\textbf{\ell^p \textit{SPACES OVER ULTRAMETRIC FIELDS}}

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\textbf{Abstract.} We consider vector spaces over a nonarchimedean valued field \( K \) with a norm that need not satisfy the strong triangle inequality. In §1 we study conditions in order that an equivalent ultranorm exists. In §2 we show that the spaces \( \ell^p \) \((1 < p < \infty) \) have topological dimension 1 making them not ultranormable. In §3 we show that a certain direct sum of finite-dimensional \( \ell^p \)-spaces is an ultrametrizable but not ultranormable Banach space. Finally we prove in §4 spherical completeness of finite-dimensional \( \ell^p \)-spaces.

\textbf{PRELIMINARIES.} Throughout \( K \) is a non-archimedean nontrivially valued field that is complete with respect to the metric induced by the valuation \( |\cdot| \). \( B_K := \{ \lambda \in K : |\lambda| \leq 1 \} \). In this note, a \textit{Norm} on a \( K \)-vector space \( E \) is a map \( ||\cdot|| : E \to [0, \infty) \) satisfying (i) \( ||x|| = 0 \iff x = 0 \) (ii) \( ||\lambda x|| = |\lambda| \cdot ||x|| \) (iii) \( ||x + y|| \leq ||x|| + ||y|| \) for all \( x, y \in E, \lambda \in K \). If \( ||\cdot|| \) satisfies (i), (ii) and (iii)' \( ||x + y|| \leq \max(||x||, ||y||) \) it is called an \textit{ultranorm}. In the same spirit, \textit{ultrasemimorm} and \textit{ultrametric} are defined. A \textit{locally convex space} is a topological \( K \)-vector space whose topology is induced by a family of ultrasemimorms in the usual way.

For an absolutely convex absorbent subset \( A \) (i.e. \( A \) is a \( B_K \)-module and \( E = \bigcup_{\lambda \in K} \lambda A \)) of a \( K \)-vector space \( E \) the Minkowski function \( p_A \) is defined by the formula

\[ p_A(x) = \inf\{ |\lambda| : \lambda \in K, x \in \lambda A \}. \]

It is an ultrasemimorm. For more explanation on non-archimedean terminology we refer to [7].
1. *(ULTRA)*NORMS AND INVARIANT *(ULTRA)*METRICS

We start off with

**Proposition 1.1.** Let \( E \) be a metrizable topological vector space over \( K \). Then the topology on \( E \) is induced by an invariant metric.

**Proof.** The proof of [6], §15.11 applies with obvious modifications.

The picture changes however when we consider the ‘natural’ ultrametric version of Proposition 1.1; it is false:

**Example 1.2.** There exists a Normed space \( E \) over \( K \) that is ultrametrizable but such that its topology is not induced by an invariant ultrametric.

To prove this we first consider the following peculiar fact.

**Theorem 1.3.** A topological vector space \( E \) over \( K \) whose topology is induced by an invariant ultrametric \( d \) is locally convex.

**Proof.** For each \( n \in \mathbb{N} \) set \( W_n := \{ x \in E : d(x,0) < \frac{1}{n} \} \) and \( V_n := \bigcap \{ \lambda W_n : \lambda \in K, |\lambda| \geq 1 \} \). The \( W_n \) form a neighbourhood base at 0 and so do the \( V_n \) by [2] Prop. 4p9. By assumption \( d(x+y,0) \leq \max(d(x+y,y),d(y,0)) = \max(d(x,0),d(y,0)) \) for all \( x,y \in E \) so that each \( W_n \) is additively closed; hence so are the \( V_n \). Since also \( \lambda V_n \subset V_n \) for all \( \lambda \in K, |\lambda| \leq 1 \) we obtain that the sets \( V_n \) are absolutely convex.

**Note to Theorem 1.3.** For a metrizable topological vector space \( E \) over \( K \) we have that \( E \) is locally convex if and only if its topology is induced by an invariant ultrametric.

For a proof, just combine Theorem 1.3 and [8], Theorem 3.12.

**Proof of Example 1.2.** Set \( E := \{(a_1,a_2,...) \in K^\mathbb{N} : a_n = 0 \text{ for large } n\} \) with the obvious operations and the Norm \( \| . \| : (a_1,a_2,...) \mapsto \Sigma |a_n| \). Then the unit vectors \( e_1 := (1,0,0,...), e_2 := (0,1,0,...), \ldots \) form a bounded set but its absolutely convex hull contains \( e_1 + e_2 + \cdots + e_n \) for each \( n \), so is unbounded. It follows that \( E \) cannot be locally convex. Hence, by Theorem 1.3, there is no invariant ultrametric defining the topology.

To show that \( E \) is ultrametrizable set \( E_n := \{(a_1,a_2,...) \in E : a_m = 0 \text{ for } m > n\} \). Then \( E_1, E_2, \ldots \) form a closed cover of \( E \). By finite dimensionality, on \( E_n \) the Norm \( \| . \| \) is equivalent to the ultranorm \( (a_1,a_2,...) \mapsto \max_m |a_m| \), so in particular \( \dim E_n = 0 \) (here \( \dim \) is the covering dimension). Then, by [3] 7.2.1, \( \dim E = 0 \). But it is proved in [5] that a metric space \( X \) with \( \dim X = 0 \) is ultrametrizable and we are done.

**Remark.** For similar examples that are in addition complete see §3, Remark 1.

We conclude this section by elaborating Theorem 1.3 for Normed spaces \( E \).
**Theorem 1.4.** For a Normed space \((E, \| \|)\) the following are equivalent.

(α) \(\| \|\) is equivalent to an ultranorm.

(β) The absolutely convex hull of a bounded set in \(E\) is bounded.

(γ) The partial sums of a bounded sequence in \(E\) are bounded.

(δ) The partial sums of a null sequence in \(E\) are bounded.

(ε) The partial sums of a null sequence in \(E\) form a Cauchy sequence.

(ζ) \(E\) is locally convex.

(η) The topology of \(E\) is defined by an invariant ultrametric.

**Proof.** Clearly (β) ⇒ (γ) ⇒ (δ); we now prove (δ) ⇒ (β). If \(\co X\) were not bounded for some bounded \(X \subset E\) we could find inductively a sequence \(z_1, z_2, \ldots\) in \(\co X\) for which \(\|z_n\| > 2^n + n \sum_{i<n} \|z_i\|\) for each \(n\). There exist \(\xi_1, \xi_2, \ldots \in B_K, x_1, x_2, \ldots \in X\) and \(1 = m_0 < m_1 < m_2 \ldots\) in \(\mathbb{N}\) such that for each \(k \in \mathbb{N}\)

\[
z_k = \sum_{i=m_k+1}^{m_{k+1}} \xi_i x_i.
\]

Choose \(\alpha_1, \alpha_2, \ldots \in B_K\) such that \(\lim_{n \to \infty} \alpha_n = 0\) but \(n|\alpha_n| \geq 1\) for each \(n\). The sequence

\[
\alpha_1 \xi_1 x_1, \alpha_1 \xi_2 x_2, \ldots, \alpha_1 \xi_{m_1} x_{m_1}, \alpha_2 \xi_{m_1+1} x_{m_1+1}, \ldots, \alpha_2 \xi_{m_2} x_{m_2}, \alpha_3 \xi_{m_2+1} x_{m_2+1}, \ldots
\]
tends to 0 since \(|\alpha_n| \to 0\) and \(\sup \|\xi_n x_n\| < \infty\) from the boundedness of \(X\). But we shall prove that the sequence of their partial sums is not. In fact, for any \(n \in \mathbb{N}\), \(\alpha_1 z_1 + \cdots + \alpha_n z_n\) is such a partial sum and

\[
\|\alpha_1 z_1 + \cdots + \alpha_n z_n\| \geq \|\alpha_n z_n\| - \|\alpha_1 z_1 + \cdots + \alpha_{n-1} z_{n-1}\|
\]

\[
\geq \frac{1}{n} \|z_n\| - \sum_{i=1}^{n-1} \|z_i\|
\]

\[
\geq \frac{1}{n} (2^n + n \sum_{i<n} \|z_i\|) - \sum_{i<n} \|z_i\|
\]

\[
\geq 2^n / n.
\]

We also have obviously (α) ⇒ (ε) ⇒ (δ); we prove (β) ⇒ (α):

The unit ball of \(E\) is bounded hence so is its absolutely convex hull \(A\). \(A\) is open. Hence the Minkowski function \(p_A\) associated to \(A\) is the requested ultranorm.

At this stage we have proved the equivalence of (α) - (ε). The implication (α) ⇒ (η) is trivial, (η) ⇒ (ζ) is Theorem 1.3, and (ζ) ⇒ (γ) is immediate.

**Corollary 1.5.** A Normed \(K\)-vector space is ultranormable if and only if each subspace of countable type is ultranormable, if and only if each countably generated subspace is ultranormable.
Proof. Consider (γ), (δ) or (ε) of the previous Theorem.

Proposition 1.6. For a Normed space \((E, \| \|)\) of countable type the following are equivalent.

(α) \(E\) is ultranormable.
(β) Every weakly bounded set in \(E\) is bounded.
(γ) Every weakly convergent sequence in \(E\) is bounded.
(δ) Every weakly convergent sequence in \(E\) is convergent.

Proof. (α) ⇒ (δ) is well known, (δ) ⇒ (γ) is trivial. To prove (γ) ⇒ (β) suppose \(X \subset E\) is an unbounded but weakly bounded set. Then we can choose a \(\lambda \in K, |\lambda| > 1\) and a weakly bounded sequence \(x_1, x_2, \ldots \in X\) such that \(\|x_n\| \geq |\lambda|^n\) for all \(n \in \mathbb{N}\). Then \(\lambda^{-n}x_n \to 0\) weakly but \(\|\lambda^{-n}x_n\| > |\lambda|^n \to \infty\) conflicting (γ). Finally we prove (β) ⇒ (α). Let \(N\) be the largest ultraseminorm that is \(\leq \|\|\). If \(X\) is an \(N\)-bounded set then \(X\) is also \(\sigma(E, (E, N)')\)-bounded. But it is well known that \((E, N)' = (E, \|\|)'\) so that \(X\) is weakly bounded and hence bounded by (β). We see that \(\|\|\) and \(N\) have the same bounded sets so they must be equivalent.

2. THE SPACE \(\ell^p\)

For \(1 \leq p < \infty\) we define \(\ell^p\) to be the set of all sequences \((x_1, x_2, \ldots) \in K^\mathbb{N}\) such that \(\sum |x_i|^p < \infty\). Like in the Archimedean case one proves that \(\ell^p\) is a \(K\)-vector space and that the formula

\[\|(x_1, x_2, \ldots)\|_p = \left(\sum |x_i|^p\right)^{1/p}\]

defines a Norm on \(\ell^p\) for which it is complete.

The completion of the space \(E\) of Example 1.2 is \(\ell^1\) so it is a natural question to ask whether \(\ell^1\) is ultrametrizable. The negative answer was given in [1], it was even proved that \(\ell^1\) is not zerodimensional! Here we make a further step by proving that \(\ell^p\) has dimension 1 in the following sense.

Theorem 2.1. Any ball in \(\ell^p(1 \leq p < \infty)\) has a zerodimensional boundary. ("\(\ell^p\) has weak inductive dimension 1").

Proof. It suffices to prove that for each \(r \in (0, 1]\) the set

\[S := \{x \in \ell^p : \|x\|_p = r\}\]

is zerodimensional. To this end we consider the inclusion map \(\varphi : S \to B_K^\mathbb{N}\) where \(B_K^\mathbb{N}\) carries the product topology and show that \(\varphi\) is a homeomorphism of \(S\) onto \(\varphi(S)\);
we need only to check the continuity of $\varphi^{-1}$. Thus, let $x^1, x^2, \ldots$ be a sequence in $S$ converging coordinatewise to $x \in S$; we shall prove that $\|x^n - x\|_p \to 0$. Write $x^n = (x^n_1, x^n_2, \ldots)$, $x = (x_1, x_2, \ldots)$ and let $\varepsilon > 0$. There is an $m \in \mathbb{N}$ such that

(1) \[ \sum_{i \leq m} |x_i|^p \geq r^p - \varepsilon. \]

There is an $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$

(2) \[ \sum_{i \leq m} |x_i - x^n_i|^p < \varepsilon \]

$|x^n_i| = |x_i|$ whenever $i \in \{1, \ldots, m\}, x_i \neq 0$.

Then $\sum_{i \leq m} |x^n_i|^p \geq \sum_{i \leq m} |x_i|^p \geq r^p - \varepsilon$ and therefore for all $n \geq N_1$

(3) \[ \sum_{i > m} |x^n_i|^p = r^p - \sum_{i \leq m} |x^n_i|^p \leq r^p - (r^p - \varepsilon) = \varepsilon. \]

Similarly we obtain from (1)

(4) \[ \sum_{i > m} |x_i|^p \leq \varepsilon. \]

Combination of (3) and (4) yields that for all $n \geq N_1$

\[ \sum_{i > m} |x_i - x^n_i|^p \leq \sum_{i > m} \max(|x_i|, |x^n_i|)^p \leq \sum_{i > m} |x_i|^p + \sum_{i > m} |x^n_i|^p < 2\varepsilon. \]

This, together with (2) results into

\[ \|x - x^n\|_p = \sum_{i=1}^{\infty} |x_i - x^n_i|^p \leq 3\varepsilon \]

and the theorem is proved.

3. DIRECT SUMS OF FINITE-DIMENSIONAL $\ell^p$ SPACES

In this section we obtain examples of spaces that are 'closer' to ultranorminable spaces than the $\ell^p$ spaces of §2. Like in the Archimedean theory we define the direct sum $\bigoplus E_n$ of Normed spaces $E_1, E_2, \ldots$ over $\mathbb{K}$ to be the space

\[ \{(x_1, x_2, \ldots) \in \prod_n E_n : \lim_{n \to \infty} \|x_n\| = 0\} \]
With the Norm given by

\[ \| \| : (x_1, x_2, \ldots) \mapsto \max_n \| x_n \| \]

\( \bigoplus E_n \) becomes a Normed space, which is complete if all \( E_n \) are complete.

For each \( n \in \mathbb{N} \) we define \( \ell^p_n \) to be the vector space \( \mathbb{K}^n \) but with the \( \ell^p \)-Norm

\[ \| (x_1, \ldots, x_n) \|_p := \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p}. \]

Clearly every \( \ell^p_n \) is ultranormable. The following lemma measures the 'distance' of \( \| \|_p \) to ultranorms.

**Lemma 3.1.** Let \( N \) be an ultranorm on \( \ell^n \). If \( c_1, c_2 > 0 \) are such that

\[ c_1 N \leq \| \|_p \leq c_2 N \]

then \( c_2/c_1 \geq \sqrt[p]{n} \).

**Proof.** Let \( e_1, e_2, \ldots, e_n \) be the unit vectors. Then \( \| \Sigma e_i \|_p = \sqrt[p]{n} \) and \( N(\Sigma e_i) \leq \max_i N(e_i) \) so that

\[ c_2 \geq \sqrt[p]{n} / \max_i N(e_i). \]

On the other hand, for each \( i \in \{1, \ldots, n\} \), \( c_i N(e_i) \leq \| e_i \|_p = 1 \) so that

\[ c_i \leq \left( \max_i N(e_i) \right)^{-1} \]

It follows that \( c_2/c_1 \geq \sqrt[p]{n} \).

Let us denote the canonical norm on \( \bigoplus_{n=1}^{\infty} \ell^p_n \) by \( \| \| \). We have

**Proposition 3.2.** \( \bigoplus \ell^p_n \) is not ultranormable.

**Proof.** If \( N \) were an equivalent ultranorm, say

\[ c_1 N \leq \| \| \leq c_2 N \]

then for every \( m \) we would have by using the obvious embedding of \( \ell^p_m \) into \( \bigoplus \ell^p_n \) and Lemma 3.1 that \( c_2/c_1 \geq \sqrt[p]{m} \). But \( \lim_{m \to \infty} \sqrt[p]{m} = \infty \), a contradiction.

On the other hand,
Proposition 3.3. $\bigoplus_{n \in \mathbb{N}} \ell^n_p$ is zerodimensional.

For the proof we need the following lemma about $\ell^n_p$.

Lemma 3.4. Let $x_1, x_2, \ldots$ be a sequence in $\ell^n_p$ converging to $x \in \ell^n_p$. Then $\|x_m\|_p \geq \|x\|_p$ for large $m$.

Proof. First observe that if $\lambda_i \in K$, $\lambda_i \to \lambda$ and $\lambda \neq 0$ then $|\lambda_i| = |\lambda|$ eventually. Thus, in general, if $\lambda_i \to \lambda$ then $|\lambda_i| \geq |\lambda|$ eventually. Hence, for each $j \in \{1, \ldots, n\}$

$$|(x_m)_j| \geq |(x)_j| \quad \text{(large } m)$$

Then

$$\|x_m\|_p = \left( \sum_{j=1}^{m} |(x_m)_j|^p \right)^{1/p} \geq \left( \sum_{j=1}^{m} |(x)_j|^p \right)^{1/p} = \|x\|_p.$$

Proof of Proposition 3.3. We shall prove that for each $\varepsilon > 0$ the open ball $B(0, \varepsilon^-) := \{x \in \bigoplus_{n=1}^{\infty} \ell^n_p : \|x\| < \varepsilon\}$ is closed. Let $x^1 := (x_1^1, x_2^1, \ldots)$, $x^2 := (x_1^2, x_2^2, \ldots, \ldots$ be a sequence in $B(0, \varepsilon^-)$ converging to $x := (x_1, x_2, \ldots)$. Then let $j \in \mathbb{N}$ be such that $\|x\| = \|x_j\|_p$. The sequence $x^1, x^2, \ldots$ converges certainly coordinatewise so $\|x_j^m\|_p \geq \|x_j\|_p$ for large $m$ by Lemma 3.4. Then for such $m$

$$\varepsilon > \|x^m\| \geq \|x^m_j\|_p \geq \|x_j\|_p = \|x\|$$

and it follows that $x \in B(0, \varepsilon^-)$.

Remark 1. If $K$ is separable then so is $\bigoplus_{n=1}^{\infty} \ell^n_p$ and hence by Proposition 3.3 the space $\bigoplus_{n=1}^{\infty} \ell^n_p$ is ultrametrizable. So here we have a complete space satisfying the conditions of Example 1.2.

Problem. Is $\bigoplus_{n=1}^{\infty} \ell^n_p$ ultrametrizable if the base field is not separable?

Remark 2. By making some modifications in the direct sum we can arrange that it is ultranormable. For example $\bigoplus_{n=1}^{\infty} \ell^n_p$ is ultranormable as soon as $n \mapsto n\sqrt{n}$ is bounded.

4. SPHERICAL COMPLETENESS OF $\ell^n_p$

Let us say that a Normed space is *spherically complete* if each nested sequence $B_1 \supset B_2 \supset \ldots$ of balls has a nonempty intersection. This concept plays a central role in the theory of ultranormed spaces.
Clearly every spherically complete Normed space is complete, the converse is not true. It was proved in [4] that $\ell^p(1 \leq p < \infty)$ is spherically complete ($K$ being spherically complete or not!) The proof uses the infinite-dimensionality of the space in an essential way. How about $\ell^n_\infty$? We shall prove (Theorem 4.5) that $\ell^n_\infty$ is spherically complete if $n > 1$ (obviously, $\ell^n_1 \simeq K$). The proof is less simple than one would expect. The case where $K$ has a discrete valuation is even more difficult than the one where the valuation is dense, which is also surprising. Simplifications of proofs would be welcome.

In the rest of this section, let $E$ be the space $\ell^n_m$ ($m \in \mathbb{N}, m > 1, p \in \mathbb{R}, p \geq 1$) i.e. $K^m$ endowed with the $\ell^p$ norm. Call $|K| = \{\lambda \mid \lambda \in K\}$ and $\|E\| = \{\|x\| : x \in E\}$. $\|\|_\infty$ will denote the $\ell^\infty$ norm over $E$. By a closed ball, we will always mean a set of the form $B[c, r] = \{x \in E : \|x - c\| \leq r\}$, where $c \in E$ is a center and $r > 0$ is a radius of the ball.

**Lemma 4.1.** Let $a, b \in E$, $r, s > 0$, and $\|a - b\| \leq r$. Assume that either $s \in |K|$ or the valuation of $K$ is dense. Then, $B[a, s] \subset B[b, r]$ if and only if

$$r^p - s^p \geq \|a - b\|^p - \min\{|a_i - b_i|^p : 1 \leq i \leq m\}.$$ 

**Proof.** Suppose $s^p > r^p - \|a - b\|^p + \min|a_i - b_i|^p$, and assume, for simplicity, that $|a_1 - b_1| = \min|a_i - b_i|$. Take $(\lambda_n) \subset K$ with $|\lambda_n| \to s$, $|\lambda_n| \leq s$ for all $n$. Then, from some $n_0$ on,

$$|\lambda_n|^p > r^p - \|a - b\|^p + |a_1 - b_1|^p \geq |a_1 - b_1|^p$$

(recall that $\|a - b\| \leq r$). Consider the point $x := (b_1 + \lambda_{n_0}, a_2, \ldots, a_m)$; then

$$\|x - a\|^p = |b_1 + \lambda_{n_0} - a_1|^p = |\lambda_{n_0}|^p \leq s^p$$

$$\|x - b\|^p = |\lambda_{n_0}|^p + |a_2 - b_2|^p + \cdots + |a_m - b_m|^p > r^p,$$

hence $x \in B[a, s]\setminus B[b, r]$.

Conversely, assume the inequality in the statement of the Lemma, and take $x$ in $B[a, s]$. Then

$$\|x - b\|^p = |(x_1 - a_1) + (a_1 - b_1)|^p + \cdots + |(x_m - a_m) + (a_m - b_m)|^p$$

is not greater than the maximum of the numbers

$$(1) \quad \sum_{j \in J}|x_j - a_j|^p + \sum_{k \in K}|a_k - b_k|^p,$$

where $\{J, K\}$ is some partition of the set $I = \{1, \ldots, m\}$ (we apologize for having given two meanings to the symbol $K$). For $j = \emptyset$, the number in (1) equals $\|a - b\|^p \leq r^p$. If
\[ J \neq \emptyset, \|x - a\| \leq s \text{ implies } \]
\[ \sum_{j \in J} |x_j - a_j|^p + \sum_{k \in K} |a_k - b_k|^p \leq s^p + \sum_{k \in K} (|a_k - b_k|^p - |x_k - a_k|^p) \leq \]
\[ \leq r^p - \sum_{i \in I} |a_i - b_i|^p + \min_{i \in I} |a_i - b_i|^p + \sum_{k \in K} |a_k - b_k|^p - \sum_{k \in K} |x_k - a_k|^p = \]
\[ = r^p - \sum_{j \in J} |a_j - b_j|^p - \sum_{k \in K} |x_k - a_k|^p + \min_{i \in I} |a_i - b_i|^p \leq r^p \]

\( (J \neq \emptyset \text{ is used for the last inequality}). \) Thus, \( B[a, s] \subseteq B[b, r] \).

**Corollary 4.2.**
1. Suppose that either \( s \in |K| \) or the valuation of \( K \) is dense. If \( B[a, s] \subseteq B[b, r] \), then \( \|a - b\|_\infty \leq r^p - s^p \). In case \( m = 2 \), the converse is also true.
2. If the valuation of \( K \) is dense, then a closed ball has only one center and only one radius.

**Proof.**
1. Apply Lemma 4.1 together with the simple observation
\[ \min |a_i - b_i|^p + \max |a_i - b_i|^p \leq \|a - b\|^p \]
(recall that \( m > 1 \)).
2. If \( K \) is densely valued and \( B[a, s] = B[b, r] \), then by last part,
\[ \|a - b\|_\infty \leq r^p - s^p \text{ and } \|a - b\|_\infty \leq s^p - r^p, \]
so \( a = b \) and \( s = r \).

**Remarks.** Let the valuation of \( K \) be discrete.
1. Centers need not be unique: take for instance as \( K \) any \( \mathbb{Q}_t \) and \( m = 2, p = 1 \); then
\[ B\left((0,0), \frac{1}{t} + \frac{1}{t^2}\right) = B\left((t^2, t^3), \frac{1}{t} + \frac{1}{t^2}\right). \]
This example illustrates the worst possible situations (see Lemma 4.4).
2. We have
\[ \|E\| = (|K|^p + \cdots + |K|^p)^{1/p} \quad (m \text{ times}). \]
(The decomposition of a number in \( \|E\|^p \) into a sum of \( m \) members of \( |K|^p \) need not be unique.)
3. Any closed ball can be written in the form \( B[a, r] \) for a unique \( r \in \|e\| \) (considering the center \( a \in E \) fixed). In the sequel, we shall always assume that closed balls have their radii in \( \|E\| \).
Lemma 4.3. Assume the valuation of $K$ is discrete. Suppose $R_n \in \|E\|$, $R^n = r^n_{1,n} + \cdots + r^n_{m,n}$, with $r_{1,n} \geq r_{2,n} \geq \cdots \geq r_{m,n}$ for all $n$ ($r_{k,n} \in |K|$ for $n \in \mathbb{N}, 1 < k \leq m$), and $R_n \rightarrow r$. Then there exists a decomposition of $r^p$, $r^p = r^p_1 + \cdots + r^p_m$, with $r_k \in |K|$ and $r_1 \geq \cdots \geq r_m$. Moreover, for $k = 1, \ldots, m$, and a certain subsequence of $R_n$,

1. $r_{k,n} \rightarrow r_k$
2. if $r_k \neq 0$, then $r_{k,n} = r_k$ for large $n$.

Proof. The sequence $(r_{m,n})_{n \in \mathbb{N}}$ is contained in $|K|$, and is bounded (since $R_n$ converges and all $r_{k,n}$ are positive), therefore either it tends to zero, or it has a constant subsequence. In the former case, we define $r_m := 0$, and in the latter, we define $r_m$ as one of the numbers that is attained infinitely many times by $r_{m,n}$.

Now consider only the convergent subsequence obtained in the last paragraph (call it $R_n$ again), and forget about the last term:

$$S^p_n := r^p_{1,n} + \cdots + r^p_{m-1,n} \rightarrow r^p - r^p_m,$$

then apply the same argument as before, showing that either $(r_{m-1,n})_{n \in \mathbb{N}}$ tends to zero (and then, $r_{m-1} := 0$) or it has some constant subsequence (and then, call that constant $r_{m-1}$).

An iteration of the same argument $m$ times gives the desired result.

Definition. For $\lambda \in K$ and $s \in |K|$, call $\Lambda(\lambda, s)$ a member of $K$ satisfying the following:

- if $|\lambda| = s$, then $\Lambda(\lambda, s) = \lambda$,
- if $|\lambda| \neq s$, then $|\Lambda(\lambda, s)| \geq |\Lambda(\lambda, s) - \lambda| = s$.

Then, $|\Lambda(\lambda, s)| \geq s \geq |\Lambda(\lambda, s) - \lambda|$.

Lemma 4.4. Assume the valuation of $K$ is discrete. Let $a, b \in E$, and $r, s \in \|E\|$, $s^p = s^p_1 + \cdots + s^p_m$, with $s_i \in |K|$ for $1 \leq i \leq m$ (see Remark 2). Then $B[a, s] \subset B[b, r]$ implies:

1. $r - s \geq 0$,
2. $r^p - (s^p - \min\{s^p_i : 1 \leq i \leq m\}) \geq \|a - b\|_\infty$.

Proof.

1. We shall define a vector $x$ in $E$ such that $\|x - b\| \geq s \geq \|x - a\|$. Then it will follow that $r \geq s$. For $i = 1, \ldots, m$, we take as $\lambda_i$ a $\Lambda(a_i - b_i, s_i)$, and define $x := (b_1 + \lambda_1, \ldots, b_m + \lambda_m)$.

2. Suppose the inequality does not hold, and take $i_1, i_2 \in \{1, \ldots, m\}$ with

$$|a_{i_1} - b_{i_1}| = \max\{|a_i - b_i| : 1 \leq i \leq m\}$$
$$s_{i_2} = \min\{s_i : 1 \leq i \leq m\};$$
then
\[ \sum_{i \neq i_2} s_i^p > r^p - |a_i - b_i|^p. \]

We take \( y := (b_1 + \lambda_1, \ldots, b_m + \lambda_m) \), where the \( \lambda_i (i = 1, \ldots, m) \) are defined as follows:

- \( \lambda_i = \Lambda(a_i - b_i, |a_i - b_i|) \)
- if \( i \neq i_1, i_2 \), then \( \lambda_i = \Lambda(a_i - b_i, s_i) \)
- if \( i_2 \neq i_1 \), then \( \lambda_i = \Lambda(a_{i_2} - b_{i_2}, s_{i_1}) \).

Thus,
\[ \|y - a\|^p = \sum_i |\lambda_i - (a_i - b_i)|^p \leq \sum_{i \neq i_2} s_i^p \leq s^p \]
\[ \|y - b\|^p = \sum_i |\lambda_i|^p \geq \sum_{i \neq i_2} s_i^p + |a_i - b_i|^p > r^p, \]
hence \( y \) belongs to \( B[a, s] \backslash B[b, r] \), which is impossible.

**Theorem 4.5.** Let \( \{B[x_n, R_n] : n \in \mathbb{N}\} \) be a shrinking collection of closed balls in \( E \), i.e., for all \( n \in \mathbb{N} \) we have

\[ B[x_n, R_n] \supset B[x_{n+1}, R_{n+1}]. \]

Then \( \bigcap_n B[x_n, R_n] \) is not empty.

**Proof.** Assume \( |K| \) is dense.

By Corollary 4.2 (1), \( \|x_n - x_{n+1}\|_{\infty} \leq R_n^p - R_{n+1}^p \to 0 \), hence \( \{x_n\} \) is a Cauchy sequence for the maximum norm, and then convergent to a certain \( x \in K^m \) in the \( \| \|_{\infty} \) norm (i.e., in the product topology). Then \( \|x_n - x\| \to 0 \), so \( x \in \bigcap_n B[x_n, R_n] \).

Now assume \( |K| \) is discrete.

By Lemma 4.4 (1) (see also Remark 3 on the way we take the radii), the sequence of radii \( \{R_n\} \) is decreasing; call \( r := \lim_n R_n \), and for each \( n \), \( R_n^p = r_{1,n}^p + \cdots + r_{m,n}^p \) with \( r_{k,n} \in |K| \) (for \( n \in \mathbb{N}, 1 \leq k \leq m \)), \( r_{1,n} \geq r_{2,n} \geq \cdots \geq r_{m,n} \).

If the limit \( r \) is attained, then \( R_n = r \) for \( n \geq n_0 \) and some \( n_0 \) in \( \mathbb{N} \). Therefore, \( x_{n_0} \) belongs to the intersection of the balls, since for \( n \geq n_0 \), \( x_n \in B[x_{n_0}, R_{n_0}] \) implies \( \|x_n - x_{n_0}\| \leq R_{n_0} = r = R_n \).

If \( r \) is never attained, then (Lemma 4.3), there is a subsequence of \( \{R_n\} \), \( \{R_{n_i}\} \), and a decomposition of \( r^p, r_{1}^p + \cdots + r_{m}^p, r_k \in |K|, r_1 \geq \cdots \geq r_m \), such that

1. \( r_{k,n_i} \to r_k \) (1 \( \leq k \leq m \))
2. If \( r_k \neq 0 \), then \( r_{k,n_i} = r_k \) for large \( i \).
We claim \( r_m = 0 \): otherwise, \( r_1, \ldots, r_{m-1} \) are also different from zero, and then (part 2 above), \( r_{1,n_i} = r_1, \ldots, r_{m,n_i} = r_m \) for a certain \( i \), so the limit \( r \) would be attained. Thus, \( r_{m,n_i} \to 0 \); therefore (apply Lemma 4.4 (2))

\[
\|x_{n_i} - x_{n+i+1}\|_\infty^p \leq R_{n_i}^p - (R_{n+i+1}^p - r_{m,n_i+1}^p) \to r_p - r_p + 0 = 0.
\]

Hence \((x_{n_i})\) is a \( \|\|_\infty \)-Cauchy sequence and then convergent to \( x \in \ell^K_\infty(m) \). Then also \( \|x_{n_i} - x\| \to 0 \), so \( x \in \bigcap_i B[x_{n_i}, R_{n_i}] = \bigcap_i B[x_n, R_n] \).

REFERENCES


