WEAK AND STRONG C'-COMPACTNESS IN NON-ARCHIMEDEAN BANACH SPACES

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ABSTRACT
Throughout $K$ is a non-archimedean complete valued field with dense valuation $| . |$. An absolutely convex set $A$ of a $K$-Banach space $E$ is called (weakly) $c'$-compact if $\max_{x \in A} p(x)$ exists for each (weakly) continuous seminorm $p$ on $E$.

Assuming the continuum hypothesis, we shall prove that, if $K$ has the cardinality of the continuum, in a strongly polar $K$-Banach space, each weakly $c'$-compact set is $c'$-compact.

INTRODUCTION
(For unexplained terms, see below and [2], [3] and [6]). It was proved in 1986 ([5], theorem 2.7) that each weakly $c'$-compact set is $c'$-compact if $E$ is a $K$-Banach space with a base. Further progress came about in 1989 ([1], theorem 5.2.13) when the same conclusion could be drawn for an arbitrary Banach space over a spherically complete $K$.

However, if $K$ is not spherically complete, the closed unit ball of (the polar space) $E^{\infty}$ is weakly $c'$-compact but not $c'$-compact. ([5], example p. 9). So, quite naturally the following problem arises:

Let $K$ be not spherically complete. Is every weakly $c'$-compact set in a strongly polar $K$-Banach space necessarily $c'$-compact?

In this note we give a partial solution (as stated in the abstract). The general problem remains open.

PRELIMINARIES
We assume that $K$ is not spherically complete and that $K$ has the cardinality of the continuum, for example $K = C_p$, the completion of the algebraic closure of $Q_p$. The residue class field of $K$ is $k$ and the canonical map \( \{ \lambda \in K \mid |\lambda| \leq 1 \} \rightarrow k \) is written $\lambda \rightarrow \bar{\lambda}$.

Let $E$ be a $K$-Banach space. Its dual is $E'$, the absolutely convex hull of a set $S \subset E$ is denoted by $\text{co}S$, the closure of $\text{co}S$ by $\overline{\text{co}}S$ and its $K$-linear span by $[S]$.

Recall that $E$ is called strongly polar if every continuous seminorm $p$ is polar. (I.e. $p = \sup \{ |f| \mid f \in E', |f| \leq p \}$). We shall need the following results which are proved in [3]. Subspaces and images under continuous linear maps of strongly polar
spaces are strongly polar. In a strongly polar space $E$ every continuous linear function
defined on a linear subspace can be extended to an element of $E'$ and for every closed
linear subspace $D$ and $x \in E \setminus D$, there exists an $f \in E'$ that vanishes on $D$ but $f(x) \neq 0$.
Spaces of countable type are strongly polar.

1. TWO IMPLICATIONS OF THE CARDINALITY OF $K$

1.1. **Theorem:** If $c_0(I)$ is strongly polar, then $I$ is at most countable.

**Proof:** Suppose that $I$ is uncountable. Then, using the continuum hypothesis, we have
$\# E^\infty = \# K \leq \# I$. Hence, there exists a surjection of $I$ onto the unit ball of $E^\infty$ which
extends to a continuous linear surjection $c_0(I) \to E^\infty$. Now $E^\infty$ is not strongly polar so
neither is $c_0(I)$.

1.2. **Corollary:** Let $t \in (0, 1]$. Any $t$-orthogonal set in a strongly polar space is at
most countable.

2. WEAK AND STRONG $C'$-COMPACTNESS

2.1. **Preliminaries:** Let $E$ be a $K$-Banach space with norm $\| \cdot \|$. For a closed and
absolutely convex subset $A$ of $E$, we put $A^1 = \{ \lambda a \mid \lambda \in K, |\lambda| < 1, a \in A \}$. Then $A^1$
and $A^\top$ are absolutely convex. The quotient $V_A = A / A^1$ is, in a natural way, a $k$-vector
space. Let $\pi : A \to V_A$ denote the quotient map.

The formula $\| \pi(x) \| = \inf \{ \| x - \lambda a \| \mid a \in A^1 \}$ defines a norm on $V_A$ for which it becomes a
$k$-Banach space ([2], proposition 3.2).

Any $k$-Banach space has, for each $t \in (0,1)$, a $t$-orthogonal base ([2], proposition 3.5).

2.2. **Lemma:** Let $A \subseteq E$ be closed, bounded and absolutely convex. Let $t \in (0,1)$ and let
$(e_i)_{i \in I}$ be a family in $A$ such that $(\pi(e_i))_{i \in I}$ is $t$-orthogonal and such that $\alpha = \inf_{i \in I} \| \pi(e_i) \| > 0$.

Then $(e_i)_{i \in I}$ is $t'$-orthogonal for some $t' \in (0,t]$.

**Proof:** Put $\beta = \sup_{i \in I} \| e_i \|$. Now, let $J \subseteq I$ be finite and put $x = \sum_{i \in J} \lambda_i e_i$ where $\lambda_i \in K^*$ for
each $i \in J$. It is no restriction to assume that $\max_{i \in J} |\lambda_i| = 1$. Then we have the following:

$$\| x \| \geq \| \pi(x) \| = \sum_{i \in J} |\lambda_i| . \| \pi(e_i) \| \geq t . \max_{i \in J} \| \pi(e_i) \| \geq t . \alpha = t . \alpha . \beta^{-1} . \sup_{i \in I} \| e_i \| \geq t . \alpha . \beta^{-1} . \max_{i \in J} \| \lambda_i e_i \| .$$

It suffices to choose $t' = t . \alpha . \beta^{-1}$ to complete the proof.

2.3. **Remark:** In the proof of lemma 2.2, the condition on the cardinality of $K$ is
redundant. (See also [2], lemma 3.11).
2.4. **Lemma**: Let \( A \) be a closed, bounded and absolutely convex subset of a strongly polar \( K \)-Banach space \( E \). Then \( V_A \) is of countable type.

**Proof**: Let \( (\pi(e_i))_{i \in I} \) be a \( t \)-orthogonal base of \( V_A \) for some \( t \in (0,1) \).

For each \( n \in \mathbb{N}_0 \), put \( I_n = \{ \ i \in I \mid \|e_i\|_t \geq \frac{1}{n} \} \). By lemma 2.2, \( (e_i)_{i \in I_n} \) is \( t' \)-orthogonal for some \( t' \in (0,1] \) and by corollary 1.2, the set \( I_n \) is at most countable. It follows that \( I = \bigcup_{n \in \mathbb{N}_0} I_n \) is countable, hence, \( V_A \) is of countable type.

2.5. **Theorem**: Let \( A \) be an absolutely convex, weakly \( c' \)-compact subset of a strongly polar \( K \)-Banach space \( E \). Then \( A \) is \( c' \)-compact.

**Proof**: We may assume that \( A \) is closed ([4], proposition 1.2). Weak \( c' \)-compactness implies weak boundedness, hence norm boundedness ([3], corollary 7.7). Choose \( t \in (0,1) \) and let \( (e_n)_{n \in \mathbb{N}_0} \subset A \) be such that \( (\pi(e_n))_{n \in \mathbb{N}_0} \) is a \( t \)-orthogonal base of \( V_A \). (Lemma 2.4).

We may assume that \( \|e_n\| \leq t^{-1} \pi(e_n) \| \) for each \( n \in \mathbb{N}_0 \). Put \( B = \overline{\text{co}}(e_n \mid n \in \mathbb{N}_0) \).

Then, as \( B \subset A \), obviously \( [B] \subset [A] \). Now, if this inclusion were strict, we could find (by strong polarness) an \( f \in E' \) that vanishes on \( [B] \) but not on \( [A] \). By weak \( c' \)-compactness, \( \alpha = \max_{x \in A} |f(x)| \) exists and is non-zero. Clearly \( |f| \leq \alpha \) on \( A^i \) and thus \( |f| < \alpha \) on \( A^i \).

On the other hand, it is not difficult to see that \( A \subset B + A^i \) (recall that \( \text{Ker} \pi = A^i \)). Hence, it follows that \( |f| < \alpha \) on \( A \) (since \( f \) vanishes on \( B \)) and this is a contradiction.

So, \( [A] = [\{ e_n \mid n \in \mathbb{N}_0 \}] \). Now, again by strong polarness, the weak topology of \( [A] \) is the restriction to \( [A] \) of the weak topology of \( E \). Hence, \( A \) is a closed, weakly \( c' \)-compact subset of a \( K \)-Banach space \( ([A]) \) of countable type.

On the other hand, since \( [A] \) is of countable type, it has a base. Now simply apply [5], theorem 2.7 to conclude that \( A \) is \( c' \)-compact in \( [A] \) and thus in \( E \).

**REFERENCES**


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