A NOTE ON \textit{p}-ADIC REFLEXIVITY

by

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ABSTRACT. For a nonarchimedean nontrivially valued complete field $K$ consider the following statements (A), (B), (C) (for terminology, see §1).

(A) If $D_1$ and $D_2$ are closed subspaces of a $K$-Banach space $E$ such that $D_1 + D_2 = E$ and $D_1 \cap D_2 = \{0\}$ and if $D_1$ and $D_2$ are (pseudo)reflexive then so is $E$.

(B) If $D$ is a finite-dimensional subspace of a $K$-Banach space $E$ and if $E/D$ is pseudoreflexive then so is $E$.

(C) If $D$ is a finite-dimensional subspace of a $K$-Banach space $E$ and if $E/D$ is reflexive then so is $E$.

The purpose of this note is to show that (A) is false and that (B) implies (C) rendering a solution of one Problem and a reduction of two other Problems of [1], §8.
§1. Preliminaries.

In this note $K$ is a nonarchimedean valued field whose valuation $| |$ is complete and non-trivial. Norms on $K$-vector spaces are always assumed to be nonarchimedean i.e. to satisfy the strong triangle inequality.

**Proposition 1.1** Let $(E, ||| ||)$ be a normed space over $K$, let $D$ be a subspace of $E$, let $q$ be a norm on $D$ satisfying

$$
\frac{1}{2}||d|| \leq q(d) \leq ||d|| \quad (d \in D).
$$

Then $q$ can be extended to a norm $||| ||_1$ on $E$ for which

$$
\frac{1}{2}||x|| \leq ||x||_1 \leq ||x|| \quad (x \in E).
$$

**Proof.** Set

$$
||x||_1 := \inf_{d \in D} \max(0, q(d) - ||x - d||).
$$

Immediate verification shows that $||| ||_1$ satisfies the requirements.

A $K$-Banach space $E$ is called pseudoreflexive if the canonical map $j_E : E \to E''$ is an isometry, reflexive if $E$ is pseudoreflexive and $j_E$ is surjective. If $K$ is spherically complete each $K$-Banach space is pseudoreflexive and the reflexive spaces are precisely the finite-dimensional ones. For such $K$ the statements (A), (B), (C) of above are trivially true. Therefore from now on we assume that $K$ is NOT SPHERICALLY COMPLETE implying that the valuation of $K$ is dense. Then it is easy to see that a $K$-Banach space $E$ is pseudoreflexive if and only if the norm is polar (recall that a seminorm $p$ is polar if $p = \sup\{|f| : f \in E^*, |f| \leq p\}$ where $E^*$ is the algebraic dual of $E$). Also, each closed subspace of a pseudoreflexive $K$-Banach space is pseudoreflexive.

A subspace $D$ of a $K$-Banach space $E$ is said to have the Weak Extension Property (WEP) if every $f \in D'$ can be extended to an element of $E'$, in other words, if the adjoint $i' : E' \to D'$ of the inclusion map $i : D \to E$ is surjective.

A subspace $D$ of a $K$-Banach space $E$ is said to have the Extension Property (EP) if for each $\epsilon > 0$ and $f \in D'$ there is an extension $\tilde{f} \in E'$ of $f$ such that $||\tilde{f}|| \leq (1 + \epsilon)||f||$, in other words, if the adjoint $i' : E' \to D'$ of the inclusion map $i : D \to E$ is a quotient map.

**Proposition 1.2.** Let $D$ be a closed subspace of a $K$-Banach space $E$, let $f \in D'$, $\epsilon > 0$, $x \in E \setminus D$. Then $f$ can be extended to an $\tilde{f} \in (Kx + D)'$ such that $||\tilde{f}|| \leq (1 + \epsilon)||f||$. 

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Proof. We may suppose that \( \text{dist}(x, D) \geq t\|x\| \) where \( t := (1 + \varepsilon)^{-1} \). Then \( \|\lambda x + d\| \geq t \max(\|\lambda x\|, \|d\|) \) for all \( \lambda \in \mathcal{K} \) and \( d \in D \). The formula \( \overline{f}(\lambda x + d) = f(d) \) defines an extension \( \overline{f} \in (Kx + D)^* \) of \( f \). For each \( \lambda \in \mathcal{K} \), \( d \in D \) we have \( |\overline{f}(\lambda x + d)| = |f(d)| \leq \|f\| \max(\|\lambda x\|, \|d\|) \leq \|f\| t^{-1} \|\lambda x + d\| \) and we see that \( \|\overline{f}\| \leq (1 + \varepsilon)\|f\| \).

**Corollary 1.3.** Any finite codimensional subspace of a K-Banach space has the EP.

**Proposition 1.4.** Let \( F, G, H \) be K-Banach spaces, let

\[
\begin{array}{c}
F \xrightarrow{T_1} G \xrightarrow{T_2} H
\end{array}
\]

be continuous linear maps such that \( \text{Im } T_1 = \text{Ker } T_2 \). Suppose that \( T_2 \mathcal{K} \) is a closed subspace of \( H \) with the WEP. Then for the adjoints

\[
\begin{array}{c}
H^* \xrightarrow{T_2^*} G^* \xrightarrow{T_1^*} F^*
\end{array}
\]

we have \( \text{Im } T_2^* = \text{Ker } T_1^* \).

**Proof.** Obviously \( T_1^* \circ T_2^* = (T_2 \circ T_1)' = 0 \) whence \( \text{Im } T_2^* \subset \text{Ker } T_1^* \). Conversely, suppose \( f \in \text{Ker } T_1 \) i.e. \( f \circ T_1 = 0 \):

\[
\begin{array}{c}
F \xrightarrow{T_2} G \xrightarrow{T_3} H
\end{array}
\]

\[
\downarrow f
\]

\[
\downarrow K
\]

Then \( f = 0 \) on \( \text{Ker } T_2 \) so there exist unique linear maps \( f_1 : G/\text{Ker } T_2 \to \mathcal{K} \) and \( i : G/\text{Ker } T_2 \to H \) making

\[
\begin{array}{c}
G \xrightarrow{T_2} H
\end{array}
\]

\[
\downarrow f
\]

\[
\downarrow \pi
\]

\[
\uparrow i
\]

\[
\begin{array}{c}
K \leftarrow f_1 \quad G/\text{Ker } T_2
\end{array}
\]

commute (here \( \pi \) is, of course, the quotient map). The maps \( f_1 \) and \( i \) are continuous and \( i \) is even a homeomorphism onto \( T_2 \mathcal{K} \) by the Banach Open Mapping Theorem. The map \( z \mapsto f_1(i^{-1}(z)) \quad (z \in T_2 \mathcal{K}) \) extends to an \( f_2 \in H' \) which obviously makes
It follows that \( f = f_2 \circ T_2 = T_2'(f_2) \in \text{Im } T_2'. \)

It is known that the spaces \( c_0 \) and \( \ell^\infty \) (whose standard norms will be denoted \( \| \| \) ) are reflexive and each others dual by means of the pairing

\[
(x, y) \mapsto \sum_{i=1}^{\infty} \xi_i \eta_i
\]

\((x = (\xi_1, \xi_2, \ldots) \in c_0, \ y = (\eta_1, \eta_2, \ldots) \in \ell^\infty)\). Then, if \( y = (\eta_1, \eta_2, \ldots) \in \ell^\infty \) then

\[
\lim_{n \to \infty} y_n = y \text{ weakly, where } y_n := (\eta_1, \eta_2, \ldots, \eta_n, 0, 0, \ldots).
\]
§2. (A) is false.

2.1. We construct an equivalent nonpolar norm $p$ on $\ell^\infty$.

Set

$$p(x) := \max\left(\frac{1}{2}\|x\|, \text{dist}(x, c_0)\).$$

We have $\frac{1}{2}\|x\| \leq p(x) \leq \|x\|$ for all $x \in \ell^\infty$ and, since $p(1,1,\ldots) = \|(1,1,\ldots)\| = 1$, $p \neq \frac{1}{2}\|\|$. To arrive at the non-polarness of $p$ we prove that $f \in (\ell^\infty)^*$, $|f| \leq p$ implies $|f| \leq \frac{1}{2}\|\|$. Let $x = (\xi_1, \xi_2, \ldots) \in \ell^\infty$. Then $x = \lim_{n \to \infty} x_n$ weakly where $x_n := (\xi_1, \xi_2, \ldots, \xi_n, 0, 0, \ldots)$. Now $f$ is in $(\ell^\infty)'$ and $x_n \in c_0$ and therefore $|f(x)| = \lim_{n \to \infty} |f(x_n)| \leq \sup_n p(x_n) = \sup_n \frac{1}{2}\|x_n\| \leq \frac{1}{2}\|\| \xi_n\| = \frac{1}{2}\|x\|.$

This way we have obtained a Banach space $E$ with two norms $\|\|_1$, $\|\|_2$, each one defining the topology while $(E, \|\|_1)$ is reflexive and $(E, \|\|_2)$ is not.

2.2. Let us denote the product norm on $\ell^\infty \times \ell^\infty$ again by $\|\|$. We construct a second norm $\|\|_1$ on $\ell^\infty \times \ell^\infty$ as follows. First define a norm $q$ on the diagonal $\Delta := \{(x, x) : x \in \ell^\infty\}$ via

$$q(x, x) := p(x)$$

where $p$ is as in 2.1. By Proposition 1.1 the formula

$$\|(x, y)\|_1 := \inf_{t \in \ell^\infty} \left(\max_{(t, t)} \|(x, y) - (t, t)\|, q(t, t)\right)$$

defines a norm $\|\|_1$ on $\ell^\infty \times \ell^\infty$. This norm is not polar since its restriction to $\Delta$ is not polar, but satisfies $\frac{1}{2}\|\| \leq \|\|_1 \leq \|\|$ for all $z \in \ell^\infty \times \ell^\infty$.

2.3. Now we show that (A) is false. Let $E := (\ell^\infty \times \ell^\infty, \|\|_1)$, and set $D_1 := \{(x, 0) : x, \ell^\infty\}$, $D_2 := \{(0, x) : x \in \ell^\infty\}$. Then $D_1 + D_2 = E$, $D_1 \cap D_2 = \{0\}$. For each $x \in \ell^\infty$ we have $\|(x, 0)\| \geq \|(x, 0)\|_1 = \inf_{t \in \ell^\infty} \max_{(x, t)} \|(x - t, -t)\|, q(t)) \geq \inf_{t \in \ell^\infty} \|(x - t, -t)\|$

$$= \inf_{t \in \ell^\infty} \max_{(x - t, |t|)} \geq \|x\| = \|(x, 0)\|.$$

It follows that $D_1$ is isometrically isomorphic to $\ell^\infty$, hence reflexive. By the same token $D_2$ is reflexive. But in 2.2 we have seen that $E$ is not even pseudoreflexive.
§3. (B) implies (C).

In the next two lemmas $D$ is a finite-dimensional subspace of a $K$-Banach space $E$ with inclusion map $i: D \to E$ and quotient map $\pi: E \to E/D$. We consider the commutative diagram

$$
\begin{array}{ccc}
D & \xrightarrow{i} & E \\
\downarrow{j_D} & & \downarrow{j_E} \\
D'' & \xrightarrow{i''} & E''
\end{array}
\quad
\begin{array}{c}
\pi\quad
\downarrow{j_{E/D}} \\
E/D
\end{array}
\quad
\begin{array}{c}
\pi''\quad
\downarrow{j_{(E/D)''}} \\
(E/D)''
\end{array}
$$

**Lemma 3.1.** $\text{Im } i'' = \text{Ker } \pi''$ and $\pi''$ is a quotient map.

*Proof.* Since $\pi$ is surjective we have by Proposition 1.4 in $(E/D)' \overset{\pi'}{\longrightarrow} E' \overset{i'}{\longrightarrow} D'$ that $\text{Im } \pi' = \text{Ker } i'$. Now $D'$ is finite-dimensional so $\text{Im } i'$ is closed and has the WEP. Hence, again by Proposition 1.4, $\text{Im } i'' = \text{Ker } \pi''$. To prove the second statement observe that $\pi'$ is an isometry whose image has finite codimension and therefore has the EP by Corollary 1.3, so $\pi''$ is a quotient map.

**Lemma 3.2.** $j_E$ is surjective if and only if $j_{E/D}$ is surjective.

*Proof.* If $j_E$ is surjective then so is $\pi'' \circ j_E = j_{E/D} \circ \pi$. Then $j_{E/D}$ must be surjective. Conversely, if $j_{E/D}$ is surjective, let $\Theta \in E''$. Then there is an $x \in E$ with $\pi''(\Theta) = j_{E/D}(\pi(x)) = \pi''(j_E(x))$. We see that $\Theta - j_E(x) \in \text{Ker } \pi'' = \text{Im } i''$ so by surjectivity of $j_D$ we can find a $d \in D$ such that

$$
\Theta - j_E(x) = i'' j_D(d) = j_E i(d)
$$

Then $\Theta = j_E(x + i(d)) \in j_E(E)$ and the surjectivity of $j_E$ is proved.

**Remark.** In the same vein one can prove $j_E$ injective $\Rightarrow j_{E/D}$ injective; a counterexample to the converse is given in [1], §8.

Also one has $j_E$ is isometrical $\Rightarrow j_{E/D}$ is isometrical. Its converse is just statement (B) the truth of which is an open problem.

If we assume (B) and if $E/D$ is reflexive we have that $j_E$ is isometrical by (B) and surjective by Lemma 3.2. We may conclude that (B) implies (C).