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WEAK AND STRONG C'-COMPACTNESS IN NON-ARCHIMEDEAN BANACH SPACES

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ABSTRACT
Throughout $K$ is a non-archimedean complete valued field with dense valuation $|.|$. An absolutely convex set $A$ of a $K$-Banach space $E$ is called (weakly) $c'$-compact if $\max_{x \in A} p(x)$ exists for each (weakly) continuous seminorm $p$ on $E$. Assuming the continuum hypothesis, we shall prove that, if $K$ has the cardinality of the continuum, in a strongly polar $K$-Banach space, each weakly $c'$-compact set is $c'$-compact.

INTRODUCTION
(For unexplained terms, see below and [2], [3] and [6]). It was proved in 1986 ([5], theorem 2.7) that each weakly $c'$-compact set is $c'$-compact if $E$ is a $K$-Banach space with a base. Further progress came about in 1989 ([1], theorem 5.2.13) when the same conclusion could be drawn for an arbitrary Banach space over a spherically complete $K$. However, if $K$ is not spherically complete, the closed unit ball of (the polar space) $E^{**}$ is weakly $c'$-compact but not $c'$-compact. ([5], example p. 9). So, quite naturally the following problem arises:

Let $K$ be not spherically complete. Is every weakly $c'$-compact set in a strongly polar $K$-Banach space necessarily $c'$-compact?

In this note we give a partial solution (as stated in the abstract). The general problem remains open.

PRELIMINARIES
We assume that $K$ is not spherically complete and that $K$ has the cardinality of the continuum, for example $K = \mathbb{C}_p$, the completion of the algebraic closure of $\mathbb{Q}_p$. The residue class field of $K$ is $k$ and the canonical map $\{ \lambda \in K | |\lambda| \leq 1 \} \rightarrow k$ is written $\lambda \rightarrow \bar{\lambda}$.

Let $E$ be a $K$-Banach space. Its dual is $E'$, the absolutely convex hull of a set $S \subseteq E$ is denoted by $\text{co}S$, the closure of $\text{co}S$ by $\overline{\text{co}}S$ and its $K$-linear span by $[S]$.

Recall that $E$ is called strongly polar if every continuous seminorm $p$ is polar. (I.e. $p = \sup \{ |f| | f \in E', |f| \leq p \}$). We shall need the following results which are proved in [3]. Subspaces and images under continuous linear maps of strongly polar
spaces are strongly polar. In a strongly polar space $E$ every continuous linear function defined on a linear subspace can be extended to an element of $E'$ and for every closed linear subspace $D$ and $x \in E \setminus D$, there exists an $f \in E'$ that vanishes on $D$ but $f(x) \neq 0$. Spaces of countable type are strongly polar.

1. TWO IMPLICATIONS OF THE CARDINALITY OF $K$

1.1. THEOREM: If $c_0(I)$ is strongly polar, then $I$ is at most countable.

Proof: Suppose that $I$ is uncountable. Then, using the continuum hypothesis, we have $\# E^\infty = \# K \leq \# I$. Hence, there exists a surjection of $I$ onto the unit ball of $E^\infty$ which extends to a continuous linear surjection $c_0(I) \to E^\infty$. Now $E^\infty$ is not strongly polar so neither is $c_0(I)$.

1.2. COROLLARY: Let $t \in (0, 1]$. Any $t$-orthogonal set in a strongly polar space is at most countable.

2. WEAK AND STRONG $C'$-COMPACTNESS

2.1. PRELIMINARIES: Let $E$ be a $K$-Banach space with norm $\| \cdot \|$. For a closed and absolutely convex subset $A$ of $E$, we put $A^i = \{ \lambda a \mid \lambda \in K, \| a \| < 1, a \in A \}$. Then $A^i$ and $\overline{A^i}$ are absolutely convex. The quotient $V_A = A / \overline{A^i}$ is, in a natural way, a $k$-vector space. Let $\pi : A \to V_A$ denote the quotient map.

The formula $\| \pi(x) \| = \inf \{ \| x - a \| \mid a \in \overline{A^i} \}$ defines a norm on $V_A$ for which it becomes a $k$-Banach space ([2], proposition 3.2).

Any $k$-Banach space has, for each $t \in (0,1)$, a $t$-orthogonal base ([2], proposition 3.5).

2.2. LEMMA: Let $A \subseteq E$ be closed, bounded and absolutely convex. Let $t \in (0,1)$ and let $(e_i)_{i \in I}$ be a family in $A$ such that $(\pi(e_i))_{i \in I}$ is $t$-orthogonal and such that $\alpha = \inf_{i \in I} \| \pi(e_i) \| > 0$.

Then $(e_i)_{i \in I}$ is $t'$-orthogonal for some $t' \in (0,t]$

Proof: Put $\beta = \sup_{i \in I} \| e_i \|$. Now, let $J \subseteq I$ be finite and put $x = \sum_{i \in J} \lambda_i e_i$ where $\lambda_i \in K^*$ for each $i \in J$. It is no restriction to assume that $\max_{i \in J} |\lambda_i| = 1$. Then we have the following:

$$\| x \| \geq \| \pi(x) \| = \| \sum_{i \in J} \lambda_i \pi(e_i) \| \geq t \max_{i \in J} \| \pi(e_i) \| \geq t \alpha = t \alpha \beta^{-1} \sup_{i \in I} \| e_i \| \geq t \alpha \beta^{-1} \max_{i \in J} \| \lambda_i, e_i \|.$$ 

It suffices to choose $t' = t \alpha \beta^{-1}$ to complete the proof.

2.3. REMARK: In the proof of lemma 2.2, the condition on the cardinality of $K$ is redundant. (See also [2], lemma 3.11).
2.4. **Lemma:** Let $A$ be a closed, bounded and absolutely convex subset of a strongly polar $K$-Banach space $E$. Then $V_A$ is of countable type.  

**Proof:** Let $(\pi(c_i))_{i \in I}$ be a $t$-orthogonal base of $V_A$ for some $t \in (0,1)$. For each $n \in \mathbb{N}_0$, put $I_n = \{ i \in I \mid \|c_i\| \geq \frac{1}{n} \}$. By lemma 2.2, $(c_i)_{i \in I_n}$ is $t'$-orthogonal for some $t' \in (0,1]$ and by corollary 1.2, the set $I_n$ is at most countable. It follows that $I = \bigcup_{n \in \mathbb{N}_0} I_n$ is countable, hence, $V_A$ is of countable type.

2.5. **Theorem:** Let $A$ be an absolutely convex, weakly $c'$-compact subset of a strongly polar $K$-Banach space $E$. Then $A$ is $c'$-compact.  

**Proof:** We may assume that $A$ is closed ([4], proposition 1.2). Weak $c'$-compactness implies weak boundedness, hence norm boundedness ([3], corollary 7.7). Choose $t \in (0,1)$ and let $(e_n)_{n \in \mathbb{N}_0} \subset A$ be such that $(\pi(e_n))_{n \in \mathbb{N}_0}$ is a $t$-orthogonal base of $V_A$. (Lemma 2.4). We may assume that $\|e_n\| \leq t^{-1} \|\pi(e_n)\|$ for each $n \in \mathbb{N}_0$. Put $B = \overline{\text{co}} (e_n \mid n \in \mathbb{N}_0)$. Then, as $B \subset A$, obviously $[B] \subset [A]$. Now, if this inclusion were strict, we could find (by strong polarness) an $f \in E'$ that vanishes on $[B]$ but not on $[A]$. By weak $c'$-compactness, $\alpha = \max_{x \in A} |f(x)|$ exists and is non-zero. Clearly $|f| < \alpha$ on $A^1$ and thus $|f| < \alpha$ on $\overline{A^1}$. On the other hand, it is not difficult to see that $A \subset B + \overline{A^1}$ (recall that $\text{Ker} \, \pi = \overline{A^1}$). Hence, it follows that $|f| < \alpha$ on $A$ (since $f$ vanishes on $B$) and this is a contradiction. So, $[A] = \overline{\{e_n \mid n \in \mathbb{N}_0\}}$. Now, again by strong polarness, the weak topology of $[A]$ is the restriction to $[A]$ of the weak topology of $E$. Hence, $A$ is a closed, weakly $c'$-compact subset of a $K$-Banach space $([A])$ of countable type. On the other hand, since $[A]$ is of countable type, it has a base. Now simply apply [5], theorem 2.7 to conclude that $A$ is $c'$-compact in $[A]$ and thus in $E$.

**REFERENCES**


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