The Orlicz-Pettis property in $p$-adic analysis

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Abstract

For a non-archimedean locally convex space $E$ the property (O.P.): "every weakly convergent sequence in $E$ is convergent" is studied. Examples are given (1.3, 2.4-2.7). If the scalar field $K$ is spherically complete every $E$ has (O.P.) (1.2). If not, the property (O.P.) is closely related to "$E$ does not contain $\ell^\infty$" (3.2).

Terminology

Throughout $K$ is a non-archimedean nontrivially valued field that is complete with respect to the metric induced by the valuation $| |$. For notations, definitions, ... we refer to [5] for normed spaces and to [6] for general locally convex spaces. However, we recall the following. Let $E, F$ be $K$-linear spaces. The $K$-linear span of a set $X \subseteq E$ is denoted $[X]$, the (algebraic) dual of $E$ is $E^*$. If $p, q$ are (non-archimedean) seminorms on $E, F$ respectively we denote by $p \otimes q$ the seminorm

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\( z \mapsto \inf \left\{ \max_{1 \leq i \leq n} p(x_i) q(y_i) : n \in \mathbb{N}, \ z = \sum_{i=1}^{n} x_i \otimes y_i : x_i \in E, \ y_i \in F \right\} \) on \( E \otimes F \). A seminorm \( p \) on \( E \) is polar if \( p = \sup\{|f| : f \in E^*, |f| \leq p\} \).

Now let \( E, F \) be locally convex spaces over \( K \). The topological dual of \( E \) is \( E' \), the weak topology on \( E \) is \( \sigma(E, E') \). \( E \) is called of countable type if for every continuous seminorm \( p \) on \( E \) the normed space \( E/\ker p \) is of countable type. \( E \) is called a polar space if the topology is generated by polar seminorms whereas \( E \) is called strongly polar if every continuous seminorm is polar. On the tensor product \( E \otimes F \) we always consider the topology generated by the seminorms \( p \otimes q \) where \( p, q \) are continuous seminorms on \( E, F \) respectively.

From now on in this paper “locally convex” will mean “Hausdorff locally convex”.

1. (O.P.)-spaces

The classical Banach space \( \ell^1 \) (over \( \mathbb{R} \) or \( \mathbb{C} \)) has the property that every weakly convergent sequence is norm convergent, which is known as the Orlicz-Pettis Theorem. In our non-archimedean theory we therefore define

**Definition 1.1.** A locally convex space over \( K \) is called Orlicz-Pettis space ((O.P.)-space) if every weakly convergent sequence is convergent.

We first consider some immediate examples. It was shown by Monna in [3] that \( c_0 \) is an (O.P.)-space (observe that in our case the dual of \( c_0 \) is no longer \( \ell^1 \) but \( \ell^\infty \)). By straightforward arguments one can prove that subspaces, products and locally convex direct sums of (O.P.)-spaces are again (O.P.)-spaces. ([4], Propositions 1.2, 1.4.) As every space of countable type is a subspace of some power of \( c_0 \) we obtain the (O.P.)-property for every space of countable type. Now let \( E \) be any locally convex space and let \( x_1, x_2, \ldots \) be a sequence in \( E \) converging weakly to 0, and set \( D = [x_1, x_2, \ldots] \). If also \( x_n \to 0 \) in \( \sigma(D, D') \) then, as \( D \) is of countable type, it would follow that \( x_n \to 0 \) in the original topology of \( E \). Now \( \sigma(E, E')|D = \sigma(D, D') \) as soon as every \( f \in D' \) has an extension \( \tilde{f} \in E' \). Such extensions exist certainly if \( K \) is spherically complete, by Ingleton’s Theorem. Hence,

**Theorem 1.2**

*If \( K \) is spherically complete every locally convex space over \( K \) is an (O.P.)-space.*
For nonspherically complete $K$ the previous reasoning produces the following:

**Theorem 1.3**

The following spaces are (O.P.)-spaces.

(i) Every strongly polar space.
(ii) Every space of countable type.
(iii) Every Banach space with a base.
(iv) Any $K$-vector space equipped with the strongest locally convex topology.

**Proof.** All the spaces indicated in (i), (ii), (iii) have the property that every continuous linear function on a subspace of countable type can be extended to a continuous linear function on the whole space ([6], Definition 3.5, Theorem 4.4 and [5], Corollary 3.18). To prove (iv) just observe that the space is linearly homeomorphic to the locally convex direct sum of a collection of onedimensional spaces. □

Because of Theorem 1.2 we assume from this point on in this paper that $K$ is not spherically complete.

To obtain a space which is not (O.P.), take $E := c_{00}$. In fact, since $(c_{00})' \simeq c_0$ (see [5], Theorem 4.17) it follows that $(1, 0, 0, \ldots), (0, 1, 0, \ldots), (0, 0, 1, 0, \ldots), \ldots$ tends weakly to 0 but, of course, is not norm convergent. This example also tells us that the class of the (O.P.)-spaces is not closed for forming of quotients: Every Banach space (in particular $c_{00}$) is a quotient of a Banach space with a base. The space $c_{00}$ will play a key role in characterizing (O.P.)-spaces (Theorem 3.2).

The following observation will be used in the sequel several times. If $E$ is an (O.P.)-space then the weak topology $\sigma(E, E')$ is Hausdorff. Indeed, if $x \in E$ and $f(x) = 0$ for all $f \in E'$ then $0, x, 0, x, \ldots$ converges weakly, hence strongly, so $x = 0$. Obviously the converse does not hold (again, take $E := c_{00}$).

2. (O.P.)-spaces of continuous functions

To establish the (O.P.)-property for some spaces of (vector valued) continuous functions we first prove three structure theorems.

**Theorem 2.1**

If $E$ and $F$ are (O.P.)-spaces then so is $E \otimes F$. 

Proof. Let $z_1, z_2, \ldots$ be a sequence in $E \otimes F$ converging weakly to 0. Let $p$ resp. $q$ be a continuous seminorm on $E$ resp. $F$. We shall prove that $(p \otimes q)(z_n) \to 0$. Let $0 < t < 1$. By an obvious modification of [5], Theorem 4.30(ii) (see also [1], Lemma 2.1) for each $n \in \mathbb{N}$ there exists $x_1^n, \ldots, x_{m_n}^n \in E$ and $y_1^n, \ldots, y_{m_n}^n \in F$ such that

(i) $z_n = \sum_{k=1}^{m_n} x_k^n \otimes y_k^n$,
(ii) $y_1^n, \ldots, y_{m_n}^n$ are $t$-orthogonal with respect to $q$,
(iii) $1 \leq q(y_k^n) \leq 2 (k \in \{1, \ldots, m_n\})$.

Now let $f \in E'$. The map $f \otimes 1 : E \otimes F \to F$ sends $z_1, z_2, \ldots$ into a weakly convergent, hence convergent, sequence in $F$, i.e.,

$$\lim_{n \to \infty} q\left(\sum_{k=1}^{m_n} f(x_k^n)y_k^n\right) = 0.$$

By $t$-orthogonality and (iii)

$$\lim_{n \to \infty} \max_{1 \leq k \leq m_n} |f(x_k^n)| = 0.$$

As the latter is true for every $f \in E'$ the sequence $x_1^1, x_2^1, \ldots, x_1^2, x_2^2, \ldots$ converges weakly to 0 in the (O.P.)-space $E$, hence with respect to $p$ so that

$$(p \otimes q)(z_n) \leq \max_{1 \leq k \leq m_n} p(x_k^n)q(y_k^n) \leq 2 \max_{1 \leq k \leq m_n} p(x_k^n) \to 0 \quad \text{for } n \to \infty$$

implying $(p \otimes q)(z_n) \to 0$. □

Theorem 2.2

Let $(E, \tau)$ be a metrizable locally convex space and let $D$ be a dense subspace of $E$. If $D$ is an (O.P.)-space then so is $E$.

Proof. There is an invariant metric $d$ on $E$ inducing $\tau$. Let $x_1, x_2, \ldots$ be a sequence in $E$ such that $\lim_{n \to \infty} x_n = 0$ in $\sigma(E, E')$. For each $n \in \mathbb{N}$, choose a $y_n \in D$ with $d(x_n, y_n) < 1/n$. Then $x_n - y_n \xrightarrow{\tau} 0$ so $x_n - y_n \to 0$ in $\sigma(E, E')$ so that $y_n = x_n - (x_n - y_n) \to 0$ in $\sigma(E, E')$, hence in $\sigma(D, D')$. Now $D$ is an (O.P.)-space, so $y_n \xrightarrow{\tau} 0$. But then also $x_n = (x_n - y_n) + y_n \xrightarrow{\tau} 0$. □

Problem. Does the conclusion hold if we drop the metrizability condition?

Theorem 2.3

Every metrizable (O.P.)-space is polar.
Proof. Let \( p_1, p_2, \ldots \) be seminorms defining the topology \( \tau \) of a (metrizable) locally convex space \( E \). For each \( n \in \mathbb{N} \) define the seminorm \( \tilde{p}_n \) by
\[
\tilde{p}_n = \sup \{ |f| : f \in E', |f| \leq p_n \}.
\]
Then the topology \( \tilde{\tau} \) induced by the \( \tilde{p}_n \) is polar. From the first inclusion in
\[
(*) \quad \sigma(E, E') \subset \tilde{\tau} \subset \tau
\]
we obtain that \( \tilde{\tau} \) is Hausdorff, so \( (E, \tilde{\tau}) \) is metrizable. Since \( (E, \tau) \) is (O.P.) we conclude from (*) that \( \tau \) and \( \tilde{\tau} \) have the same convergent sequences implying \( \tau = \tilde{\tau} \) by metrizability. We see that \( \tau \) is polar. \( \square \)

Remark. There exist (nonmetrizable) (O.P.)-spaces that are not polar ([4], Proposition 4.1).

Now we turn to function spaces. For a Hausdorff zerodimensional topological space \( X \) and a locally convex space \( E \) over \( K \) we define
\[
PC(X, E): \text{The space of all continuous functions } f : X \to E \text{ for which } f(X) \text{ is precompact, endowed with the topology } \tau_u \text{ of uniform convergence,}
\]
\[
C(X, E): \text{The space of all continuous functions } X \to E, \text{ endowed with the topology } \tau_c \text{ of compact convergence,}
\]
\[
C_b(X, E): \text{The space of all bounded continuous functions } X \to E, \text{ endowed with the strict topology } \tau_s. \text{ This is the topology generated by all seminorms } f \mapsto \sup_{x \in X} |\phi(x)|p(f(x)) \quad (f \in C_b(X, E)) \text{ where } \phi : X \to K \text{ is a bounded function vanishing at infinity and } p \text{ is a continuous seminorm on } E.
\]

In the sequel we shall often restrict to Fréchet spaces \( E \) as we need the Theorems 2.2 and 2.3.

**Theorem 2.4**

Let \( E \) be a Fréchet space. Then \( PC(X, E) \) is an (O.P.)-space if and only if \( E \) is an (O.P.)-space.

Proof. The constant functions form a subspace of \( PC(X, E) \) that is linearly homeomorphic to \( E \) which proves the "only if". To complete the proof, let \( E \) be an (O.P.)-space. Then, as \( PC(X, K) \) has a base ([5], Cor. 5.23), we have by Theorems 1.3(iii) and 2.1 that \( PC(X, K) \otimes E \) is a metrizable (O.P.)-space. It is easily seen that the map \( T : PC(X, K) \otimes E \to PC(X, E) \), given by the formula
\[
T \left( \sum_{j=1}^{n} f_j \otimes a_j \right)(x) = \sum_{j=1}^{n} f_j(x) a_j \quad (x \in X)
\]
is a linear homeomorphism onto a dense subspace of \( PC(X, E) \). Then \( PC(X, E) \) is an (O.P.)-space by Theorem 2.2. \( \square \)
Corollary 2.5

Let $E$ be a Fréchet space. Then $C(X,E)$ is an (O.P.)-space if and only if $E$ is an (O.P.)-space.

Proof. We prove the "if". Let $E$ be an (O.P.)-space, let $f_1, f_2, \ldots$ be a sequence in $C(X,E)$ converging weakly to 0. Then, for every compact set $H$ in $X$, the sequence $n \mapsto f_n|H$ converges weakly to 0 in $PC(H,E)$. By Theorem 2.4 $f_n \to 0$ uniformly on $H$ and the conclusion follows. □

Corollary 2.6

Let $E$ be a Fréchet space. Then $C_b(X,E)$ is an (O.P.)-space if and only if $E$ is an (O.P.)-space.

Proof. Again we prove the "if". Let $E$ be an (O.P.)-space, let $f_1, f_2, \ldots$ converge weakly to 0 in $C_b(X,E)$. Since $\tau_c \subset \tau_\beta$ this sequence converges also weakly to zero in $(C_b(X,E), \tau_\beta)$. The latter, being a subspace of $(C(X,E), \tau_c)$ is (O.P.) by Corollary 2.5, so $f_n \to 0$ uniformly on compacts. Further, $E$ is polar and so is $(C_b(X,E), \tau_\beta)$ implying that $\{f_1, f_2, \ldots\}$ is $\tau_\beta$-bounded. Now apply Proposition 2.11 and Corollary 2.12 of [2] to conclude that $f_n \to 0$ in $\tau_\beta$. □

The picture changes if we endow $C_b(X,E)$ with the uniform topology $\tau_u$:

Corollary 2.7

Let $E$ be a Fréchet space. Then $(C_b(X,E), \tau_u)$ is an (O.P.)-space if and only if $X$ is pseudocompact and $E$ is an (O.P.)-space.

Proof. $\Rightarrow$. If $X$ is not pseudocompact we can find a countably infinite clopen partition $X = \bigcup_n X_n$. Choose $e \in E$ and define $T: \ell^\infty \to C_b(X,E)$ by the formula

$$T(\alpha_1, \alpha_2, \ldots)(x) = \alpha_n e \quad \text{if } n \in \mathbb{N}, x \in X_n.$$ 

It is easily seen that $T$ is a linear homeomorphism of $\ell^\infty$ onto a subspace of $(C_b(X,E), \tau_u)$ which yields a contradiction as $\ell^\infty$ is not (O.P.). For the other conclusion, consider again the constant functions.

$\Leftarrow$. One verifies that, if $X$ is pseudocompact, then $C_b(X,E) = PC(X,E)$. Now apply Theorem 2.4. □
3. Fréchet (O.P.)-spaces

An (O.P.)-space cannot contain a copy of $\ell^\infty$ as $\ell^\infty$ itself is not (O.P.). This simple observation is the starting point of Theorem 3.2 that characterizes Fréchet (O.P.)-spaces. The key result is Proposition 3.1 in which we describe polar spaces that do not contain $\ell^\infty$.

**Proposition 3.1**

For a polar locally convex space $E$ the following are equivalent.

(a) $E$ does not contain a subspace linearly homeomorphic to $\ell^\infty$.

(b) Every continuous linear map $\ell^\infty \to E$ is compact.

(c) $E$ does not contain a complemented subspace linearly homeomorphic to $\ell^\infty$.

**Proof.** (a) $\Rightarrow$ (b). Let $T : \ell^\infty \to E$ be a noncompact continuous linear map; we derive a contradiction. Let $e_1, e_2, \ldots$ be the unit vectors of $\ell^\infty$. Then $\{T e_1, T e_2, \ldots\}$ is not a compactoid (otherwise the weak closure of the absolutely convex hull of $T e_1, T e_2, \ldots$ would be a compactoid ([6], Theorem 5.13) hence so would its subset $T(\{x \in \ell^\infty : \|x\| \leq 1\})$ implying compactness of $T$). So, there exists a continuous polar seminorm $p$ such that $\{T e_1, T e_2, \ldots\}$ is not a $p$-compactoid. By [7], Theorem 2 there exists a $t \in (0, 1)$ and a subsequence $z_1, z_2, \ldots$ of $T e_1, T e_2, \ldots$ that is $t$-orthogonal with respect to $p$ and such that $\inf_n p(z_n) > 0$. Without loss, assume $p(z_n) \geq 1$ for each $n$.

Now, inductively we shall construct a subsequence $u_1, u_2, \ldots$ of $z_1, z_2, \ldots$ and $f_1, f_2, \ldots \in E'$ such that $|f_n| \leq 2t^{-1}p$ for all $n$ and

$$|f_m(u_n)| = \begin{cases} 0 & \text{if } m > n \\ 1 & \text{if } m = n \end{cases} \quad |f_m(u_n)| \leq \frac{1}{2} \quad \text{if } m < n$$

To do that, observe that the function $h_1 : \lambda z_1 \mapsto \lambda$ ($\lambda \in K$) satisfies $|h_1| \leq p$. By polarity it can be extended to an $f_1 \in E'$ such that $|f_1| \leq 2p$. Set $u_1 := z_1$. Suppose $f_1, \ldots, f_{m-1}$ and $u_1, \ldots, u_{m-1}$ are chosen with the required properties. Since $T e_n \rightharpoonup 0$ weakly we have $z_n \rightharpoonup 0$ weakly. So we can find a $k$ (larger than the indexes with respect to $z$ of $u_1, \ldots, u_{m-1}$) such that $|f_1(z_n)| \leq 1/2, \ldots, |f_{m-1}(z_n)| \leq 1/2$ for $n \geq k$. Choose $u_m := z_k$. The function $h_m : \lambda_1 u_1 + \ldots + \lambda_m u_m \mapsto \lambda_m$ ($\lambda_1, \ldots, \lambda_m \in K$) satisfies $|h_m| \leq t^{-1}p$ so it can be extended to a function $f_m \in E'$ such that $|f_m| \leq 2t^{-1}p$. We see that $f_1, \ldots, f_m$ and $u_1, \ldots, u_m$ have the required properties.
Now we have that \( u_1, u_2, \ldots \) is a subsequence, say \( T e_{i_1}, T e_{i_2}, \ldots \) of \( T e_1, T e_2, \ldots \). Define a linear isometry \( \Omega : \ell^\infty \to \ell^\infty \) by the formula

\[
(\Omega(y_1, y_2, \ldots))_n = \begin{cases} 
0 & \text{if } n \not\in \{i_1, i_2, \ldots\} \\
y_{i_m} & \text{if } m \in \mathbb{N}, n = i_m
\end{cases}
\]

and set \( S := T \circ \Omega \). Then obviously \( S \) is continuous and \( S \) is described by the formula

\[
S(y_1, y_2, \ldots) = \sigma(E, E') - \sum_{n=1}^{\infty} y_n u_n.
\]

Finally let \( y = (y_1, y_2, \ldots) \in \ell^\infty, y \neq 0 \). There is an \( m \in \mathbb{N} \) such that \( |y_m| > \frac{1}{2} \|y\| \). We have \( p(Sy) \geq \frac{1}{2} t|f_m(Sy)| = \frac{1}{2} t|\sum_{n \geq m} y_n f_m(u_n)| \). If \( n > m \) we have \( |y_n f_m(u_n)| \leq \frac{1}{2} \|y\| \leq \frac{1}{2} \|y\| \) whereas \( |y_m f_m(u_m)| = |y_m| > \frac{1}{2} \|y\| \) so \( p(Sy) \geq \frac{1}{2} \|y\| \) implying that \( S \) is a linear homeomorphism from \( \ell^\infty \) onto \( S(\ell^\infty) \subset E \) which gives the desired contradiction.

\((\beta) \Rightarrow (\gamma)\) is obvious. The implication \((\gamma) \Rightarrow (\alpha)\) was proved in [8], Theorem 1.2 for (polar) Banach spaces \( E \). From here the step to locally convex \( E \) is easy ([4], Lemma 4.6). \( \square \)

**Theorem 3.2**

For a Fréchet space \( E \) the following are equivalent.

(\(\alpha\)) \( E \) is an (O.P.)-space.

(\(\beta\)) \( E \) is polar, weakly sequentially complete and \( E \) does not contain a subspace linearly homeomorphic to \( \ell^\infty \).

(\(\gamma\)) \( E \) is polar, weakly sequentially complete and \( E \) does not contain a complemented subspace linearly homeomorphic to \( \ell^\infty \).

(\(\delta\)) \( E \) is polar, weakly sequentially complete and every continuous linear map \( \ell^\infty \to E \) is compact.

**Proof.** The equivalence of \((\beta), (\gamma), (\delta)\) follows from Proposition 3.1.

(\(\alpha\)) \( \Rightarrow (\beta) \). Theorem 2.3 yields polarness of \( E \). Now let \( x_1, x_2, \ldots \) be a weakly Cauchy sequence. Then \( x_{n+1} - x_n \to 0 \) weakly, hence strongly. As \( E \) is complete, there is an \( x \in E \) with \( x_n \to x \) strongly, hence weakly. Thus, \( E \) is weakly sequentially complete. Obviously, \( E \) does not contain \( \ell^\infty \).
\((\delta) \Rightarrow (\alpha)\). Let \(x_1, x_2, \ldots\) be a sequence in \(E\) tending weakly to 0. By weak sequential completeness the formula

\[
(\eta_1, \eta_2, \ldots) \overset{T}{\rightarrow} \sigma(E, E') - \lim_{n \rightarrow \infty} \sum_{i=1}^{n} \eta_i x_i
\]

defines a linear map \(T : \ell^\infty \rightarrow E\). \(E\) is polar and \(\{x_1, x_2, \ldots\}\) is weakly bounded hence bounded \(([6], \text{Cor. 7.7})\) and it follows that \(T\) is continuous. By \((\delta)\), \(T\) is compact. Then \(\{x_1, x_2, \ldots\}\) is a compactoid on which the weak and strong topologies coincide \(([6], \text{Theorem 5.12})\). Thus, \(x_n \rightarrow 0\) strongly and the theorem is proved. \(\square\)

**Problem.** Does there exist a Banach (or Fréchet) space that is polar, is not (O.P.), and does not contain \(\ell^\infty\)? In other words, may we drop the condition of weak sequential completeness in Theorem 3.2?

**References**


