WEAK AND STRONG C'-COMPACTNESS IN NON-ARCHIMEDEAN BANACH SPACES

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ABSTRACT
Throughout $K$ is a non-archimedean complete valued field with dense valuation $|.|$. An absolutely convex set $A$ of a $K$-Banach space $E$ is called (weakly) $c'$-compact if $\max_{x \in A} p(x)$ exists for each (weakly) continuous seminorm $p$ on $E$.

Assuming the continuum hypothesis, we shall prove that, if $K$ has the cardinality of the continuum, in a strongly polar $K$-Banach space, each weakly $c'$-compact set is $c'$-compact.

INTRODUCTION
(For unexplained terms, see below and [2], [3] and [6]). It was proved in 1986 ([5], theorem 2.7) that each weakly $c'$-compact set is $c'$-compact if $E$ is a $K$-Banach space with a base. Further progress came about in 1989 ([1], theorem 5*2.13) when the same conclusion could be drawn for an arbitrary Banach space over a spherically complete $K$.

However, if $K$ is not spherically complete, the closed unit ball of (the polar space) $E^{**}$ is weakly $c'$-compact but not $c'$-compact. ([5], example p. 9). So, quite naturally the following problem arises:

Let $K$ be not spherically complete. Is every weakly $c'$-compact set in a strongly polar $K$-Banach space necessarily $c'$-compact?

In this note we give a partial solution (as stated in the abstract). The general problem remains open.

PRELIMINARIES
We assume that $K$ is not spherically complete and that $K$ has the cardinality of the continuum, for example $K = C_p$, the completion of the algebraic closure of $Q_p$. The residue class field of $K$ is $k$ and the canonical map $\{ \lambda \in K \mid |\lambda| \leq 1 \} \rightarrow k$ is written $\lambda \rightarrow \overline{\lambda}$.

Let $E$ be a $K$-Banach space. Its dual is $E'$, the absolutely convex hull of a set $S \subseteq E$ is denoted by $\text{co}S$, the closure of $\text{co}S$ by $\overline{\text{co}}S$ and its $K$-linear span by $[S]$.

Recall that $E$ is called strongly polar if every continuous seminorm $p$ is polar. (I.e. $p = \sup \{ |f(\lambda)| f \in E', |\lambda| \leq 1 \}$). We shall need the following results which are proved in [3]. Subspaces and images under continuous linear maps of strongly polar
spaces are strongly polar. In a strongly polar space \( E \) every continuous linear function defined on a linear subspace can be extended to an element of \( E' \) and for every closed linear subspace \( D \) and \( x \in E \setminus D \), there exists an \( f \in E' \) that vanishes on \( D \) but \( f(x) \neq 0 \). Spaces of countable type are strongly polar.

1. TWO IMPLICATIONS OF THE CARDINALITY OF \( K \)

1.1. THEOREM: If \( c_0(I) \) is strongly polar, then \( I \) is at most countable.

Proof: Suppose that \( I \) is uncountable. Then, using the continuum hypothesis, we have \( \# E^\infty = \# K < \# I \). Hence, there exists a surjection of \( I \) onto the unit ball of \( E^\infty \) which extends to a continuous linear surjection \( c_0(I) \to E^\infty \). Now \( E^\infty \) is not strongly polar so neither is \( c_0(I) \).

1.2. COROLLARY: Let \( t \in (0,1) \). Any \( t \)-orthogonal set in a strongly polar space is at most countable.

2. WEAK AND STRONG \( C' \)-COMPACTNESS

2.1. PRELIMINARIES: Let \( E \) be a \( K \)-Banach space with norm \( \| \cdot \| \). For a closed and absolutely convex subset \( A \) of \( E \), we put \( A^i = \{ \lambda.a \mid \lambda \in K, |\lambda| < 1, a \in A \} \). Then \( A^i \) and \( \overline{A^i} \) are absolutely convex. The quotient \( V_A = A / A^i \) is, in a natural way, a \( k \)-vector space. Let \( \pi: A \to V_A \) denote the quotient map.

The formula \( \| \pi(x) \| = \inf \{ \| x - a \| \mid a \in \overline{A^i} \} \) defines a norm on \( V_A \) for which it becomes a \( k \)-Banach space ([2], proposition 3.2).

Any \( k \)-Banach space has, for each \( t \in (0,1) \), a \( t \)-orthogonal base ([2], proposition 3.5).

2.2. LEMMA: Let \( A \subset E \) be closed, bounded and absolutely convex. Let \( t \in (0,1) \) and let \( (e_i)_{i \in I} \) be a family in \( A \) such that \( (\pi(e_i))_{i \in I} \) is \( t \)-orthogonal and such that \( \alpha = \inf_{i \in I} \| \pi(e_i) \| > 0 \).

Then \( (e_i)_{i \in I} \) is \( t' \)-orthogonal for some \( t' \in (0,t) \).

Proof: Put \( \beta = \sup_{i \in I} \| e_i \| \). Now, let \( J \subset I \) be finite and put \( x = \sum_{i \in J} \lambda_i e_i \) where \( \lambda_i \in K^* \) for each \( i \in J \). It is no restriction to assume that \( \max_{i \in J} |\lambda_i| = 1 \). Then we have the following:

\[ \| x \| \geq \| \pi(x) \| = \| \sum_{i \in J} \lambda_i \pi(e_i) \| \geq t. \max_{i \in J} \| \pi(e_i) \| \geq t. \alpha = t. \alpha. \beta^{-1}. \sup_{i \in I} \| e_i \| \geq t. \alpha. \beta^{-1}. \max_{i \in I} \| e_i \| \].

It suffices to choose \( t' = t. \alpha. \beta^{-1} \) to complete the proof.

2.3. REMARK: In the proof of lemma 2.2, the condition on the cardinality of \( K \) is redundant. (See also [2], lemma 3.11).
2.4. **Lemma:** Let $A$ be a closed, bounded and absolutely convex subset of a strongly polar K-Banach space $E$. Then $V_A$ is of countable type.

**Proof:** Let $(\pi(e_i))_{i \in I}$ be a $t$-orthogonal base of $V_A$ for some $t \in (0,1)$.

For each $n \in N_0$, put $I_n = \{ i \in I : \| e_i \| \geq \frac{1}{n} \}$. By lemma 2.2, $(e_i)_{i \in I_n}$ is $t'$-orthogonal for some $t' \in (0,1]$ and by corollary 1.2, the set $I_n$ is at most countable. It follows that $I = \bigcup_{n \in N_0} I_n$ is countable, hence, $V_A$ is of countable type.

2.5. **Theorem:** Let $A$ be an absolutely convex, weakly $c'$-compact subset of a strongly polar K-Banach space $E$. Then $A$ is $c'$-compact.

**Proof:** We may assume that $A$ is closed ([4], proposition 1.2). Weak $c'$-compactness implies weak boundedness, hence norm boundedness ([3], corollary 7.7). Choose $t \in (0,1)$ and let $(e_n)_{n \in N_0} \subset A$ be such that $(\pi(e_n))_{n \in N_0}$ is a $t$-orthogonal base of $V_A$. (Lemma 2.4).

We may assume that $\| e_n \| \leq t^{-1} \| \pi(e_n) \|$ for each $n \in N_0$. Put $B = \overline{\operatorname{co}}(e_n \mid n \in N_0)$.

Then, as $B \subset A$, obviously $[B] \subset [A]$. Now, if this inclusion were strict, we could find (by strong polarness) an $f \in E'$ that vanishes on $[B]$ but not on $[A]$. By weak $c'$-compactness, $\alpha = \max_{x \in A} |f(x)|$ exists and is non-zero. Clearly $\| f \| < \alpha$ on $A^i$ and thus $\| f \| < \alpha$ on $\overline{A^i}$.

On the other hand, it is not difficult to see that $A \subset B + \overline{A^i}$ (recall that $\operatorname{Ker} \pi = \overline{A^i}$).

Hence, it follows that $\| f \| < \alpha$ on $A$ (since $f$ vanishes on $B$) and this is a contradiction.

So, $[A] = \{ [e_n] \mid n \in N_0 \}$. Now, again by strong polarness, the weak topology of $[A]$ is the restriction to $[A]$ of the weak topology of $E$. Hence, $A$ is a closed, weakly $c'$-compact subset of a K-Banach space $([A])$ of countable type.

On the other hand, since $[A]$ is of countable type, it has a base. Now simply apply [5], theorem 2.7 to conclude that $A$ is $c'$-compact in $[A]$ and thus in $E$.

**REFERENCES**


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