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WEAK AND STRONG C'-COMPACTNESS IN NON-ARCHIMEDEAN
BANACH SPACES

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ABSTRACT
Throughout \( K \) is a non-archimedean complete valued field with dense valuation \( |.| \). An absolutely convex set \( A \) of a \( K \)-Banach space \( E \) is called (weakly) \( c' \)-compact if \( \max_{x \in A} p(x) \) exists for each (weakly) continuous seminorm \( p \) on \( E \).
Assuming the continuum hypothesis, we shall prove that, if \( K \) has the cardinality of the continuum, in a strongly polar \( K \)-Banach space, each weakly \( c' \)-compact set is \( c' \)-compact.

INTRODUCTION
(For unexplained terms, see below and [2], [3] and [6]). It was proved in 1986 ([5], theorem 2.7) that each weakly \( c' \)-compact set is \( c' \)-compact if \( E \) is a \( K \)-Banach space with a base. Further progress came about in 1989 ([1], theorem 5.2.13) when the same conclusion could be drawn for an arbitrary Banach space over a spherically complete \( K \). However, if \( K \) is not spherically complete, the closed unit ball of (the polar space) \( E^* \) is weakly \( c' \)-compact but not \( c' \)-compact. ([5], example p. 9). So, quite naturally the following problem arises:
Let \( K \) be not spherically complete. Is every weakly \( c' \)-compact set in a strongly polar \( K \)-Banach space necessarily \( c' \)-compact?

In this note we give a partial solution (as stated in the abstract). The general problem remains open.

PRELIMINARIES
We assume that \( K \) is not spherically complete and that \( K \) has the cardinality of the continuum, for example \( K = C_p \), the completion of the algebraic closure of \( Q_p \). The residue class field of \( K \) is \( \mathbb{k} \) and the canonical map \( \{ \lambda \in K \mid |\lambda| \leq 1 \} \rightarrow \mathbb{k} \) is written \( \lambda \rightarrow \bar{\lambda} \).
Let \( E \) be a \( K \)-Banach space. Its dual is \( E^* \), the absolutely convex hull of a set \( S \subset E \) is denoted by \( \text{co}S \), the closure of \( \text{co}S \) by \( \bar{\text{co}}S \) and its \( K \)-linear span by \( [S] \).
Recall that \( E \) is called strongly polar if every continuous seminorm \( p \) is polar.
(I.e. \( p = \sup \{ |f| \mid f \in E^*, |f| \leq p \} \)). We shall need the following results which are proved in [3]. Subspaces and images under continuous linear maps of strongly polar...
spaces are strongly polar. In a strongly polar space $E$ every continuous linear function defined on a linear subspace can be extended to an element of $E'$ and for every closed linear subspace $D$ and $x \in E \setminus D$, there exists an $f \in E'$ that vanishes on $D$ but $f(x) \neq 0$. Spaces of countable type are strongly polar.

1. TWO IMPLICATIONS OF THE CARDINALITY OF $K$

1.1. **THEOREM**: If $c_0(I)$ is strongly polar, then $I$ is at most countable.

*Proof*: Suppose that $I$ is uncountable. Then, using the continuum hypothesis, we have $\# E^\infty = \# K \leq \# I$. Hence, there exists a surjection of $I$ onto the unit ball of $L^\infty$ which extends to a continuous linear surjection $c_0(I) \to L^\infty$. Now $L^\infty$ is not strongly polar so neither is $c_0(I)$.

1.2. **COROLLARY**: Let $t \in (0, 1]$. Any $t$-orthogonal set in a strongly polar space is at most countable.

2. WEAK AND STRONG $C'$-COMPACTNESS

2.1. **PRELIMINARIES**: Let $E$ be a $K$-Banach space with norm $\|\cdot\|$. For a closed and absolutely convex subset $A$ of $E$, we put $A^i = \{ \lambda a \mid \lambda \in K, |\lambda| < 1, a \in A \}$. Then $A^i$ and $\overline{A^i}$ are absolutely convex. The quotient $V_A = A / \overline{A^i}$ is, in a natural way, a $k$-vector space. Let $\pi : A \to V_A$ denote the quotient map.

The formula $\|\pi(x)\| = \inf \{ \|x - a\| \mid a \in \overline{A^i} \}$ defines a norm on $V_A$ for which it becomes a $k$-Banach space ([2], proposition 3.2).

Any $k$-Banach space has, for each $t \in (0,1)$, a $t$-orthogonal base ([2], proposition 3.5).

2.2. **LEMMA**: Let $A \subset E$ be closed, bounded and absolutely convex. Let $t \in (0,1)$ and let $(e_i)_{i \in I}$ be a family in $A$ such that $(\pi(e_i))_{i \in I}$ is $t$-orthogonal and such that $\alpha = \inf_{i \in I} \|\pi(e_i)\| > 0$.

Then $(e_i)_{i \in I}$ is $t'$-orthogonal for some $t' \in (0,t]$.

*Proof*: Put $\beta = \sup_{i \in I} \|e_i\|$. Now, let $J \subset I$ be finite and put $x = \sum_{i \in J} \lambda_i e_i$ where $\lambda_i \in K^*$ for each $i \in J$. It is no restriction to assume that $\max_{i \in J} |\lambda_i| = 1$. Then we have the following:

$$\|x\| \geq \|\pi(x)\| = \|\sum_{i \in J} \overline{\lambda_i} \pi(e_i)\| \geq \lambda_{\max_{i \in J}} \|\pi(e_i)\| \geq \max_{i \in I} \|\pi(e_i)\| \geq \max_{i \in J} \|e_i\| \geq \alpha = \max_{i \in I} \|\lambda_i e_i\|. $$

It suffices to choose $t' = t \alpha \beta^{-1}$ to complete the proof.

2.3. **REMARK**: In the proof of lemma 2.2, the condition on the cardinality of $K$ is redundant. (See also [2], lemma 3.11).
2.4. LEMMA: Let \( A \) be a closed, bounded and absolutely convex subset of a strongly polar \( K \)-Banach space \( E \). Then \( V_A \) is of countable type.

\textit{Proof}: Let \( (\pi(e_i))_{i \in I} \) be a \( t \)-orthogonal base of \( V_A \) for some \( t \in (0,1) \).

For each \( n \in \mathbb{N}_0 \), put \( I_n = \{ i \in I \mid \|e_i\| \geq \frac{1}{n} \} \). By lemma 2.2, \( (e_i)_{i \in I_n} \) is \( t' \)-orthogonal for some \( t' \in (0,1] \) and by corollary 1.2, the set \( I_n \) is at most countable. It follows that \( I = \bigcup_{n \in \mathbb{N}_0} I_n \) is countable, hence, \( V_A \) is of countable type.

2.5. THEOREM: Let \( A \) be an absolutely convex, weakly \( c' \)-compact subset of a strongly polar \( K \)-Banach space \( E \). Then \( A \) is \( c' \)-compact.

\textit{Proof}: We may assume that \( A \) is closed ([4], proposition 1.2). Weak \( c' \)-compactness implies weak boundedness, hence norm boundedness ([3], corollary 7.7). Choose \( t \in (0,1) \) and let \( (e_n)_{n \in \mathbb{N}_0} \subseteq A \) be such that \( (\pi(e_n))_{n \in \mathbb{N}_0} \) is a \( t \)-orthogonal base of \( V_A \). (Lemma 2.4).

We may assume that \( \|e_n\| \leq t^{-1} \pi(e_n) \) for each \( n \in \mathbb{N}_0 \). Put \( B = \overline{\text{co}}(e_n \mid n \in \mathbb{N}_0) \).

Then, as \( B \subseteq A \), obviously \( \overline{B} \subseteq \overline{A} \). Now, if this inclusion were strict, we could find (by strong polarmess) an \( f \in E' \) that vanishes on \( \overline{B} \) but not on \( \overline{A} \). By weak \( c' \)-compactness, \( \alpha = \max_{x \in A} |f(x)| \) exists and is non-zero. Clearly \( |f| < \alpha \) on \( A^1 \) and thus \( \|f\| < \alpha \) on \( \overline{A^1} \).

On the other hand, it is not difficult to see that \( A \subseteq B + \overline{A^1} \) (recall that \( \text{Ker} \pi = \overline{A^1} \)).

Hence, it follows that \( \|f\| < \alpha \) on \( A \) (since \( f \) vanishes on \( B \)) and this is a contradiction.

So, \( \overline{A} = \{ e_n \mid n \in \mathbb{N}_0 \} \). Now, again by strong polarmess, the weak topology of \( \overline{A} \) is the restriction to \( \overline{A} \) of the weak topology of \( E \). Hence, \( A \) is a closed, weakly \( c' \)-compact subset of a \( K \)-Banach space \( \overline{A} \) of countable type.

On the other hand, since \( \overline{A} \) is of countable type, it has a base. Now simply apply [5], theorem 2.7 to conclude that \( A \) is \( c' \)-compact in \( \overline{A} \) and thus in \( E \).

REFERENCES


[2] \textsc{Borrey, S.}: Weak \( c' \)-compactness in (strongly) polar Banach spaces over a non-archimedean, densely valued field. To appear in P-adic functional analysis. (Marcel Dekker Inc.)


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