THE $p$-ADIC KREIN-ŠMULIAN THEOREM

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ABSTRACT. The natural non-archimedean version of the Krein-Šmulian Theorem holds essentially only when the base field is spherically complete (Corollary 1.6). For Banach spaces over nonspherically complete scalar fields two restricted versions of the Krein-Šmulian Theorem (Theorems 2.2 and 3.1) are proved.

INTRODUCTION. (For unexplained terms see below*) Consider the following statement (*).

Let $E$ be a $K$-Banach space and let $A \subseteq E'$ be convex. If $A \cap B$ is $w'$-closed in $B$ for each bounded set $B \subseteq E'$ then $A$ is $w'$-closed.

If $K = \mathbb{R}$ or $\mathbb{C}$ then (*) is known as the Krein-Šmulian Theorem. By modifying classical techniques (in particular by using $c$-compactness arguments) (*) can also be proved if $K$ is a spherically complete non-archimedean valued field ([4], Theorem 5.1). Now see the Abstract.

PRELIMINARIES. (For terms still unexplained see [2].) Throughout $K$ is a nonarchimedean nontrivially valued field that is complete under the metric induced by the valuation $| |$. We always ASSUME THAT $K$ IS NOT SPHERICALLY COMPLETE, so $|K| := \{|\lambda| : \lambda \in K\}$ is dense. If $X \subseteq K$ is bounded, nonempty, we set diam $X = \sup\{|x - y| : x, y \in X\}$.

Let $E$ be a $K$-vector space. A nonempty subset $A$ of $E$ is absolutely convex if $x, y \in A$, $\lambda, \mu \in K$, $|\lambda| \leq 1$, $|\mu| \leq 1$ implies $\lambda x + \mu y \in A$. For such $A$ we set $A^e := \{\lambda A : \lambda \in K, |\lambda| > 1\}$. $A$ is edged if $A = A^e$. The smallest absolutely convex set containing $X \subseteq E$ is denoted $\text{co } X$. A nonempty set in $E$ is convex (edged convex)
if it is an additive coset of an absolutely convex (edged absolutely convex) set. By
definition, the empty set is convex. The algebraic dual of $E$ is the vector space $E^*$
consisting of all linear functions $E \to K$. The weakest topology on $E$ for which all
$f \in E^*$ are continuous is denoted $\sigma(E, E^*)$.

A seminorm on $E$ is a map $p : E \to [0, \infty)$ such that $p(x) \geq 0$, $p(\lambda x) = |\lambda|p(x)$,
$p(x+y) \leq \max(p(x), p(y))$ for all $x, y \in E$, $\lambda \in K$. We shall use expressions such as ‘$p$
A seminorm $p$ is of finite type if $\text{Ker } p$ has finite codimension, of countable type if $E/\text{Ker } p$ with the norm induced by $p$ is of countable type. A seminorm $p$ is polar if $p = \sup\{ |f| : f \in E^*, \ |f| \leq p \}$. A seminorm $p$ is a norm if $p(x) = 0$ implies $x = 0$.
Norms are usually denoted $||$ rather than $p$.

Let $(E, || ||)$ be a normed space over $K$. Let $a \in E$, $r > 0$. We write $B_E(a, r) := \{x \in E : ||x-a|| \leq r \}$ and $B_E := B_E(0, 1)$. The dual space $E'$ is the Banach space consisting of all continuous linear functions $E \to K$, normed by $f \mapsto ||f|| := \sup_{B_E} |f|$. The natural map $j_E : E \to E''$ is continuous. $E$ is pseudoreflexive if $j_E$ is an isometry (which is equivalent to polarity of the norm on $E$). A linear map $T$ from a $K$-Banach space $E$ to a $K$-Banach space $F$ is a quotient map if $T$ maps $\{x \in E : ||x|| < 1 \}$ onto $\{x \in F, ||x|| < 1 \}$.

Let $(E, \tau)$ be a locally convex space over $K$. It is called of finite (countable) type if every continuous seminorm is of finite (countable) type. $(E, \tau)$ is strongly polar if each continuous seminorm is polar, polar if there exists a base of polar continuous seminorms. Let $E' = (E, \tau)'$ be the space of all continuous linear functions $E \to K$. The weak topology $w = \sigma(E, E')$ is the weakest topology on $E$ such that all $f \in E'$ are continuous. Similarly, the weak-star topology $w' = \sigma(E', E)$ is the weakest topology on $E'$ such that for each $x \in E$ the evaluation $f \mapsto f(x)$ ($f \in E'$) is continuous. It is well known (see [5]) that the natural map $E \to (E', \sigma(E', E)')$ is surjective. Let $X \subset E$, $Y \subset E'$. We set $X^0 := \{f \in E' : |f(x)| \leq 1 \text{ for all } x \in X \}$ and $Y_0 := \{x \in E : |f(x)| \leq 1 \text{ for all } f \in Y \}$. $X$ is a polar set if $X_0^0 = X$. For a ball $B_E(0, r)$ in a normed space $E$ we have $B_E(0, r)^0 = B_{E'}(0, 1/r)$. If $E$ is pseudoreflexive, $B_{E'}(0, r)_0 = B_E(0, 1/r)$. The closure of a set $X \subset E$ is $\overline{X}$. Instead of $\text{co } X$ we write $\overline{\text{co } X}$. Let $E, F$ be locally convex spaces over $K$. The adjoint of a continuous linear map $T : E \to F$ is the map $T' : F' \to E'$ defined by $f \mapsto f \circ T$. Following [1] we say that a subspace $D$ of $E$ has the Weak Extension Property (WEP) if the adjoint
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$E' \rightarrow D'$ of the inclusion map $D \hookrightarrow E$ is surjective.

§1. FAILURE OF THE KREIN-ŠMULIAN THEOREM

The key theorem of this section is the following. Recall that $K$ is not spherically complete.

**Theorem 1.1.** Let $\tau_1, \tau_2$ be locally convex topologies on a $K$-vector space $E$ such that $\tau_2$ is of finite type while $\tau_1$ is not. Then there exists a $\tau_1$-closed absolutely convex set in $E$ that is not $\tau_2$-closed.

We signal the following corollary which is in sharp contrast to the theory over spherically complete base fields.

**Corollary 1.2.** Let $E$ be a locally $K$-convex space whose topology is not the weak topology. Then there exists a closed absolutely convex set in $E$ that is not weakly closed.

The proof of Theorem 1.1 runs in a few steps. Let us say that a seminorm $q$ on a $K$-vector space is *special* if $q(x) \in |K|$ for each $x \in E$ and if for all $x, y \in E$

$$x \perp y \text{ in the sense of } q \implies q(x) = 0 \text{ or } q(y) = 0$$

**Lemma 1.3.** On a normed space of countable type over $K$ there exists an equivalent special norm.

**Proof.** Let $(\bar{K}, |\cdot|)$ be the spherical completion of $(K, |\cdot|)$ in the sense of [2], Theorem 4.49. Then $|\cdot|$, considered as a norm on the $K$-vector space $\bar{K}$ is special. (Indeed, we have $|\bar{K}| = |K|$. If $x, y \in \bar{K}$, $x \perp y$, $y \neq 0$ then $xy^{-1} \perp 1$ so $xy^{-1} \perp K$. But $\bar{K}$ is an immediate extension of $K$ so $xy^{-1} = 0$ i.e. $x = 0$.) As $\bar{K}$ is infinite dimensional over $K$ we can, for a given normed space $E$ of countable type over $K$, make a $K$-linear homeomorphism $T$ of $E$ into $\bar{K}$. Then $x \mapsto |Tx|$ is the required norm.

**Lemma 1.4.** Let $E$ be a strongly polar locally convex space over $K$. If $E$ is not of finite type then there exists a continuous special seminorm $q$ on $E$, $q$ not of finite type.

**Proof.** There is a continuous seminorm of infinite type $p$ on $E$. The $p$-continuous linear functions form an infinite dimensional space so we can find linearly independent
$f_1, f_2, \ldots \in E'$ such that $n|f_n| \leq p$ for each $n \in \mathbb{N}$. The formula $\tilde{p}(x) = \max_n |f_n(x)|$ defines a continuous seminorm $\tilde{p}$ on $E$, of infinite countable type. Now Lemma 1.3 (applied to $E/Ker \tilde{p}$) leads to a special seminorm $q$ equivalent to $\tilde{p}$.

**Remark.** The conclusion of Lemma 1.4 holds for any polar space $E$ that is not of finite type.

**Lemma 1.5.** Let $q$ be a special seminorm on a $K$-vector space $E$. If $q$ is not of finite type then $\{x \in E : q(x) < 1\}$ is not $\sigma(E, E^*)$-closed.

**Proof.** Let $x \in E$, $q(x) = 1$ (such $x$ exist!). We shall prove that $x$ is in the $\sigma(E, E^*)$-closure of $A := \{x \in E : q(x) < 1\}$ by producing, for given $f_1, \ldots, f_n \in E^*$, a point $a \in A$ such that $f_i(x-a) = 0$ for $i \in \{1, \ldots, n\}$.

(i) Suppose $f_i(x-a) \neq 0$ for all $a \in A$. Then $f_1(x) \notin f_1(A)$ so, by convexity, $f_1(A)$ is bounded and $f_1$ is $q$-continuous. We have

$$|f_1(x)| \geq \text{diam } f_1(A) = \sup\{|f_1(a)| : q(a) < 1\} = \|f_1\|$$

where $\|f_1\|$ is the operator seminorm of $f_1$ with respect to $q$. For each $y \in \text{Ker } f_1$

$$\|f_1\| ||q(x-y) \geq |f_1(x-y)| = |f_1(x)| \geq \|f_1\|$$

and we find $x \perp y$ in the sense of $q$. As $q$ is special and $q(x) = 1$ we must have $q = 0$ on $\text{Ker } f_1$ implying that $q$ is of finite type, a contradiction. Thus, we may conclude that there exists an $a_1 \in A$ with $f_1(x-a_1) = 0$.

(ii) Now we repeat the argument in (i) where $E$ is replaced by $\text{Ker } f_1$, $q$ by $q|\text{Ker } f_1$, $x$ by $x-a_1$, $A$ by $A \cap \text{Ker } f_1$ and $f_1$ by $f_2|\text{Ker } f_1$. (Indeed, $q|\text{Ker } f_1$ is special, of infinite type and $q(x-a_1) = 1$). So there exists an $a_2 \in A \cap \text{Ker } f_1$ such that $f_2(x-a_1-a_2) = 0$. Observe that also $f_1(x-a_1-a_2) = 0$. In this spirit we arrive inductively at points $a_1, a_2, \ldots, a_n \in A$ such that $f_1(x-a) = 0$ ($i \in \{1, \ldots, n\}$) where $a := \sum_{i=1}^n a_i \in A$.

**Proof of Theorem 1.1.** If $(E, \tau_1)$ is not strongly polar, choose any nonpolar continuous seminorm $q$ and set $A := \{x \in E : q(x) \leq 1\}$. $A$ is $\tau_1$-closed but, as $q$ is not polar and $A$ is edged, $A$ is not $\sigma(E, E^*)$-closed so certainly $A$ is not $\tau_2$-closed. If $(E, \tau_1)$ is strongly polar, let $q$ be as in Lemma 1.4. By Lemma 1.5 the set $A := \{x \in E : q(x) < 1\}$ is not $\sigma(E, E^*)$-closed, so not $\tau_2$-closed.
Part (i) of the next corollary demonstrates the failure of the Krein-Šmulian Theorem for nonspherically complete base fields.

**Corollary 1.6.** Let $E$ be a normed space over $K$ such that $E'$ is infinite dimensional.

(i) There exists an absolutely convex set $A \subset E'$ such that $A \cap B$ is $w'$-closed in $B$ for each bounded set $B \subset E'$ while $A$ is not $w'$-closed.

(ii) There exists an absolutely convex set $A \subset E$ such that $A \cap B$ is $w$-closed in $B$ for each bounded set $B \subset E$ while $A$ is not $w$-closed.

**Proof.** (i) $(E', w')$ is an infinite dimensional Hausdorff space of countable type so its dual (which is $j_E(E)$) is infinite dimensional. Thus, we can choose $x_1, x_2, \ldots$ in $E$ such that $j_E(x_1), j_E(x_2), \ldots$ are linearly independent and $\lim_{n \to \infty} \|x_n\| = 0$. The seminorm $p$ on $E'$ defined by

$$p(f) = \max_n |f(x_n)| = \max_n |j_E(x_n)(f)|$$

is therefore not of finite type. By Theorem 1.1 there exists an absolutely convex set $A \subset E'$ which is $p$-closed but not $w'$-closed. But it is easily seen that, on any bounded set $B \subset E'$, $w'$-convergence implies $p$-convergence. Thus, the $p$-closedness of $A$ implies that $A \cap B$ is $w'$-closed in $B$.

(ii) Similar to the above proof but now with the seminorm $x \mapsto \max_n |f_n(x)|$ ($x \in E$), where $f_1, f_2, \ldots$ is a linearly independent sequence in $E'$ for which $\lim_{n \to \infty} \|f_n\| = 0$. We leave the details to the reader.

§2. SAVE THE KREIN-ŠMULIAN THEOREM! (PART ONE)

To save the Krein-Šmulian Theorem we shall concentrate on *edged* convex sets. As such sets are translates of edged absolutely convex sets no harm is done by considering only the latter. Thus, we arrive at

**Definition 2.1.** A normed space $E$ over $K$ is a Krein-Šmulian space if the following holds. If $A \subset E'$ is absolutely convex and edged and if $A \cap B$ is $w'$-closed in $B$ for each bounded set $B \subset E'$ then $A$ is $w'$-closed.

Observe that, for an absolutely convex $A \subset E'$, the expression `$A \cap B$ is $w'$-closed in $B$ for each bounded set $B \subset E'$' is equivalent to `for each $n \in \mathbb{N}$ the set $A \cap B_{E'}(0, n)$
is $w'$-closed' and, if $A$ is a subspace, to ‘$A \cap B_{E'}$ is $w'$-closed’.

The main result of this section is

**Theorem 2.2.** A strongly polar Banach space is a Krein–Šmulian space.

For the proof we need first a lemma on Banach spaces. Let us call a sequence $X_1 \supset X_2 \supset \ldots$ of closed absolutely convex subsets of a $K$-Banach space $E$ quasi Cauchy if for each $\lambda \in K$, $|\lambda| > 1$ and $N \in \mathbb{N}$

$$X_n \subset \lambda(X_m + B_E(0, \frac{1}{N})) \quad (m, n \geq N)$$

**Lemma 2.3.** Let $X_1 \supset X_2 \supset \ldots$ be a quasi Cauchy sequence in a $K$-Banach space. Set $X := \bigcap_n X_n$. Then, for each $n \in \mathbb{N}$ and $x_n \in X_n$, and each $\lambda \in K$, $|\lambda| > 1$ there is an $x \in \lambda X$ such that $\|x_n - x\| \leq \frac{|\lambda|}{n}$.

Proof. Choose $\lambda_1, \lambda_2, \ldots \in K$ with $|\lambda_i| > 1$ for each $i$, $\prod_i |\lambda_i| = |\lambda|$. We have

$$X_n \subset \lambda_1(X_{n+1} + B_E(0, \frac{1}{n})) \quad \text{whence} \quad X_n \subset \lambda_1X_{n+1} + B_E(0, \frac{|\lambda|}{n})$$

$$X_{n+1} \subset \lambda_2(X_{n+1} + B(0, \frac{1}{n+1})) \quad \text{whence} \quad \lambda_1X_{n+1} \subset \lambda_1\lambda_2X_{n+2} + B_E(0, \frac{|\lambda|}{n+1})$$

etc.

So, given $x_n \in X_n$, we can find a sequence $x_{n+1}, x_{n+2}, \ldots$ where $x_{n+1} \in \lambda_1X_{n+1}$, $x_{n+2} \in \lambda_1\lambda_2X_{n+2}$, $\ldots$ such that for all $k \in \{0, 1, 2, \ldots\}$

$$\|x_{n+k} - x_{n+k+1}\| \leq \frac{|\lambda|}{n+k}.$$ 

By completeness $x := \lim_{k \to \infty} x_{n+k}$ exists. We have $\lambda^{-1}x_{n+1} \in \lambda^{-1}\lambda_1X_{n+1} \subset X_{n+1}$; $\lambda^{-1}x_{n+2} \in \lambda^{-1}\lambda_1\lambda_2X_{n+2} \subset X_{n+2}$, etc., so $\lambda^{-1}x = \lim_{k \to \infty} \lambda^{-1}x_{n+k} \in \bigcap_{i \geq n+1} X_i = X$ and it follows that $x \in \lambda X$. Further, we have

$$\|x_n - x\| \leq \max\{\|x_n - x_{n+1}\|, \|x_{n+1} - x_{n+2}\|, \ldots\} \leq \max\left(\frac{|\lambda|}{n}, \frac{|\lambda|}{n+1}, \ldots\right) \leq \frac{|\lambda|}{n}.$$ 

Proof of Theorem 2.2. Let $A \subset E'$ be absolutely convex, edged and assume that $A \cap B_{E'}(0, n)$ is $w'$-closed for each $n \in \mathbb{N}$. Then $(A \cap B_{E'}(0, n)$ is also edged) $A \cap B_{E'}(0, n)$ is a polar set. Setting

$$X_n := (A \cap B_{E'}(0, n))_0 \quad (n \in \mathbb{N})$$

$$X := \bigcap_n X_n$$
one verifies immediately (i), (ii), (iii), (iv) below.

(i) Each $X_n$ is a polar subset of $E$.

(ii) $X^0 = A \cap B_E'(0, n)$ for each $n \in \mathbb{N}$.

(iii) $X_1 \supset X_2 \supset \ldots$.

(iv) $X = A_0$.

(v) For each $N \in \mathbb{N}$ and $m, n \geq N$

$$X_n \subset (X_m + B_E(0, \frac{1}{N}))^0.$$

(Proof: $(X_m + B_E(0, \frac{1}{N}))^0 = X_m^0 \cap B_E(0, \frac{1}{N})^0 = A \cap B_E'(0, m) \cap B_E'(0, N) = A \cap B_E'(0, N)$, so $X_n \subset X_N = (A \cap B_E'(0, N))^0 = (X_m + B_E(0, \frac{1}{N}))^0$.)

(vi) $X_1, X_2, \ldots$ is quasi Cauchy. (Proof. Let $\lambda \in K, |\lambda| > 1, N \in \mathbb{N}, m, n \geq N$.

The set $X_m + B_E(0, \frac{1}{N})$ is norm open hence norm closed. So $(X_m + B_E(0, \frac{1}{N}))^e$ is norm closed and edged, hence polar (as $E$ is strongly polar). It follows via (v), that $X_n \subset (X_m + B_E(0, \frac{1}{N}))^0 \subset \lambda(X_m + B_E(0, \frac{1}{N}))$.)

(vii) $X^0 \subset A$. (Proof. Let $f \in X^0, \lambda \in K, |\lambda| > 1$. It suffices to prove that $f \in \lambda A$.

Let $n \in \mathbb{N}$ be such that $\|f\| \leq n$. Choose any $x \in X_n$. By Lemma 2.3 there is a $y \in \lambda X$ with $\|x - y\| \leq \frac{|\lambda|}{n}$. We have

$$|f(x)| \leq |f(x-y)| \vee |f(y)|$$

$$\leq \|f\| \|x-y\| \vee |\lambda| \leq n \cdot \frac{|\lambda|}{n} \vee |\lambda| = |\lambda|$$

and we see that $|\lambda^{-1} f| \leq 1$ on $X_n$, so $\lambda^{-1} f \in X^0 = A \cap B_E'(0, n) \subset A$ i.e. $f \in \lambda A$.)

Now (iv) combined with (vii) yields $A = X^0$ is $w'$-closed.

**Corollary 2.4.** A subspace of the dual of a strongly polar Banach space is $w'$-closed as soon as its intersection with the closed unit ball is $w'$-closed.

**Corollary 2.5.** An edged absolutely convex subset $A$ of $\ell^\infty$ is $\sigma(\ell^\infty, c_0)$-closed as soon as $A \cap B_{\ell^\infty}(0, n)$ is $\sigma(\ell^\infty, c_0)$-closed for each $n \in \mathbb{N}$.

Proof. $c_0$ is a (reflexive) strongly polar space.

We also have:

**Theorem 2.6.** If $E$ is a Krein-Šmulian space and $D \subset E$ is a closed subspace then $E/D$ is a Krein-Šmulian space.
Proof. Let $i : (E/D)' \to E'$ be the adjoint of the quotient map $E \to E/D$. It is easily seen that $i$ is an isometry, that $\text{Im } i$ is $w'$-closed in $E'$ and that $i$ is a $w'$ to $w'$ homeomorphism $(E/D)' \to \text{Im } i$.

Now let $A$ be an edged absolutely convex subset of $(E/D)'$ such that $A \cap B$ is $w'$-closed in $B$ for each bounded set $B$ in $(E/D)'$. Then $i(A)$ is edged. If $X \subset E'$ is bounded then $i(A) \cap X$ is $w'$-closed in $X$. (Proof. Let $j \mapsto a_j$ be a net in $A$ such that $i(a_j) \in X$ for all $j$ and let $w' - \lim_j i(a_j) = b \in X$. As $\text{Im } i$ is $w'$-closed $b = i(a)$ for some $a \in i^{-1}(X)$. Then $w' - \lim_j a_j = a$. Now $a_j \in A \cap i^{-1}(X)$ for all $j$, $a \in i^{-1}(X)$ and $i^{-1}(X)$ is bounded, so by assumption on $A$ we have $a \in A$, so $b \in i(A) \cap X$.) Since $E$ is a Krein-Šmulian space, $i(A)$ is $w'$-closed in $E'$ so that $A = i^{-1}(i(A))$ is $w'$-closed in $(E/D)'$.

**Theorem 2.7.** If $E$ is a Krein-Šmulian space and if $D \subset E$ is a weakly closed subspace having the WEP then $D$ is a Krein-Šmulian space.

Proof. Let $\pi : E' \to D'$ be the adjoint of the inclusion map $D \hookrightarrow E$. Then $\pi$ is surjective and $w'$ to $w'$ continuous. If $A$ is an edged absolutely convex set in $D'$ and $\pi^{-1}(A)$ is $w'$-closed then $A$ is $w'$-closed. (Proof. Let $g \in D'$, $g \notin A$. There is an $f \in E'$ with $\pi(f) = g$. Then $f \notin \pi^{-1}(A)$. Now $\pi^{-1}(A)$ is $w'$-closed and edged so there exists an $x \in E$ such that $f(x) = 1$ and $|h(x)| < 1$ for all $h \in \pi^{-1}(A)$. In particular, $|h(x)| < 1$ for all $h \in \text{Ker } \pi = D^0$ i.e. $h(x) = 0$ for all $h \in D^0$ so $x \in D^0 = D$. Then $g(x) = f(x) = 1$ and $|h(x)| < 1$ for all $h \in A$.)

Now let $A$ be an absolutely convex edged subset of $D'$ such that $A \cap B$ is $\sigma(D', D)$-closed in $B$ for each bounded set $B \subset D'$. Then for such $B$, $\pi^{-1}(A) \cap \pi^{-1}(B)$ is $\sigma(E', E)$-closed in $\pi^{-1}(B)$. If $X \subset E'$ is bounded then $\pi(X)$ is bounded and $X \subset \pi^{-1}(\pi(X))$ so it follows that $\pi^{-1}(A) \cap X$ is $w'$-closed in $X$ for each bounded set $X \subset E'$. Since $E$ is Krein-Šmulian we have that $\pi^{-1}(A)$ is $w'$-closed, so by the remark above, $A$ is $w'$-closed.

**Remark.** Not every Krein-Šmulian polar space is strongly polar; $\ell^\infty$ is an easy example. In §3 we will see that, if $I$ is large enough, $c_0(I)$ is not Krein-Šmulian. This leads to the

**Problem.** Characterize the class of Krein-Šmulian spaces.

A concrete help would be the answer to the following two questions.
- Is $c_0 \times \ell^\infty$ a Krein-Šmulian space? (More generally, if $E_1$ and $E_2$ are Krein-Šmulian spaces then does it follow that $E_1 \times E_2$ is Krein-Šmulian?)
- Is the subspace of $D$ of $\ell^\infty$ constructed in [2], Ex. 4.K Krein-Šmulian?

§3. SAVE THE KREIN-ŠMULIAN THEOREM! (PART TWO)

In this section we shall prove the following version of the Krein-Šmulian Theorem. Observe that ($\alpha$) holds for any polar $K$-Banach space.

**Theorem 3.1.** For a normed space $E$ over $K$ the following are equivalent.

($\alpha$) $j_E(E)$ is norm closed in $E''$.

($\beta$) If $H \subset E'$ is a subspace of finite codimension and if $H \cap B_{E'}$ is $w'$-closed then so is $H$.

For a normed space $E$ over $K$ the $bw'$-topology (the 'bounded-weak-star topology') is by definition the strongest locally convex topology on $E'$ that coincides with $w'$ on bounded subsets of $E'$.

**Proposition 3.2.** Let $E$ be a normed space over $K$.

(i) $bw'$ is stronger than $w'$ but weaker than the norm topology on $E'$.

(ii) $(E',bw')$ is of countable type.

(iii) A seminorm $p$ on $E'$ is $bw'$-continuous if and only if $p|B_{E'}$ is $w'$-continuous.

(iv) For any locally convex space $(X,\tau)$ and any linear map $T : E' \to X$ we have that $T$ is $bw'$ to $\tau$ continuous if and only if $T|B_{E'}$ is $w'$ to $\tau$ continuous.

Proof. $E' = [B_{E'}]$ and $B_{E'}$ is a $w'$-compactoid, hence a $bw'$-compactoid. This implies (ii). The other proofs are straightforward.

We know that $(E',w')' = j_E(E)$ ([5])). We now prove

**Proposition 3.3.** For a normed space $E$ over $K$ the dual of $(E',bw')$ is the norm closure of $j_E(E)$ in $E''$.

Proof. Every $\theta \in \overline{j_E(E)}$ is, on $B_{E'}$, the uniform limit of a sequence in $j_E(E)$ so $\theta|B_{E'}$ is $w'$-continuous and $\theta$ is $bw'$-continuous (Proposition 3.2 (iv)). Thus $\overline{j_E(E)} \subset (E',bw')'$. Conversely, let $\theta \in (E',bw')'$. Then (Proposition 3.2 (i)) $\theta \in E''$. Let $\varepsilon > 0$; we shall
find an $x \in E$ such that $\|\theta - j_E(x)\| < \epsilon$. Let $\alpha \in K$, $0 < |\alpha| < \epsilon$. The $w'$-continuity of $\theta|B_{E'}$ yields a finite set $F \subset E$ such that $f \in F^0 \cap B_{E'}$ implies $|\theta(f)| \leq |\alpha|$, in other words

$$f \in j_E(F)_0 \cap (B_{E''})_0 \implies |(\alpha^{-1}\theta)(f)| \leq 1.$$ 

So we see that $\alpha^{-1}\theta \in (j_E(F)_0 \cap (B_{E''})_0)^0 = (A + B_{E''})^0$, where $A = j_E(\text{co } F)$. Now $B_{E''}$ is $w'$-closed and $A$ is finite dimensional so by [3], 1.4, $(A + B_{E''})^0 = (A + B_{E''})^e$.

For any $\lambda \in K$ such that $|\lambda| > 1$ and $|\lambda\alpha| < \epsilon$ we have $\alpha^{-1}\theta \in \lambda A + \lambda B_{E''}$, hence $\theta \in j_E(E) + \alpha \lambda B_{E''}$ and there is an $x \in E$ with $\theta - j_E(x) \in \alpha \lambda B_{E''}$ i.e. $\|\theta - j_E(x)\| < \epsilon$.

**Corollary 3.4.** Let $j_E(E)$ be closed in $E''$, let $A \subset E'$ be absolutely convex and edged. Then $A$ is $w'$-closed if and only if $A$ is $bw'$-closed.

**Proof.** Let $A$ be $bw'$ closed. As $A$ is also edged and $(E',bw')$ is strongly polar (Proposition 3.2 (ii)), $A$ is a polar set i.e. $A = S_0$ for some $S \subset (E',bw')'$. But by Proposition 3.3 $S \subset (E',w')'$ so that $A$ is $w'$-closed.

Further, we need the following general lemma.

**Lemma 3.5.** Let $A$ be a closed absolutely convex subset of a Hausdorff locally convex space over $K$; let $D$ be a finite dimensional subspace such that $A \cap D = \{0\}$. Then $A + D$ is closed and the addition map is a homeomorphism $A \times D \rightarrow A + D$.

**Proof.** (i) If addition is homeomorphic then $A + D$ is closed. In fact, let $i \rightarrow a_i + d_i$ be a net in $A + D$ (where $a_i \in A$, $d_i \in D$ for each $i$), converging to some $z$. Then $(i,j) \rightarrow a_i - a_j + d_i - d_j$ converges to $0$. By homeomorphism, $d_i - d_j \rightarrow 0$, by completeness of $D$, $d_i \rightarrow d$ for some $d \in D$. Then $a_i \rightarrow z - d$ and, by closedness of $A$, $z - d \in A$. We see that $z \in A + D$.

(ii) Assume $n := \dim D = 1$, say $D = Kx$ for some nonzero $x$. Let $i \rightarrow a_i + \lambda_i x$ $(a_i \in A, \lambda_i \in K)$ be a net in $A + D$ converging to $0$. If not $\lambda_i \rightarrow 0$ we may assume $|\lambda_i| \geq |\alpha| > 0$ for all $i$ and some $\alpha \in K$. Then $\alpha \lambda_i^{-1}(a_i + \lambda_i x) \rightarrow 0$ so $\alpha x = - \lim_i \alpha \lambda_i^{-1} a_i \in \overline{A} = A$ conflicting $Kx \cap A = \{0\}$. Thus, addition is homeomorphic and via (i) the lemma is proved if $n = 1$.

(iii) The proof of the induction step $n-1 \rightarrow n$ is now standard and left to the reader.

**Proof of Theorem 3.1.**

(i) Suppose $(\alpha)$, and let $H \subset E'$ be a subspace of finite codimension such that $H \cap B_{E'}$ is $w'$-closed. Then $H \cap B_{E'}$ is norm closed, hence so is $H$. For some $t \in (0,1)$ $H$ has a
t-orthogonal complement $D$. Let $P : E' \rightarrow D$ be the obvious projection. For $\lambda \in K$, $|\lambda| \geq t^{-1}$ we have

$$B_{E'} \subset \lambda(H \cap B_{E'}) + \lambda(D \cap B_{E'}) \subset \lambda(H \cap B_{E'}) + D.$$ 

Let $i \mapsto f_i$ be a net in $B_{E'}$, $w' - \lim_i f_i = 0$. Then, by Lemma 3.5, $\lim_i Pf_i = 0$. We see that $P|B_{E'}$ is continuous, so (Proposition 3.2 (iv)) $P$ is $bw'$ to norm continuous and $\text{Ker } P = H$ is $bw'$-closed, hence $w'$-closed by Corollary 3.4, and $(\beta)$ is proved.

(ii) Suppose $(\alpha)$ is not true. Choose $\theta \in j_{E'}(E) \setminus j_E(E)$. Then $\theta$ is not $w'$-continuous so $H := \text{Ker } \theta$ is not $w'$-closed. But $\theta$ is $bw'$-continuous by Proposition 3.3. so $H \cap B_{E'}$ is $w'$-closed.

The results of this section yield the existence of polar non-Krein-Šmulian spaces (see §2).

**Corollary 3.6.** If $m$ is a cardinality $\geq \#K$ then $c_0(m)$ is not a Krein-Šmulian space.

**Proof.** In [2], Exercise 4.N a Banach space $E$ is constructed such that $j_E(E)$ is a proper dense subset of $E''$. From this construction it is easily seen that $\#E = \#\ell^\infty \leq \#K^N = \#K$. Now let $I$ be a set with cardinality $\geq \#K$ and let $\{e_i : i \in I\}$ be the natural orthonormal base of $c_0(I)$. There is a surjection $\{e_i : i \in I\} \rightarrow B_E$, it extends to a quotient map $c_0(I) \rightarrow E$. Now $E$ is not a Krein-Šmulian space by Theorem 3.1, neither is $c_0(I)$ by Theorem 2.6. (It is not hard to see by looking at the proof of Theorem 2.6 that one can even find a subspace $D \subset \ell^\infty(I)$ that is not $w'$-closed while $D \cap B_{\ell^\infty(I)}$ is.)

**Corollary 3.7.** If $m$ is a nonmeasurable cardinality $\geq \#K^K$ then $\ell^\infty(m)$ is not a Krein-Šmulian space.

**Proof.** In the spirit of the previous proof one constructs a quotient map $\pi : c_0(m) \rightarrow \ell^\infty(n)$ where $n = \#K$. By reflexivity ([2], Theorem 4.21) the adjoint $\pi' : c_0(n) \rightarrow \ell^\infty(m)$ is an isometry and $\pi'(c_0(n))$ has the WEP in $\ell^\infty(m)$. From [4], Lemma 2.2 we obtain that $\pi'(c_0(n))$ is also weakly closed in $\ell^\infty(m)$. By the previous corollary $c_0(n)$ is not a Krein-Šmulian space, neither is $\ell^\infty(m)$ by Theorem 2.7.

**Problem.** Determine the smallest cardinality $m$ for which $c_0(m)$ ($\ell^\infty(m)$ if $\#K$ is nonmeasurable) is not a Krein-Šmulian space.
As a further application we now prove a nonarchimedean version of a classical reflexivity criterion (Theorem 3.8). First we 'dualize' the notion of a polar seminorm as follows. A seminorm \( p \) on the dual \( E' \) of a locally \( K \)-convex space \( E \) is a **dual seminorm** if there exists an \( X \subset E \) such that \( p(f) = \sup \{|f(x)| : x \in X\} \) for all \( f \in E' \). An easy exercise shows that \( p \) is dual if and only if \( \{f \in E' : p(f) \leq 1\} \) is \( \sigma(E', E) \)-closed. Dual seminorms are automatically polar.

**Theorem 3.8.** Let \( E \) be a pseudoreflexive \( K \)-Banach space. Then \( E \) is reflexive if and only if each polar norm on \( E' \) inducing the topology is dual.

**Proof.** Let \( E \) be reflexive and let \( \nu \) be a polar norm on \( E' \) inducing the topology. Then \( \{f \in E' : \nu(f) \leq 1\} \) is weakly closed so, by reflexivity, \( w' \)-closed. Hence \( \nu \) is dual by the above remark. Conversely, suppose each polar norm on \( E' \) inducing the topology is dual. To prove reflexivity of \( E \) it suffices (by pseudoreflexivity) to show that any \( \theta \in E'' \) is \( w' \)-continuous. For each \( n \in \mathbb{N} \) the norm \( f \mapsto n|\theta(f)| \vee \|f\| \) (where \( \| \| \) is the 'natural' norm on \( E' \)) is easily seen to be polar and it is obviously equivalent to \( \| \| \). By assumption its closed unit ball

\[
B_n := \{f \in E' : |\theta(f)| \leq \frac{1}{n}, \|f\| \leq 1\}
\]

is \( w' \)-closed. Hence so is \( \bigcap_n B_n \) which is \( \text{Ker } \theta \cap B_{E'} \). By Theorem 3.1 \( \text{Ker } \theta \) is \( w' \)-closed implying that \( \theta \) is \( w' \)-continuous.

**Remark.** One also may consider a 'predual form' of the Krein-Šmulian property (compare Definition 1.1, see also Corollary 1.6 (ii)) as follows. A normed space \( E \) is \( \mathcal{P}K\mathcal{S} \) space if for each absolutely convex edged \( A \subset E \):

\[
A \cap B \text{ is } w\text{-closed in } B \text{ for each bounded } B \subset E \quad \implies \quad A \text{ is } w\text{-closed.}
\]

(Obviously this notion is of no use in classical Banach space theory.) The reader will not have difficulties in proving results about \( \mathcal{P}K\mathcal{S} \) spaces similar to the one of \( \mathcal{K}\mathcal{S} \)-spaces of this paper. More precisely, we have

(i) A strongly polar normed space is \( \mathcal{P}K\mathcal{S} \).

(ii) Let \( E \) be a normed \( \mathcal{P}K\mathcal{S} \) space, let \( D \) be a closed subspace. Then \( E/D \) is \( \mathcal{P}K\mathcal{S} \).

\[\text{If } D, \text{ in addition, is weakly closed and has the WEP then } D \text{ has } \mathcal{P}K\mathcal{S}.\]

(iii) Let \( E \) be a normed space. If \( H \subset E \) is a subspace with finite codimension and \( H \cap B_E \) is weakly closed then \( H \) is weakly closed.
REFERENCES


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