THE $p$-ADIC KREIN-ŠMULIAN THEOREM

W.H. Schikhof

ABSTRACT. The natural non-archimedean version of the Krein-Šmulian Theorem holds essentially only when the base field is spherically complete (Corollary 1.6). For Banach spaces over nonspherically complete scalar fields two restricted versions of the Krein-Šmulian Theorem (Theorems 2.2 and 3.1) are proved.

INTRODUCTION. (For unexplained terms see below*) Consider the following statement (*).

\[
(*) \quad \text{Let } E \text{ be a } K\text{-Banach space and let } A \subseteq E' \text{ be convex. If } A \cap B \text{ is } w'\text{-closed in } B \text{ for each bounded set } B \subseteq E' \text{ then } A \text{ is } w'\text{-closed.}
\]

If $K = \mathbb{R}$ or $\mathbb{C}$ then (*) is known as the Krein-Šmulian Theorem. By modifying classical techniques (in particular by using c-compactness arguments) (*) can also be proved if $K$ is a spherically complete non-archimedean valued field ([4], Theorem 5.1). Now see the Abstract.

PRELIMINARIES. (For terms still unexplained see [2].) Throughout $K$ is a nonarchimedean nontrivially valued field that is complete under the metric induced by the valuation $|\cdot|$. We always ASSUME THAT $K$ IS NOT SPHERICALLY COMPLETE, so $|K| := \{|\lambda| : \lambda \in K\}$ is dense. If $X \subseteq K$ is bounded, nonempty, we set $\text{diam } X = \sup\{|x - y| : x, y \in X\}$.

Let $E$ be a $K$-vector space. A nonempty subset $A$ of $E$ is absolutely convex if $x, y \in A, \lambda, \mu \in K, |\lambda| \leq 1, |\mu| \leq 1$ implies $\lambda x + \mu y \in A$. For such $A$ we set $A^* := \cap\{\lambda A : \lambda \in K, |\lambda| > 1\}$. $A$ is edged if $A = A^*$. The smallest absolutely convex set containing $X \subseteq E$ is denoted $\text{co } X$. A nonempty set in $E$ is convex (edged convex)
if it is an additive coset of an absolutely convex (edged absolutely convex) set. By
definition, the empty set is convex. The algebraic dual of $E$ is the vector space $E^*$
consisting of all linear functions $E \to K$. The weakest topology on $E$ for which all
$f \in E^*$ are continuous is denoted $\sigma(E, E^*)$.

A seminorm on $E$ is a map $p : E \to [0, \infty)$ such that $p(x) \geq 0$, $p(\lambda x) = |\lambda| p(x)$,
$p(x+y) \leq \max(p(x), p(y))$ for all $x, y \in E$, $\lambda \in K$. We shall use expressions such as ‘p-
A seminorm $p$ is of finite type if Ker $p$ has finite codimension, of countable type if $E/$Ker $p$ with the norm induced by $p$ is of countable type. A seminorm $p$ is polar if
$p = \sup\{|f| : f \in E^*, |f| \leq p\}$. A seminorm $p$ is a norm if $p(x) = 0$ implies $x = 0$.
Norms are usually denoted $\| \|$ rather than $p$.

Let $(E, \| \|)$ be a normed space over $K$. Let $a \in E$, $r > 0$. We write $B_E(a, r) := \{x \in E : \|x-a\| \leq r\}$ and $B_E := B_E(0, 1)$. The dual space $E'$ is the Banach space
consisting of all continuous linear functions $E \to K$, normed by $f \mapsto \|f\| := \sup_{B_E} |f|$. The natural map $j_E : E \to E''$ is continuous. $E$ is pseudoreflexive if $j_E$ is an isometry
(which is equivalent to polarity of the norm on $E$). A linear map $T$ from a $K$-Banach
space $E$ to a $K$-Banach space $F$ is a quotient map if $T$ maps $\{x \in E : \|x\| < 1\}$ onto
$\{x \in F, \|x\| < 1\}$.

Let $(E, \tau)$ be a locally convex space over $K$. It is called of finite (countable) type
if every continuous seminorm is of finite (countable) type. $(E, \tau)$ is strongly polar if
each continuous seminorm is polar, polar if there exists a base of polar continuous
seminorms. Let $E' = (E, \tau)'$ be the space of all continuous linear functions $E \to K$.
The weak topology $w = \sigma(E, E')$ is the weakest topology on $E$ such that all $f \in E'$ are
continuous. Similarly, the weak-star topology $w' = \sigma(E', E)$ is the weakest topology on $E'$ such that for each $x \in E$ the evaluation $f \mapsto f(x)$ ($f \in E'$) is continuous.
It is well known (see [5]) that the natural map $E \to (E', \sigma(E', E))'$ is surjective.
Let $X \subset E$, $Y \subset E'$. We set $X^0 := \{f \in E' : |f(x)| \leq 1$ for all $x \in X\}$ and
$Y_0 := \{x \in E : |f(x)| \leq 1$ for all $f \in Y\}$. $X$ is a polar set if $X^0_0 = X$. For a ball $B_E(0, r)$ in a normed space $E$ we have $B_E(0, r)^0 = B_{E'}(0, 1/r)$. If $E$ is pseudoreflexive,
$B_{E'}(0, r)_0 = B_E(0, 1/r)$. The closure of a set $X \subset E$ is $\overline{X}$. Instead of $\co X$ we write
$\overline{\co} X$. Let $E, F$ be locally convex spaces over $K$. The adjoint of a continuous linear
map $T : E \to F$ is the map $T' : F' \to E'$ defined by $f \mapsto f \circ T$. Following [1] we
say that a subspace $D$ of $E$ has the Weak Extension Property (WEP) if the adjoint
$E' \rightarrow D'$ of the inclusion map $D \hookrightarrow E$ is surjective.

§1. FAILURE OF THE KREIN-ŠMULIAN THEOREM

The key theorem of this section is the following. Recall that $K$ is not spherically complete.

**Theorem 1.1.** Let $\tau_1, \tau_2$ be locally convex topologies on a $K$-vector space $E$ such that $\tau_2$ is of finite type while $\tau_1$ is not. Then there exists a $\tau_1$-closed absolutely convex set in $E$ that is not $\tau_2$-closed.

We signal the following corollary which is in sharp contrast to the theory over spherically complete base fields.

**Corollary 1.2.** Let $E$ be a locally $K$-convex space whose topology is not the weak topology. Then there exists a closed absolutely convex set in $E$ that is not weakly closed.

The proof of Theorem 1.1 runs in a few steps. Let us say that a seminorm $q$ on a $K$-vector space is *special* if $q(x) \in |K|$ for each $x \in E$ and if for all $x, y \in E$

$$x \perp y \text{ in the sense of } q \implies q(x) = 0 \text{ or } q(y) = 0$$

**Lemma 1.3.** On a normed space of countable type over $K$ there exists an equivalent special norm.

**Proof.** Let $(\hat{K}, |\cdot|)$ be the spherical completion of $(K, |\cdot|)$ in the sense of [2], Theorem 4.49. Then $|\cdot|$, considered as a norm on the $K$-vector space $\hat{K}$ is special. (Indeed, we have $|\hat{K}| = |K|$. If $x, y \in \hat{K}$, $x \perp y$, $y \neq 0$ then $xy^{-1} \perp 1$ so $xy^{-1} \perp K$. But $\hat{K}$ is an immediate extension of $K$ so $xy^{-1} = 0$ i.e. $x = 0$.) As $\hat{K}$ is infinite dimensional over $K$ we can, for a given normed space $E$ of countable type over $K$, make a $K$-linear homeomorphism $T$ of $E$ into $\hat{K}$. Then $x \mapsto |Tx|$ is the required norm.

**Lemma 1.4.** Let $E$ be a strongly polar locally convex space over $K$. If $E$ is not of finite type then there exists a continuous special seminorm $q$ on $E$, $q$ not of finite type.

**Proof.** There is a continuous seminorm of infinite type $p$ on $E$. The $p$-continuous linear functions form an infinite dimensional space so we can find linearly independent
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\[ f_1, f_2, \ldots \in E' \text{ such that } n|f_n| \leq p \text{ for each } n \in \mathbb{N}. \]  
The formula \( \tilde{p}(x) = \max_n |f_n(x)| \) defines a continuous seminorm \( \tilde{p} \) on \( E \), of infinite countable type. Now Lemma 1.3 (applied to \( E/\ker \tilde{p} \)) leads to a special seminorm \( q \) equivalent to \( \tilde{p} \).

**Remark.** The conclusion of Lemma 1.4 holds for any polar space \( E \) that is not of finite type.

**Lemma 1.5.** Let \( q \) be a special seminorm on a \( K \)-vector space \( E \). If \( q \) is not of finite type then \( \{ x \in E : q(x) < 1 \} \) is not \( \sigma(E, E^*) \)-closed.

**Proof.** Let \( x \in E, q(x) = 1 \) (such \( x \) exist!). We shall prove that \( x \) is in the \( \sigma(E, E^*) \)-closure of \( A := \{ x \in E : q(x) < 1 \} \) by producing, for given \( f_1, \ldots, f_n \in E^* \), a point \( a \in A \) such that \( f_i(x-a) = 0 \) for \( i \in \{1, \ldots, n\} \).

(i) Suppose \( f_1(x-a) \neq 0 \) for all \( a \in A \). Then \( f_1(x) \notin f_1(A) \) so, by convexity, \( f_1(A) \) is bounded and \( f_1 \) is \( q \)-continuous. We have

\[ |f_1(x)| \geq \text{diam } f_1(A) = \sup \{|f_1(a)| : q(a) < 1\} = \|f_1\| \]

where \( \|f_1\| \) is the operator seminorm of \( f_1 \) with respect to \( q \). For each \( y \in \ker f_1 \)

\[ \|f_1\|q(x-y) \geq |f_1(x-y)| = |f_1(x)| \geq \|f_1\| \]

and we find \( x \perp y \) in the sense of \( q \). As \( q \) is special and \( q(x) = 1 \) we must have \( q = 0 \) on \( \ker f_1 \) implying that \( q \) is of finite type, a contradiction. Thus, we may conclude that there exists an \( a_1 \in A \) with \( f_1(x-a_1) = 0 \).

(ii) Now we repeat the argument in (i) where \( E \) is replaced by \( \ker f_1, q \) by \( q|\ker f_1, x \) by \( x-a_1, A \) by \( A \cap \ker f_1 \) and \( f_1 \) by \( f_2|\ker f_1 \). (Indeed, \( q|\ker f_1 \) is special, of infinite type and \( q(x-a_1) = 1 \).) So there exists an \( a_2 \in A \cap \ker f_1 \) such that \( f_2(x-a_1-a_2) = 0 \).

Observe that also \( f_1(x-a_1-a_2) = 0 \). In this spirit we arrive inductively at points \( a_1, a_2, \ldots, a_n \in A \) such that \( f_i(x-a) = 0 \) \( (i \in \{1, \ldots, n\}) \) where \( a := \sum_{i=1}^{n} a_i \in A \).

**Proof of Theorem 1.1.** If \( (E, \tau_1) \) is not strongly polar, choose any nonpolar continuous seminorm \( q \) and set \( A := \{ x \in E : q(x) \leq 1 \} \). \( A \) is \( \tau_1 \)-closed but, as \( q \) is not polar and \( A \) is edged, \( A \) is not \( \sigma(E, E^*) \)-closed so certainly \( A \) is not \( \tau_2 \)-closed. If \( (E, \tau_1) \) is strongly polar, let \( q \) be as in Lemma 1.4. By Lemma 1.5 the set \( A := \{ x \in E : q(x) < 1 \} \) is not \( \sigma(E, E^*) \)-closed, so not \( \tau_2 \)-closed.
Part (i) of the next corollary demonstrates the failure of the Krein-Šmulian Theorem for nonspherically complete base fields.

**COROLLARY 1.6.** Let $E$ be a normed space over $K$ such that $E'$ is infinite dimensional.

(i) There exists an absolutely convex set $A \subset E'$ such that $A \cap B$ is $w'$-closed in $B$ for each bounded set $B \subset E'$ while $A$ is not $w'$-closed.

(ii) There exists an absolutely convex set $A \subset E$ such that $A \cap B$ is $w$-closed in $B$ for each bounded set $B \subset E$ while $A$ is not $w$-closed.

**Proof.** (i) $(E', w')$ is an infinite dimensional Hausdorff space of countable type so its dual (which is $j_E(E)$) is infinite dimensional. Thus, we can choose $x_1, x_2, \ldots$ in $E$ such that $j_E(x_1), j_E(x_2), \ldots$ are linearly independent and $\lim_{n \to \infty} \|x_n\| = 0$. The seminorm $p$ on $E'$ defined by

$$p(f) = \max_{n} |f(x_n)| = \max_{n} |j_E(x_n)(f)|$$

is therefore not of finite type. By Theorem 1.1 there exists an absolutely convex set $A \subset E'$ which is $p$-closed but not $w'$-closed. But it is easily seen that, on any bounded set $B \subset E'$, $w'$-convergence implies $p$-convergence. Thus, the $p$-closedness of $A$ implies that $A \cap B$ is $w'$-closed in $B$.

(ii) Similar to the above proof but now with the seminorm $x \mapsto \max_{n} |f_n(x)|$ ($x \in E$), where $f_1, f_2, \ldots$ is a linearly independent sequence in $E'$ for which $\lim_{n \to \infty} \|f_n\| = 0$. We leave the details to the reader.

§2. SAVE THE KREIN-ŠMULIAN THEOREM! (PART ONE)

To save the Krein-Šmulian Theorem we shall concentrate on *edged* convex sets. As such sets are translates of edged absolutely convex sets no harm is done by considering only the latter. Thus, we arrive at

**DEFINITION 2.1.** A normed space $E$ over $K$ is a **Krein-Šmulian space** if the following holds. If $A \subset E'$ is absolutely convex and edged and if $A \cap B$ is $w'$-closed in $B$ for each bounded set $B \subset E'$ then $A$ is $w'$-closed.

Observe that, for an absolutely convex $A \subset E'$, the expression '$A \cap B$ is $w'$-closed in $B$ for each bounded set $B \subset E'$' is equivalent to 'for each $n \in \mathbb{N}$ the set $A \cap B_{E'}(0,n)$
is $w'$-closed' and, if $A$ is a subspace, to 'A ∩ $B_{E'}$ is $w'$-closed'.

The main result of this section is

**Theorem 2.2.** A strongly polar Banach space is a Krein–Smulian space.

For the proof we need first a lemma on Banach spaces. Let us call a sequence $X_1 ⊃ X_2 ⊃ \ldots$ of closed absolutely convex subsets of a $K$-Banach space $E$ quasi Cauchy if for each $\lambda \in K$, $|\lambda| > 1$ and $N \in \mathbb{N}$

$$X_n \subset \lambda(X_m + B_E(0, \frac{1}{N})) \quad (m, n \geq N)$$

**Lemma 2.3.** Let $X_1 \supset X_2 \supset \ldots$ be a quasi Cauchy sequence in a $K$-Banach space. Set $X := \bigcap X_n$. Then, for each $n \in \mathbb{N}$ and $x_n \in X_n$, and each $\lambda \in K$, $|\lambda| > 1$ there is an $x \in \lambda X$ such that $\|x_n - x\| \leq |\lambda|/n$.

Proof. Choose $\lambda_1, \lambda_2, \ldots \in K$ with $|\lambda_i| > 1$ for each $i$, $\prod_{i} |\lambda_i| = |\lambda|$. We have

$$X_n \subset \lambda_1(X_{n+1} + B_E(0, \frac{1}{n}))$$

whence

$$X_n \subset \lambda_1 X_{n+1} + B_E(0, \frac{|\lambda|}{n})$$

$$X_{n+1} \subset \lambda_2(X_{n+1} + B(0, \frac{1}{n+1}))$$

whence

$$\lambda_1 X_{n+1} \subset \lambda_1 \lambda_2 X_{n+2} + B_E(0, \frac{|\lambda|}{n+1})$$

etc.

So, given $x_n \in X_n$, we can find a sequence $x_{n+1}, x_{n+2}, \ldots$ where $x_{n+1} \in \lambda_1 X_{n+1}, x_{n+2} \in \lambda_1 \lambda_2 X_{n+2}, \ldots$ such that for all $k \in \{0, 1, 2, \ldots\}$

$$\|x_{n+k} - x_{n+k+1}\| \leq \frac{|\lambda|}{n+k}.$$  

By completeness $x := \lim_{k \to \infty} x_{n+k}$ exists. We have $\lambda^{-1} x_{n+1} \in \lambda^{-1} \lambda_1 X_{n+1} \subset X_{n+1}$; $\lambda^{-1} x_{n+2} \in \lambda^{-1} \lambda_1 \lambda_2 X_{n+2} \subset X_{n+2}$, etc., so $\lambda^{-1} x = \lim_{k \to \infty} \lambda^{-1} x_{n+k} \in \bigcap_{i \geq n+1} X_i = X$ and it follows that $x \in \lambda X$. Further, we have

$$\|x_n - x\| \leq \max(\|x_n - x_{n+1}\|, \|x_{n+1} - x_{n+2}\|, \ldots) \leq \max(\frac{|\lambda|}{n}, \frac{|\lambda|}{n+1}, \ldots) \leq \frac{|\lambda|}{n}.$$  

**Proof of Theorem 2.2.** Let $A \subset E'$ be absolutely convex, edged and assume that $A \cap B_{E'}(0, n)$ is $w'$-closed for each $n \in \mathbb{N}$. Then $(A \cap B_{E'}(0, n))$ is also edged) $A \cap B_{E'}(0, n)$ is a polar set. Setting

$$X_n := (A \cap B_{E'}(0, n))_0 \quad (n \in \mathbb{N})$$

$$X := \bigcap_n X_n$$
one verifies immediately (i), (ii), (iii), (iv) below.

(i) Each \( X_n \) is a polar subset of \( E \).

(ii) \( X_n^0 = A \cap B_{E'}(0, n) \) for each \( n \in \mathbb{N} \).

(iii) \( X_1 \supset X_2 \supset \ldots \).

(iv) \( X = A_0 \).

(v) For each \( N \in \mathbb{N} \) and \( m, n \geq N \)

\[ X_n \subset (X_m + B_E(0, \frac{1}{N}))^0. \]

(Proof: \( (X_m + B_E(0, \frac{1}{N}))^0 = X_m^0 \cap B_E(0, \frac{1}{N})^0 = A \cap B_{E'}(0, m) \cap B_{E'}(0, N) = A \cap B_{E'}(0, N) \), so \( X_n \subset X_N = (A \cap B_{E'}(0, N))^0 = (X_m + B_E(0, \frac{1}{N}))^0. \)

(vi) \( X_1, X_2, \ldots \) is quasi Cauchy. (Proof. Let \( \lambda \in K, |\lambda| > 1, N \in \mathbb{N}, m, n \geq N \). The set \( X_m + B_E(0, \frac{1}{N}) \) is norm open hence norm closed. So \( (X_m + B_E(0, \frac{1}{N}))^o \) is norm closed and edged, hence polar (as \( E \) is strongly polar). It follows via (v), that \( X_n \subset (X_m + B_E(0, \frac{1}{N}))^0 \subset \lambda(X_m + B_E(0, \frac{1}{N})). \))

(vii) \( X^0 \subset A \). (Proof. Let \( f \in X^0, \lambda \in K, |\lambda| > 1 \). It suffices to prove that \( f \in \lambda A \).

Let \( n \in \mathbb{N} \) be such that \( ||f|| \leq n \). Choose any \( x \in X_n \). By Lemma 2.3 there is a \( y \in \lambda X \) with \( ||x - y|| \leq \frac{|\lambda|}{n} \). We have

\[ |f(x)| \leq |f(x - y)| + |f(y)| \]

\[ \leq ||f|| \cdot ||x - y|| + |\lambda| \cdot \frac{|\lambda|}{n} \leq |\lambda| \]

and we see that \( |\lambda^{-1}f| \leq 1 \) on \( X_n \), so \( \lambda^{-1}f \in X_n^0 = A \cap B_{E'}(0, n) \subset A \) i.e. \( f \in \lambda A \).

Now (iv) combined with (vii) yields \( A = X^0 \) is \( w' \)-closed.

**COROLLARY 2.4.** A subspace of the dual of a strongly polar Banach space is \( w' \)-closed as soon as its intersection with the closed unit ball is \( w' \)-closed.

**COROLLARY 2.5.** An edged absolutely convex subset \( A \) of \( \ell^\infty \) is \( \sigma(\ell^\infty, c_0) \)-closed as soon as \( A \cap B_{\ell^\infty}(0, n) \) is \( \sigma(\ell^\infty, c_0) \)-closed for each \( n \in \mathbb{N} \).

Proof. \( c_0 \) is a (reflexive) strongly polar space.

We also have:

**THEOREM 2.6.** If \( E \) is a Krein-Šmulian space and \( D \subset E \) is a closed subspace then \( E/D \) is a Krein-Šmulian space.
Proof. Let \( i : (E/D)' \to E' \) be the adjoint of the quotient map \( E \to E/D \). It is easily seen that \( i \) is an isometry, that \( \text{Im} \ i \) is \( w' \)-closed in \( E' \) and that \( i \) is a \( w' \) to \( w' \) homeomorphism \( (E/D)' \to \text{Im} \ i \).

Now let \( A \) be an edged absolutely convex subset of \( (E/D)' \) such that \( A \cap B \) is \( w' \)-closed in \( B \) for each bounded set \( B \) in \( (E/D)' \). Then \( i(A) \) is edged. If \( X \subset E' \) is bounded then \( i(A) \cap X \) is \( w' \)-closed in \( X \). (Proof. Let \( j \mapsto a_j \) be a net in \( A \) such that \( i(a_j) \in X \) for all \( j \) and let \( w' - \lim_j a_j = b \in X \). As \( \text{Im} \ i \) is \( w' \)-closed \( b = i(a) \) for some \( a \in i^{-1}(X) \). Then \( w' - \lim_j a_j = a \). Now \( a_j \in A \cap i^{-1}(X) \) for all \( j \), \( a \in i^{-1}(X) \) and \( i^{-1}(X) \) is bounded, so by assumption on \( A \) we have \( a \in A \), so \( b \in i(A) \cap X \).)

Since \( E \) is a Krein-Šmulian space, \( i(A) \) is \( w' \)-closed in \( E' \) so that \( A = i^{-1}(i(A)) \) is \( w' \)-closed in \( (E/D)' \).

THEOREM 2.7. If \( E \) is a Krein-Šmulian space and if \( D \subset E \) is a weakly closed subspace having the WEP then \( D \) is a Krein-Šmulian space.

Proof. Let \( \pi : E' \to D' \) be the adjoint of the inclusion map \( D \to E \). Then \( \pi \) is surjective and \( w' \) to \( w' \) continuous. If \( A \) is an edged absolutely convex set in \( D' \) and \( \pi^{-1}(A) \) is \( w' \)-closed then \( A \) is \( w' \)-closed. (Proof. Let \( g \in D' \), \( g \notin A \). There is an \( f \in E' \) with \( \pi(f) = g \). Then \( f \notin \pi^{-1}(A) \). Now \( \pi^{-1}(A) \) is \( w' \)-closed and edged so there exists an \( x \in E \) such that \( f(x) = 1 \) and \( |h(x)| < 1 \) for all \( h \in \pi^{-1}(A) \). In particular, \( |h(x)| < 1 \) for all \( h \in \text{Ker} \ \pi = D^0 \) i.e. \( h(x) = 0 \) for all \( h \in D^0 \) so \( x \in D_0^0 = D \). Then \( g(x) = f(x) = 1 \) and \( |h(x)| < 1 \) for all \( h \in A \).

Now let \( A \) be an absolutely convex edged subset of \( D' \) such that \( A \cap B \) is \( \sigma(D', D) \)-closed in \( B \) for each bounded set \( B \subset D' \). Then for such \( B \), \( \pi^{-1}(A) \cap \pi^{-1}(B) \) is \( \sigma(E', E) \)-closed in \( \pi^{-1}(B) \). If \( X \subset E' \) is bounded then \( \pi(X) \) is bounded and \( X \subset \pi^{-1}(\pi(X)) \) so it follows that \( \pi^{-1}(A) \cap X \) is \( w' \)-closed in \( X \) for each bounded set \( X \subset E' \). Since \( E \) is Krein-Šmulian we have that \( \pi^{-1}(A) \) is \( w' \)-closed, so by the remark above, \( A \) is \( w' \)-closed.

Remark. Not every Krein-Šmulian polar space is strongly polar; \( \ell^\infty \) is an easy example. In §3 we will see that, if \( I \) is large enough, \( c_0(I) \) is not Krein-Šmulian. This leads to the

PROBLEM. Characterize the class of Krein-Šmulian spaces.

A concrete help would be the answer to the following two questions.
- Is $c_0 \times c^\infty$ a Krein-Šmulian space? (More generally, if $E_1$ and $E_2$ are Krein-Šmulian spaces then does it follow that $E_1 \times E_2$ is Krein-Šmulian?)
- Is the subspace of $D$ of $c^\infty$ constructed in [2], Ex. 4.J Krein-Šmulian?

§3. SAVE THE KREIN-ŠMULIAN THEOREM! (PART TWO)

In this section we shall prove the following version of the Krein-Šmulian Theorem. Observe that (a) holds for any polar $K$-Banach space.

**THEOREM 3.1.** For a normed space $E$ over $K$ the following are equivalent.

(a) $j_E(E)$ is norm closed in $E''$.

(b) If $H \subseteq E'$ is a subspace of finite codimension and if $H \cap B_{E'}$ is $w'$-closed then so is $H$.

For a normed space $E$ over $K$ the $bw'$-topology (the 'bounded-weak-star topology') is by definition the strongest locally convex topology on $E'$ that coincides with $w'$ on bounded subsets of $E'$.

**PROPOSITION 3.2.** Let $E$ be a normed space over $K$.

(i) $bw'$ is stronger than $w'$ but weaker than the norm topology on $E'$.

(ii) $(E', bw')$ is of countable type.

(iii) A seminorm $p$ on $E'$ is $bw'$-continuous if and only if $p|B_{E'}$ is $w'$-continuous.

(iv) For any locally convex space $(X, \tau)$ and any linear map $T : E' \to X$ we have that $T$ is $bw'$ to $\tau$ continuous if and only if $T|B_{E'}$ is $w'$ to $\tau$ continuous.

**Proof.** $E' = [B_{E'}]$ and $B_{E'}$ is a $w'$-compactoid, hence a $bw'$-compactoid. This implies (ii). The other proofs are straightforward.

We know that $(E', w')' = j_E(E)$ ([5])). We now prove

**PROPOSITION 3.3.** For a normed space $E$ over $K$ the dual of $(E', bw')$ is the norm closure of $j_E(E)$ in $E''$.

**Proof.** Every $\theta \in \overline{j_E(E)}$ is, on $B_{E'}$, the uniform limit of a sequence in $j_E(E)$ so $\theta|B_{E'}$ is $w'$-continuous and $\theta$ is $bw'$-continuous (Proposition 3.2 (iv)). Thus $\overline{j_E(E)} \subseteq (E', bw')'$. Conversely, let $\theta \in (E', bw')'$. Then (Proposition 3.2 (i)) $\theta \in E''$. Let $\varepsilon > 0$; we shall
find an $x \in E$ such that $\|\theta - j_E(x)\| < \epsilon$. Let $\alpha \in K$, $0 < |\alpha| < \epsilon$. The $w'$-continuity of $\theta|B_{E'}$ yields a finite set $F \subset E$ such that $f \in F^0 \cap B_{E'}$ implies $|\theta(f)| \leq |\alpha|$, in other words

$$f \in j_E(F)_0 \cap (B_{E''})_0 \quad \implies \quad |(\alpha^{-1}\theta)(f)| \leq 1.$$ 

So we see that $\alpha^{-1}\theta \in (j_E(F)_0 \cap (B_{E''})_0)^0 = (A + B_{E''})_0^0$, where $A = j_E(\text{co } F)$. Now $B_{E''}$ is $w'$-closed and $A$ is finite dimensional so by [3], 1.4, $(A + B_{E''})_0^0 = (A + B_{E''})^e$. For any $\lambda \in K$ such that $|\lambda| > 1$ and $|\lambda \alpha| < \epsilon$ we have $\alpha^{-1}\theta \in \lambda A + \lambda B_{E''}$, hence $\theta \in j_E(E) + \alpha \lambda B_{E''}$ and there is an $x \in E$ with $\theta - j_E(x) \in \alpha \lambda B_{E''}$ i.e. $\|\theta - j_E(x)\| < \epsilon$.

**Corollary 3.4.** Let $j_E(E)$ be closed in $E''$, let $A \subset E'$ be absolutely convex and edged. Then $A$ is $w'$-closed if and only if $A$ is $bw'$-closed.

**Proof.** Let $A$ be $bw'$ closed. As $A$ is also edged and $(E', bw')$ is strongly polar (Proposition 3.2 (ii)), $A$ is a polar set i.e. $A = S_0$ for some $S \subset (E', bw')'$. But by Proposition 3.3 $S \subset (E', w')'$ so that $A$ is $w'$-closed.

Further, we need the following general lemma.

**Lemma 3.5.** Let $A$ be a closed absolutely convex subset of a Hausdorff locally convex space over $K$; let $D$ be a finite dimensional subspace such that $A \cap D = \{0\}$. Then $A + D$ is closed and the addition map is a homeomorphism $A \times D \to A + D$.

**Proof.** (i) If addition is homeomorphic then $A + D$ is closed. In fact, let $i \to a_i + d_i$ be a net in $A + D$ (where $a_i \in A$, $d_i \in D$ for each $i$), converging to some $z$. Then $(i,j) \mapsto a_i - a_j + d_i - d_j$ converges to 0. By homeomorphism, $d_i - d_j \to 0$, by completeness of $D$, $d_i \to d$ for some $d \in D$. Then $a_i \to z - d$ and, by closedness of $A$, $z - d \in A$. We see that $z \in A + D$.

(ii) Assume $n := \dim D = 1$, say $D = Kx$ for some nonzero $x$. Let $i \to a_i + \lambda_i x$ ($a_i \in A$, $\lambda_i \in K$) be a net in $A + D$ converging to 0. If not $\lambda_i \to 0$ we may assume $|\lambda_i| \geq |\alpha| > 0$ for all $i$ and some $\alpha \in K$. Then $\alpha \lambda_i^{-1}(a_i + \lambda_i x) \to 0$ so $\alpha x = -\lim_i \alpha \lambda_i^{-1} a_i \in \overline{A} = A$ conflicting $Kx \cap A = \{0\}$. Thus, addition is homeomorphic and via (i) the lemma is proved if $n = 1$.

(iii) The proof of the induction step $n - 1 \to n$ is now standard and left to the reader.

**Proof of Theorem 3.1.**

(i) Suppose $(\alpha)$, and let $H \subset E'$ be a subspace of finite codimension such that $H \cap B_{E'}$ is $w'$-closed. Then $H \cap B_{E'}$ is norm closed, hence so is $H$. For some $t \in (0,1)$ $H$ has a
t-orthogonal complement $D$. Let $P : E^t \to D$ be the obvious projection. For $\lambda \in K$, $|\lambda| \geq t^{-1}$ we have

$$B_{E'} \subset \lambda(H \cap B_{E'}) + \lambda(D \cap B_{E'}) \subset \lambda(H \cap B_{E'}) + D.$$ 

Let $i \mapsto f_i$ be a net in $B_{E'}$, $w^t - \lim f_i = 0$. Then, by Lemma 3.5, $\lim P f_i = 0$. We see that $P|B_{E'}$ is continuous, so (Proposition 3.2 (iv)) $P$ is bw' to norm continuous and $\text{Ker } P = H$ is bw'-closed, hence $w'$-closed by Corollary 3.4, and (β) is proved.

(ii) Suppose (α) is not true. Choose $\theta \in j_E(E) \setminus j_E(E)$. Then $\theta$ is not $w'$-continuous so $H := \text{Ker } \theta$ is not $w'$-closed. But $\theta$ is bw'-continuous by Proposition 3.3. so $H \cap B_{E'}$ is $w'$-closed.

The results of this section yield the existence of polar non-Krein-Šmulian spaces (see §2).

**Corollary 3.6.** If $m$ is a cardinality $\geq \# K$ then $c_0(m)$ is not a Krein-Šmulian space.

**Proof.** In [2], Exercise 4.N a Banach space $E$ is constructed such that $j_E(E)$ is a proper dense subset of $E''$. From this construction it is easily seen that $\#E = \# \ell^\infty \leq \# K^N = \# K$. Now let $I$ be a set with cardinality $\geq \# K$ and let $\{e_i : i \in I\}$ be the natural orthonormal base of $c_0(I)$. There is a surjection $\{e_i : i \in I\} \to B_E$, it extends to a quotient map $c_0(I) \to E$. Now $E$ is not a Krein-Šmulian space by Theorem 3.1, neither is $c_0(I)$ by Theorem 2.6. (It is not hard to see by looking at the proof of Theorem 2.6 that one can even find a subspace $D \subset \ell^\infty(I)$ that is not $w'$-closed while $D \cap B_{\ell^\infty(I)}$ is.)

**Corollary 3.7.** If $m$ is a nonmeasurable cardinality $\geq \# K^K$ then $\ell^\infty(m)$ is not a Krein-Šmulian space.

**Proof.** In the spirit of the previous proof one constructs a quotient map $\pi : c_0(m) \to \ell^\infty(n)$ where $n = \# K$. By reflexivity ([2], Theorem 4.21) the adjoint $\pi' : c_0(n) \to \ell^\infty(m)$ is an isometry and $\pi'(c_0(n))$ has the WEP in $\ell^\infty(m)$. From [4], Lemma 2.2 we obtain that $\pi'(c_0(n))$ is also weakly closed in $\ell^\infty(m)$. By the previous corollary $c_0(n)$ is not a Krein-Šmulian space, neither is $\ell^\infty(m)$ by Theorem 2.7.

**Problem.** Determine the smallest cardinality $m$ for which $c_0(m)$ ($\ell^\infty(m)$ if $\# K$ is nonmeasurable) is not a Krein-Šmulian space.
As a further application we now prove a nonarchimedean version of a classical reflexivity criterion (Theorem 3.8). First we 'dualize' the notion of a polar seminorm as follows. A seminorm \( p \) on the dual \( E' \) of a locally \( K \)-convex space \( E \) is a dual seminorm if there exists an \( X \subset E \) such that \( p(f) = \sup \{|f(x)| : x \in X \} \) for all \( f \in E' \). An easy exercise shows that \( p \) is dual if and only if \( \{f \in E' : p(f) \leq 1\} \) is \( \sigma(E',E) \)-closed. Dual seminorms are automatically polar.

**Theorem 3.8.** Let \( E \) be a pseudoreflexive \( K \)-Banach space. Then \( E \) is reflexive if and only if each polar norm on \( E' \) inducing the topology is dual.

Proof. Let \( E \) be reflexive and let \( \nu \) be a polar norm on \( E' \) inducing the topology. Then \( \{f \in E' : \nu(f) \leq 1\} \) is weakly closed so, by reflexivity, \( \nu' \)-closed. Hence \( \nu \) is dual by the above remark. Conversely, suppose each polar norm on \( E' \) inducing the topology is dual. To prove reflexivity of \( E \) it suffices (by pseudoreflexivity) to show that any \( \theta \in E'' \) is \( \nu' \)-continuous. For each \( n \in \mathbb{N} \) the norm \( f \mapsto n|\theta(f)| \vee \|f\| \) (where \( \|\| \) is the 'natural' norm on \( E' \)) is easily seen to be polar and it is obviously equivalent to \( \|\| \). By assumption its closed unit ball

\[
B_n := \{f \in E' : |\theta(f)| \leq \frac{1}{n}, \|f\| \leq 1\}
\]

is \( \nu' \)-closed. Hence so is \( \bigcap_n B_n \) which is Ker \( \theta \cap B_{E'} \). By Theorem 3.1 Ker \( \theta \) is \( \nu' \)-closed implying that \( \theta \) is \( \nu' \)-continuous.

**Remark.** One also may consider a 'predual form' of the Krein-Šmulian property (compare Definition 1.1, see also Corollary 1.6 (ii)) as follows. A normed space \( E \) is PKŠ space if for each absolutely convex edged \( A \subset E \):

\[
A \cap B \text{ is } \nu\text{-closed in } B \text{ for each bounded } B \subset E \implies A \text{ is } \nu\text{-closed}.
\]

(Obviously this notion is of no use in classical Banach space theory.) The reader will not have difficulties in proving results about PKŠ spaces similar to the one of KŠ-spaces of this paper. More precisely, we have

(i) A strongly polar normed space is PKŠ.

(ii) Let \( E \) be a normed PKŠ space, let \( D \) be a closed subspace. Then \( E/D \) is PKŠ. If \( D \), in addition, is weakly closed and has the WEP then \( D \) has PKŠ.

(iii) Let \( E \) be a normed space. If \( H \subset E \) is a subspace with finite codimension and \( H \cap B_E \) is weakly closed then \( H \) is weakly closed.
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Katholieke Universiteit
Mathematisch Instituut
Toernooiveld
6525 ED Nijmegen, The Netherlands