NON-ARCHIMEDEAN $r$-FRAMES AND FM-SPACES

N. DE GRANDE-DE KIMPE, C. PEREZ-GARCIA$^1$ AND W. H. SCHIKHOF

**Abstract.** We generalize the notion of $r$-orthogonality in $p$-adic Banach spaces by introducing $r$-frames ($\S$2). This we use to prove that a Fréchet-Montel (FM-)space is of countable type (Theorem 3.1), the non-archimedean counterpart of a well known theorem in functional analysis over $\mathbb{R}$ or $\mathbb{C}$ ([6], p. 231). We obtain several characterizations of FM-spaces (Theorem 3.3) and characterize the nuclear spaces among them ($\S$4).

1. Preliminaries. Throughout this paper $K$ is a non-archimedean non-trivially valued complete field with valuation $|\cdot|$. For the basic notions and properties concerning normed and locally convex spaces over $K$ we refer to [11] and [7]. However we recall the following.

1. Let $E$ be a $K$-vector space. Let $X \subset E$. The absolutely convex hull of $X$ is denoted by $\text{co}X$, its linear hull by $[X]$. For a (non-archimedean) seminorm $p$ on $E$ we denote by $E_p$ the vector space $E/\text{Ker}p$ and by $\pi_p: E \to E_p$ the canonical surjection. The formula $\|\pi_p(x)\| = p(x)$ defines a norm on $E_p$.

2. Let $(E, \|\cdot\|)$ be a normed space over $K$. For $r > 0$ we write $B(0, r) := \{x \in E : \|x\| \leq r\}$. Let $a \in E$, $X \subset E$. Then $\text{dist}(a, X) := \inf\{\|a - x\| : x \in X\}$. For $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in E$ we consider $\text{Vol}(x_1, \ldots, x_n) := \|x_1\| \cdot \text{dist}(x_2, [x_1]) \cdot \text{dist}(x_3, [x_1, x_2]) \cdots \text{dist}(x_n, [x_1, \ldots, x_{n-1}])$. For properties of this Volume Function (in particular, its symmetry), we refer to [10]. A linear continuous map $E \to F$, where $F$ is a normed space, is said to be compact if it sends the unit ball of $E$ into a compactoid set (see below).

3. Now let $E$ be a Hausdorff locally convex space over $K$. A subset $X$ of $E$ is called compactoid if for every zero-neighbourhood $U$ in $E$ there exists a finite set $S$ of $E$ such that $X \subset \text{co} S + U$. $E$ is said to be of countable type if for each continuous seminorm $p$ the normed space $E_p$ is of countable type (Recall that a normed space is called of countable type if it is the closed linear hull of a countable set). $E$ is called nuclear if for every continuous seminorm $p$ on $E$ there exists a continuous seminorm $q$ on $E$ with $p \leq q$, and such that $\Phi_{pq}$ is compact, where $\Phi_{pq}$ is the unique map making the diagram

$$
\begin{array}{ccc}
\pi_q & \nearrow & \pi_p \\
E_q & \xrightarrow{\Phi_{pq}} & E_p \\
\end{array}
$$

$^1$Research partially supported by the Spanish Dirección General de Investigación Científica y Técnica (DGICYT, PS87-0094).

Received by the editors September 18, 1991.

AMS subject classification: 46S10.

commute. \( E \) is called Montel if it is polar, polarly barrelled and if each closed bounded subset is a complete compactoid. A Fréchet space which is Montel is called an FM-space.

The closure of a set \( X \subset E \) is denoted by \( \overline{X} \).

2. \( t \)-frames in \( p \)-adic Banach spaces. Throughout \( \S 2 \) \( E \) is a normed space over \( K \).

We introduce a concept which generalizes the notion of \( t \)-orthogonality and it allows us to prove one of the main Theorems in the paper (Theorem 3.1).

**Definition 2.1.** Let \( t \in (0,1] \) and let \( X \subset E \) be a subset not containing 0. We call \( X \) a \( t \)-frame if for every \( n \in \mathbb{N} \) and distinct \( x_1, \ldots, x_n \in X \) we have \( \text{Vol}(x_1, \ldots, x_n) \geq t^{n-1} \cdot \|x_1\| \cdot \cdots \cdot \|x_n\| \).

We make the following simple observations. Let \( t \in (0,1] \).

1. Any \( t \)-orthogonal set in \( E \) is a \( t \)-frame. (Let \( \{e_i : i \in I\} \) be a \( t \)-orthogonal set in \( E \), let \( i_1, \ldots, i_n \) be \( n \) distinct elements of \( I \). Then, by the definition of the Volume Function and by \( t \)-orthogonality,

\[
\text{Vol}(e_{i_1}, \ldots, e_{i_n}) = \|e_{i_1}\| \cdot \text{dist}(e_{i_2}, [e_{i_1}]) \cdot \cdots \cdot \text{dist}(e_{i_n}, [e_{i_1}, \ldots, e_{i_{n-1}}]) \\
\geq \|e_{i_1}\| \cdot t \cdot \|e_{i_2}\| \cdot \cdots \cdot t \cdot \|e_{i_n}\| = t^{n-1} \cdot \|e_{i_1}\| \cdots \|e_{i_n}\|. 
\]

2. Every \( t \)-frame in \( E \) is a linearly independent set.
3. Every subset of a \( t \)-frame is itself a \( t \)-frame.
4. Every \( t \)-frame in \( E \) can be extended to a maximal \( t \)-frame.

By a \( t \)-frame sequence we shall mean a sequence \( a_1, a_2, \ldots \) in \( E \) such that \( \{x_1, x_2, \ldots\} \) is a \( t \)-frame.

**Proposition 2.2 (Compare [8], Theorem 2).** A bounded subset \( X \) of \( E \) is a compactoid if and only if for every \( t \in (0,1] \) every \( t \)-frame sequence in \( X \) tends to 0.

**Proof.** Suppose \( X \) is a compactoid. Suppose, for some \( t \in (0,1] \), and some \( \alpha > 0 \), \( X \) contains a \( t \)-frame sequence \( x_1, x_2, \ldots \) for which \( \|x_n\| \geq \alpha \) for all \( n \). Then, for each \( n \in \mathbb{N} \),

\[
\text{Vol}(x_1, \ldots, x_n) \geq t^{n-1} \cdot \|x_1\| \cdot \cdots \cdot \|x_n\| \geq \alpha^n t^{n-1}
\]

implying \( \lim_{n \to \infty} \inf \sqrt[n]{\text{Vol}(x_1, \ldots, x_n)} \geq \alpha t > 0 \) conflicting the compactoidity of \( X \) ([8], §2). This proves one half of the statement. The other half is obvious.

The following two Propositions are crucial for Theorem 2.5.

**Proposition 2.3.** Let \( 0 < t < 1 \); let \( X \) be a maximal \( t \)-frame in \( E \). Then \( \overline{X} = E \).

**Proof.** Let \( D := \overline{X} \). If \( D \neq E \) then we can find a nonzero \( a \in E \) with \( \text{dist}(a, D) \geq t \cdot \|a\| \) ([11], Lemma 3.14, here we use that \( t \neq 1 \)). So we shall prove that \( \text{dist}(a, D) < t \cdot \|a\| \) for every \( a \in E - D \). By maximality \( \{a\} \cup X \) is no longer a \( t \)-frame, yielding the existence of a \( k \in \mathbb{N} \) and distinct \( x_1, \ldots, x_k \in X \) such that

\[
\text{Vol}(a, x_1, \ldots, x_k) < t^k \cdot \|a\| \cdot \|x_1\| \cdot \cdots \cdot \|x_k\|.
\]
On the other hand we have

\[
\text{Vol}(a, x_1, \ldots, x_k) = \text{dist}(a, [x_1, \ldots, x_k]) \cdot \text{Vol}(x_1, \ldots, x_k) \\
\geq \text{dist}(a, D) \cdot t^{k-1} \cdot \|x_1\| \cdots \|x_k\|.
\]

So \(\text{dist}(a, D) < t \cdot \|a\|\).

**Remark.** We now can easily find examples of \(t\)-frames \(X\) that are \(s\)-orthogonal for no \(s \in (0, 1]\): Let \(0 < t < 1\), let \(E\) have no base, choose for \(X\) a maximal \(t\)-frame (Observe that the clause \(t \neq 1\) is essential!).

**Proposition 2.4.** Every uncountable subset of \(c_0\) contains an infinite compactoid.

**Proof.** Let \(X\) be an uncountable subset of \(c_0\); it has a bounded uncountable subset \(Y\). Let \(e_1, e_2, \ldots\) be the standard basis of \(c_0\). We have \(B(0, 1) + [e_1, e_2, \ldots] = c_0\) so there exists an \(n_1 \in N\) such that

\[Y_1 := Y \cap (B(0, 1) + [e_1, e_2, \ldots, e_{n_1}])\]

is uncountable. In its turn, there exists an \(n_2 \in N\) such that

\[Y_2 := Y_1 \cap (B(0, 1/2) + [e_1, e_2, \ldots, e_{n_2}])\]

is uncountable. We obtain uncountable sets \(Y_1 \supset Y_2 \supset \cdots\) such that \(Y_n \subset B(0, 1/n) + D_n\) for each \(n\) where \(D_n\) is a finite-dimensional space. Choose distinct \(x_1, x_2, \ldots\) where \(x_n \in Y_n\) for each \(n\), and set \(Z := \{x_1, x_2, \ldots\}\). Then \(Z\) is infinite, bounded, in \(X\). Also, for each \(n \in N\) we have

\[Z \subset \{x_1, \ldots, x_{n-1}\} \cup Y_n \subset [x_1, \ldots, x_{n-1}] + B(0, 1/n) + D_n \subset B(0, 1/n) + D_n\]

where \(D_n\) is a finite-dimensional space. It follows that \(Z\) is a compactoid.

**Theorem 2.5.** The following assertions about the normed space \(E\) are equivalent.

(i) \(E\) is of countable type.

(ii) For every \(t \in (0, 1)\), every \(t\)-frame in \(E\) is countable.

(iii) For some \(t \in (0, 1)\), every \(t\)-frame in \(E\) is countable.

**Proof.** (i) \(\Rightarrow\) (ii). We may assume \(E = c_0\). Let \(X\) be a \(t\)-frame in \(E\). For each \(n \in N\) set \(X_n := \{x \in X : \|x\| \geq 1/n\}\). If, for some \(n\), \(X_n\) were uncountable it would contain an infinite compactoid \(\{x_1, x_2, \ldots\}\) by Proposition 2.4. Then from Proposition 2.2 \(\lim_{k \to \infty} x_k = 0\), a contradiction.

(ii) \(\Rightarrow\) (iii) is obvious.

(iii) \(\Rightarrow\) (i). Let \(X\) be a maximal \(t\)-frame in \(E\). By assumption \(X\) is countable. By Proposition 2.3, \(E = [X]\) is of countable type.

**Remark.** The question if Theorem 2.5 remains true when we consider in (i) and (ii) \(t\)-orthogonal sets instead \(t\)-frames is an open problem in non-archimedean analysis ([11], p. 199).
3. Characterizations of FM-spaces among F-spaces. From now on in this paper $E$ is a polar Hausdorff locally convex space over $K$.

It is proved in [6], Theorem 11.6.2, that a Fréchet Montel space over $\mathbb{R}$ or $\mathbb{C}$ is separable. It does not simply carry over the non-archimedean case because $K$ may be not locally compact; so we have to deal with compactoids (§1.3) rather than compact sets. This modification is obstructing the classical proof which is essentially based upon separability. It is here where the $t$-frames of §2 come to the rescue as will be demonstrated in the following theorem (for other applications of $t$-frames in $p$-adic analysis, see [9], p. 51–57).

**Theorem 3.1.** An FM-space is of countable type.

**Proof.** Let the topology of the FM-space $E$ be defined by the sequence of seminorms $p_1 \leq p_2 \leq \cdots$. Set $U_n = \{ x \in E : p_n(x) \leq 1 \}$. Choose $\lambda \in K$, $|\lambda| > 1$.

It suffices to show that $E_1 := E_{p_1}$ is of countable type. Let $X$ be a $t$-frame in $(E_1, \| \cdot \|_1)$ for some $t \in (0, 1)$; we show (Theorem 2.5) that $X$ is countable. Suppose not. We may assume that $\inf\{\|x\|_1 : x \in X\} > 0$. Choose an $A_1 \subset E$ such that $\pi_{p_1}(A_1) = X$. Since $E = U_n \lambda^n U_2$ there exists an $n_2$ such that $A_2 := A_1 \cap \lambda^{n_2} U_2$ is uncountable. Inductively we arrive at uncountable sets $A_1 \supset A_2 \supset \cdots$ such that $A_n$ is $p_n$-bounded for each $n \geq 2$. Choose distinct $a_1, a_2, \ldots$ with $a_n \in A_n$ for each $n$. Then $\{a_1, a_2, \ldots\}$ is bounded in $E$. As $E$ is Montel, it is a compactoid. By Proposition 2.2, $\lim_{n \to \infty} \pi_{p_n}(a_n) = 0$ conflicting $\inf\{\|x\|_1 : x \in X\} > 0$.

**Lemma 3.2.** Every bounded subset $B$ of a Fréchet space $E$, is compactoid for the topology of uniform convergence on the $\beta(E', E)$-compactoid subsets of $E'$ (where $\beta(E', E)$ denotes the strong topology on $E'$ with respect to the dual pair $(E, E')$).

**Proof.** Consider the canonical map $J_E : E \to E'' = (E', \beta(E', E))^\prime$. It is easy to see that the set $J_E(B)$ is equicontinuous on $(E'', \beta(E'', E))$. By [7] Lemma 10.6 we have that on $J_E(B)$ the topology $\tau_{kk}$ (on $E''$) of the uniform convergence on the $\beta(E', E)$-compactoid subsets of $E'$, coincides with the weak topology $\sigma(E'', E')$. Hence $J_E(B)$ is $\tau_{kk}$-compactoid in $E''$. Since $J_E$ is an homeomorphism from $E$ onto a subspace of $E''$ ([7], Lemmas 9.2, 9.3) we are done.

**Theorem 3.3.** For a Fréchet space $E$, the following properties are equivalent.

(i) $E$ is an FM-space.

(ii) Every bounded subset of $E$ is compactoid.

(iii) In $E$ every weakly convergent sequence is convergent and $(E', \beta(E', E))$ is of countable type.

(iv) In $E'$ every $\sigma(E', E)$-convergent sequence is $\beta(E', E)$-convergent and $E$ is of countable type.

(v) Both $E$ and $(E', \beta(E', E))$ are of countable type.

(vi) $(E', \beta(E', E))$ is nuclear.

(vii) $(E', \beta(E', E))$ is Montel.
Every \(\sigma(E', E)\)-bounded subset of \(E'\) is \(\beta(E', E)\)-compactoid.

**PROOF.** The implications (i) \(\Leftrightarrow\) (ii) \(\Leftrightarrow\) (iii), (i) \(\Rightarrow\) (vi) \(\Rightarrow\) (viii) and (i) \(\Rightarrow\) (vii) \(\Rightarrow\) (viii) are known (see [7]) or easy. Also, from Theorem 3.1 we can easily prove (i) \(\Rightarrow\) (iv) and (i) \(\Rightarrow\) (v).

Now we prove (viii) \(\Rightarrow\) (ii): Since \(E\) is a polar Fréchet space, its topology \(\tau\) is the topology of uniform convergence on the \(\sigma(E', E)\)-bounded subsets of \(E'\). By (viii) these subsets are \(\beta(E', E)\)-compactoid. Now apply Lemma 3.2.

The implication (v) \(\Rightarrow\) (iii) follows from [7] Proposition 4.11.

Finally, for the proof of (iv) \(\Rightarrow\) (ii) observe that the topology on a polar Fréchet space of countable type is the topology of uniform convergence on the \(\sigma(E', E)\)-null sequences in \(E'\) (see [4], Theorem 3.2). By (iv) these sequences are \(\beta(E', E)\)-convergent. Now apply Lemma 3.2. 

**REMARK.** It is known that a Fréchet space \(E\) over \(\mathbb{R}\) over \(\mathbb{C}\) is nuclear if and only if \((E', \beta(E', E))\) is nuclear ([6], p. 491).

In the non-archimedean case the situation is essentially different. Indeed, in 4.1 we will give an example of an FM-space which is not nuclear (while its strong dual is by (i) \(\Leftrightarrow\) (vi)). To do that we need some preliminary concepts and results.

**DEFINITION 3.4.** Let \(A = (a_{ij})\) be a matrix of strictly positive real numbers such that \(a_{ij}^{k+1} > a_{ij}^k\) for all \(i\) and all \(k\). Then the corresponding Köthe sequence space \(K(A)\) is defined by

\[K(A) = \{\alpha = (\alpha_i) : \lim_{i} |\alpha_i| \cdot a_{i}^{k} = 0 \text{ for all } k\}\.

On \(K(A)\) we consider the sequence of norms \((p_k)\), where

\[p_k(\alpha) = \max_{i} |\alpha_i| \cdot a_{i}^{k}, \quad k = 1, 2, \ldots; \quad \alpha \in K(A)\.

It is known that \(K(A)\) is a polar Fréchet space of countable type. For the importance of this class of spaces and for their further properties we refer to [3].

We then have:

**PROPOSITION 3.5.** Let \(\Lambda = K(A)\) be a Köthe space and let \(\Lambda^*\) the corresponding Köthe dual space. Then the following properties are equivalent:

(i) \(\Lambda\) is an FM-space.

(ii) \((\Lambda^*, \beta(\Lambda^*, \Lambda))\) is of countable type.

(iii) \((\Lambda^*, \beta(\Lambda^*, \Lambda))\) is nuclear.

(iv) \((\Lambda^*, \beta(\Lambda^*, \Lambda))\) is Montel.

(v) The unit vectors \(e_1, e_2, \ldots\) form a Schauder basis for \(\Lambda^*, \beta(\Lambda^*, \Lambda)\).

(vi) \(n(\Lambda, \Lambda) = \beta(\Lambda^*, \Lambda)\) (where \(n(\Lambda, \Lambda)\) is the natural topology on \(\Lambda^*\)).

(vii) No subspace of \(\Lambda\) is isomorphic (linearly homeomorphic) to \(c_0\).

(viii) The sequence of coordinate projections \((P_i)\), where \(P_i : \Lambda \rightarrow \Lambda : \alpha = (\alpha_i) \rightarrow \alpha_i e_i\), converges to the zero-map uniformly on every bounded subset of \(\Lambda\).
(ix) The sequence of sections-maps \( (S_n) \), where \( S_n: \Lambda \rightarrow \Lambda : \alpha = (\alpha_i) \rightarrow (\alpha_1, \alpha_2, ..., \alpha_n, 0, 0, ...) \) converges to the identity map \( \text{Id} \) uniformly on every bounded subset of \( \Lambda \).

**Proof.** We only have to prove (i) \( \Rightarrow \) (v), (vi) \( \Rightarrow \) (viii) and (ix) \( \Rightarrow \) (i). The other implications are easy.

(i) \( \Rightarrow \) (v): The unit vectors \( e_1, e_2, ... \) form a Schauder basis for \( (\Lambda^*, \sigma(\Lambda^*, \Lambda)) \). Then, apply (i) \( \Rightarrow \) (iv) in 3.3.

(v) \( \Rightarrow \) (vi): By [4], p. 21 it suffices to prove that \( \beta(\Lambda^*, \Lambda) \) is compatible with the duality \( (\Lambda^*, \Lambda) \) and this is done as in [1], Proposition 20.

(vii) \( \Rightarrow \) (viii): Suppose \( \Lambda \) contains a bounded subset \( D \) on which \( (P_i) \) does not converge uniformly to the zero-map. We show that \( \Lambda \) contains a subspace isomorphic to \( c_0 \).

From the assumption it follows that there exist \( \varepsilon > 0, k \in \mathbb{N} \) and an increasing sequence of indices \( (i_n) \) such that, for all \( n \), there exists \( \alpha^n = (\alpha^n_i) \in D \) with \( |\alpha^n_i| \cdot a^n_h > \varepsilon, n = 1, 2, ... \). We put \( z_{i_n} = \alpha^n_i \cdot e_{i_n}, n = 1, 2, ... \). Then, the sequence \( (z_{i_n}) \) is bounded in \( \Lambda \).

Now we can define a linear map

\[
T: c_0 \rightarrow \Lambda : \sigma = (\sigma_n) \rightarrow \sum_n \sigma_n z_{i_n}.
\]

We prove that \( T \) is an isomorphism from \( c_0 \) into \( \Lambda \). It is easy to see that \( T \) is injective and continuous. Also, \( T: c_0 \rightarrow \text{Im} T \) is open.

Indeed, for \( \sigma = (\sigma_n) \in c_0 \), we have \( p_k(T(\sigma)) = \max_{n} \frac{1}{n} |\sigma_n a^n_k| \cdot e^n_k \geq \varepsilon \cdot ||\sigma||_{c_0}. \)

(ix) \( \Rightarrow \) (i): We prove that \( \text{Id}: \Lambda \rightarrow \Lambda \) transforms bounded subsets into compactoid subsets. Observe that (ix) means that \( \lim_n S_n = \text{Id} \) in \( L_{\infty}(\Lambda, \Lambda) \). Then apply Proposition 4 in [2].

The next corollary is for later use.

**Corollary 3.6.** If for every \( k \in \mathbb{N} \) and every subsequence \( (i_n) \) of the indices there exists \( h > k \) such that the sequence \( (a^n_h / a^n_k) \) is bounded, then \( K(\Lambda) \) is an FM-space.

**Proof.** An analysis of the proof of (vii) \( \Rightarrow \) (viii) shows that if \( K(\Lambda) \) is not an FM-space, there exist a subsequence of the indices \( (i_n) \) and elements \( \eta_{i_n} \in \mathbb{K}, n = 1, 2, ... \) such that the linear map \( T: c_0 \rightarrow \text{Im} T : (\sigma_n) \rightarrow (\sigma_n \eta_{i_n}) \) is an isomorphism of \( c_0 \) into \( \Lambda \).

Consider now in \( c_0 \) the subspace \( c^{(0)} \) generated by the unit vectors \( e_1, e_2, ... \). Then \( c^{(0)} \) is isomorphic to the subspace \( F \) of \( K(\Lambda) \) generated by \( e_1, e_2, ... \). Therefore the topology induced by \( K(\Lambda) \) on \( F \) is normable. This means that there exists \( k \) such that for all \( h > k \) there exists \( t_h > 0 \) with \( p_h(\delta) \leq t_h \cdot p_k(\delta) \) for all \( \delta \in K(\Lambda) \). In particular, for \( \delta = e_{i_n}, \)
For $n = 1, 2, \ldots$, we have that there is a $k$ such that for all $h > k$, there exists $t_h > 0$ with $a_{ih}^h \leq t_h \cdot a_{ik}^k$ for all $n$, and we are done. 

4. Characterizations of nuclear spaces among FM-spaces. We start this section with the construction of an FM-space which is not nuclear.

**Example 4.1.** For $k = 1, 2, \ldots$, consider the infinite matrix

$$A^k = (a_{ij}^k) = \begin{pmatrix}
1^k & \cdots & 2^k & \cdots & j^k & \cdots \\
1^k & \cdots & 2^k & \cdots & j^k & \cdots \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\
(k+1)^k & \cdots & (k+1)^k & \cdots & (k+1)^k & \cdots \\
(k+2)^k & \cdots & (k+2)^k & \cdots & (k+2)^k & \cdots \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots 
\end{pmatrix} \rightarrow (k+1)$$

We can think of $A^k$ as a sequence for some order, $k = 1, 2, \ldots$ (we fix the same order for all $k$). We then consider the Köthe space $K(A) = \{ \beta = (\beta_{ij}) : \lim_{i,j} |\beta_{ij}| \cdot a_{ij}^k = 0, k = 1, 2, \ldots \}$ equipped with the sequence of norms $(p_k)$ where $p_k(\beta) = \max_{i,j} |\beta_{ij}| \cdot a_{ij}^k$.

We first show that $K(A)$ is not nuclear. If $k > 1$, then the sequence $(a_{ij}^1/a_{ij}^k)$ contains a constant sequence. Then by [3] Proposition 3.5 the conclusion follows.

We now apply Corollary 3.6 in order to prove that $K(A)$ is an FM-space.

Choose $k$ and any subsequence of the indices $(i_n, j_m)_{n,m}$. We consider the corresponding elements $a_{i_n, j_m}^k$ of $A^k$. There are several possibilities.

a) The subsequence $(a_{i_n, j_m}^k)_{n,m}$ contains an infinite number of elements of some row of $A^k$.

If this row is between the rows $1, \ldots, k$, take $h = k + 1$. Then the sequence of the quotients $(a_{i_n, j_m}^h/a_{i_n, j_m}^k)_{n,m}$ is unbounded.

If this row is the $(k + r)$-th row for some $r \geq 1$, then take $h = k + r$.

b) The subsequence $(a_{i_n, j_m}^k)_{n,m}$ consists of finitely many elements of an infinite number of rows. Consider then a subsequence with one element in an infinite number of rows below the $k$th row. Such a subsequence looks like

$$(k + l_1)^k, (k + l_2)^k, (k + l_3)^k, \ldots$$

with $(l_n)_n$ increasing to infinity. Take now $h = k + 1$. 


Finally we investigate what the situation exactly is.

**Definition 4.2.** A locally convex space $X$ is said to be *quasinormable* if for every zero-neighbourhood $U$ in $X$ there exists a zero-neighbourhood $V$ in $X$, $V \subset U$, such that on $U'$ the topology $\beta(X', X)$ coincides with norm topology of $X'_{U'}$.

**Definition 4.3.** Let $X$ be a locally convex space. A sequence $(a_n) \subset X'$ is said to be *locally convergent to zero* if there exists a zero-neighbourhood $U$ in $X$ such that $(a_n) \subset X'_{U}$ and $\lim_{n} \|a_n\|_{U} = 0$.

**Theorem 4.4.** For an FM-space $E$ the following properties are equivalent.

(i) $E$ is nuclear.

(ii) $E$ is quasinormable.

(iii) Every $\beta(E', E)$-convergent sequence in $E'$ is locally convergent.

**Proof.** The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) follow by [2], Proposition 14 and [5], 5.2 respectively.

(iii) $\Rightarrow$ (i) Since $E$ is of countable type (Theorem 3.1) its topology can be described by the $\sigma(E', E)$-null sequences on $E'$ ([4], Theorem 3.2). By Theorem 3.3 (i) $\Rightarrow$ (iv) these sequences are null-sequences in $\beta(E', E)$ and by (iii) they are locally convergent to zero. The conclusion then follows from [5], 4.6.i).

**Corollary 4.5.** The Köthe space in 4.1 is also an example of an FM-space which is not quasinormable.

**References**


