NON-ARCHIMEDEAN $t$-FRAMES AND FM-SPACES

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Abstract. We generalize the notion of $t$-orthogonality in $p$-adic Banach spaces by introducing $t$-frames (§2). This we use to prove that a Fréchet-Montel (FM-)space is of countable type (Theorem 3.1), the non-archimedean counterpart of a well known theorem in functional analysis over $\mathbb{R}$ or $\mathbb{C}$ ([6], p. 231). We obtain several characterizations of FM-spaces (Theorem 3.3) and characterize the nuclear spaces among them (§4).

1. Preliminaries. Throughout this paper $K$ is a non-archimedean non-trivially valued complete field with valuation $|.|$. For the basic notions and properties concerning normed and locally convex spaces over $K$ we refer to [11] and [7]. However we recall the following.

1. Let $E$ be a $K$-vector space. Let $X \subseteq E$. The absolutely convex hull of $X$ is denoted by $\text{co} X$, its linear hull by $[X]$. For a (non-archimedean) seminorm $p$ on $E$ we denote by $E_p$ the vector space $E/\text{Ker } p$ and by $\pi_p: E \to E_p$ the canonical surjection. The formula $\|\pi_p(x)\| = p(x)$ defines a norm on $E_p$.

2. Let $(E, \|\cdot\|)$ be a normed space over $K$. For $r > 0$ we write $B(0, r) := \{x \in E : \|x\| \leq r\}$. Let $a \in E$, $X \subseteq E$. Then $\text{dist}(a, X) := \inf\{\|a - x\| : x \in X\}$. For $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in E$ we consider $\text{Vol}(x_1, \ldots, x_n) := \|x_1\| \cdot \text{dist}(x_2, [x_1]) \cdot \text{dist}(x_3, [x_1, x_2]) \cdots \text{dist}(x_n, [x_1, \ldots, x_{n-1}])$. For properties of this Volume Function (in particular, its symmetry), we refer to [10]. A linear continuous map $E \to F$, where $F$ is a normed space, is said to be compact if it sends the unit ball of $E$ into a compactoid set (see below).

3. Now let $E$ be a Hausdorff locally convex space over $K$. A subset $X$ of $E$ is called compactoid if for every zero-neighbourhood $U$ in $E$ there exists a finite set $S$ of $E$ such that $X \subseteq \text{co } S + U$. $E$ is said to be of countable type if for each continuous seminorm $p$ the normed space $E_p$ is of countable type (Recall that a normed space is called of countable type if it is the closed linear hull of a countable set). $E$ is called nuclear if for every continuous seminorm $p$ on $E$ there exists a continuous seminorm $q$ on $E$ with $p \leq q$, and such that $\Phi_{pq}$ is compact, where $\Phi_{pq}$ is the unique map making the diagram

\[
\begin{array}{ccc}
E & \xrightarrow{\pi_p} & E_p \\
\Phi_{pq} & \downarrow & \downarrow \\
E_q & \xrightarrow{\pi_q} & E_p
\end{array}
\]
commute. $E$ is called Montel if it is polar, polarly barrelled and if each closed bounded subset is a complete compactoid. A Fréchet space which is Montel is called an FM-space.

The closure of a set $X \subseteq E$ is denoted by $\overline{X}$.

2. $t$-frames in $p$-adic Banach spaces. Throughout §2 $E$ is a normed space over $K$. We introduce a concept which generalizes the notion of $t$-orthogonality and it allows us to prove one of the main Theorems in the paper (Theorem 3.1).

**Definition 2.1.** Let $t \in (0, 1]$, and let $X \subseteq E$ be a subset not containing 0. We call $X$ a $t$-frame if for every $n \in \mathbb{N}$ and distinct $x_1, \ldots, x_n \in X$ we have $	ext{Vol}(x_1, \ldots, x_n) \geq t^{n-1} \cdot \|x_1\| \cdot \cdots \cdot \|x_n\|.

We make the following simple observations. Let $t \in (0, 1]$.

1. Any $t$-orthogonal set in $E$ is a $t$-frame. (Let $\{e_i : i \in I\}$ be a $t$-orthogonal set in $E$, let $i_1, \ldots, i_n$ be $n$ distinct elements of $I$. Then, by the definition of the Volume Function and by $t$-orthogonality,

$$\text{Vol}(e_{i_1}, \ldots, e_{i_n}) = \|e_{i_1}\| \cdot \text{dist}(e_{i_2}, [e_{i_1}]) \cdot \cdots \cdot \text{dist}(e_{i_n}, [e_{i_1}, \ldots, e_{i_{n-1}}]) \geq \|e_{i_1}\| \cdot t \cdot \|e_{i_2}\| \cdot \cdots \cdot t \cdot \|e_{i_n}\| = t^{n-1} \cdot \|e_{i_1}\| \cdot \cdots \cdot \|e_{i_n}\|.)$$

2. Every $t$-frame in $E$ is a linearly independent set.
3. Every subset of a $t$-frame is itself a $t$-frame.
4. Every $t$-frame in $E$ can be extended to a maximal $t$-frame.

By a $t$-frame sequence we shall mean a sequence $x_1, x_2, \ldots$ in $E$ such that $\{x_1, x_2, \ldots\}$ is a $t$-frame.

**Proposition 2.2** (Compare [8], Theorem 2). A bounded subset $X$ of $E$ is a compactoid if and only if for every $t \in (0, 1]$ every $t$-frame sequence in $X$ tends to 0.

**Proof.** Suppose $X$ is a compactoid. Suppose, for some $t \in (0, 1]$, and some $\alpha > 0$, $X$ contains a $t$-frame sequence $x_1, x_2, \ldots$ for which $\|x_n\| \geq \alpha$ for all $n$. Then, for each $n \in \mathbb{N},$

$$\text{Vol}(x_1, \ldots, x_n) \geq t^{n-1} \cdot \|x_1\| \cdot \cdots \cdot \|x_n\| \geq \alpha^n \cdot t^{n-1}$$

implying $\lim_{n \to \infty} \inf \sqrt[n]{\text{Vol}(x_1, \ldots, x_n)} \geq \alpha t > 0$ conflicting the compactoidity of $X$ ([8], §2). This proves one half of the statement. The other half is obvious.$\blacksquare$

The following two Propositions are crucial for Theorem 2.5.

**Proposition 2.3.** Let $0 < t < 1$; let $X$ be a maximal $t$-frame in $E$. Then $[\overline{X}] = E$.

**Proof.** Let $D := \overline{X}$. If $D \neq E$ then we can find a nonzero $a \in E$ with $\text{dist}(a, D) \geq t \cdot \|a\|$ ([11], Lemma 3.14, here we use that $t \neq 1$). So we shall prove that $\text{dist}(a, D) < t \cdot \|a\|$ for every $a \in E - D$. By maximality $\{a\} \cup X$ is no longer a $t$-frame, yielding the existence of a $k \in \mathbb{N}$ and distinct $x_1, \ldots, x_k \in X$ such that

$$\text{Vol}(a, x_1, \ldots, x_k) < t^k \cdot \|a\| \cdot \|x_1\| \cdot \cdots \cdot \|x_k\|.$$
On the other hand we have
\[
\text{Vol}(a, x_1, \ldots, x_k) = \text{dist}(a, [x_1, \ldots, x_k]) \cdot \text{Vol}(x_1, \ldots, x_k) \\
\geq \text{dist}(a, D) \cdot t^{k-1} \cdot \|x_1\| \cdots \|x_k\|.
\]
So dist(a, D) < t \cdot ||a||.

**Remark.** We now can easily find examples of \( t \)-frames \( X \) that are \( s \)-orthogonal for no \( s \in (0, 1] \): Let \( 0 < t < 1 \), let \( E \) have no base, choose for \( X \) a maximal \( t \)-frame (Observe that the clause \( t \neq 1 \) is essential!).

**Proposition 2.4.** Every uncountable subset of \( c_0 \) contains an infinite compactoid.

**Proof.** Let \( X \) be an uncountable subset of \( c_0 \); it has a bounded uncountable subset \( Y \). Let \( e_1, e_2, \ldots \) be the standard basis of \( c_0 \). We have \( B(0, 1) + [e_1, e_2, \ldots] = c_0 \) so there exists an \( n_1 \in \mathbb{N} \) such that
\[
Y_1 := Y \cap \left( B(0, 1) + [e_1, e_2, \ldots e_{n_1}] \right)
\]
is uncountable. In its turn, there exists an \( n_2 \in \mathbb{N} \) such that
\[
Y_2 := Y_1 \cap \left( B(0, 1/2) + [e_1, e_2, \ldots, e_{n_2}] \right)
\]
is uncountable. We obtain uncountable sets \( Y_1 \supset Y_2 \supset \cdots \) such that \( Y_n \subset B(0, 1/n) + D_n \) for each \( n \) where \( D_n \) is a finite-dimensional space. Choose distinct \( x_1, x_2, \ldots \) where \( x_n \in Y_n \) for each \( n \), and set \( Z := \{x_1, x_2, \ldots\} \). Then \( Z \) is infinite, bounded, in \( X \). Also, for each \( n \in \mathbb{N} \) we have
\[
Z \subset \{x_1, \ldots, x_{n-1}\} \cup Y_n \subset [x_1, \ldots, x_{n-1}] + B(0, 1/n) + D_n \subset B(0, 1/n) + \hat{D}_n
\]
where \( \hat{D}_n \) is a finite-dimensional space. It follows that \( Z \) is a compactoid.

**Theorem 2.5.** The following assertions about the normed space \( E \) are equivalent.

(i) \( E \) is of countable type.

(ii) For every \( t \in (0, 1) \), every \( t \)-frame in \( E \) is countable.

(iii) For some \( t \in (0, 1) \), every \( t \)-frame in \( E \) is countable.

**Proof.** (i) \( \Rightarrow \) (ii). We may assume \( E = c_0 \). Let \( X \) be a \( t \)-frame in \( E \). For each \( n \in \mathbb{N} \) set \( X_n := \{x \in X : \|x\| \geq 1/n\} \). If, for some \( n \), \( X_n \) were uncountable it would contain an infinite compactoid \( \{x_1, x_2, \ldots\} \) by Proposition 2.4. Then from Proposition 2.2 \( \lim_{k \to \infty} x_k = 0 \), a contradiction.

(ii) \( \Rightarrow \) (iii) is obvious.

(iii) \( \Rightarrow \) (i). Let \( X \) be a maximal \( t \)-frame in \( E \). By assumption \( X \) is countable. By Proposition 2.3, \( E = [X] \) is of countable type.

**Remark.** The question if Theorem 2.5 remains true when we consider in (i) and (ii) \( t \)-orthogonal sets instead \( t \)-frames is an open problem in non-archimedean analysis ([11], p. 199).
3. Characterizations of FM-spaces among F-spaces. From now on in this paper E is a polar Hausdorff locally convex space over K.

It is proved in [6], Theorem 11.6.2, that a Fréchet Montel space over R or C is separable. It does not simply carry over the non-archimedean case because K may be not locally compact; so we have to deal with compactoids (§1.3) rather than compact sets. This modification is obstructing the classical proof which is essentially based upon separability. It is here where the \( t \)-frames of §2 come to the rescue as will be demonstrated in the following theorem (for other applications of \( t \)-frames in \( p \)-adic analysis, see [9], p. 51–57).

**Theorem 3.1.** An FM-space is of countable type.

**Proof.** Let the topology of the FM-space \( E \) be defined by the sequence of seminorms \( p_1 \leq p_2 \leq \cdots \). Set \( U_n = \{ x \in E : p_n(x) \leq 1 \} \). Choose \( \lambda \in K, |\lambda| > 1 \).

It suffices to show that \( E_1 := E_{p_1} \) is of countable type. Let \( X \) be a \( t \)-frame in \( (E, \| \cdot \|_1) \) for some \( t \in (0, 1) \); we show (Theorem 2.5) that \( X \) is countable. Suppose not. We may assume that \( \inf\{ \| x \|_1 : x \in X \} > 0 \). Choose an \( A_1 \subset E \) such that \( \pi_{p_1}(A_1) = X \). Since \( E = \bigcup_n \lambda^n U_2 \) there exists an \( n_2 \) such that \( A_2 := A_1 \cap \lambda^{n_2} U_2 \) is uncountable. Inductively we arrive at uncountable sets \( A_1 \supset A_2 \supset \cdots \) such that \( A_n \) is \( p_n \)-bounded for each \( n \geq 2 \). Choose distinct \( a_1, a_2, \ldots \) with \( a_n \in A_n \) for each \( n \). Then \( \{ a_1, a_2, \ldots \} \) is bounded in \( E \). As \( E \) is Montel, it is a compactoid. By Proposition 2.2, \( \lim_{n \to \infty} \pi_{p_n}(a_n) = 0 \) conflicting \( \inf\{ \| x \|_1 : x \in X \} > 0 \).

**Lemma 3.2.** Every bounded subset \( B \) of a Fréchet space \( E \), is compactoid for the topology of uniform convergence on the \( \beta(E', E) \)-compactoid subsets of \( E' \) (where \( \beta(E', E) \) denotes the strong topology on \( E' \) with respect to the dual pair \( (E, E') \)).

**Proof.** Consider the canonical map \( J_E : E \to E'' = (E', \beta(E', E))' \). It is easy to see that the set \( J_E(B) \) is equicontinuous on \( (E', \beta(E', E)) \). By [7] Lemma 10.6 we have that on \( J_E(B) \) the topology \( \tau_{bk} \) (on \( E'' \)) of the uniform convergence on the \( \beta(E', E) \)-compactoid subsets of \( E' \), coincides with the weak topology \( \sigma(E'', E') \). Hence \( J_E(B) \) is \( \tau_{bk} \)-compactoid in \( E'' \). Since \( J_E \) is an homeomorphism from \( E \) onto a subspace of \( E'' \) ([7], Lemmas 9.2, 9.3) we are done.

**Theorem 3.3.** For a Fréchet space \( E \), the following properties are equivalent.

(i) \( E \) is an FM-space.

(ii) Every bounded subset of \( E \) is compactoid.

(iii) In \( E \) every weakly convergent sequence is convergent and \( (E', \beta(E', E)) \) is of countable type.

(iv) In \( E' \) every \( \sigma(E', E) \)-convergent sequence is \( \beta(E', E) \)-convergent and \( E \) is of countable type.

(v) Both \( E \) and \( (E', \beta(E', E)) \) are of countable type.

(vi) \( (E', \beta(E', E)) \) is nuclear.

(vii) \( (E', \beta(E', E)) \) is Montel.
Every $\sigma(E', E)$-bounded subset of $E'$ is $\beta(E', E)$-compactoid.

**Proof.** The implications (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii), (i) $\Rightarrow$ (vi) $\Rightarrow$ (viii) and (i) $\Rightarrow$ (vii) $\Rightarrow$ (viii) are known (see [7]) or easy. Also, from Theorem 3.1 we can easily prove (i) $\Rightarrow$ (iv) and (i) $\Rightarrow$ (v).

Now we prove (viii) $\Rightarrow$ (ii): Since $E$ is a polar Fréchet space, its topology $\tau$ is the topology of uniform convergence on the $\sigma(E', E)$-bounded subsets of $E'$. By (viii) these subsets are $\beta(E', E)$-compactoid. Now apply Lemma 3.2.

The implication (v) $\Rightarrow$ (iii) follows from [7] Proposition 4.11.

Finally, for the proof of (iv) $\Rightarrow$ (ii) observe that the topology on a polar Fréchet space of countable type is the topology of uniform convergence on the $\sigma(E', E)$-null sequences in $E'$ (see [4], Theorem 3.2). By (iv) these sequences are $\beta(E', E)$-convergent. Now apply Lemma 3.2.

**Remark.** It is known that a Fréchet space $E$ over $\mathbb{R}$ over $\mathbb{C}$ is nuclear if and only if $(E', \beta(E', E))$ is nuclear ([6], p. 491).

In the non-archimedean case the situation is essentially different. Indeed, in 4.1 we will give an example of an FM-space which is not nuclear (while its strong dual is by (i) $\Leftrightarrow$ (vi)). To do that we need some preliminary concepts and results.

**Definition 3.4.** Let $A = (a_{ik})$ be a matrix of strictly positive real numbers such that $a_{i,k+1} > a_{i,k}$ for all $i$ and all $k$. Then the corresponding Köthe sequence space $K(A)$ is defined by

$$K(A) = \{ \alpha = (\alpha_i) : \lim_{i} |\alpha_i| \cdot a_{i,k} = 0 \text{ for all } k \}.$$ 

On $K(A)$ we consider the sequence of norms $(p_k)$, where

$$p_k(\alpha) = \max_i |\alpha_i| \cdot a_{i,k}, \quad k = 1, 2, \ldots; \quad \alpha \in K(A).$$

It is known that $K(A)$ is a polar Fréchet space of countable type. For the importance of this class of spaces and for their further properties we refer to [3].

We then have:

**Proposition 3.5.** Let $\Lambda = K(A)$ be a Köthe space and let $\Lambda^*$ the corresponding Köthe dual space. Then the following properties are equivalent:

(i) $\Lambda$ is an FM-space.

(ii) $(\Lambda^*, \beta(\Lambda^*, \Lambda))$ is of countable type.

(iii) $(\Lambda^*, \beta(\Lambda^*, \Lambda))$ is nuclear.

(iv) $(\Lambda^*, \beta(\Lambda^*, \Lambda))$ is Montel.

(v) The unit vectors $e_1, e_2, \ldots$ form a Schauder basis for $\Lambda^*, \beta(\Lambda^*, \Lambda)$.

(vi) $n(\Lambda^*, \Lambda) = \beta(\Lambda^*, \Lambda)$ (where $n(\Lambda^*, \Lambda)\text{ is the natural topology on } \Lambda^*$).

(vii) No subspace of $\Lambda$ is isomorphic (linearly homeomorphic) to $c_0$.

(viii) The sequence of coordinate projections $(P_i)$, where $P_i : \Lambda \to \Lambda : \alpha = (\alpha_i) \mapsto \alpha_i e_i$, converges to the zero-map uniformly on every bounded subset of $\Lambda$. 
(ix) The sequence of sections-maps \((S_n)\), where \(S_n : \Lambda \rightarrow \Lambda : \alpha = (\alpha_i) \rightarrow (\alpha_1, \alpha_2, \ldots, \alpha_n, 0, 0, \ldots)\) converges to the identity map \(\text{Id}\) uniformly on every bounded subset of \(\Lambda\).

**Proof.** We only have to prove \((i) \Rightarrow (v) \Rightarrow (vi), (vii) \Rightarrow (viii)\) and \((ix) \Rightarrow (i)\). The other implications are easy.

\((i) \Rightarrow (v)\): The unit vectors \(e_1, e_2, \ldots\) form a Schauder basis for \((\Lambda^*, \sigma(\Lambda^*, \Lambda))\). Then, apply \((i) \Rightarrow (iv)\) in 3.3.

\((v) \Rightarrow (vi)\): By [4], p. 21 it suffices to prove that \(\beta(\Lambda^*, \Lambda)\) is compatible with the duality \((\Lambda^*, \Lambda)\) and this is done as in [1], Proposition 20.

\((vii) \Rightarrow (viii)\): Suppose \(\Lambda\) contains a bounded subset \(D\) on which \((P_i)\) does not converge uniformly to the zero-map. We show that \(\Lambda\) contains a subspace isomorphic to \(c_0\).

From the assumption it follows that there exist \(\varepsilon > 0, k \in \mathbb{N}\) and an increasing sequence of indices \((i_n)\) such that, for all \(n\), there exists \(\alpha^n = (\alpha^n_i) \in D\) with \(|\alpha^n_i| \cdot a^n_k > \varepsilon\), \(n = 1, 2, \ldots\). We put \(z_{i_n} = \alpha^n_i \cdot e_{i_n}, n = 1, 2, \ldots\). Then, the sequence \((z_{i_n})\) is bounded in \(\Lambda\).

Now we can define a linear map

\[ T : c_0 \rightarrow \Lambda : \sigma = (\sigma_n) \rightarrow \sum_n \sigma_n z_{i_n}. \]

We prove that \(T\) is an isomorphism from \(c_0\) into \(\Lambda\). It is easy to see that \(T\) is injective and continuous. Also, \(T : c_0 \rightarrow \text{Im} T\) is open.

Indeed, for \(\sigma = (\sigma_n) \in c_0\), we have

\[ p_k \left( T(\sigma) \right) = \max_n |\sigma_n a^n_k| \cdot a^n_k \geq \varepsilon \cdot \|\sigma\|_{c_0}. \]

\((ix) \Rightarrow (i)\): We prove that \(\text{Id} : \Lambda \rightarrow \Lambda\) transforms bounded subsets into compactoid subsets. Observe that \((ix)\) means that \(\lim_n S_n = \text{Id} \in L_c(\Lambda, \Lambda)\). Then apply Proposition 4 in [2].

The next corollary is for later use.

**Corollary 3.6.** If for every \(k \in \mathbb{N}\) and every subsequence \((i_n)\) of the indices there exists \(h > k\) such that the sequence \((a^n_h / a^n_k)\) is bounded, then \(K(\Lambda)\) is an FM-space.

**Proof.** An analysis of the proof of \((vii) \Rightarrow (viii)\) shows that if \(K(\Lambda)\) is not an FM-space, there exist a subsequence of the indices \((i_n)\) and elements \(\eta_{i_n}\) in \(K, n = 1, 2, \ldots\) such that the linear map \(T : c_0 \rightarrow \text{Im} T : (\sigma_n) \rightarrow (\sigma_n \eta_{i_n})\) is an isomorphism of \(c_0\) into \(\Lambda\).

Consider now in \(c_0\) the subspace \(c_{00}\) generated by the unit vectors \(e_1, e_2, \ldots\). Then \(c_{00}\) is isomorphic to the subspace \(F\) of \(K(\Lambda)\) generated by \(e_1, e_2, \ldots\). Therefore the topology induced by \(K(\Lambda)\) on \(F\) is normable. This means that there exists \(k\) such that for all \(h > k\) there exists \(t_h > 0\) with \(p_h(\delta) \leq t_h \cdot p_k(\delta)\) for all \(\delta \in K(\Lambda)\). In particular, for \(\delta = e_1\),
4. Characterizations of nuclear spaces among FM-spaces. We start this section with the construction of an FM-space which is not nuclear.

Example 4.1. For \( k = 1, 2, \ldots \), consider the infinite matrix

\[
A^k = (a_{ij}^k) = \begin{pmatrix}
1^k & 2^k & \cdots & j^k & \cdots \\
1^k & 2^k & \cdots & j^k & \cdots \\
\vdots & \vdots & & \vdots & \vdots \\
(k+1)^k & (k+1)^k & \cdots & (k+1)^k & \cdots \\
(k+2)^k & (k+2)^k & \cdots & (k+2)^k & \cdots \\
\vdots & \vdots & & \vdots & \vdots \\
\end{pmatrix} \to (k+1)
\]

We can think of \( A^k \) as a sequence for some order, \( k = 1, 2, \ldots \) (we fix the same order for all \( k \)). We then consider the Köthe space

\[
K(A) = \{ \beta = (\beta_{ij}) : \lim_{i,j} |\beta_{ij}| \cdot a_{ij}^k = 0, k = 1, 2, \ldots \}
\]

equipped with the sequence of norms \( (p_k) \) where \( p_k(\beta) = \max_{i,j} |\beta_{ij}| \cdot a_{ij}^k \).

We first show that \( K(A) \) is not nuclear. If \( k > 1 \), then the sequence \( (a_{ij}^k/a_{ij}^k) \) contains a constant sequence. Then by [3] Proposition 3.5 the conclusion follows.

We now apply Corollary 3.6 in order to prove that \( K(A) \) is an FM-space.

Choose \( k \) and any subsequence of the indices \( (l_n,j_m)_{n,m} \). We consider the corresponding elements \( a_{l_m,j_m}^k \) of \( A^k \). There are several possibilities.

a) The subsequence \( (a_{l_m,j_m}^k)_{n,m} \) contains an infinite number of elements of some row of \( A^k \).

If this row is between the rows 1, \ldots, \( k \), take \( h = k + 1 \). Then the sequence of the quotients \( (a_{l_m,j_m}^h/a_{l_m,j_m}^k)_{n,m} \) is unbounded.

If this row is the \( (k+r) \)-th row for some \( r \geq 1 \), then take \( h = k + r \).

b) The subsequence \( (a_{l_m,j_m}^k)_{n,m} \) consists of finitely many elements of an infinite number of rows. Consider then a subsequence with one element in an infinite number of rows below the \( k \)-th row. Such a subsequence looks like

\[
(k + l_1)^k, (k + l_2)^k, (k + l_2)^k, \ldots
\]

with \( (l_n) \) increasing to infinity. Take now \( h = k + 1 \).
Finally we investigate what the situation exactly is.

**Definition 4.2.** A locally convex space $X$ is said to be *quasinormable* if for every zero-neighbourhood $U$ in $X$ there exists a zero-neighbourhood $V$ in $X$, $V \subset U$, such that on $U'$ the topology $\beta(X', X)$ coincides with norm topology of $X'_{U'}$.

**Definition 4.3.** Let $X$ be a locally convex space. A sequence $(a_n) \subset X'$ is said to be **locally convergent to zero** if there exists a zero-neighbourhood $U$ in $X$ such that $(a_n) \subset X'_{U'}$ and $\lim_{n} \|a_n\|_{U'} = 0$.

**Theorem 4.4.** For an FM-space $E$ the following properties are equivalent.

(i) $E$ is nuclear.

(ii) $E$ is quasinormable.

(iii) Every $\beta(E', E)$-convergent sequence in $E'$ is locally convergent.

**Proof.** The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) follow by [2], Proposition 14 and [5], 5.2 respectively.

(iii) $\Rightarrow$ (i) Since $E$ is of countable type (Theorem 3.1) its topology can be described by the $\sigma(E', E)$-null sequences on $E'$ ([4], Theorem 3.2). By Theorem 3.3 (i) $\Rightarrow$ (iv) these sequences are null-sequences in $\beta(E', E)$ and by (iii) they are locally convergent to zero. The conclusion then follows from [5], 4.6.i).

**Corollary 4.5.** The Köthe space in 4.1 is also an example of an FM-space which is not quasinormable.

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