A NOTE ON $p$-ADIC REFLEXIVITY

by

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**ABSTRACT.** For a nonarchimedean nontrivially valued complete field $K$ consider the following statements (A), (B), (C) (for terminology, see §1).

(A) If $D_1$ and $D_2$ are closed subspaces of a $K$-Banach space $E$ such that $D_1 + D_2 = E$ and $D_1 \cap D_2 = \{0\}$ and if $D_1$ and $D_2$ are (pseudo)reflexive then so is $E$.

(B) If $D$ is a finite-dimensional subspace of a $K$-Banach space $E$ and if $E/D$ is pseudoreflexive then so is $E$.

(C) If $D$ is a finite-dimensional subspace of a $K$-Banach space $E$ and if $E/D$ is reflexive then so is $E$.

The purpose of this note is to show that (A) is false and that (B) implies (C) rendering a solution of one Problem and a reduction of two other Problems of [1], §8.
§1. Preliminaries.

In this note $K$ is a nonarchimedean valued field whose valuation $| |$ is complete and non-trivial. Norms on $K$-vector spaces are always assumed to be nonarchimedean i.e. to satisfy the strong triangle inequality.

**Proposition 1.1** Let $(E, || ||)$ be a normed space over $K$, let $D$ be a subspace of $E$, let $q$ be a norm on $D$ satisfying

$$
\frac{1}{2}d \leq q(d) \leq d \quad (d \in D).
$$

Then $q$ can be extended to a norm $|| ||_1$ on $E$ for which

$$
\frac{1}{2}x \leq ||x||_1 \leq x \quad (x \in E).
$$

**Proof.** Set

$$
||x||_1 := \inf_{d \in D} \max(||x - d||, q(d)).
$$

Immediate verification shows that $|| ||_1$ satisfies the requirements.

A $K$-Banach space $E$ is called *pseudoreflexive* if the canonical map $j_E : E \to E''$ is an isometry, *reflexive* if $E$ is pseudoreflexive and $j_E$ is surjective. If $K$ is spherically complete each $K$-Banach space is pseudoreflexive and the reflexive spaces are precisely the finite-dimensional ones. For such $K$ the statements (A), (B), (C) of above are trivially true. Therefore from now on we assume that $K$ is NOT SPHERICALLY COMPLETE implying that the valuation of $K$ is dense. Then it is easy to see that a $K$-Banach space $E$ is pseudoreflexive if and only if the norm is polar (recall that a seminorm $p$ is polar if $p = \sup \{|f| : f \in E^*, |f| \leq p\}$ where $E^*$ is the algebraic dual of $E$). Also, each closed subspace of a pseudoreflexive $K$-Banach space is pseudoreflexive.

A subspace $D$ of a $K$-Banach space $E$ is said to have the Weak Extension Property (WEP) if every $f \in D'$ can be extended to an element of $E'$, in other words, if the adjoint $i' : E' \to D'$ of the inclusion map $i : D \to E$ is surjective.

A subspace $D$ of a $K$-Banach space $E$ is said to have the Extension Property (EP) if for each $\varepsilon > 0$ and $f \in D'$ there is an extension $\tilde{f} \in E'$ of $f$ such that $||\tilde{f}|| \leq (1+\varepsilon)||f||$, in other words, if the adjoint $i' : E' \to D'$ of the inclusion map $i : D \to E$ is a quotient map.

**Proposition 1.2.** Let $D$ be a closed subspace of a $K$-Banach space $E$, let $f \in D'$, $\varepsilon > 0$, $x \in E \setminus D$. Then $f$ can be extended to an $\tilde{f} \in (Kx + D)'$ such that $||\tilde{f}|| \leq (1+\varepsilon)||f||$. 

2
Proof. We may suppose that \( \text{dist}(x, D) > t\|x\| \) where \( t := (1 + \varepsilon)^{-1} \). Then \( \|\lambda x + d\| \geq t \max(||\lambda x||, ||d||) \) for all \( \lambda \in K \) and \( d \in D \). The formula \( \tilde{f}(\lambda x + d) = f(d) \) defines an extension \( \tilde{f} \in (Kx + D)^* \) of \( f \). For each \( \lambda \in K, d \in D \) we have \( |\tilde{f}(\lambda x + d)| = |f(d)| \leq \|f\|\|d\| \leq \|f\|\max(||\lambda x||, ||d||) \) \( \leq \|f\|t^{-1}\|\lambda x + d\| \) and we see that \( \|\tilde{f}\| \leq (1 + \varepsilon)\|f\| \).

**COROLLARY 1.3.** Any finite codimensional subspace of a \( K \)-Banach space has the EP.

**PROPOSITION 1.4.** Let \( F, G, H \) be \( K \)-Banach spaces, let

\[
F \xrightarrow{T_1} G \xrightarrow{T_2} H
\]

be continuous linear maps such that \( \text{Im} \, T_1 = \text{Ker} \, T_2 \). Suppose that \( T_2G \) is a closed subspace of \( H \) with the WEP. Then for the adjoints

\[
H^1 \xrightarrow{T_1'} G^* \xrightarrow{T_2'} F^*
\]

we have \( \text{Im} \, T_2' = \text{Ker} \, T_1' \).

**Proof.** Obviously \( T_1' \circ T_2' = (T_2 \circ T_1)' = 0 \) whence \( \text{Im} \, T_2' \subset \text{Ker} \, T_1' \). Conversely, suppose \( f \in \text{Ker} \, T_1 \) i.e. \( f \circ T_1 = 0 \):

\[
\begin{array}{ccc}
F & \xrightarrow{T_1} & G & \xrightarrow{T_2} & H \\
\downarrow f & & \downarrow j & & \downarrow K \\
K & & & &
\end{array}
\]

Then \( f = 0 \) on \( \text{Ker} \, T_2 \) so there exist unique linear maps \( f_1 : G/\text{Ker} \, T_2 \to K \) and \( i : G/\text{Ker} \, T_2 \to H \) making

\[
\begin{array}{ccc}
G & \xrightarrow{T_2} & H \\
\downarrow f & \downarrow \pi & \uparrow i \\
K & \xleftarrow{f_1} & G/\text{Ker} \, T_2
\end{array}
\]

commute (here \( \pi \) is, of course, the quotient map). The maps \( f_1 \) and \( i \) are continuous and \( i \) is even a homeomorphism onto \( T_2G \) by the Banach Open Mapping Theorem. The map \( z \mapsto f_1(i^{-1}(z)) \) \( (z \in T_2G) \) extends to an \( f_2 \in H' \) which obviously makes
It is known that the spaces $c_0$ and $\ell^\infty$ (whose standard norms will be denoted $\| \|$) are reflexive and each others dual by means of the pairing

$$(x, y) \mapsto \sum_{i=1}^{\infty} \xi_i \eta_i$$

($x = (\xi_1, \xi_2, \ldots) \in c_0$, $y = (\eta_1, \eta_2, \ldots) \in \ell^\infty$). Then, if $y = (\eta_1, \eta_2, \ldots) \in \ell^\infty$ then $\lim_{n \to \infty} y_n = y$ weakly, where $y_n := (\eta_1, \eta_2, \ldots, \eta_n, 0, 0, \ldots)$. 

\[ G \xrightarrow{T_2} H \]
\[ f \downarrow \not\sim f_2 \]
\[ K \]
§2. \( (A) \) is false.

2.1. We construct an equivalent nonpolar norm \( p \) on \( \ell^\infty \).

Set
\[
    p(x) := \max \left( \frac{1}{2} \|x\|, \text{dist}(x, c_0) \right).
\]

We have \( \frac{1}{2} \|x\| \leq p(x) \leq \|x\| \) for all \( x \in \ell^\infty \) and, since \( p(1, 1, 1, \ldots) = \|(1, 1, \ldots)\| = 1 \), \( p \neq \frac{1}{2} \|\cdot\| \). To arrive at the non-polarness of \( p \) we prove that \( f \in (\ell^\infty)^*, \ |f| \leq p \) implies \( |f| \leq \frac{1}{2} \|\cdot\| \). Let \( x = (\xi_1, \xi_2, \ldots) \in \ell^\infty \). Then \( x = \lim_{n \to \infty} x_n \) weakly where \( x_n := (\xi_1, \xi_2, \ldots, \xi_n, 0, 0, \ldots) \). Now \( f \) is in \((\ell^\infty)^*\) and \( x_n \in c_0 \) and therefore \( |f(x)| = \lim_{n \to \infty} |f(x_n)| \leq \sup_n p(x_n) = \sup_n \frac{1}{2} \|x_n\| \leq \frac{1}{2} \sup_n |\xi_n| = \frac{1}{2} \|x\| \).

This way we have obtained a Banach space \( E \) with two norms \( \|\cdot\|_1, \|\cdot\|_2 \), each one defining the topology while \((E, \|\cdot\|_1)\) is reflexive and \((E, \|\cdot\|_2)\) is not.

2.2. Let us denote the product norm on \( \ell^\infty \times \ell^\infty \) again by \( \|\cdot\| \). We construct a second norm \( \|\cdot\|_1 \) on \( \ell^\infty \times \ell^\infty \) as follows. First define a norm \( q \) on the diagonal \( \Delta := \{(x, x) : x \in \ell^\infty\} \) via
\[
    q(x, x) := p(x)
\]

where \( p \) is as in 2.1. By Proposition 1.1 the formula
\[
    \|((x, y), (t, t))\|_1 := \inf_{(t, t) \in \ell^\infty} \left( \max_{t \in \ell^\infty} \|((x, y) - (t, t))\|, q(t, t) \right)
\]
defines a norm \( \|\cdot\|_1 \) on \( \ell^\infty \times \ell^\infty \). This norm is not polar since its restriction to \( \Delta \) is not polar, but satisfies \( \frac{1}{2} \|x\| \leq \|x\|_1 \leq \|x\| \) for all \( z \in \ell^\infty \times \ell^\infty \).

2.3. Now we show that \( (A) \) is false. Let \( E := (\ell^\infty \times \ell^\infty, \|\cdot\|_1) \), and set \( D_1 := \{(x, 0) : x \in \ell^\infty\}, D_2 := \{(0, x) : x \in \ell^\infty\} \). Then \( D_1 + D_2 = E, D_1 \cap D_2 = \{0\} \). For each \( x \in \ell^\infty \) we have
\[
    \|(x, 0)\| \geq \|(x, 0)\|_1 = \inf_{t \in \ell^\infty} \max_{t \in \ell^\infty} \|((x - t, -t))\|, q(t) \geq \inf_{t \in \ell^\infty} \|((x - t, -t))\| = \inf_{t \in \ell^\infty} \max_{t \in \ell^\infty} \|((x - t, -t))\| \geq \|x\| = \|(x, 0)\|.
\]
It follows that \( D_1 \) is isometrically isomorphic to \( \ell^\infty \), hence reflexive. By the same token \( D_2 \) is reflexive. But in 2.2 we have seen that \( E \) is not even pseudoreflexive.
§3. (B) implies (C).

In the next two lemmas $D$ is a finite-dimensional subspace of a $K$-Banach space $E$ with inclusion map $i : D \rightarrow E$ and quotient map $\pi : E \rightarrow E/D$. We consider the commutative diagram

\[
\begin{array}{ccc}
D & \xrightarrow{i} & E & \xrightarrow{\pi} & E/D \\
\downarrow{j_D} & & \downarrow{j_E} & & \downarrow{j_{E/D}} \\
D'' & \xrightarrow{i''} & E'' & \xrightarrow{\pi''} & (E/D)''
\end{array}
\]

**Lemma 3.1.** $\text{Im } i'' = \text{Ker } \pi''$ and $\pi''$ is a quotient map.

*Proof.* Since $\pi$ is surjective we have by Proposition 1.4 in $(E/D)' \xrightarrow{\pi'} E' \xrightarrow{i'} D'$ that $\text{Im } \pi' = \text{Ker } i'$. Now $D'$ is finite-dimensional so $\text{Im } i'$ is closed and has the WEP. Hence, again by Proposition 1.4, $\text{Im } i'' = \text{Ker } \pi''$. To prove the second statement observe that $\pi'$ is an isometry whose image has finite codimension and therefore has the EP by Corollary 1.3, so $\pi''$ is a quotient map.

**Lemma 3.2.** $j_E$ is surjective if and only if $j_{E/D}$ is surjective.

*Proof.* If $j_E$ is surjective then so is $\pi'' \circ j_E = j_{E/D} \circ \pi$. Then $j_{E/D}$ must be surjective. Conversely, if $j_{E/D}$ is surjective, let $\Theta \in E''$.

Then there is an $x \in E$ with $\pi''(\Theta) = j_{E/D}(\pi(x)) = \pi''(j_E(x))$. We see that $\Theta - j_E(x) \in \text{Ker } \pi'' = \text{Im } i''$ so by surjectivity of $j_D$ we can find a $d \in D$ such that

$$\Theta - j_E(x) = i'' j_D(d) = j_E i(d)$$

Then $\Theta = j_E(x + i(d)) \in j_E(E)$ and the surjectivity of $j_E$ is proved.

**Remark.** In the same vein one can prove $j_E$ injective $\Rightarrow j_{E/D}$ injective; a counterexample to the converse is given in [1], §8.

Also one has $j_E$ is isometrical $\Rightarrow j_{E/D}$ is isometrical. Its converse is just statement (B) the truth of which is an open problem.

If we assume (B) and if $E/D$ is reflexive we have that $j_E$ is isometrical by (B) and surjective by Lemma 3.2. We may conclude that (B) implies (C).
REFERENCE