THE \( p \)-ADIC KREIN-ŠMULIAN THEOREM

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ABSTRACT. The natural non-archimedean version of the Krein-Šmulian Theorem holds essentially only when the base field is spherically complete (Corollary 1.6). For Banach spaces over nonspherically complete scalar fields two restricted versions of the Krein-Šmulian Theorem (Theorems 2.2 and 3.1) are proved.

INTRODUCTION. (For unexplained terms see below.) Consider the following statement (*).

\(*)\begin{align*}
\text{Let } E \text{ be a } K\text{-Banach space and let } A \subseteq E' \text{ be convex. If } A \cap B \\
is \text{w}'\text{-closed in } B \text{ for each bounded set } B \subseteq E' \text{ then } A \text{ is w}'\text{-closed.}
\end{align*}

If \( K = \mathbb{R} \) or \( \mathbb{C} \) then (*) is known as the Krein-Šmulian Theorem. By modifying classical techniques (in particular by using \( c \)-compactness arguments) (*) can also be proved if \( K \) is a spherically complete non-archimedean valued field ([4], Theorem 5.1). Now see the Abstract.

PRELIMINARIES. (For terms still unexplained see [2].) Throughout \( K \) is a nonarchimedean nontrivially valued field that is complete under the metric induced by the valuation \( | \cdot | \). We always assume that \( K \) is not spherically complete, so \( |K| := \{ |\lambda| : \lambda \in K \} \) is dense. If \( X \subseteq K \) is bounded, nonempty, we set \( \text{diam } X = \sup \{|x - y| : x, y \in X\} \).

Let \( E \) be a \( K \)-vector space. A nonempty subset \( A \) of \( E \) is absolutely convex if \( x, y \in A, \lambda, \mu \in K, |\lambda| \leq 1, |\mu| \leq 1 \) implies \( \lambda x + \mu y \in A \). For such \( A \) we set \( A^\ast := \bigcap\{ \lambda A : \lambda \in K, |\lambda| > 1 \} \). \( A \) is edged if \( A = A^\ast \). The smallest absolutely convex set containing \( X \subseteq E \) is denoted \( \text{co } X \). A nonempty set in \( E \) is convex (edged convex)
if it is an additive coset of an absolutely convex (edged absolutely convex) set. By definition, the empty set is convex. The algebraic dual of $E$ is the vector space $E^*$ consisting of all linear functions $E \to K$. The weakest topology on $E$ for which all $f \in E^*$ are continuous is denoted $\sigma(E, E^*)$.

A seminorm on $E$ is a map $p : E \to [0, \infty)$ such that $p(x) \geq 0$, $p(\lambda x) = |\lambda|p(x)$, $p(x+y) \leq \max(p(x), p(y))$ for all $x, y \in E$, $\lambda \in K$. We shall use expressions such as ‘$p$-convergence’, ‘$p$-closure’, ‘$p$-compactoid’, ‘$p$-orthogonal’ without further explanation. A seminorm $p$ is of finite type if $\text{Ker } p$ has finite codimension, of countable type if $E/\text{Ker } p$ with the norm induced by $p$ is of countable type. A seminorm $p$ is polar if $p = \sup\{|f| : f \in E^*, |f| \leq p\}$. A seminorm $p$ is a norm if $p(x) = 0$ implies $x = 0$. Norms are usually denoted $\| \|$ rather than $p$.

Let $(E, \| \|)$ be a normed space over $K$. Let $a \in E$, $r > 0$. We write $B_E(a, r) := \{x \in E : \|x-a\| \leq r\}$ and $B_E := B_E(0, 1)$. The dual space $E'$ is the Banach space consisting of all continuous linear functions $E \to K$, normed by $f \mapsto \|f\| := \sup_{B_E} |f|$. The natural map $j_E : E \to E''$ is continuous. $E$ is pseudoreflexive if $j_E$ is an isometry (which is equivalent to polarity of the norm on $E$). A linear map $T$ from a $K$-Banach space $E$ to a $K$-Banach space $F$ is a quotient map if $T$ maps $\{x \in E : \|x\| < 1\}$ onto $\{x \in F, \|x\| < 1\}$.

Let $(E, \tau)$ be a locally convex space over $K$. It is called of finite (countable) type if every continuous seminorm is of finite (countable) type. $(E, \tau)$ is strongly polar if each continuous seminorm is polar, polar if there exists a base of polar continuous seminorms. Let $E' = (E, \tau)'$ be the space of all continuous linear functions $E \to K$. The weak topology $w = \sigma(E, E')$ is the weakest topology on $E$ such that all $f \in E'$ are continuous. Similarly, the weak-star topology $w' = \sigma(E', E)$ is the weakest topology on $E'$ such that for each $x \in E$ the evaluation $f \mapsto f(x)$ ($f \in E'$) is continuous. It is well known (see [5]) that the natural map $E \to (E', \sigma(E', E))'$ is surjective. Let $X \subset E$, $Y \subset E'$. We set $X^0 := \{f \in E' : |f(x)| \leq 1 \text{ for all } x \in X\}$ and $Y_0 := \{x \in E : |f(x)| \leq 1 \text{ for all } f \in Y\}$. $X$ is a polar set if $X_0 = X$. For a ball $B_E(0, r)$ in a normed space $E$ we have $B_E(0, r)^0 = B_E(0, 1/r)$. If $E$ is pseudoreflexive, $B_{E'}(0, r)_0 = B_E(0, 1/r)$. The closure of a set $X \subset E$ is $\overline{X}$. Instead of $\overline{\text{co } X}$ we write $\overline{\text{co } X}$. Let $E, F$ be locally convex spaces over $K$. The adjoint of a continuous linear map $T : E \to F$ is the map $T' : F' \to E'$ defined by $f \mapsto f \circ T$. Following [1] we say that a subspace $D$ of $E$ has the Weak Extension Property (WEP) if the adjoint
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$E' \to D'$ of the inclusion map $D \to E$ is surjective.

§1. FAILURE OF THE KREIN-ŠMULIAN THEOREM

The key theorem of this section is the following. Recall that $K$ is not spherically complete.

**THEOREM 1.1.** Let $\tau_1, \tau_2$ be locally convex topologies on a $K$-vector space $E$ such that $\tau_2$ is of finite type while $\tau_1$ is not. Then there exists a $\tau_1$-closed absolutely convex set in $E$ that is not $\tau_2$-closed.

We signal the following corollary which is in sharp contrast to the theory over spherically complete base fields.

**COROLLARY 1.2.** Let $E$ be a locally $K$-convex space whose topology is not the weak topology. Then there exists a closed absolutely convex set in $E$ that is not weakly closed.

The proof of Theorem 1.1 runs in a few steps. Let us say that a seminorm $q$ on a $K$-vector space is **special** if $q(x) \in |K|$ for each $x \in E$ and if for all $x, y \in E$

$$x \perp y \text{ in the sense of } q \implies q(x) = 0 \text{ or } q(y) = 0$$

**LEMMA 1.3.** On a normed space of countable type over $K$ there exists an equivalent special norm.

**Proof.** Let $(\bar{K}, | |)$ be the spherical completion of $(K, | |)$ in the sense of [2], Theorem 4.49. Then $| |$, considered as a norm on the $K$-vector space $\bar{K}$ is special. (Indeed, we have $|\bar{K}| = |K|$. If $x, y \in \bar{K}$, $x \perp y$, $y \neq 0$ then $xy^{-1} \perp 1$ so $xy^{-1} \perp K$. But $\bar{K}$ is an immediate extension of $K$ so $xy^{-1} = 0$ i.e. $x = 0$.) As $\bar{K}$ is infinite dimensional over $K$ we can, for a given normed space $E$ of countable type over $K$, make a $K$-linear homeomorphism $T$ of $E$ into $\bar{K}$. Then $x \mapsto |Tx|$ is the required norm.

**LEMMA 1.4.** Let $E$ be a strongly polar locally convex space over $K$. If $E$ is not of finite type then there exists a continuous special seminorm $q$ on $E$, $q$ not of finite type.

**Proof.** There is a continuous seminorm of infinite type $p$ on $E$. The $p$-continuous linear functions form an infinite dimensional space so we can find linearly independent
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\( f_1, f_2, \ldots \in E' \) such that \( n|f_n| \leq \rho \) for each \( n \in \mathbb{N} \). The formula \( \hat{p}(x) = \max_n |f_n(x)| \) defines a continuous seminorm \( \hat{p} \) on \( E \), of infinite countable type. Now Lemma 1.3 (applied to \( E/\text{Ker} \hat{p} \)) leads to a special seminorm \( q \) equivalent to \( \hat{p} \).

**Remark.** The conclusion of Lemma 1.4 holds for any polar space \( E \) that is not of finite type.

**LEMMA 1.5.** Let \( q \) be a special seminorm on a \( K \)-vector space \( E \). If \( q \) is not of finite type then \( \{ x \in E : q(x) < 1 \} \) is not \( \sigma(E, E^*) \)-closed.

**Proof.** Let \( x \in E, q(x) = 1 \) (such \( x \) exist!). We shall prove that \( x \) is in the \( \sigma(E, E^*) \)-closure of \( A := \{ x \in E : q(x) < 1 \} \) by producing, for given \( f_1, \ldots, f_n \in E^* \), a point \( a \in A \) such that \( f_i(x-a) = 0 \) for \( i \in \{1, \ldots, n\} \).

(i) Suppose \( f_1(x-a) \neq 0 \) for all \( a \in A \). Then \( f_1(x) \notin f_1(A) \) so, by convexity, \( f_1(A) \) is bounded and \( f_1 \) is \( q \)-continuous. We have

\[
|f_1(x)| \geq \text{diam } f_1(A) = \sup \{|f_1(a)| : q(a) < 1\} = \|f_1\|
\]

where \( \|f_1\| \) is the operator seminorm of \( f_1 \) with respect to \( q \). For each \( y \in \text{Ker} f_1 \)

\[
\|f_1\| q(x-y) \geq |f_1(x-y)| = |f_1(x)| \geq \|f_1\|
\]

and we find \( x \perp y \) in the sense of \( q \). As \( q \) is special and \( q(x) = 1 \) we must have \( q = 0 \) on \( \text{Ker} f_1 \) implying that \( q \) is of finite type, a contradiction. Thus, we may conclude that there exists an \( a_1 \in A \) with \( f_1(x-a_1) = 0 \).

(ii) Now we repeat the argument in (i) where \( E \) is replaced by \( \text{Ker} f_1, q \) by \( q|\text{Ker} f_1 \), \( x \) by \( x-a_1 \), \( A \) by \( A \cap \text{Ker} f_1 \) and \( f_1 \) by \( f_2|\text{Ker} f_1 \). (Indeed, \( q|\text{Ker} f_1 \) is special, of infinite type and \( q(x-a_1) = 1 \)). So there exists an \( a_2 \in A \cap \text{Ker} f_1 \) such that \( f_2(x-a_1-a_2) = 0 \). Observe that also \( f_1(x-a_1-a_2) = 0 \). In this spirit we arrive inductively at points \( a_1, a_2, \ldots, a_n \in A \) such that \( f_1(x-a) = 0 \) \((i \in \{1, \ldots, n\})\) where \( a := \sum_{i=1}^n a_i \in A \).

**Proof of Theorem 1.1.** If \( (E, \tau_1) \) is not strongly polar, choose any nonpolar continuous seminorm \( q \) and set \( A := \{ x \in E : q(x) \leq 1 \} \). \( A \) is \( \tau_1 \)-closed but, as \( q \) is not polar and \( A \) is edged, \( A \) is not \( \sigma(E, E^*) \)-closed so certainly \( A \) is not \( \tau_2 \)-closed. If \( (E, \tau_1) \) is strongly polar, let \( q \) be as in Lemma 1.4. By Lemma 1.5 the set \( A := \{ x \in E : q(x) < 1 \} \) is not \( \sigma(E, E^*) \)-closed, so not \( \tau_2 \)-closed.
Part (i) of the next corollary demonstrates the failure of the Krein-Šmulian Theorem for nonspherically complete base fields.

**COROLLARY 1.6.** Let $E$ be a normed space over $K$ such that $E'$ is infinite dimensional.

(i) There exists an absolutely convex set $A \subset E'$ such that $A \cap B$ is $w'$-closed in $B$ for each bounded set $B \subset E'$ while $A$ is not $w'$-closed.

(ii) There exists an absolutely convex set $A \subset E$ such that $A \cap B$ is $w$-closed in $B$ for each bounded set $B \subset E$ while $A$ is not $w$-closed.

**Proof.** (i) $(E', w')$ is an infinite dimensional Hausdorff space of countable type so its dual (which is $j_E(E)$) is infinite dimensional. Thus, we can choose $x_1, x_2, \ldots$ in $E$ such that $j_E(x_1), j_E(x_2), \ldots$ are linearly independent and $\lim_{n \to \infty} \|x_n\| = 0$. The seminorm $p$ on $E'$ defined by

$$p(f) = \max_n |f(x_n)| = \max_n |j_E(x_n)(f)|$$

is therefore not of finite type. By Theorem 1.1 there exists an absolutely convex set $A \subset E'$ which is $p$-closed but not $w'$-closed. But it is easily seen that, on any bounded set $B \subset E'$, $w'$-convergence implies $p$-convergence. Thus, the $p$-closedness of $A$ implies that $A \cap B$ is $w'$-closed in $B$.

(ii) Similar to the above proof but now with the seminorm $x \mapsto \max_n |f_n(x)|$ ($x \in E$), where $f_1, f_2, \ldots$ is a linearly independent sequence in $E'$ for which $\lim_{n \to \infty} \|f_n\| = 0$. We leave the details to the reader.

§2. SAVE THE KREIN-ŠMULIAN THEOREM! (PART ONE)

To save the Krein-Šmulian Theorem we shall concentrate on *edged* convex sets. As such sets are translates of edged absolutely convex sets no harm is done by considering only the latter. Thus, we arrive at

**DEFINITION 2.1.** A normed space $E$ over $K$ is a *Krein-Šmulian space* if the following holds. If $A \subset E'$ is absolutely convex and edged and if $A \cap B$ is $w'$-closed in $B$ for each bounded set $B \subset E'$ then $A$ is $w'$-closed.

Observe that, for an absolutely convex $A \subset E'$, the expression `$A \cap B$ is $w'$-closed in $B$ for each bounded set $B \subset E'$' is equivalent to `for each $n \in \mathbb{N}$ the set $A \cap B_{E'}(0, n)$
is $w'$-closed' and, if $A$ is a subspace, to 'A ∩ $B_{E'}$ is $w'$-closed'.

The main result of this section is

**THEOREM 2.2.** A strongly polar Banach space is a Krein-Šmulian space.

For the proof we need first a lemma on Banach spaces. Let us call a sequence $X_1 ⊃ X_2 ⊃ \ldots$ of closed absolutely convex subsets of a $K$-Banach space $E$ quasi Cauchy if for each $\lambda \in K$, $|\lambda| > 1$ and $N \in \mathbb{N}$

$$X_n ⊂ \lambda(X_m + B_E(0, \frac{1}{N})) \quad (m, n \geq N)$$

**LEMMA 2.3.** Let $X_1 ⊃ X_2 ⊃ \ldots$ be a quasi Cauchy sequence in a $K$-Banach space. Set $X := \bigcap X_n$. Then, for each $n \in \mathbb{N}$ and $x_n \in X_n$, and each $\lambda \in K$, $|\lambda| > 1$ there is an $x \in \lambda X$ such that $\|x_n - x\| \leq \frac{|\lambda|}{n}$.

Proof. Choose $\lambda_1, \lambda_2, \ldots \in K$ with $|\lambda_i| > 1$ for each $i$, $\prod_{1}^{i} |\lambda_i| = |\lambda|$. We have

$$X_n ⊂ \lambda_1(X_{n+1} + B_E(0, \frac{1}{n})) \text{ whence } X_n ⊂ \lambda_1 X_{n+1} + B_E(0, \frac{|\lambda|}{n})$$

$$X_{n+1} ⊂ \lambda_2(X_{n+1} + B(0, \frac{1}{n+1})) \text{ whence } \lambda_1 X_{n+1} ⊂ \lambda_1 \lambda_2 X_{n+2} + B_E(0, \frac{|\lambda|}{n+1})$$

etc.

So, given $x_n \in X_n$, we can find a sequence $x_{n+1}, x_{n+2}, \ldots$ where $x_{n+1} \in \lambda_1 X_{n+1}$, $x_{n+2} \in \lambda_1 \lambda_2 X_{n+2}, \ldots$ such that for all $k \in \{0,1,2,\ldots\}$

$$\|x_{n+k} - x_{n+k+1}\| \leq \frac{|\lambda|}{n + k}.$$

By completeness $x := \lim_{k \to \infty} x_{n+k}$ exists. We have $\lambda^{-1} x_{n+1} \in \lambda^{-1} \lambda_1 X_{n+1} \subset X_{n+1}$; $\lambda^{-1} x_{n+2} \in \lambda^{-1} \lambda_1 \lambda_2 X_{n+2} \subset X_{n+2}$, etc., so $\lambda^{-1} x = \lim_{k \to \infty} \lambda^{-1} x_{n+k} \in \bigcap_{i \geq n+1} X_i = X$ and it follows that $x \in \lambda X$. Further, we have

$$\|x_n - x\| \leq \max\{\|x_n - x_{n+1}\|, \|x_{n+1} - x_{n+2}\|, \ldots\} \leq \max\{\frac{|\lambda|}{n}, \frac{|\lambda|}{n+1}, \ldots\} \leq \frac{|\lambda|}{n}.$$

**Proof of Theorem 2.2.** Let $A \subset E'$ be absolutely convex, edged and assume that $A \cap B_{E'}(0,n)$ is $w'$-closed for each $n \in \mathbb{N}$. Then $(A \cap B_{E'}(0,n)$ is also edged) $A \cap B_{E'}(0,n)$ is a polar set. Setting

$$X_n := (A \cap B_{E'}(0,n))_0 \quad (n \in \mathbb{N})$$

$$X := \bigcap_n X_n$$
one verifies immediately (i), (ii), (iii), (iv) below.

(i) Each $X_n$ is a polar subset of $E$.

(ii) $X_n^0 = A \cap B_{E'}(0, n)$ for each $n \in \mathbb{N}$.

(iii) $X_1 \supset X_2 \supset \ldots$.

(iv) $X = A_0$.

(v) For each $N \in \mathbb{N}$ and $m, n \geq N$

$$X_n \subset (X_m + B_E(0, \frac{1}{N}))^0.$$

(Proof: $(X_m + B_E(0, \frac{1}{N}))^0 = X_m^0 \cap B_E(0, \frac{1}{N})^0 = A \cap B_{E'}(0, m) \cap B_{E'}(0, N) = A \cap B_{E'}(0, N)$, so $X_n \subset X_N = (A \cap B_{E'}(0, N))^0 = (X_m + B_E(0, \frac{1}{N}))^0$.)

(vi) $X_1, X_2, \ldots$ is quasi Cauchy. (Proof. Let $\lambda \in K, |\lambda| > 1, N \in \mathbb{N}, m, n \geq N$.

The set $X_m + B_E(0, \frac{1}{N})$ is norm open hence norm closed. So $(X_m + B_E(0, \frac{1}{N}))^\circ$ is

norm closed and edged, hence polar (as $E$ is strongly polar). It follows via (v), that

$X_n \subset (X_m + B_E(0, \frac{1}{N}))^0 \subset \lambda(X_m + B_E(0, \frac{1}{N})).$)

(vii) $X^0 \subset A$. (Proof. Let $f \in X^0, \lambda \in K, |\lambda| > 1$. It suffices to prove that $f \in \lambda A$.

Let $n \in \mathbb{N}$ be such that $\|f\| \leq n$. Choose any $x \in X_n$. By Lemma 2.3 there is a $y \in \lambda X$ with $\|x - y\| \leq \frac{|\lambda|}{n}$. We have

$$|f(x)| \leq |f(x-y)| + |f(y)|$$

$$\leq \|f\||x-y| + |\lambda| \leq n \cdot \frac{|\lambda|}{n} + |\lambda| = |\lambda|$$

and we see that $|\lambda^{-1} f| \leq 1$ on $X_n$, so $\lambda^{-1} f \in X_n^0 = A \cap B_{E'}(0, n) \subset A$ i.e. $f \in \lambda A$.)

Now (iv) combined with (vii) yields $A = X^0$ is $w'$-closed.

**Corollary 2.4.** A subspace of the dual of a strongly polar Banach space is $w'$-closed

as soon as its intersection with the closed unit ball is $w'$-closed.

**Corollary 2.5.** An edged absolutely convex subset $A$ of $\ell^\infty$ is $\sigma(\ell^\infty, c_0)$-closed as

soon as $A \cap B_{\ell^\infty}(0, n)$ is $\sigma(\ell^\infty, c_0)$-closed for each $n \in \mathbb{N}$.

**Proof.** $c_0$ is a (reflexive) strongly polar space.

We also have:

**Theorem 2.6.** If $E$ is a Krein-Šmulian space and $D \subset E$ is a closed subspace then

$E/D$ is a Krein-Šmulian space.
Proof. Let $i : (E/D)' \to E'$ be the adjoint of the quotient map $E \to E/D$. It is easily seen that $i$ is an isometry, that $\text{Im } i$ is $w'$-closed in $E'$ and that $i$ is a $w'$ to $w'$ homeomorphism $(E/D)' \to \text{Im } i$.

Now let $A$ be an edged absolutely convex subset of $(E/D)'$ such that $A \cap B$ is $w'$-closed in $B$ for each bounded set $B$ in $(E/D)'$. Then $i(A)$ is edged. If $X \subset E'$ is bounded then $i(A) \cap X$ is $w'$-closed in $X$. (Proof. Let $j \mapsto a_j$ be a net in $A$ such that $i(a_j) \in X$ for all $j$ and let $w' - \lim_j a_j = b \in X$. As $\text{Im } i$ is $w'$-closed $b = i(a)$ for some $a \in i^{-1}(X)$. Then $w' - \lim_j a_j = a$. Now $a_j \in A \cap i^{-1}(X)$ for all $j$, $a \in i^{-1}(X)$ and $i^{-1}(X)$ is bounded, so by assumption on $A$ we have $a \in A$, so $b \in i(A) \cap X.$) Since $E$ is a Krein-Šmulian space, $i(A)$ is $w'$-closed in $E'$ so that $A = i^{-1}(i(A))$ is $w'$-closed in $(E/D)'$.

**Theorem 2.7.** If $E$ is a Krein-Šmulian space and if $D \subset E$ is a weakly closed subspace having the WEP then $D$ is a Krein-Šmulian space.

Proof. Let $\pi : E' \to D'$ be the adjoint of the inclusion map $D \hookrightarrow E$. Then $\pi$ is surjective and $w'$ to $w'$ continuous. If $A$ is an edged absolutely convex set in $D'$ and $\pi^{-1}(A)$ is $w'$-closed then $A$ is $w'$-closed. (Proof. Let $g \in D'$, $g \notin A$. There is an $f \in E'$ with $\pi(f) = g$. Then $f \notin \pi^{-1}(A)$. Now $\pi^{-1}(A)$ is $w'$-closed and edged so there exists an $x \in E$ such that $f(x) = 1$ and $|h(x)| < 1$ for all $h \in \pi^{-1}(A)$. In particular, $|h(x)| < 1$ for all $h \in \text{Ker } \pi = D^0$ i.e. $h(x) = 0$ for all $h \in D^0$ so $x \in D_0^0 = D$. Then $g(x) = f(x) = 1$ and $|h(x)| < 1$ for all $h \in A.$)

Now let $A$ be an absolutely convex edged subset of $D'$ such that $A \cap B$ is $\sigma(D', D)$-closed in $B$ for each bounded set $B \subset D'$. Then for such $B$, $\pi^{-1}(A) \cap \pi^{-1}(B)$ is $\sigma(E', E)$-closed in $\pi^{-1}(B)$. If $X \subset E'$ is bounded then $\pi(X)$ is bounded and $X \subset \pi^{-1}(\pi(X))$ so it follows that $\pi^{-1}(A) \cap X$ is $w'$-closed in $X$ for each bounded set $X \subset E'$. Since $E$ is Krein-Šmulian we have that $\pi^{-1}(A)$ is $w'$-closed, so by the remark above, $A$ is $w'$-closed.

**Remark.** Not every Krein-Šmulian polar space is strongly polar; $\ell^\infty$ is an easy example. In §3 we will see that, if $I$ is large enough, $c_0(I)$ is not Krein-Šmulian. This leads to the

**Problem.** Characterize the class of Krein-Šmulian spaces.

A concrete help would be the answer to the following two questions.
- Is \( c_0 \times c^\infty \) a Krein-Šmulian space? (More generally, if \( E_1 \) and \( E_2 \) are Krein-Šmulian spaces then does it follow that \( E_1 \times E_2 \) is Krein-Šmulian?)

- Is the subspace of \( D \) of \( c^\infty \) constructed in [2], Ex. 4J Krein-Šmulian?

§3. SAVE THE KREIN-ŠMULIAN THEOREM! (PART TWO)

In this section we shall prove the following version of the Krein-Šmulian Theorem. Observe that \((\alpha)\) holds for any polar \( K \)-Banach space.

**THEOREM 3.1.** For a normed space \( E \) over \( K \) the following are equivalent.

\((\alpha)\) \( J_E(E) \) is norm closed in \( E'' \).

\((\beta)\) If \( H \subset E' \) is a subspace of finite codimension and if \( H \cap B_{E'} \) is \( w' \)-closed then so is \( H \).

For a normed space \( E \) over \( K \) the \( bw' \)-topology (the 'bounded-weak-star topology') is by definition the strongest locally convex topology on \( E' \) that coincides with \( w' \) on bounded subsets of \( E' \).

**PROPOSITION 3.2.** Let \( E \) be a normed space over \( K \).

\((i)\) \( bw' \) is stronger than \( w' \) but weaker than the norm topology on \( E' \).

\((ii)\) \( (E', bw') \) is of countable type.

\((iii)\) A seminorm \( p \) on \( E' \) is \( bw' \)-continuous if and only if \( p|B_{E'} \) is \( w' \)-continuous.

\((iv)\) For any locally convex space \( (X, \tau) \) and any linear map \( T : E' \to X \) we have that \( T \) is \( bw' \) to \( \tau \) continuous if and only if \( T|B_{E'} \) is \( w' \) to \( \tau \) continuous.

**Proof.** \( E' = [B_{E'}] \) and \( B_{E'} \) is a \( w' \)-compactoid, hence a \( bw' \)-compactoid. This implies (ii). The other proofs are straightforward.

We know that \((E', w')' = j_E(E) ([5]))\). We now prove

**PROPOSITION 3.3.** For a normed space \( E \) over \( K \) the dual of \((E', bw')\) is the norm closure of \( j_E(E) \) in \( E'' \).

**Proof.** Every \( \theta \in \overline{j_E(E)} \) is, on \( B_{E'} \), the uniform limit of a sequence in \( j_E(E) \) so \( \theta|B_{E'} \) is \( w' \)-continuous and \( \theta \) is \( bw' \)-continuous (Proposition 3.2 (iv)). Thus \( \overline{j_E(E)} \subset (E', bw')' \). Conversely, let \( \theta \in (E', bw')' \). Then (Proposition 3.2 (i)) \( \theta \in E'' \). Let \( \varepsilon > 0 \); we shall
find an \( x \in E \) such that \( \|\theta - j_E(x)\| < \varepsilon \). Let \( \alpha \in K \), \( 0 < |\alpha| < \varepsilon \). The \( w' \)-continuity of \( \theta|B_{E'} \) yields a finite set \( F \subset E \) such that \( f \in F^0 \cap B_{E'} \) implies \( |\theta(f)| \leq |\alpha| \), in other words

\[
f \in j_E(F)_0 \cap (B_{E''})_0 \implies |(\alpha^{-1}\theta)(f)| \leq 1.
\]

So we see that \( \alpha^{-1}\theta \in (j_E(F)_0 \cap (B_{E''})_0)^0 = (A + B_{E''})^0 \), where \( A = j_E(\operatorname{co} F) \). Now \( B_{E''} \) is \( w' \)-closed and \( A \) is finite dimensional so by [3], 1.4, \( (A + B_{E''})^0 = (A + B_{E''})^e \).

For any \( \lambda \in K \) such that \( |\lambda| > 1 \) and \( |\lambda\alpha| < \varepsilon \) we have \( \alpha^{-1}\theta \in \lambda A + \lambda B_{E''} \), hence \( \theta \in j_E(E) + \alpha\lambda B_{E''} \) and there is an \( x \in E \) with \( \theta - j_E(x) \in \alpha\lambda B_{E''} \) i.e. \( \|\theta - j_E(x)\| < \varepsilon \).

**Corollary 3.4.** Let \( j_E(E) \) be closed in \( E'' \), let \( A \subset E' \) be absolutely convex and edged. Then \( A \) is \( w' \)-closed if and only if \( A \) is \( bw' \)-closed.

**Proof.** Let \( A \) be \( bw' \)-closed. As \( A \) is also edged and \( (E', bw') \) is strongly polar (Proposition 3.2 (ii)), \( A \) is a polar set i.e. \( A = S_0 \) for some \( S \subset (E', bw')' \). But by Proposition 3.3 \( S \subset (E', w')' \) so that \( A \) is \( w' \)-closed.

Further, we need the following general lemma.

**Lemma 3.5.** Let \( A \) be a closed absolutely convex subset of a Hausdorff locally convex space over \( K \); let \( D \) be a finite dimensional subspace such that \( A \cap D = \{0\} \). Then \( A + D \) is closed and the addition map is a homeomorphism \( A \times D \to A + D \).

**Proof.** (i) If addition is homeomorphic then \( A + D \) is closed. In fact, let \( i \to a_i + d_i \) be a net in \( A + D \) (where \( a_i \in A \), \( d_i \in D \) for each \( i \)), converging to some \( z \). Then \( (i,j) \to a_i - a_j + d_i - d_j \) converges to 0. By homeomorphism, \( d_i - d_j \to 0 \), by completeness of \( D \), \( d_i \to d \) for some \( d \in D \). Then \( a_i \to z - d \) and, by closedness of \( A \), \( z - d \in A \). We see that \( z \in A + D \).

(ii) Assume \( n := \dim D = 1 \), say \( D = Kx \) for some nonzero \( x \). Let \( i \to a_i + \lambda_i x \) (\( a_i \in A \), \( \lambda_i \in K \)) be a net in \( A + D \) converging to 0. If not \( \lambda_i \to 0 \) we may assume \( |\lambda_i| \geq |\alpha| > 0 \) for all \( i \) and some \( \alpha \in K \). Then \( \alpha \lambda_i^{-1}(a_i + \lambda_i x) \to 0 \) so \( \alpha x = -\lim_{i} \alpha \lambda_i^{-1} a_i \in A = A \) conflicting \( Kx \cap A = \{0\} \). Thus, addition is homeomorphic and via (i) the lemma is proved if \( n = 1 \).

(iii) The proof of the induction step \( n-1 \to n \) is now standard and left to the reader.

**Proof of Theorem 3.1.**

(i) Suppose \((\alpha)\), and let \( H \subset E' \) be a subspace of finite codimension such that \( H \cap B_{E'} \) is \( w' \)-closed. Then \( H \cap B_{E'} \) is norm closed, hence so is \( H \). For some \( t \in (0,1) \) \( H \) has a
t-orthogonal complement $D$. Let $P : E' \to D$ be the obvious projection. For $\lambda \in K$, $|\lambda| \geq t^{-1}$ we have

$$B_{E'} \subset \lambda(H \cap B_{E'}) + \lambda(D \cap B_{E'}) \subset \lambda(H \cap B_{E'}) + D.$$ 

Let $i \mapsto f_i$ be a net in $B_{E'}$, $w' - \lim f_i = 0$. Then, by Lemma 3.5, $\lim P f_i = 0$. We see that $P|B_{E'}$ is continuous, so (Proposition 3.2 (iv)) $P$ is bw' to norm continuous and Ker $P = H$ is bw'-closed, hence $w'$-closed by Corollary 3.4, and ($\beta$) is proved.

(ii) Suppose ($\alpha$) is not true. Choose $\theta \in \overline{j_E(E) \setminus j_E(E)}$. Then $\theta$ is not $w'$-continuous so $H := \text{Ker } \theta$ is not $w'$-closed. But $\theta$ is bw'-continuous by Proposition 3.3. so $H \cap B_{E'}$ is $w'$-closed.

The results of this section yield the existence of polar non-Krein-Šmulian spaces (see §2).

**Corollary 3.6.** If $m$ is a cardinality $\geq \# K$ then $c_0(m)$ is not a Krein-Šmulian space.

**Proof.** In [2], Exercise 4.N a Banach space $E$ is constructed such that $j_E(E)$ is a proper dense subset of $E''$. From this construction it is easily seen that $\# E = \# c_0 \leq \# K^N = \# K$. Now let $I$ be a set with cardinality $\geq \# K$ and let $\{e_i : i \in I\}$ be the natural orthonormal base of $c_0(I)$. There is a surjection $\{e_i : i \in I\} \to B_E$, it extends to a quotient map $c_0(I) \to E$. Now $E$ is not a Krein-Šmulian space by Theorem 3.1, neither is $c_0(I)$ by Theorem 2.6. (It is not hard to see by looking at the proof of Theorem 2.6 that one can even find a subspace $D \subset c_0(I)$ that is not $w'$-closed while $D \cap B_{c_0(I)}$ is.)

**Corollary 3.7.** If $m$ is a nonmeasurable cardinality $\geq \# K^K$ then $\ell^\infty(m)$ is not a Krein-Šmulian space.

**Proof.** In the spirit of the previous proof one constructs a quotient map $\pi : c_0(n) \to \ell^\infty(n)$ where $n = \# K$. By reflexivity ([2], Theorem 4.21) the adjoint $\pi' : c_0(n) \to \ell^\infty(m)$ is an isometry and $\pi'(c_0(n))$ has the WEP in $\ell^\infty(m)$. From [4], Lemma 2.2 we obtain that $\pi'(c_0(n))$ is also weakly closed in $\ell^\infty(m)$. By the previous corollary $c_0(n)$ is not a Krein-Šmulian space, neither is $\ell^\infty(m)$ by Theorem 2.7.

**Problem.** Determine the smallest cardinality $m$ for which $c_0(m)$ ($\ell^\infty(m)$ if $\# K$ is nonmeasurable) is not a Krein-Šmulian space.
As a further application we now prove a nonarchimedean version of a classical reflexivity criterion (Theorem 3.8). First we 'dualize' the notion of a polar seminorm as follows. A seminorm $p$ on the dual $E'$ of a locally $K$-convex space $E$ is a dual seminorm if there exists an $X \subset E$ such that $p(f) = \sup \{|f(x)| : x \in X\}$ for all $f \in E'$. An easy exercise shows that $p$ is dual if and only if $\{f \in E' : p(f) \leq 1\}$ is $\sigma(E', E)$-closed. Dual seminorms are automatically polar.

**Theorem 3.8.** Let $E$ be a pseudoreflexive $K$-Banach space. Then $E$ is reflexive if and only if each polar norm on $E'$ inducing the topology is dual.

**Proof.** Let $E$ be reflexive and let $\nu$ be a polar norm on $E'$ inducing the topology. Then $\{f \in E' : \nu(f) \leq 1\}$ is weakly closed so, by reflexivity, $w'$-closed. Hence $\nu$ is dual by the above remark. Conversely, suppose each polar norm on $E'$ inducing the topology is dual. To prove reflexivity of $E$ it suffices (by pseudoreflexivity) to show that any $\theta \in E''$ is $w'$-continuous. For each $n \in \mathbb{N}$ the norm $f \mapsto n|\theta(f)| \vee \|f\|$ (where $\|\| \|$ is the 'natural' norm on $E'$) is easily seen to be polar and it is obviously equivalent to $\|\|$. By assumption its closed unit ball

$$B_n := \{f \in E' : |\theta(f)| \leq \frac{1}{n}, \|f\| \leq 1\}$$

is $w'$-closed. Hence so is $\bigcap_n B_n$ which is Ker $\theta \cap B_{E'}$. By Theorem 3.1 Ker $\theta$ is $w'$-closed implying that $\theta$ is $w'$-continuous.

**Remark.** One also may consider a 'predual form' of the Krein-Šmulian property (compare Definition 1.1, see also Corollary 1.6 (ii)) as follows. A normed space $E$ is PKŠ-space if for each absolutely convex edged $A \subset E$:

$$A \cap B \text{ is } w\text{-closed in } B \text{ for each bounded } B \subset E \implies A \text{ is } w\text{-closed.}$$

(Obviously this notion is of no use in classical Banach space theory.) The reader will not have difficulties in proving results about PKŠ spaces similar to the one of KŠ-spaces of this paper. More precisely, we have

(i) A strongly polar normed space is PKŠ.

(ii) Let $E$ be a normed PKŠ space, let $D$ be a closed subspace. Then $E/D$ is PKŠ. If $D$, in addition, is weakly closed and has the WEP then $D$ has PKŠ.

(iii) Let $E$ be a normed space. If $H \subset E$ is a subspace with finite codimension and $H \cap B_E$ is weakly closed then $H$ is weakly closed.
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