NON-ARCHIMEDEAN $t$-FRAMES AND FM-SPACES

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Abstract. We generalize the notion of $t$-orthogonality in $p$-adic Banach spaces by introducing $t$-frames ($\S$2). This we use to prove that a Fréchet-Montel (FM-)space is of countable type (Theorem 3.1), the non-archimedean counterpart of a well known theorem in functional analysis over $R$ or $C$ ([6], p. 231). We obtain several characterizations of FM-spaces (Theorem 3.3) and characterize the nuclear spaces among them ($\S$4).

1. Preliminaries. Throughout this paper $K$ is a non-archimedean non-trivially valued complete field with valuation $| . |$. For the basic notions and properties concerning normed and locally convex spaces over $K$ we refer to [11] and [7]. However we recall the following.

1. Let $E$ be a $K$-vector space. Let $X \subset E$. The absolutely convex hull of $X$ is denoted by $coX$, its linear hull by $[X]$. For a (non-archimedean) seminorm $p$ on $E$ we denote by $E_p$ the vector space $E/\text{Ker } p$ and by $\pi_p: E \rightarrow E_p$ the canonical surjection. The formula $\| \pi_p(x) \| = p(x)$ defines a norm on $E_p$.

2. Let $(E, \| \cdot \|)$ be a normed space over $K$. For $r > 0$ we write $B(0, r) := \{x \in E : \|x\| \leq r\}$. Let $a \in E$, $X \subset E$. Then $\text{dist}(a, X) := \inf\{\|a - x\| : x \in X\}$. For $n \in N$ and $x_1, \ldots, x_n \in E$ we consider $\text{Vol}(x_1, \ldots, x_n) := \|x_1\| \cdot \text{dist}(x_2, [x_1]) \cdot \text{dist}(x_3, [x_1, x_2]) \cdots \text{dist}(x_n, [x_1, \ldots, x_{n-1}])$. For properties of this Volume Function (in particular, its symmetry), we refer to [10]. A linear continuous map $E \rightarrow F$, where $F$ is a normed space, is said to be compact if it sends the unit ball of $E$ into a compactoid set (see below).

3. Now let $E$ be a Hausdorff locally convex space over $K$. A subset $X$ of $E$ is called compactoid if for every zero-neighbourhood $U$ in $E$ there exists a finite set $S$ of $E$ such that $X \subset \text{co } S + U$. $E$ is said to be of countable type if for each continuous seminorm $p$ the normed space $E_p$ is of countable type (Recall that a normed space is called of countable type if it is the closed linear hull of a countable set). $E$ is called nuclear if for every continuous seminorm $p$ on $E$ there exists a continuous seminorm $q$ on $E$ with $p \leq q$, and such that $\Phi_{pq}$ is compact, where $\Phi_{pq}$ is the unique map making the diagram

$$
\begin{array}{ccc}
E_q & \xrightarrow{\pi_q} & E \\
E_{pq} & \xrightarrow{\Phi_{pq}} & E_p
\end{array}
$$

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commute. \( E \) is called Montel if it is polar, polarly barrelled and if each closed bounded subset is a complete compactoid. A Fréchet space which is Montel is called an FM-space.

The closure of a set \( X \subset E \) is denoted by \( \overline{X} \).

2. \( t \)-frames in \( p \)-adic Banach spaces. Throughout §2 \( E \) is a normed space over \( K \). We introduce a concept which generalizes the notion of \( t \)-orthogonality and it allows us to prove one of the main Theorems in the paper (Theorem 3.1).

**Definition 2.1.** Let \( t \in (0, 1] \), and let \( X \subset E \) be a subset not containing 0. We call \( X \) a \( t \)-frame if for every \( n \in N \) and distinct \( x_1, \ldots, x_n \in X \) we have \( \text{Vol}(x_1, \ldots, x_n) \geq t^{n-1} \cdot \|x_1\| \cdot \cdots \cdot \|x_n\| \).

We make the following simple observations. Let \( t \in (0, 1] \).

1. Any \( t \)-orthogonal set in \( E \) is a \( t \)-frame. (Let \( \{e_i : i \in I\} \) be a \( t \)-orthogonal set in \( E \), let \( i_1, \ldots, i_n \) be \( n \) distinct elements of \( I \). Then, by the definition of the Volume Function and by \( t \)-orthogonality,

\[
\text{Vol}(e_{i_1}, \ldots, e_{i_n}) = \|e_{i_1}\| \cdot \text{dist}(e_{i_2}, [e_{i_1}]) \cdot \cdots \cdot \text{dist}(e_{i_n}, [e_{i_1}, \ldots, e_{i_{n-1}}]) \\
\geq \|e_{i_1}\| \cdot t \cdot \|e_{i_2}\| \cdot \cdots \cdot t \cdot \|e_{i_n}\| = t^{n-1} \cdot \|e_{i_1}\| \cdots \cdot \|e_{i_n}\|.
\]

2. Every \( t \)-frame in \( E \) is a linearly independent set.
3. Every subset of a \( t \)-frame is itself a \( t \)-frame.
4. Every \( t \)-frame in \( E \) can be extended to a maximal \( t \)-frame.

By a \( t \)-frame sequence we shall mean a sequence \( x_1, x_2, \ldots \) in \( E \) such that \( \{x_1, x_2, \ldots \} \) is a \( t \)-frame.

**Proposition 2.2 (Compare [8], Theorem 2).** A bounded subset \( X \) of \( E \) is a compactoid if and only if for every \( t \in (0, 1] \) every \( t \)-frame sequence in \( X \) tends to 0.

**Proof.** Suppose \( X \) is a compactoid. Suppose, for some \( t \in (0, 1] \), and some \( \alpha > 0 \), \( X \) contains a \( t \)-frame sequence \( x_1, x_2, \ldots \) for which \( \|x_n\| \geq \alpha \) for all \( n \). Then, for each \( n \in N \),

\[
\text{Vol}(x_1, \ldots, x_n) \geq t^{n-1} \cdot \|x_1\| \cdot \cdots \cdot \|x_n\| \geq \alpha^n \cdot t^{n-1}
\]

implying \( \lim_{n \to \infty} \inf \sqrt[n]{\text{Vol}(x_1, \ldots, x_n)} \geq \alpha > 0 \) conflicting the compactoidity of \( X \) ([8], §2). This proves one half of the statement. The other half is obvious. ■

The following two Propositions are crucial for Theorem 2.5.

**Proposition 2.3.** Let \( 0 < t < 1 \); let \( X \) be a maximal \( t \)-frame in \( E \). Then \( \overline{X} = E \).

**Proof.** Let \( D := \overline{X} \). If \( D \neq E \) then we can find a nonzero \( a \in E \) with \( \text{dist}(a, D) \geq t \cdot \|a\| \) ([11], Lemma 3.14, here we use that \( t \neq 1 \)). So we shall prove that \( \text{dist}(a, D) < t \cdot \|a\| \) for every \( a \in E - D \). By maximality \( \{a\} \cup X \) is no longer a \( t \)-frame, yielding the existence of a \( k \in N \) and distinct \( x_1, \ldots, x_k \in X \) such that

\[
\text{Vol}(a, x_1, \ldots, x_k) < t^k \cdot \|a\| \cdot \|x_1\| \cdot \cdots \cdot \|x_k\|.
\]
On the other hand we have
\[
\text{Vol}(a,x_1,\ldots,x_k) = \text{dist}(a,[x_1,\ldots,x_k]) \cdot \text{Vol}(x_1,\ldots,x_k) \\
\geq \text{dist}(a,D) \cdot t^{k-1} \cdot \|x_1\| \cdots \|x_k\|.
\]
So \(\text{dist}(a,D) < t \cdot \|a\|\).

Remark. We now can easily find examples of \(t\)-frames \(X\) that are \(s\)-orthogonal for no \(s \in (0,1]\): Let \(0 < t < 1\), let \(E\) have no base, choose for \(X\) a maximal \(t\)-frame (Observe that the clause \(t \neq 1\) is essential!).

Proposition 2.4. Every uncountable subset of \(c_0\) contains an infinite compactoid.

Proof. Let \(X\) be an uncountable subset of \(c_0\); it has a bounded uncountable subset \(Y\). Let \(e_1,e_2,\ldots\) be the standard basis of \(c_0\). We have \(B(0,1) + [e_1,e_2,\ldots] = c_0\) so there exists an \(n_1 \in \mathbb{N}\) such that
\[
Y_1 := Y \cap (B(0,1) + [e_1,e_2,\ldots,e_{n_1}])
\]
is uncountable. In its turn, there exists an \(n_2 \in \mathbb{N}\) such that
\[
Y_2 := Y_1 \cap (B(0,1/2) + [e_1,e_2,\ldots,e_{n_2}])
\]
is uncountable. We obtain uncountable sets \(Y_1 \supset Y_2 \supset \cdots\) such that \(Y_n \subset B(0,1/n) + D_n\) for each \(n\) where \(D_n\) is a finite-dimensional space. Choose distinct \(x_1,x_2,\ldots\) where \(x_n \in Y_n\) for each \(n\), and set \(Z := \{x_1,x_2,\ldots\}\). Then \(Z\) is infinite, bounded, in \(X\). Also, for each \(n \in \mathbb{N}\) we have
\[
Z \subset \{x_1,\ldots,x_{n-1}\} \cup Y_n \subset [x_1,\ldots,x_{n-1}] + B(0,1/n) + D_n \subset B(0,1/n) + \hat{D}_n
\]
where \(\hat{D}_n\) is a finite-dimensional space. It follows that \(Z\) is a compactoid.

Theorem 2.5. The following assertions about the normed space \(E\) are equivalent.

(i) \(E\) is of countable type.

(ii) For every \(t \in (0,1)\), every \(t\)-frame in \(E\) is countable.

(iii) For some \(t \in (0,1)\), every \(t\)-frame in \(E\) is countable.

Proof. (i) \(\Rightarrow\) (ii). We may assume \(E = c_0\). Let \(X\) be a \(t\)-frame in \(E\). For each \(n \in \mathbb{N}\) set \(X_n := \{x \in X : \|x\| \geq 1/n\}\). If, for some \(n\), \(X_n\) were uncountable it would contain an infinite compactoid \(\{x_1,x_2,\ldots\}\) by Proposition 2.4. Then from Proposition 2.2 \(\lim_{k \to \infty} x_k = 0\), a contradiction.

(ii) \(\Rightarrow\) (iii) is obvious.

(iii) \(\Rightarrow\) (i). Let \(X\) be a maximal \(t\)-frame in \(E\). By assumption \(X\) is countable. By Proposition 2.3, \(E = [X]\) is of countable type.

Remark. The question if Theorem 2.5 remains true when we consider in (i) and (ii) \(t\)-orthogonal sets instead \(t\)-frames is an open problem in non-archimedean analysis ([11], p. 199).
3. Characterizations of FM-spaces among F-spaces. From now on in this paper E is a polar Hausdorff locally convex space over K.

It is proved in [6], Theorem 11.6.2, that a Fréchet Montel space over R or C is separable. It does not simply carry over the non-archimedean case because K may be not locally compact; so we have to deal with compactoids (§1.3) rather than compact sets. This modification is obstructing the classical proof which is essentially based upon separability. It is here where the t-frames of §2 come to the rescue as will be demonstrated in the following theorem (for other applications of t-frames in p-adic analysis, see [9], p. 51–57).

**Theorem 3.1.** An FM-space is of countable type.

**Proof.** Let the topology of the FM-space E be defined by the sequence of seminorms \( p_1 \leq p_2 \leq \cdots \). Set \( U_n = \{ x \in E : p_n(x) \leq 1 \} \). Choose \( \lambda \in K, |\lambda| > 1 \).

It suffices to show that \( E_n := E_{p_n} \) is of countable type. Let \( X \) be a t-frame in \((E_1, \| \cdot \|_1)\) for some \( t \in (0, 1) \); we show (Theorem 2.5) that \( X \) is countable. Suppose not. We may assume that \( \inf \{ \| x \|_1 : x \in X \} > 0 \). Choose an \( A_1 \subset E \) such that \( \pi_{p_n}(A_1) = X \). Since \( E = \bigcup_n \lambda^n U_2 \) there exists an \( n_2 \) such that \( A_2 := A_1 \cap \lambda^{n_2} U_2 \) is uncountable. Inductively we arrive at uncountable sets \( A_1 \supset A_2 \supset \cdots \) such that \( A_n \) is \( p_n \)-bounded for each \( n \geq 2 \). Choose distinct \( a_1, a_2, \ldots \) with \( a_n \in A_n \) for each \( n \). Then \( \{ a_1, a_2, \ldots \} \) is bounded in \( E \).

As \( E \) is Montel, it is a compactoid. By Proposition 2.2, \( \lim_n \| a_n \| = 0 \) conflicting \( \inf \{ \| x \|_1 : x \in X \} > 0 \) ■

**Lemma 3.2.** Every bounded subset \( B \) of a Fréchet space \( E \), is compactoid for the topology of uniform convergence on the \( \beta(E', E) \)-compactoid subsets of \( E' \) (where \( \beta(E', E) \) denotes the strong topology on \( E' \) with respect to the dual pair \((E, E')\)).

**Proof.** Consider the canonical map \( J_E : E \to E'' = (E', \beta(E', E))' \). It is easy to see that the set \( J_E(B) \) is equicontinuous on \((E', \beta(E', E))\). By [7] Lemma 10.6 we have that on \( J_E(B) \) the topology \( \tau_{uk} \) (on \( E'' \)) of the uniform convergence on the \( \beta(E', E) \)-compactoid subsets of \( E' \), coincides with the weak topology \( \sigma(E'', E') \). Hence \( J_E(B) \) is \( \tau_{uk} \)-compactoid in \( E'' \). Since \( J_E \) is an homeomorphism from \( E \) onto a subspace of \( E'' \) ([7], Lemmas 9.2, 9.3) we are done. ■

**Theorem 3.3.** For a Fréchet space \( E \), the following properties are equivalent.

(i) \( E \) is an FM-space.

(ii) Every bounded subset of \( E \) is compactoid.

(iii) In \( E \) every weakly convergent sequence is convergent and \((E', \beta(E', E))\) is of countable type.

(iv) In \( E' \) every \( \sigma(E', E) \)-convergent sequence is \( \beta(E', E) \)-convergent and \( E \) is of countable type.

(v) Both \( E \) and \((E', \beta(E', E))\) are of countable type.

(vi) \((E', \beta(E', E))\) is nuclear.

(vii) \((E', \beta(E', E))\) is Montel.
(viii) Every $\sigma(E', E)$-bounded subset of $E'$ is $\beta(E', E)$-compactoid.

**Proof.** The implications (i) ⇔ (ii) ⇔ (iii), (i) ⇒ (vi) ⇒ (viii) and (i) ⇒ (vii) ⇒ (viii) are known (see [7]) or easy. Also, from Theorem 3.1 we can easily prove (i) ⇒ (iv) and (i) ⇒ (v).

Now we prove (viii) ⇒ (ii): Since $E$ is a polar Fréchet space, its topology $\tau$ is the topology of uniform convergence on the $\sigma(E', E)$-bounded subsets of $E'$. By (viii) these subsets are $\beta(E', E)$-compactoid. Now apply Lemma 3.2.

The implication (v) ⇒ (iii) follows from [7] Proposition 4.11.

Finally, for the proof of (iv) ⇒ (ii) observe that the topology on a polar Fréchet space of countable type is the topology of uniform convergence on the $\sigma(E, E)$-null sequences in $E'$ (see [4], Theorem 3.2). By (iv) these sequences are $\beta(E', E)$-convergent. Now apply Lemma 3.2. ■

**Remark.** It is known that a Fréchet space $E$ over $\mathbb{R}$ over $\mathbb{C}$ is nuclear if and only if $(E', \beta(E', E))$ is nuclear ([16], p. 491).

In the non-archimedean case the situation is essentially different. Indeed, in 4.1 we will give an example of an FM-space which is not nuclear (while its strong dual is by (i) ⇔ (vi)). To do that we need some preliminary concepts and results.

**Definition 3.4.** Let $A = (a_{ik})$ be a matrix of strictly positive real numbers such that $a_{i+1,k} > a_{i,k}$ for all $i$ and all $k$. Then the corresponding Köthe sequence space $K(A)$ is defined by

$$K(A) = \{ \alpha = (\alpha_i) : \lim_i |\alpha_i| \cdot a_{i,k} = 0 \text{ for all } k \}.$$ 

On $K(A)$ we consider the sequence of norms $(p_k)$, where

$$p_k(\alpha) = \max_i |\alpha_i| \cdot a_{i,k}, \quad k = 1, 2, \ldots; \quad \alpha \in K(A).$$

It is known that $K(A)$ is a polar Fréchet space of countable type. For the importance of this class of spaces and for their further properties we refer to [3].

We then have:

**Proposition 3.5.** Let $\Lambda = K(A)$ be a Köthe space and let $\Lambda^*$ the corresponding Köthe dual space. Then the following properties are equivalent:

(i) $\Lambda$ is an FM-space.

(ii) $(\Lambda^*, \beta(\Lambda^*, \Lambda))$ is of countable type.

(iii) $(\Lambda^*, \beta(\Lambda^*, \Lambda))$ is nuclear.

(iv) $(\Lambda^*, \beta(\Lambda^*, \Lambda))$ is Montel.

(v) The unit vectors $e_1, e_2, \ldots$ form a Schauder basis for $\Lambda^*, \beta(\Lambda^*, \Lambda)$.

(vi) $n(\Lambda, \Lambda) = \beta(\Lambda^*, \Lambda)$ (where $n(\Lambda, \Lambda)$ is the natural topology on $\Lambda^*$).

(vii) No subspace of $\Lambda$ is isomorphic (linearly homeomorphic) to $c_0$.

(viii) The sequence of coordinate projections $(P_i)$, where $P_i : \Lambda \to \Lambda : \alpha = (\alpha_i) \mapsto \alpha_ie_i$, converges to the zero-map uniformly on every bounded subset of $\Lambda$. 

(ix) The sequence of sections-maps \((S_n)\), where \(S_n : \Lambda \to \Lambda : \alpha = (\alpha_i) \to (\alpha_1, \alpha_2, \ldots, \alpha_n, 0, 0, \ldots)\) converges to the identity map \(\text{Id}\) uniformly on every bounded subset of \(\Lambda\).

**Proof.** We only have to prove \((i) \Rightarrow (v) \Rightarrow (vi), (vii) \Rightarrow (viii)\) and \((ix) \Rightarrow (i)\). The other implications are easy.

\((i) \Rightarrow (v)\): The unit vectors \(e_1, e_2, \ldots\) form a Schauder basis for \((\Lambda^*, \sigma(\Lambda^*, \Lambda))\). Then, apply \((i) \Rightarrow (iv)\) in 3.3.

\((v) \Rightarrow (vi)\): By [4], p. 21 it suffices to prove that \(\beta(\Lambda^*, \Lambda)\) is compatible with the duality \((\Lambda^*, \Lambda)\) and this is done as in [1], Proposition 20.

\((vii) \Rightarrow (viii)\): Suppose \(\Lambda\) contains a bounded subset \(D\) on which \((P_i)\) does not converge uniformly to the zero-map. We show that \(\Lambda\) contains a subspace isomorphic to \(c_0\).

From the assumption it follows that there exist \(\varepsilon > 0, k \in \mathbb{N}\) and an increasing sequence of indices \((i_n)\) such that, for all \(n\), there exists \(\alpha^n = (\alpha_n^i) \in D\) with \(|\alpha_n^i| \cdot a_n^k > \varepsilon, n = 1, 2, \ldots\). We put \(z_{i_n} = \alpha_n^i \cdot e_i, n = 1, 2, \ldots\). Then, the sequence \((z_{i_n})\) is bounded in \(\Lambda\).

Now we can define a linear map

\[ T : c_0 \to \Lambda : \sigma = (\sigma_n) \to \sum_n \sigma_n z_{i_n}. \]

We prove that \(T\) is an isomorphism from \(c_0\) into \(\Lambda\). It is easy to see that \(T\) is injective and continuous. Also, \(T : c_0 \to \text{Im} T\) is open.

Indeed, for \(\sigma = (\sigma_n) \in c_0\), we have \(p_k(T(\sigma)) = \max_{n \geq 1} |\sigma_n\alpha_n^i| \cdot a_n^k \geq \varepsilon \cdot ||\sigma||_{c_0} \).

\((ix) \Rightarrow (i)\): We prove that \(\text{Id} : \Lambda \to \Lambda\) transforms bounded subsets into compactoid subsets. Observe that \((ix)\) means that \(\lim_n S_n = \text{Id}\) in \(L_c(\Lambda, \Lambda)\). Then apply Proposition 4 in [2].

The next corollary is for later use.

**Corollary 3.6.** If for every \(k \in \mathbb{N}\) and every subsequence \((i_n)\) of the indices there exists \(h > k\) such that the sequence \((\alpha_n^i / a_n^k)\) is bounded, then \(K(\Lambda)\) is an FM-space.

**Proof.** An analysis of the proof of \((vii) \Rightarrow (viii)\) shows that if \(K(\Lambda)\) is not an FM-space, there exist a subsequence of the indices \((i_n)\) and elements \(\eta_{i_n}\) in \(\Lambda, n = 1, 2, \ldots\) such that the linear map \(T : c_0 \to \text{Im} T : (\sigma_n) \to (\sigma_n\eta_{i_n})\) is an isomorphism of \(c_0\) into \(\Lambda\).

Consider now in \(c_0\) the subspace \(c_{00}\) generated by the unit vectors \(e_1, e_2, \ldots\). Then \(c_{00}\) is isomorphic to the subspace \(F\) of \(K(\Lambda)\) generated by \(e_1, e_2, \ldots\). Therefore, the topology induced by \(K(\Lambda)\) on \(F\) is normable. This means that there exists \(k\) such that for all \(h > k\) there exists \(t_h > 0\) with \(p_h(\delta) \leq t_h \cdot p_k(\delta)\) for all \(\delta \in K(\Lambda)\). In particular, for \(\delta = e_i\),
n = 1, 2, ..., we have that there is a $k$ such that for all $h > k$, there exists $t_h > 0$ with $a^h_i \leq t_h \cdot a^k_i$ for all $n$, and we are done.

4. Characterizations of nuclear spaces among FM-spaces. We start this section with the construction of an FM-space which is not nuclear.

**Example 4.1.** For $k = 1, 2, ..., k$, consider the infinite matrix

$$
A^k = (a^k_{ij}) =
\begin{pmatrix}
1^k & \cdots & 2^k & \cdots & j^k & \cdots \\
1^k & \cdots & 2^k & \cdots & j^k & \cdots \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\
(k+1)^k & \cdots & (k+1)^k & \cdots & (k+1)^k & \cdots \\
(k+2)^k & \cdots & (k+2)^k & \cdots & (k+2)^k & \cdots \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\
\end{pmatrix}
\rightarrow (k+1)
$$

We can think of $A^k$ as a sequence for some order, $k = 1, 2, ...$ (we fix the same order for all $k$). We then consider the Köthe space

$$
K(A) = \{ \beta = (\beta_{ij}) : \lim_{ij} |\beta_{ij}| \cdot a^k_{ij} = 0, k = 1, 2, ... \}
$$
equipped with the sequence of norms $(p_k)$ where $p_k(\beta) = \max_{ij} |\beta_{ij}| \cdot a^k_{ij}$.

We first show that $K(A)$ is not nuclear. If $k > 1$, then the sequence $(a^k_{ij} / a^k_{ij})$ contains a constant sequence. Then by [3] Proposition 3.5 the conclusion follows.

We now apply Corollary 3.6 in order to prove that $K(A)$ is an FM-space.

Choose $k$ and any subsequence of the indices $(l_n, m_n)_{n,m}$. We consider the corresponding elements $a^k_{l_n,m_n}$ of $A^k$. There are several possibilities.

a) The subsequence $(a^k_{l_n,m_n})_{n,m}$ contains an infinite number of elements of some row of $A^k$.

If this row is between the rows 1, ..., $k$, take $h = k + 1$. Then the sequence of the quotients $(a^{h}_{l_n,m_n} / a^{k}_{l_n,m_n})_{n,m}$ is unbounded.

If this row is the $(k + r)$-th row for some $r \geq 1$, then take $h = k + r$.

b) The subsequence $(a^k_{l_n,m_n})_{n,m}$ consists of finitely many elements of an infinite number of rows. Consider then a subsequence with one element in an infinite number of rows below the $k$th row. Such a subsequence looks like

$$(k+1)^k, (k+l_2)^k, (k+l_3)^k, ...$$

with $(l_n)_n$ increasing to infinity. Take now $h = k + 1$. ■
Finally we investigate what the situation exactly is.

**Definition 4.2.** A locally convex space $X$ is said to be quasinormable if for every zero-neighbourhood $U$ in $X$ there exists a zero-neighbourhood $V$ in $X$, $V \subset U$, such that on $U'$ the topology $\beta(X', X)$ coincides with norm topology of $X'_\|\cdot\|_U$.

**Definition 4.3.** Let $X$ be a locally convex space. A sequence $(a_n) \subset X'$ is said to be locally convergent to zero if there exists a zero-neighbourhood $U$ in $X$ such that $(a_n) \subset X'_U$ and $\lim_n \|a_n\|_U = 0$.

**Theorem 4.4.** For an FM-space $E$ the following properties are equivalent.

(i) $E$ is nuclear.
(ii) $E$ is quasinormable.
(iii) Every $\beta(E', E)$-convergent sequence in $E'$ is locally convergent.

**Proof.** The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) follow by [2], Proposition 14 and [5], 5.2 respectively.

(iii) $\Rightarrow$ (i) Since $E$ is of countable type (Theorem 3.1) its topology can be described by the $\sigma(E', E)$-null sequences on $E'$ ([4], Theorem 3.2). By Theorem 3.3 (i) $\Rightarrow$ (iv) these sequences are null-sequences in $\beta(E', E)$ and by (iii) they are locally convergent to zero. The conclusion then follows from [5], 4.6.1).

**Corollary 4.5.** The Köthe space in 4.1 is also an example of an FM-space which is not quasinormable.

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