A note on $p$-adic reflexivity

by

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**ABSTRACT.** For a nonarchimedean nontrivially valued complete field $K$ consider the following statements (A), (B), (C) (for terminology, see §1).

(A) If $D_1$ and $D_2$ are closed subspaces of a $K$-Banach space $E$ such that $D_1 + D_2 = E$ and $D_1 \cap D_2 = \{0\}$ and if $D_1$ and $D_2$ are (pseudo)reflexive then so is $E$.

(B) If $D$ is a finite-dimensional subspace of a $K$-Banach space $E$ and if $E/D$ is pseudoreflexive then so is $E$.

(C) If $D$ is a finite-dimensional subspace of a $K$-Banach space $E$ and if $E/D$ is reflexive then so is $E$.

The purpose of this note is to show that (A) is false and that (B) implies (C) rendering a solution of one Problem and a reduction of two other Problems of [1], §8.
§1. Preliminaries.

In this note $K$ is a nonarchimedean valued field whose valuation $|\cdot|$ is complete and non-trivial. Norms on $K$-vector spaces are always assumed to be nonarchimedean i.e. to satisfy the strong triangle inequality.

**Proposition 1.1** Let $(E, \|\|)$ be a normed space over $K$, let $D$ be a subspace of $E$, let $q$ be a norm on $D$ satisfying

$$\frac{1}{2}\|d\| \leq q(d) \leq \|d\| \quad (d \in D).$$

Then $q$ can be extended to a norm $\|\|_1$ on $E$ for which

$$\frac{1}{2}\|x\| \leq \|x\|_1 \leq \|x\| \quad (x \in E).$$

**Proof.** Set

$$\|x\|_1 := \inf_{d \in D} \max(\|x - d\|, q(d)).$$

Immediate verification shows that $\|\|_1$ satisfies the requirements.

A $K$-Banach space $E$ is called *pseudoreflexive* if the canonical map $j_E : E \to E''$ is an isometry, *reflexive* if $E$ is pseudoreflexive and $j_E$ is surjective. If $K$ is spherically complete each $K$-Banach space is pseudoreflexive and the reflexive spaces are precisely the finite-dimensional ones. For such $K$ the statements (A), (B), (C) of above are trivially true. Therefore from now on we assume that $K$ is *NOT SPHERICALLY COMPLETE* implying that the valuation of $K$ is dense. Then it is easy to see that a $K$-Banach space $E$ is pseudoreflexive if and only if the norm is polar (recall that a seminorm $p$ is polar if $p = \sup\{|f| : f \in E^*, |f| \leq p\}$ where $E^*$ is the algebraic dual of $E$). Also, each closed subspace of a pseudoreflexive $K$-Banach space is pseudoreflexive.

A subspace $D$ of a $K$-Banach space $E$ is said to have the *Weak Extension Property* (WEP) if every $f \in D'$ can be extended to an element of $E'$, in other words, if the adjoint $i' : E' \to D'$ of the inclusion map $i : D \to E$ is surjective.

A subspace $D$ of a $K$-Banach space $E$ is said to have the *Extension Property* (EP) if for each $\varepsilon > 0$ and $f \in D'$ there is an extension $\overline{f} \in E'$ of $f$ such that $\|\overline{f}\| \leq (1 + \varepsilon)\|f\|$, in other words, if the adjoint $i' : E' \to D'$ of the inclusion map $i : D \to E$ is a quotient map.

**Proposition 1.2.** Let $D$ be a closed subspace of a $K$-Banach space $E$, let $f \in D'$, $\varepsilon > 0$, $x \in E \setminus D$. Then $f$ can be extended to an $\overline{f} \in (Kx + D)'$ such that $\|\overline{f}\| \leq (1 + \varepsilon)\|f\|$.
Proof. We may suppose that \( \text{dist} (x, D) \geq t\|x\| \) where \( t := (1 + \epsilon)^{-1} \). Then \( \|\lambda x + d\| \geq t \max(\|\lambda x\|, \|d\|) \) for all \( \lambda \in K \) and \( d \in D \). The formula \( \overline{f}(\lambda x + d) = f(d) \) defines an extension \( \overline{f} \in (Kx + D)^* \) of \( f \). For each \( \lambda \in K \), \( d \in D \) we have \( |\overline{f}(\lambda x + d)| = |f(d)| \leq \|f\| \|d\| \leq \|f\| \max(\|\lambda x\|, \|d\|) \leq \|f\| t^{-1} \|\lambda x + d\| \) and we see that \( \|\overline{f}\| \leq (1 + \epsilon)\|f\| \).

**COROLLARY 1.3.** Any finite codimensional subspace of a \( K \)-Banach space has the EP.

**PROPOSITION 1.4.** Let \( F, G, H \) be \( K \)-Banach spaces, let

\[
F \xrightarrow{T_1} G \xrightarrow{T_2} H
\]

be continuous linear maps such that \( \text{Im} T_1 = \text{Ker} T_2 \). Suppose that \( T_2G \) is a closed subspace of \( H \) with the WEP. Then for the adjoints

\[
H^1 \xrightarrow{T_2'} G' \xrightarrow{T_1'} F'
\]

we have \( \text{Im} T_2' = \text{Ker} T_1' \).

**Proof.** Obviously \( T_1' \circ T_2' = (T_2 \circ T_1)' = 0 \) whence \( \text{Im} T_2' \subset \text{Ker} T_1' \). Conversely, suppose \( f \in \text{Ker} T_1' \) i.e. \( f \circ T_1 = 0 \):

\[
\begin{array}{ccc}
F & \xrightarrow{T_2} & G \\
& \downarrow f & \\
& K & \xrightarrow{T_1} H
\end{array}
\]

Then \( f = 0 \) on \( \text{Ker} T_2 \) so there exist unique linear maps \( f_1 : G/\text{Ker} T_2 \to K \) and \( i : G/\text{Ker} T_2 \to H \) making

\[
\begin{array}{ccc}
G & \xrightarrow{T_2} & H \\
\downarrow f & \swarrow \pi & \nwarrow i \\
K & \xleftarrow{f_1} & G/\text{Ker} T_2
\end{array}
\]

commute (here \( \pi \) is, of course, the quotient map). The maps \( f_1 \) and \( i \) are continuous and \( i \) is even a homeomorphism onto \( T_2G \) by the Banach Open Mapping Theorem. The map \( z \mapsto f_1(i^{-1}(z)) \) \((z \in T_2G)\) extends to an \( f_2 \in H' \) which obviously makes
commute. It follows that \( f = f_2 \circ T_2 = T_2'(f_2) \in \text{Im } T_2' \).

It is known that the spaces \( c_0 \) and \( \ell^\infty \) (whose standard norms will be denoted \( \| \| \)) are reflexive and each others dual by means of the pairing

\[
(x, y) \mapsto \sum_{i=1}^{\infty} \xi_i \eta_i
\]

\( (x = (\xi_1, \xi_2, \ldots) \in c_0, \ y = (\eta_1, \eta_2, \ldots) \in \ell^\infty ) \). Then, if \( y = (\eta_1, \eta_2, \ldots) \in \ell^\infty \) then

\[
\lim_{n \to \infty} y_n = y \text{ weakly, where } y_n := (\eta_1, \eta_2, \ldots, \eta_n, 0, 0, \ldots).
\]
§2. (A) is false.

2.1. We construct an equivalent nonpolar norm \( p \) on \( \ell^\infty \).
Set
\[
p(x) := \max \left( \frac{1}{2} \|x\|, \text{dist}(x, c_0) \right).
\]
We have \( \frac{1}{2} \|x\| \leq p(x) \leq \|x\| \) for all \( x \in \ell^\infty \) and, since \( p(1,1,1,\ldots) = \|(1,1,\ldots)\| = 1 \), \( p \neq \frac{1}{2} \|\cdot\| \). To arrive at the non-polarity of \( p \) we prove that \( f \in (\ell^\infty)^* \), \( |f| \leq p \) implies \( |f| \leq \frac{1}{2} \|\cdot\| \). Let \( x = (\xi_1, \xi_2, \ldots) \in \ell^\infty \). Then \( x = \lim_{n \to \infty} x_n \) weakly where \( x_n := (\xi_1, \xi_2, \ldots, \xi_n, 0, 0, \ldots) \). Now \( f \) is in \( (\ell^\infty)^* \) and \( x_n \in c_0 \) and therefore \( |f(x)| = \lim_{n \to \infty} |f(x_n)| \leq \sup_n p(x_n) = \sup_n \frac{1}{2} \|x_n\| \leq \frac{1}{2} \sup_n |\xi_n| = \frac{1}{2} \|x\| \).

This way we have obtained a Banach space \( E \) with two norms \( \| \|_1, \| \|_2 \), each one defining the topology while \( (E, \| \|_1) \) is reflexive and \( (E, \| \|_2) \) is not.

2.2. Let us denote the product norm on \( \ell^\infty \times \ell^\infty \) again by \( \| \| \). We construct a second norm \( \| \|_1 \) on \( \ell^\infty \times \ell^\infty \) as follows. First define a norm \( q \) on the diagonal \( \Delta := \{(x, x) : x \in \ell^\infty \} \) via
\[
q(x, x) := p(x)
\]
where \( p \) is as in 2.1. By Proposition 1.1 the formula
\[
\|(x, y)\|_1 := \inf_{t \in \ell^\infty} \left( \max \|\langle x, y \rangle - (t, t)\|, q(t, t) \right)
\]
defines a norm \( \| \|_1 \) on \( \ell^\infty \times \ell^\infty \). This norm is not polar since its restriction to \( \Delta \) is not polar, but satisfies \( \frac{1}{2} \|z\| \leq \|z\|_1 \leq \|z\| \) for all \( z \in \ell^\infty \times \ell^\infty \).

2.3. Now we show that (A) is false. Let \( E := (\ell^\infty \times \ell^\infty, \| \|_1) \), and set \( D_1 := \{(x, 0) : x \in \ell^\infty \}, D_2 := \{(0, x) : x \in \ell^\infty \} \). Then \( D_1 + D_2 = E, D_1 \cap D_2 = \{0\} \). For each \( x \in \ell^\infty \) we have
\[
\|(x, 0)\| \geq \|(x, 0)\|_1 = \inf_{t \in \ell^\infty} \max \|\langle x - t, -t \rangle\|, q(t) \geq \inf_{t \in \ell^\infty} \|\langle x - t, -t \rangle\|
\]
\[
= \inf_{t \in \ell^\infty} \max \|x - t\|, \|t\| \geq \|x\| = \|(x, 0)\|.
\]
It follows that \( D_1 \) is isometrically isomorphic to \( \ell^\infty \), hence reflexive. By the same token \( D_2 \) is reflexive. But in 2.2 we have seen that \( E \) is not even pseudoreflexive.
§3. (B) implies (C).

In the next two lemmas $D$ is a finite-dimensional subspace of a $K$-Banach space $E$ with inclusion map $i : D \to E$ and quotient map $\pi : E \to E/D$. We consider the commutative diagram

$$
\begin{array}{ccc}
D & \xrightarrow{i} & E \\
\downarrow{i_D} & & \downarrow{i_E} \\
D'' & \xrightarrow{i''} & E''
\end{array}
\quad
\begin{array}{c}
\xrightarrow{\pi} \\
\downarrow{j_{E/D}} \\
(E/D)''
\end{array}
$$

**Lemma 3.1.** $\text{Im } i'' = \text{Ker } \pi''$ and $\pi''$ is a quotient map.

*Proof.* Since $\pi$ is surjective we have by Proposition 1.4 in $(E/D)' \xrightarrow{\pi'} E' \xrightarrow{i'} D'$ that $\text{Im } \pi' = \text{Ker } i'$. Now $D'$ is finite-dimensional so $\text{Im } i'$ is closed and has the WEP. Hence, again by Proposition 1.4, $\text{Im } i'' = \text{Ker } \pi''$. To prove the second statement observe that $\pi'$ is an isometry whose image has finite codimension and therefore has the EP by Corollary 1.3, so $\pi''$ is a quotient map.

**Lemma 3.2.** $j_E$ is surjective if and only if $j_{E/D}$ is surjective.

*Proof.* If $j_E$ is surjective then so is $\pi'' \circ j_E = j_{E/D} \circ \pi$. Then $j_{E/D}$ must be surjective. Conversely, if $j_{E/D}$ is surjective, let $\Theta \in E''$. Then there is an $x \in E$ with $\pi''(\Theta) = j_{E/D}(\pi(x)) = \pi''(j_E(x))$. We see that $\Theta - j_E(x) \in \text{Ker } \pi'' = \text{Im } i''$ so by surjectivity of $j_D$ we can find a $d \in D$ such that

$$
\Theta - j_E(x) = i'' j_D(d) = j_E i(d)
$$

Then $\Theta = j_E(x + i(d)) \in j_E(E)$ and the surjectivity of $j_E$ is proved.

**Remark.** In the same vein one can prove $j_E$ injective $\Rightarrow j_{E/D}$ injective; a counterexample to the converse is given in [1], §8.

Also one has $j_E$ is isometrical $\Rightarrow j_{E/D}$ is isometrical. Its converse is just statement (B) the truth of which is an open problem.

If we assume (B) and if $E/D$ is reflexive we have that $j_E$ is isometrical by (B) and surjective by Lemma 3.2. We may conclude that (B) implies (C).