The Orlicz-Pettis property in $p$-adic analysis

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Abstract
For a non-archimedean locally convex space $E$ the property (O.P.): “every weakly convergent sequence in $E$ is convergent” is studied. Examples are given (1.3, 2.4-2.7). If the scalar field $K$ is spherically complete every $E$ has (O.P.) (1.2). If not, the property (O.P.) is closely related to “$E$ does not contain $\ell^\infty$” (3.2).

Terminology
Throughout $K$ is a non-archimedean nontrivially valued field that is complete with respect to the metric induced by the valuation $|\cdot|$. For notations, definitions, ... we refer to [5] for normed spaces and to [6] for general locally convex spaces. However, we recall the following. Let $E, F$ be $K$-linear spaces. The $K$-linear span of a set $X \subseteq E$ is denoted $[X]$, the (algebraic) dual of $E$ is $E^*$. If $p, q$ are (non-archimedean) seminorms on $E, F$ respectively we denote by $p \otimes q$ the seminorm

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$z \mapsto \inf \\{ \max_{1 \leq i \leq n} p(x_i) q(y_i) : n \in \mathbb{N}, \ z = \sum_{i=1}^{n} x_i \otimes y_i : x_i \in E, \ y_i \in F \}$ on $E \otimes F$. A seminorm $p$ on $E$ is polar if $p = \sup\{ |f| : f \in E^*, |f| \leq p \}$.

Now let $E, F$ be locally convex spaces over $K$. The topological dual of $E$ is $E'$, the weak topology on $E$ is $\sigma(E, E')$. $E$ is called of countable type if for every continuous seminorm $p$ on $E$ the normed space $E/\ker p$ is of countable type. $E$ is called a polar space if the topology is generated by polar seminorms whereas $E$ is called strongly polar if every continuous seminorm is polar. On the tensor product $E \otimes F$ we always consider the topology generated by the seminorms $p \otimes q$ where $p, q$ are continuous seminorms on $E, F$ respectively.

From now on in this paper "locally convex" will mean "Hausdorff locally convex".

1. (O.P.)-spaces

The classical Banach space $\ell^1$ (over $\mathbb{R}$ or $\mathbb{C}$) has the property that every weakly convergent sequence is norm convergent, which is known as the Orlicz-Pettis Theorem. In our non-archimedean theory we therefore define

**Definition 1.1.** A locally convex space over $K$ is called Orlicz-Pettis space ((O.P.)-space) if every weakly convergent sequence is convergent.

We first consider some immediate examples. It was shown by Monna in [3] that $c_0$ is an (O.P.)-space (observe that in our case the dual of $c_0$ is no longer $\ell^1$ but $\ell^\infty$). By straightforward arguments one can prove that subspaces, products and locally convex direct sums of (O.P.)-spaces are again (O.P.)-spaces. ([4], Propositions 1.2, 1.4.) As every space of countable type is a subspace of some power of $c_0$ we obtain the (O.P.)-property for every space of countable type. Now let $E$ be any locally convex space and let $x_1, x_2, \ldots$ be a sequence in $E$ converging weakly to 0, and set $D = [x_1, x_2, \ldots]$. If also $x_n \to 0$ in $\sigma(D, D')$ then, as $D$ is of countable type, it would follow that $x_n \to 0$ in the original topology of $E$. Now $\sigma(E, E')|D = \sigma(D, D')$ as soon as every $f \in D'$ has an extension $\tilde{f} \in E'$. Such extensions exist certainly if $K$ is spherically complete, by Ingleton's Theorem. Hence,

**Theorem 1.2**

If $K$ is spherically complete every locally convex space over $K$ is an (O.P.)-space.
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For nonspherically complete $K$ the previous reasoning produces the following:

**Theorem 1.3**

*The following spaces are (O.P.)-spaces.*

(i) Every strongly polar space.
(ii) Every space of countable type.
(iii) Every Banach space with a base.
(iv) Any $K$-vector space equipped with the strongest locally convex topology.

*Proof.* All the spaces indicated in (i), (ii), (iii) have the property that every continuous linear function on a subspace of countable type can be extended to a continuous linear function on the whole space ([6], Definition 3.5, Theorem 4.4 and [5], Corollary 3.18). To prove (iv) just observe that the space is linearly homeomorphic to the locally convex direct sum of a collection of one-dimensional spaces. □

Because of Theorem 1.2 we assume from this point on in this paper that $K$ is not spherically complete.

To obtain a space which is not (O.P.), take $E := l^{\infty}$. In fact, since $(l^{\infty})' \simeq c_0$ (see [5], Theorem 4.17) it follows that $(1,0,0,...), (0,1,0,...), (0,0,1,0,...), ...$ tends weakly to $0$ but, of course, is not norm convergent. This example also tells us that the class of the (O.P.)-spaces is not closed for forming of quotients: Every Banach space (in particular $l^{\infty}$) is a quotient of a Banach space with a base. The space $l^{\infty}$ will play a key role in characterizing (O.P.)-spaces (Theorem 3.2).

The following observation will be used in the sequel several times. If $E$ is an (O.P.)-space then the weak topology $\sigma(E, E')$ is Hausdorff. Indeed, if $x \in E$ and $f(x) = 0$ for all $f \in E'$ then $0, x, 0, x, ...$ converges weakly, hence strongly, so $x = 0$. Obviously the converse does not hold (again, take $E := l^{\infty}$).

2. (O.P.)-spaces of continuous functions

To establish the (O.P.)-property for some spaces of (vector valued) continuous functions we first prove three structure theorems.

**Theorem 2.1**

*If $E$ and $F$ are (O.P.)-spaces then so is $E \otimes F$.***
Proof. Let \( z_1, z_2, \ldots \) be a sequence in \( E \otimes F \) converging weakly to 0. Let \( p \) resp. \( q \) be a continuous seminorm on \( E \) resp. \( F \). We shall prove that \((p \otimes q)(z_n) \to 0\). Let \( 0 < t < 1 \). By an obvious modification of [5], Theorem 4.30(ii) (see also [1], Lemma 2.1) for each \( n \in \mathbb{N} \) there exists \( x^n_1, \ldots, x^n_{m_n} \in E \) and \( y^n_1, \ldots, y^n_{m_n} \in F \) such that

(i) \( z_n = \sum_{k=1}^{m_n} x^n_k \otimes y^n_k \),
(ii) \( y^n_1, \ldots, y^n_{m_n} \) are \( t \)-orthogonal with respect to \( q \),
(iii) \( 1 \leq q(y^n_k) \leq 2 \) (\( k \in \{1, \ldots, m_n \} \)).

Now let \( f \in E' \). The map \( f \otimes 1 : E \otimes F \to F \) sends \( z_1, z_2, \ldots \) into a weakly convergent, hence convergent, sequence in \( F \), i.e.,

\[
\lim_{n \to \infty} q \left( \sum_{k=1}^{m_n} f(x^n_k)y^n_k \right) = 0.
\]

By \( t \)-orthogonality and (iii)

\[
\lim_{n \to \infty} \max_{1 \leq k \leq m_n} |f(x^n_k)| = 0.
\]

As the latter is true for every \( f \in E' \) the sequence \( x^1_1, x^1_2, \ldots, x^1_{m_1}, x^2_1, \ldots, x^2_{m_2}, \ldots \) converges weakly to 0 in the (O.P.)-space \( E \), hence with respect to \( p \) so that

\[
(p \otimes q)(z_n) \leq \max_{1 \leq k \leq m_n} p(x^n_k)q(y^n_k) \leq 2 \max_{1 \leq k \leq m_n} p(x^n_k) \to 0 \quad \text{for } n \to \infty
\]

implying \((p \otimes q)(z_n) \to 0\). □

**Theorem 2.2**

Let \((E, \tau)\) be a metrizable locally convex space and let \( D \) be a dense subspace of \( E \). If \( D \) is an (O.P.)-space then so is \( E \).

**Proof.** There is an invariant metric \( d \) on \( E \) inducing \( \tau \). Let \( x_1, x_2, \ldots \) be a sequence in \( E \) such that \( \lim_{n \to \infty} x_n = 0 \) in \( \sigma(E, E') \). For each \( n \in \mathbb{N} \), choose a \( y_n \in D \) with \( d(x_n, y_n) < 1/n \). Then \( x_n - y_n \rightharpoonup 0 \) so \( x_n - y_n \to 0 \) in \( \sigma(E, E') \) so that \( y_n = x_n - (x_n - y_n) \to 0 \) in \( \sigma(E, E') \), hence in \( \sigma(D, D') \). Now \( D \) is an (O.P.)-space, so \( y_n \rightharpoonup 0 \). But then also \( x_n = (x_n - y_n) + y_n \rightharpoonup 0 \). □

**Problem.** Does the conclusion hold if we drop the metrizability condition?

**Theorem 2.3**

Every metrizable (O.P.)-space is polar.
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Proof. Let \( p_1, p_2, \ldots \) be seminorms defining the topology \( \tau \) of a (metrizable) locally convex space \( E \). For each \( n \in \mathbb{N} \) define the seminorm \( \tilde{p}_n \) by

\[
\tilde{p}_n = \sup \{|f| : f \in E', |f| \leq p_n \}.
\]

Then the topology \( \tilde{\tau} \) induced by the \( \tilde{p}_n \) is polar. From the first inclusion in

\[
(*) \quad \sigma(E, E') \subset \tilde{\tau} \subset \tau
\]

we obtain that \( \tilde{\tau} \) is Hausdorff, so \((E, \tilde{\tau})\) is metrizable. Since \((E, \tau)\) is (O.P.) we conclude from (*) that \( \tau \) and \( \tilde{\tau} \) have the same convergent sequences implying \( \tau = \tilde{\tau} \) by metrizability. We see that \( \tau \) is polar. □

Remark. There exist (nonmetrizable) (O.P.)-spaces that are not polar ([4], Proposition 4.1).

Now we turn to function spaces. For a Hausdorff zerodimensional topological space \( X \) and a locally convex space \( E \) over \( K \) we define

\( PC(X, E) \): The space of all continuous functions \( f : X \to E \) for which \( f(X) \) is precompact, endowed with the topology \( \tau_u \) of uniform convergence,

\( C(X, E) \): The space of all continuous functions \( X \to E \), endowed with the topology \( \tau_c \) of compact convergence,

\( C_b(X, E) \): The space of all bounded continuous functions \( X \to E \), endowed with the strict topology \( \tau_\beta \). This is the topology generated by all seminorms \( f \mapsto \sup_{x \in X} |\phi(x)|p(f(x)) \) \((f \in C_b(X, E))\) where \( \phi : X \to K \) is a bounded function vanishing at infinity and \( p \) is a continuous seminorm on \( E \).

In the sequel we shall often restrict to Fréchet spaces \( E \) as we need the Theorems 2.2 and 2.3.

Theorem 2.4

Let \( E \) be a Fréchet space. Then \( PC(X, E) \) is an (O.P.)-space if and only if \( E \) is an (O.P.)-space.

Proof. The constant functions form a subspace of \( PC(X, E) \) that is linearly homeomorphic to \( E \) which proves the "only if". To complete the proof, let \( E \) be an (O.P.)-space. Then, as \( PC(X, K) \) has a base ([5], Cor. 5.23), we have by Theorems 1.3(iii) and 2.1 that \( PC(X, K) \otimes E \) is a metrizable (O.P.)-space. It is easily seen that the map \( T : PC(X, K) \otimes E \to PC(X, E) \), given by the formula

\[
T \left( \sum_{j=1}^{n} f_j \otimes a_j \right)(x) = \sum_{j=1}^{n} f_j(x) a_j \quad (x \in X)
\]

is a linear homeomorphism onto a dense subspace of \( PC(X, E) \). Then \( PC(X, E) \) is an (O.P.)-space by Theorem 2.2. □
Corollary 2.5

Let $E$ be a Fréchet space. Then $C(X, E)$ is an (O.P.)-space if and only if $E$ is an (O.P.)-space.

Proof. We prove the “if”. Let $E$ be an (O.P.)-space, let $f_1, f_2, \ldots$ be a sequence in $C(X, E)$ converging weakly to 0. Then, for every compact set $H$ in $X$, the sequence $n \mapsto f_n|H$ converges weakly to 0 in $PC(H, E)$. By Theorem 2.4 $f_n \to 0$ uniformly on $H$ and the conclusion follows. □

Corollary 2.6

Let $E$ be a Fréchet space. Then $C_b(X, E)$ is an (O.P.)-space if and only if $E$ is an (O.P.)-space.

Proof. Again we prove the “if”. Let $E$ be an (O.P.)-space, let $f_1, f_2, \ldots$ converge weakly to 0 in $C_b(X, E)$. Since $\tau_c \subset \tau_\beta$ this sequence converges also weakly to zero in $(C_b(X, E), \tau_c)$. The latter, being a subspace of $(C(X, E), \tau_c)$ is (O.P.) by Corollary 2.5, so $f_n \to 0$ uniformly on compacts. Further, $E$ is polar and so is $(C_b(X, E), \tau_\beta)$ implying that $\{f_1, f_2, \ldots\}$ is $\tau_\beta$-bounded. Now apply Proposition 2.11 and Corollary 2.12 of [2] to conclude that $f_n \to 0$ in $\tau_\beta$. □

The picture changes if we endow $C_b(X, E)$ with the uniform topology $\tau_u$:

Corollary 2.7

Let $E$ be a Fréchet space. Then $(C_b(X, E), \tau_u)$ is an (O.P.)-space $\iff$ $X$ is pseudocompact and $E$ is an (O.P.)-space.

Proof. $\Rightarrow$. If $X$ is not pseudocompact we can find a countably infinite clopen partition $X = \bigcup_n X_n$. Choose $e \in E$ and define $T: \ell^\infty \to C_b(X, E)$ by the formula

$$T(\alpha_1, \alpha_2, \ldots)(x) = \alpha_n e \quad \text{if } n \in \mathbb{N}, x \in X_n.$$ 

It is easily seen that $T$ is a linear homeomorphism of $\ell^\infty$ onto a subspace of $(C_b(X, E), \tau_u)$ which yields a contradiction as $\ell^\infty$ is not (O.P.). For the other conclusion, consider again the constant functions.

$\Leftarrow$. One verifies that, if $X$ is pseudocompact, then $C_b(X, E) = PC(X, E)$. Now apply Theorem 2.4. □
3. Fréchet (O.P.)-spaces

An (O.P.)-space cannot contain a copy of $\ell^\infty$ as $\ell^\infty$ itself is not (O.P.). This simple observation is the starting point of Theorem 3.2 that characterizes Fréchet (O.P.)-spaces. The key result is Proposition 3.1 in which we describe polar spaces that do not contain $\ell^\infty$.

**Proposition 3.1**

For a polar locally convex space $E$ the following are equivalent.

(a) $E$ does not contain a subspace linearly homeomorphic to $\ell^\infty$.

(b) Every continuous linear map $\ell^\infty \to E$ is compact.

(γ) $E$ does not contain a complemented subspace linearly homeomorphic to $\ell^\infty$.

**Proof.** (α) $\Rightarrow$ (β). Let $T : \ell^\infty \to E$ be a noncompact continuous linear map; we derive a contradiction. Let $e_1, e_2, \ldots$ be the unit vectors of $\ell^\infty$. Then $\{Te_1, Te_2, \ldots\}$ is not a compactoid (otherwise the weak closure of the absolutely convex hull of $Te_1, Te_2, \ldots$ would be a compactoid ([6], Theorem 5.13) hence so would its subset $T(\{x \in \ell^\infty : \|x\| \leq 1\})$ implying compactness of $T$). So, there exists a continuous polar seminorm $p$ such that $\{Te_1, Te_2, \ldots\}$ is not a $p$-compactoid. By [7], Theorem 2 there exists a $t \in (0, 1)$ and a subsequence $z_1, z_2, \ldots$ of $Te_1, Te_2, \ldots$ that is $t$-orthogonal with respect to $p$ and such that $\inf_n p(z_n) > 0$. Without loss, assume $p(z_n) \geq 1$ for each $n$.

Now, inductively we shall construct a subsequence $u_1, u_2, \ldots$ of $z_1, z_2, \ldots$ and $f_1, f_2, \ldots \in E'$ such that $|f_n| \leq 2t^{-1}p$ for all $n$ and

$$
|f_m(u_n)| = \begin{cases} 
0 & \text{if } m > n \\
1 & \text{if } m = n
\end{cases}

|f_m(u_n)| \leq \frac{1}{2} \quad \text{if } m < n
$$

To do that, observe that the function $h_1 : \lambda z_1 \mapsto \lambda (\lambda \in \mathbb{K})$ satisfies $|h_1| \leq \frac{1}{2}$. By polarity it can be extended to an $f_1 \in E'$ such that $|f_1| \leq 2p$. Set $u_1 := z_1$. Suppose $f_1, \ldots, f_{m-1}$ and $u_1, \ldots, u_{m-1}$ are chosen with the required properties. Since $Te_n \to 0$ weakly we have $z_n \to 0$ weakly. So we can find a $k$ (larger than the indexes with respect to $z$ of $u_1, \ldots, u_{m-1}$) such that $|f_1(z_n)| \leq 1/2$, $|f_{m-1}(z_n)| \leq 1/2$ for $n \geq k$. Choose $u_m := z_k$. The function $h_m : \lambda_1 u_1 + \ldots + \lambda_m u_m \mapsto \lambda_m (\lambda_1, \ldots, \lambda_m \in \mathbb{K})$ satisfies $|h_m| \leq t^{-1}p$ so it can be extended to a function $f_m \in E'$ such that $|f_m| \leq 2t^{-1}p$. We see that $f_1, \ldots, f_m$ and $u_1, \ldots, u_m$ have the required properties.
Now we have that \( u_1, u_2, \ldots \) is a subsequence, say \( T e_{i_1}, T e_{i_2}, \ldots \) of \( T e_1, T e_2, \ldots \). Define a linear isometry \( \Omega : \ell^\infty \to \ell^\infty \) by the formula

\[
(\Omega(y_1, y_2, \ldots))_n = \begin{cases} 
0 & \text{if } n \not\in \{i_1, i_2, \ldots\} \\
y_{i_m} & \text{if } m \in \mathbb{N}, n = i_m
\end{cases}
\]

and set \( S := T \circ \Omega \). Then obviously \( S \) is continuous and \( S \) is described by the formula

\[
S(y_1, y_2, \ldots) = \sigma(E, E') - \sum_{n=1}^{\infty} y_n u_n.
\]

Finally let \( y = (y_1, y_2, \ldots) \in \ell^\infty, y \neq 0 \). There is an \( m \in \mathbb{N} \) such that \( |y_m| > \frac{1}{2} \|y\| \). We have \( p(Sy) \geq \frac{1}{2} t|f_m(Sy)| = \frac{1}{2} t|\sum_{n \geq m} y_n f_m(u_n)| \). If \( n > m \) we have \( |y_n f_m(u_n)| \leq \frac{1}{2} |y_n| \leq \frac{1}{2} \|y\| \) whereas \( |y_m f_m(v_m)| = |y_m| > \frac{1}{2} \|y\| \) so \( p(Sy) \geq \frac{1}{2} \|y\| \) implying that \( S \) is a linear homeomorphism from \( \ell^\infty \) onto \( S(\ell^\infty) \subseteq E \) which gives the desired contradiction.

\((\beta) \Rightarrow (\gamma)\) is obvious. The implication \((\gamma) \Rightarrow (\alpha)\) was proved in [8], Theorem 1.2 for (polar) Banach spaces \( E \). From here the step to locally convex \( E \) is easy ([4], Lemma 4.6).

**Theorem 3.2**

For a Fréchet space \( E \) the following are equivalent.

- \((\alpha)\) \( E \) is an (O.P.)-space.
- \((\beta)\) \( E \) is polar, weakly sequentially complete and \( E \) does not contain a subspace linearly homeomorphic to \( \ell^\infty \).
- \((\gamma)\) \( E \) is polar, weakly sequentially complete and \( E \) does not contain a complemented subspace linearly homeomorphic to \( \ell^\infty \).
- \((\delta)\) \( E \) is polar, weakly sequentially complete and every continuous linear map \( \ell^\infty \to E \) is compact.

**Proof.** The equivalence of \((\beta), (\gamma), (\delta)\) follows from Proposition 3.1.

\((\alpha) \Rightarrow (\beta)\). Theorem 2.3 yields polararness of \( E \). Now let \( x_1, x_2, \ldots \) be a weakly Cauchy sequence. Then \( x_{n+1} - x_n \to 0 \) weakly, hence strongly. As \( E \) is complete, there is an \( x \in E \) with \( x_n \to x \) strongly, hence weakly. Thus, \( E \) is weakly sequentially complete. Obviously, \( E \) does not contain \( \ell^\infty \).
(δ) ⇒ (α). Let $x_1, x_2, \ldots$ be a sequence in $E$ tending weakly to 0. By weak sequential completeness the formula

$$(\eta_1, \eta_2, \ldots) \overset{T}{\rightarrow} \sigma(E, E') - \lim_{n \to \infty} \sum_{i=1}^{n} \eta_i x_i$$

defines a linear map $T : \ell^\infty \rightarrow E$. $E$ is polar and $\{x_1, x_2, \ldots\}$ is weakly bounded hence bounded ((6), Cor. 7.7) and it follows that $T$ is continuous. By (δ), $T$ is compact. Then $\{x_1, x_2, \ldots\}$ is a compactoid on which the weak and strong topologies coincide (6, Theorem 5.12). Thus, $x_n \to 0$ strongly and the theorem is proved. □

Problem. Does there exist a Banach (or Fréchet) space that is polar, is not (O.P.), and does not contain $\ell^\infty$? In other words, may we drop the condition of weak sequential completeness in Theorem 3.2?

References