product Rule). Let $n \in \mathbb{N}$, let $h, t : \nabla^n X \rightarrow K$. Then for all $(x_1, x_2, \ldots, x_{n+1}) \in \nabla^{n+1} X$ we have
\[
\Delta(ht)(x_1, x_2, \ldots, x_{n+1}) = h(x_2, x_3, \ldots, x_{n+1})\Delta t(x_1, x_2, \ldots, x_{n+1}) + t(x_1, x_3, \ldots, x_{n+1})\Delta h(x_1, x_2, \ldots, x_{n+1}).
\]

Proof. Straightforward.

Lemma 2.2. Let $f : X \rightarrow K$, $n \in \mathbb{N}_0$. Let $S_n$ be the set of the following functions defined on $\nabla^{n+1} X$.
\[
(x_1, \ldots, x_{n+1}) \mapsto \Phi_1 f(x_{i_1}, x_{i_2}) \quad (1 \leq i_1 < i_2 \leq n + 1)
\]
\[
(x_1, \ldots, x_{n+1}) \mapsto \Phi_2 f(x_{i_1}, x_{i_2}, x_{i_3}) \quad (1 \leq i_1 < i_2 < i_3 \leq n + 1)
\]
\[ \vdots \]
\[
(x_1, \ldots, x_{n+1}) \mapsto \Phi_n f(x_1, \ldots, x_{n+1}).
\]
For $k \in \mathbb{N}$, let $R_n^k$ be the additive group generated by $S_n, S_n^2, \ldots, S_n^k$ where, for each $j \in \{1, \ldots, k\}$, $S_n^j$ is the product set $\{h_1 h_2 \ldots h_j : h_i \in S_n \text{ for each } i \in \{1, \ldots, j\}\}$.
Then, for all $k, n \in \mathbb{N}$, $\Delta R_n^k \subseteq R_n^{k+1}$.

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Proof. We use induction with respect to $k$. For the case $k = 1$ it suffices to prove $h \in \mathcal{S}_n \Rightarrow \Delta h \in R_{n+1}^1$. Then $h$ has the form

$$(x_1, \ldots, x_{n+1}) \mapsto \Phi_j f(x_{i_1}, x_{i_2}, \ldots, x_{i_j+1})$$

for some $j \in \{2, 3, \ldots, n+1\}$ and so

$$\Delta h(x_1, x_2, \ldots, x_{n+1}) = \frac{h(x_1, x_3, \ldots, x_{n+2}) - h(x_2, x_3, \ldots, x_{n+2})}{x_1 - x_2}$$

vanishes if $i_1 > 1$ (and then $\Delta h$ is the null function), while if $i_1 = 1$ it equals

$$= \frac{\Phi_j f(x_1, x_{i_2+1}, \ldots, x_{i_j+1+1}) - \Phi_j f(x_2, x_{i_2+1}, \ldots, x_{i_j+1+1})}{x_1 - x_2}$$

and it follows that $\Delta h \in \mathcal{S}_{n+1} \subset R_{n+1}^1$. For the induction step assume $\Delta R_{n}^{k-1} \subset R_{n+1}^{k-1}$; it suffices to prove that $\Delta \mathcal{S}_{n}^{k} \subset R_{n+1}^{k}$. So let $h \in \mathcal{S}_{n}^{k}$ and write $h = h_1 H$, where $h_1 \in \mathcal{S}_n$, $H \in \mathcal{S}_{n}^{k-1}$. By the Product Rule 2.1 we have

$$\Delta h(x_1, \ldots, x_{n+2}) = h_1(x_2, x_3, \ldots, x_{n+2}) \Delta H(x_1, x_2, \ldots, x_{n+2}) +$$

$$+ H(x_1, x_3, \ldots, x_{n+2}) \Delta h_1(x_1, x_2, \ldots, x_{n+2}).$$

The fact that $h_1 \in \mathcal{S}_n$ makes

$$(x_1, x_2, \ldots, x_{n+2}) \mapsto h_1(x_1, x_3, \ldots, x_{n+2})$$

into an element of $\mathcal{S}_{n+1}$. Similarly, since $H \in \mathcal{S}_{n}^{k-1}$, the function

$$(x_1, x_2, \ldots, x_{n+2}) \mapsto H(x_2, x_3, \ldots, x_{n+2})$$

is in $\mathcal{S}_{n+1}$. By our first induction step, $\Delta h_1 \in R_{n+1}^1$ and by the induction hypothesis $\Delta H \in R_{n+1}^{k-1}$. Hence,

$$\Delta h \in \mathcal{S}_{n+1} R_{n+1}^{k-1} + \mathcal{S}_{n+1}^1 R_{n+1}^1$$

$$\subset R_{n+1}^1 R_{n+1}^{k-1} + R_{n+1}^{k-1} R_{n+1}^1 \subset R_{n+1}^k.$$  

Lemma 2.3. Let $f, n, S_n, k, R_n^k$ be as in the previous lemma. Let $f(X) \subset Y \subset K$ where $Y$ has no isolated points. Let $g : Y \rightarrow K$ be a $C^n$-function. Let $B_n$ be the set of the following functions defined on $\nabla^{n+1} X$.

$$(x_1, \ldots, x_{n+1}) \mapsto \overline{\Phi}_1 g(f(x_{i_1}), f(x_{i_2})) \quad  \quad (1 \leq i_1 < i_2 \leq n + 1)$$

$$(x_1, \ldots, x_{n+1}) \mapsto \overline{\Phi}_2 g(f(x_{i_1}), f(x_{i_2}), f(x_{i_3})) \quad  \quad (1 \leq i_1 < i_2 < i_3 \leq n + 1)$$

$$\vdots$$

$$(x_1, \ldots, x_{n+1}) \mapsto \overline{\Phi}_n g(f(x_1), f(x_2), \ldots, f(x_{n+1})).$$
Let $A_n$ be the additive group generated by $B_nR^n_n$. Then

$$\Delta A_n \subseteq A_{n+1}.$$ 

**Proof.** We prove: $h \in B_nR^n_n \Rightarrow \Delta h \in A_{n+1}$. Write $h = br$ where $b \in B_n$, $r \in R^n_n$. By the Product Rule 2.1 we have for all $(x_1, x_2, \ldots, x_{n+2}) \in \nabla^{n+2}X$

$$\Delta h(x_1, x_2, \ldots, x_{n+2}) = b(x_2, x_3, \ldots, x_{n+2})\Delta r(x_1, x_2, \ldots, x_{n+2}) + r(x_1, x_3, \ldots, x_{n+2})\Delta b(x_1, x_2, \ldots, x_{n+2}).$$

We have:

(i) $b \in B_n$ so $(x_1, \ldots, x_{n+2}) \mapsto b(x_2, x_3, \ldots, x_{n+1})$ is in $B_{n+1}$.

(ii) $r \in R^n_n$ so $(x_1, \ldots, x_{n+2}) \mapsto r(x_1, x_3, \ldots, x_{n+2})$ is in $R^n_{n+1}$ (in the previous proof we had $r \in S^i_n \Rightarrow$ the map $(x_1, \ldots, x_{n+2}) \mapsto r(x_1, x_3, \ldots, x_{n+1})$ is in $S^i_{n+1}$, and (ii) follows from this).

(iii) $r \in R^n_n$ so $\Delta r \in R^n_{n+1}$ (Previous Lemma).

(iv) $b$ has the form

$$(x_1, x_2, \ldots, x_{n+1}) \mapsto \Phi_j g(f(x_i), \ldots, f(x_{i+1}))$$

for some $j \in \{2, \ldots, n+1\}$ and so

$$\Delta b(x_1, x_2, \ldots, x_{n+2}) = \frac{b(x_1, x_3, x_4, \ldots, x_{n+2}) - b(x_2, x_3, \ldots, x_{n+2})}{x_1 - x_2}$$

vanishes if $i_1 > 1$ (and then $\Delta b$ is the null function), while if $i_1 = 1$ it equals

$$\Phi_j g(f(x_1), f(x_{i_2+1}), \ldots, f(x_{i_{i+1}+1})) - \Phi_j g(f(x_2), f(x_{i_2+1}), \ldots, f(x_{i_{i+1}+1}))$$

$$= \frac{\Phi_{j+1} g(f(x_1), f(x_2), f(x_{i_2+1}), \ldots, f(x_{i_{i+1}+1}))}{x_1 - x_2} \Phi_1 f(x_1, x_2).$$

(if $f(x_1) = f(x_2)$ we have 0 at both sides). So we see that $\Delta b \in B_{n+1}R^n_{n+1}$. Combining (i) - (iv) we get $\Delta h \in B_{n+1}R^n_{n+1} + R^n_{n+1}B_{n+1}R^n_{n+1} \subseteq B_{n+1}R^n_{n+1} + B_{n+1} \cdot R^n_{n+1} \subseteq A_{n+1}$.

**Corollary 2.4.** With the notations as in the previous lemma we have $\Phi_n(g \circ f) \in A_n$ ($n \in \mathbb{N}$).

**Proof.** We proceed by induction on $n$. For the case $n = 1$ we write, for $(x_1, x_2) \in \nabla^2 X$,

$$\Phi_1(g \circ f)(x_1, x_2) = (x_1 - x_2)^{-1} \left( g(f(x_1)) - g(f(x_2)) \right) = \Phi_1 g(f(x_1), f(x_2)) \Phi_1 f(x_1, x_2).$$
Hence, $ \Phi_1(g \circ f) \in B_1S_1 \subseteq B_1R_1 \subseteq A_1$. To prove the step $n \to n+1$ observe that by the induction hypothesis, $\Phi_n(g \circ f) \in A_n$. By Lemma 2.3, $\Phi_{n+1}(g \circ f) = \Delta \Phi_n(g \circ f) \in A_{n+1}$.

**Remark.** From Corollary 2.4 it follows easily that the composition of two $C^n$-functions is again a $C^n$-function, a result that already was obtained in [3], 77.5.

**Proposition 2.5.** (Continuity of $g \mapsto g \circ f$) Let $n \in \mathbb{N}_0$, let $f \in C^n(X \to K)$ and let $g \in C^n(Y \to K)$ where $Y$ has no isolated points, $Y \supset f(X)$. Then $\|g \circ f\|_{n,X} \leq \|g\|_{n,Y} \max_{0 \leq i \leq n} \|f\|_i^{j,X}.$

**Proof.** We may assume $\|g\|_{n,Y} < \infty$. It suffices to prove $\|\Phi_n(g \circ f)\|_{\mathcal{V}^{n+1},X} \leq \|g\|_{n,Y} \|f\|_{n,X}^{n,X}$. Now $\|\Phi_0(g \circ f)\|_{\mathcal{V}^1,X} = \max_{x \in X} |g(f(x))| \leq \|g\|_{0,Y} \|f\|_{0,X}^0$ which proves the case $n = 0$. For $n \geq 1$ we apply Corollary 2.4 which says that $\Phi_n(g \circ f) \in A_n$ i.e. $\Phi_n(g \circ f)$ is a sum of functions in $B_nS^n$. By the definition of $B_n$ we have

\[(*) \quad h \in B_n \Rightarrow \|h\|_{\mathcal{V}^{n+1},X} \leq \|g\|_{n,Y}\]

Similarly

\[k \in S_n \Rightarrow \|k\|_{\mathcal{V}^{n+1},X} \leq \max_{0 \leq i \leq n} \|\Phi_if\|_{\mathcal{V}^{i+1},X} \leq \|f\|_{n,X}\]

so that

\[(**) \quad k \in S^n_n \Rightarrow \|k\|_{\mathcal{V}^{n+1},X} \leq \|f\|_{n,X}^n\]

Combination of $(*)$ and $(**)$ yields $\|\Phi_n(g \circ f)\|_{\mathcal{V}^{n+1},X} \leq \|g\|_{n,Y} \|f\|_{n,X}^n$.

Proposition 2.5 enables us to prove

**Proposition 2.6.** Let $n \in \mathbb{N}_0$ and let $A$ be a closed subalgebra of $C^n(X \to K)$. Suppose $A$ separates the points of $X$ and contains the constant functions. Then $A$ contains all locally constant functions $X \to K$.

**Proof.** 1. We first prove that $f \in A$, $U \subseteq K$, $U$ clopen implies $\xi_{f^{-1}(U)} \in A$. In fact, $f(U)$ is compact so there exist a $\delta \in (0,1)$ and finitely many disjoint balls $B_1, \ldots, B_m$ in $U$ of radius $\delta$ covering $f(U)$. Let $\varepsilon > 0$. By the Key Lemma 1.2 there exists, for each $i \in \{1, \ldots, m\}$ a polynomial $P_i$ such that $\|\xi_{B_i} - P_i\|_{n,B} < \varepsilon$, where $B := \bigcup B_i$. Then $P := \Sigma P_i$ is a polynomial and $\|P - \xi_U\|_{n,B} = \|P - \xi_B\|_{n,B} = \|\Sigma (P_i - \xi_{B_i})\| < \varepsilon$.

By Proposition 2.5

\[\|(P - \xi_U) \circ f\|_{n,X} \leq \|P - \xi_U\|_{n,B} \max_{0 \leq j \leq n} \|f\|_j^{j,X} \leq \varepsilon \max_{0 \leq j \leq n} \|f\|_j^{j,X}\]

and we see that there exists a sequence $P_1, P_2, \ldots$ of polynomials such that
\[ \|P_n \circ f - \xi U \circ f\|_{n, X} \to 0. \] Since \( A \) is an algebra with an identity we have \( P_n \circ f \in A \) for all \( n \). Then \( \xi_{f^{-1}(U)} = \xi_U \circ f = \lim_{n \to \infty} P_n \circ f \in A. \)

2. Now consider

\[ B := \{ V \subset X, \xi_V \in A \}. \]

It is very easy to see that \( B \) is a ring of clopen subsets of \( X \) and that \( B \) covers \( X \). To show that \( B \) separates the points of \( X \) let \( x \in X, y \in X, x \neq y \). Then there is an \( f \in A \) for which \( f(x) \neq f(y) \). Set \( U := \{ \lambda \in K : |\lambda - f(x)| < |f(x) - f(y)| \} \). Then \( U \) is clopen in \( K \). By the first part of the proof, \( f^{-1}(U) \in B \). But \( x \notin f^{-1}(U) \) whereas \( y \notin f^{-1}(U) \).

By [1], Exercise 2.H \( B \) is the ring of all clopens of \( X \). It follows easily that all locally constant functions are in \( A \).

To arrive at the Weierstrass-Stone Theorem 2.10 we need a final technical lemma.

**Lemma 2.7.** Let \( a_1, \ldots, a_m \in X \), let \( \delta_1, \ldots, \delta_m \) be in \((0,1)\) such that \( B(a_1, \delta_1), \ldots, B(a_m, \delta_m) \) form a disjoint covering of \( X \). Let \( n \in \mathbb{N}_0, h \in \mathcal{C}^n(X \to K) \) and suppose

\[ D_j h(a_i) = 0 \quad \text{and} \quad |\Phi_{n-j} D_j h(x_1, \ldots, x_{n-j+1})| \leq \varepsilon \]

for all \( i \in \{1, \ldots, m\}, x_1, \ldots, x_{n+1} \in B(a_i, \delta_i) \cap X, j \in \{0,1, \ldots, n\} \). Then \( \|h\|_{n, X} \leq \varepsilon \).

**Proof.** We first prove that \( \|h\|_{n, X} \leq \varepsilon \) (see Proposition 0.4(iii)). Let \( i \in \{1, \ldots, m\} \). By Taylor's formula (Proposition 0.3(iv)) we have for \( x \in X \cap B_i : |h(x)| = \left| \sum_{s=0}^{n-1} (x - a_i)^s D_s h(a_i) + (x - a_i)^n \rho_1 h(x, a_i) \right| = |x - a_i|^n |\Phi_n h(x, a_i, a_i, \ldots, a_i)| \leq \delta^n \varepsilon.

Similarly we have for \( j \in \{0, \ldots, n-1\} \) and \( x \in X \cap B_i : |D_j h(x)| = \left| \sum_{s=0}^{n-1} (x - a_i)^s D_s h(a_i) + (x - a_i)^n \rho_1 (D_j h)(x, a_i) \right|. \]

Now using Proposition 0.3(iii) we see that \( D_i D_j h(a_i) = 0 \) so that

\[ |D_j(x)| = |x - a_i|^{n-j} |\Phi_{n-j} D_j h(x, a_i, \ldots, a_i)| \leq \delta^{n-j} \varepsilon. \]

It follows that \( \|h\|_X, \|D_1 h\|_X, \ldots, \|D_{n-1} h\|_X \) are all \( \leq \varepsilon \). Now let \( x, y \in X \). If \( x, y \) are in the same \( B_i \) then \( |\rho_1 h(x, y)| = |\Phi_n h(x, y, y, \ldots, y)| \leq \varepsilon \) by assumption. If \( x \in B_i, y \in B_j \) and \( i \neq s \) then \( |x - y| \geq \delta := \max(\delta_i, \delta_s) \) and by Taylor's formula

\[ h(x) = \sum_{l=0}^{n-1} (x - y)^l D_l h(y) + (x - y)^n \rho_1 h(x, y) \]

we obtain, using (\( \ast \)),

\[ |\rho_1 h(x, y)| \leq \frac{|h(x) - h(y)|}{|(x - y)^n|} \vee \frac{|D_1 h(y)|}{|x - y|^{n-1}} \vee \cdots \vee \frac{|D_{n-1} h(y)|}{|x - y|} \leq \frac{\delta^n \varepsilon}{\delta^n} \vee \frac{\delta^{n-1} \varepsilon}{\delta^n} \vee \cdots \vee \frac{\delta \varepsilon}{\delta} \leq \varepsilon \]

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and we have proved $\|h\|_{n,X}^\sim \leq \varepsilon$

Now to prove that even $\|h\|_{n,X} \leq \varepsilon$ observe that by Proposition 0.4(iii)

$$\|h\|_{n,X} = \|h\|_{n,X}^\sim \lor \|D_1 h\|_{n-1,X}^\sim \lor \cdots \lor \|D_n h\|_{0,X}^\sim.$$ 

To prove, for example, that $\|D_1 h\|_{n-1,X}^\sim \leq \varepsilon$ we observe that $D_1 h \in C^{n-1}(X \to K)$ and that for $i \in \{1, \ldots, m\}$ and $j \in \{0, 1, \ldots, n-2\}$ we have $D_j D_1 h(a_i) = (j+1)D_{j+1} h(a_i) = 0$ and for all $x_1, \ldots, x_n \in B(a_i, \delta_i)$ and $j \in \{0, 1, \ldots, n-2\}$

$$|\Phi_{n-1-j} D_j(D_1 h)(x_1, \ldots, x_{n-j})| = |(j+1)|\Phi_{n-1-j} D_{j+1} h(x_1, \ldots, x_{n-j})| \leq \varepsilon$$

by assumption. So the conditions of our Lemma (with $D_1 h$, $n-1$ in place of $h$, $n$ respectively) are satisfied and by the first part of the proof we may conclude that $\|D_1 h\|_{n-1,X}^\sim \leq \varepsilon$. In a similar way we prove that $\|D_2 h\|_{n-2,X}^\sim \leq \varepsilon$, $\ldots$, $\|D_n f\|_{0,X}^\sim \leq \varepsilon$ and it follows that $\|h\|_{n,X} \leq \varepsilon$.

**Proposition 2.8.** Let $n \in \mathbb{N}_0$ and let $A$ be a closed subalgebra of $C^n(X \to K)$ containing the locally constant functions. Let $g \in C^n(X \to K)$ and suppose for each $a \in X$ there exists an $f_a \in A$ with $D_i g(a) = D_i f_a(a)$ for $i \in \{0, 1, \ldots, n\}$. Then $g \in A$.

**Proof.** Let $\varepsilon > 0$. For each $a \in X$ choose an $f_a \in A$ with $f_a(a) = g(a)$, $D_1 f_a(a) = D_1 g(a), \ldots, D_n f_a(a) = D_n g(a)$. By continuity there exists a $\delta_a > 0$ such that, with $h_a := f_a - g$, $|\Phi_{n-j} D_j h_a(x_1, \ldots, x_{n-j+1})| \leq \varepsilon$ for all $j \in \{0, 1, \ldots, n\}$ and $x_1, \ldots, x_{n-j+1} \in B(a, \delta_a)$. The $B(a, \delta_a)$ cover $X$ and by compactness there exists a finite disjoint subcovering $B(a_1, \delta_{a_1}), \ldots, B(a_m, \delta_{a_m})$. Set

$$f := \sum_{i=1}^m f_{a_i} \chi_{B(a_i, \delta_{a_i}) \cap X}$$

Then, by our assumption on $A$, $f \in A$. By Lemma 2.7, applied to $h := f - g$ and where $\delta_1, \ldots, \delta_m$ are replaced by $\delta_{a_1}, \ldots, \delta_{a_m}$ respectively, we then have $\|f - g\|_{n,X} \leq \varepsilon$. We see that $g \in A = A$.

**Remark.** It follows directly that the local polynomial functions $X \to K$ form a dense subset of $C^n(X \to K)$.

**Proposition 2.9.** Let $n \in \mathbb{N}$ and let $A$ be a $K$-subalgebra of $C^n(X \to K)$ containing the constant functions. Suppose $f'(a) \neq 0$ for some $f \in A$, $a \in X$. Then there is a $g \in A$ with $g(a) = 0$, $g'(a) = 1$ and $D_2 g(a) = D_3 g(a) = \cdots = D_n g(a) = 0$.

**Proof.** By considering the function $f'(a)^{-1}(f - f(a))$ it follows that we may assume that $f(a) = 0$, $f'(a) = 1$. Then

$$f = (X - a) h$$

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where \( h \) is continuous, \( h(a) = 1 \). To obtain the statement by induction with respect to \( n \) we only have to consider the induction step \( n - 1 \rightarrow n \) and, to prove that, we may assume that \( D_2 f(a) = \cdots = D_{n-1} f(a) = 0 \). From (*) we obtain

\[
f^n = (X - a)^n h^n
\]

and by uniqueness of the Taylor expansion of the \( C^n \)-function \( f^n \) we obtain \( f^n(a) = D_1 f^n(a) = \cdots = D_{n-1} f^n(a) = 0 \) and \( D_n f^n(a) = h^n(a) = 1 \). We see that \( g := f - D_n f(a) f^n \) is in \( A \) and that \( g(a) = 0, g'(a) = 1, D_2 g(a) = \cdots = D_{n-1} g(a) = 0 \) and \( D_n g(a) = D_n f(a) - D_n f(a) D_n f^n(a) = 0 \).

**Theorem 2.10. (Weierstrass-Stone Theorem for \( C^n \)-functions).** Let \( n \in \mathbb{N}_0 \) and let \( A \) be a closed subalgebra that separates the points of \( X \) and that contains the constant functions. Suppose also that for each \( a \in X \) there exists an \( f \in A \) with \( f(a) \neq 0 \). Then \( A = C^n(X \rightarrow K) \).

**Proof.** By Proposition 2.9, for each \( a \in X \) there exists an \( f \in A \) with \( f(a) = 0, f'(a) = 1, D_i f(a) = 0 \) for \( i \in \{2, \ldots, n\} \). The function \( g := X \) satisfies \( g(a) = 0, g'(a) = 1, D_i g(a) = 0 \) for \( i \in \{2, \ldots, n\} \) so applying Proposition 2.8 (observe that \( A \) contains the locally constant functions by Proposition 2.6) we obtain that \( X \in A \). But then all polynomials are in \( A \) and \( A = C^n(X \rightarrow K) \) by the Weierstrass Theorem 1.4.

**Remarks.**

1. The case \( n = 0 \) yields, at least for those \( X \) that are embeddable into \( K \), the well known Kaplansky Theorem proved in [1], 6.15.

2. We leave it to the reader to establish a \( C^\infty \)-version of Theorem 2.10.

**REFERENCES**

