OPEN PROBLEMS

A.C.M. van Rooij and W.H. Schikhof

It was during the Conference when the idea came up to call upon our colleagues to attack open problems that so far had gotten little attention. The result is a selection of 19 problems all centered around Banach space theory. We do not claim that they reflect the state of the art in $p$-adic Functional Analysis. In fact, in many papers, including the ones of these Proceedings, one may find questions that are at least equally interesting. We also wish to point out that—as far as we know—none of the problems stated in [5] has been solved yet!

NOTATIONS & TERMINOLOGY

We follow the conventions of [5]. Throughout $K$ is a nonarchimedean nontrivially valued field that is complete under the metric induced by the valuation $| \cdot |$. Norms and seminorms are non-archimedean. $E, F$ are $K$-Banach spaces; $E \oplus F$ is the orthogonal direct sum of $E$ and $F$; the formula $E \sim F$ means that $E, F$ are isometrically isomorphic. The ‘open’ (‘closed’) ball with radius $r$ and center $a$ is denoted $B(a, r^-)$ ($B(a, r)$). For $a \in E$ we write $[a]$ for the linear subspace generated by $a$. An absolutely convex subset $A$ of $E$ is edged if for each $a \in E$ the set $\{ |\lambda| : \lambda \in K, \lambda a \in A \}$ is closed in $|K|$.  

1. FINITE DIMENSIONAL PRIME DECOMPOSITION

Let us call an $E \neq \{0\}$ prime if there do not exist non-zero Banach spaces $E_1, E_2$ such that $E \sim E_1 \oplus E_2$. For a finite dimensional Banach space $E$, $E \neq \{0\}$, clearly there exist prime spaces $E_1, \ldots, E_N$ such that $E \sim E_1 \oplus \ldots \oplus E_N$.  

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Problem. Is the prime decomposition of a finite dimensional Banach space in the natural sense unique?

(One can show: if \( E, F \) are finite dimensional and \( K \oplus E \sim K \oplus F \) then \( E \sim F \). Also, if \( E_1, \ldots E_N \) are prime, \( D \) is a Banach space and \( E_1 \oplus \cdots \oplus E_N \sim K \oplus D \) then \( E_n \sim K \) for some \( n \).)

Problem. If \( E \) is prime, must \( \{ \| x \| : x \in E, x \neq 0 \} \) be just one coset of the value group of \( K \) in \((0, \infty)\)?

2. NORMS ON \( K^2 \)

Let \( K \) not be spherically complete. Let \( \mathcal{N} \) be the set of all norms \( \nu \) on \( K^2 \) for which

1. \( \nu(x) \in |K| \) \( (x \in K^2) \);
2. \( K^2 \) has no base that is orthogonal relative to \( \nu \).

For \( \mu, \nu \in \mathcal{N} \) write

\[ \mu \sim \nu \]

if there is a linear bijection \( A : K^2 \to K^2 \) such that \( \mu = \nu \circ A \).

Problem. Describe the set of all \( \sim \)-equivalence classes. E.g., can it be finite?

The following consideration may be of use.

By a hole we mean a maximal chain of balls of \( K \) with empty intersection. From a hole we can make other holes by translation and by multiplication with a nonzero scalar. Let us call two holes, \( \Omega \) and \( \Omega' \) equivalent,

\[ \Omega \sim \Omega' \]

if there exist \( \alpha \in K \) and \( \beta \in K \setminus \{0\} \) for which \( \Omega' = \alpha + \beta \Omega \).

A hole \( \Omega \) determines a norm \( \nu_\Omega \) on \( K^2 \) in the following fashion. Let \( |\Omega| \) be the infimum of the radii of the balls belonging to \( \Omega \). For \( r \in (|\Omega|, \infty) \) let \( \Omega_r \) be the element of \( \Omega \) whose radius is \( r \). If \( (\alpha, \beta) \in K^2 \), then for sufficiently small \( r \in (|\Omega|, \infty) \) the function \( \lambda \mapsto |\alpha - \beta \lambda| \) is constant on \( \Omega_r \); let \( \nu_\Omega(\alpha, \beta) \) be its value.

It turns out that the map \( \Omega \mapsto \nu_\Omega \) establishes a bijective correspondence between the equivalence classes of norms and the equivalence classes of holes: every \( \nu_\Omega \) lies in
N, for every $\nu \in N$ there is a hole $\Omega$ with $\nu \sim \nu_\Omega$, and for any two holes, $\Omega$ and $\Omega'$, one has

$$\nu_\Omega \sim \nu_{\Omega'} \iff \Omega \sim \Omega'.$$

Thus, our problem may also be formulated as: describe the set of all equivalence classes of the holes.

3. MULTI-ORTHOGONAL BASES

The following observation was made already in 1964 by Carpentier [2]. (It also follows easily from [6], Theorem 1.11).

Let \( \| \|_1, \| \|_2 \) be norms on a finite-dimensional space \( E \) such that \( (E, \| \|_1) \), \( (E, \| \|_2) \) both have orthogonal bases. Then there exists a base of \( E \) which is orthogonal to both \( \| \|_1 \) and \( \| \|_2 \).

**Problem.** Do we have a similar result for three (finitely many) norms? For which \( K \) is the 'natural' infinite dimensional version of the above true?

4. CARTESIAN SPACES

In the terminology of S. Bosch et al. [1], a normed vector space is Cartesian if every finite dimensional subspace has an orthogonal base.

Let us call a normed space Hilbertian of every 1-dimensional linear subspace has an orthogonal complement. Every finite dimensional linear subspace of a Hilbertian space has an orthogonal complement; it follows that a Hilbertian space is Cartesian.

**Problem.** Is every Cartesian normed space Hilbertian?

If \( K \) is spherically complete, then every normed space is Hilbertian and Cartesian. In general, every normed space that has an orthogonal base is Hilbertian and Cartesian. Conversely, a Cartesian space of countable type has an orthogonal base, hence is Hilbertian.
However, if the valuation of $K$ is dense, there exists a Cartesian space without orthogonal base. We sketch a proof of this statement. Let $\hat{c}_0$ be the spherical completion of $c_0$. By Zorn’s Lemma, there is a maximal Cartesian subspace $E$ of $\hat{c}_0$ containing $c_0$. Suppose $E$ has an orthogonal base. Then $E$ has a countable orthonormal base $\{a_1, a_2, \ldots\}$. Choose $\varepsilon_1, \varepsilon_2, \ldots \in K$ with $|\varepsilon_1| > |\varepsilon_2| > \ldots \to 1$. Take $u \in \hat{c}_0$ such that

$$
\|u - \sum_{n=1}^{N} \varepsilon_n a_n\| \leq |\varepsilon_{N+1}| \quad (N = 1, 2, \ldots).
$$

Then $u \notin E$. Set $F = E + [u]$. Putting

$$
b_N = u - \sum_{n=1}^{N} \varepsilon_n a_n \quad (N = 1, 2, \ldots)
$$

one can show that $\{u, b_1, b_2, \ldots\}$ is an orthogonal base for $F$, so that $F$ is Cartesian. This contradicts the maximality of $E$.

5. ORTHOGONAL ALMOST COMPLEMENTS

A closed subspace $S$ of $E$ is said to be an orthogonal almost complement (o.a.c.) of a closed subspace $T$ if $S \perp T$ and $S + T$ has finite codimension in $E$. (To show that this concept does not appear out of the blue consider the following easily proved Proposition. The ‘open’ balls $B(0, r^-)$ are weakly closed for each $r > 0$ if and only if each onedimensional subspace has an o.a.c.)

**Problem.** (Compare [5], Lemma 4.35 (iii).) If each onedimensional subspace has an o.a.c. then does it follow that each finite dimensional subspace has an o.a.c.?

6. IMAGES OF COMPACTOIDS

If $A \subset E$ is a compactoid and $T \in \mathcal{L}(E, F)$ is injective then $T$ maps $A$ homeomorphically onto $TA$. This leads to the

**Problem.* Let $A \subset E$ be an absolutely convex compactoid and let $T \in \mathcal{L}(E, F)$. If

*Note of the editors: A solution to this problem was obtained during the production of this book; it appears in this same volume as Appendix B.*
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Consider a sequence in $TA$ with $\lim_{n \to \infty} y_n = 0$ does there exist a sequence $x_1, x_2, \ldots$ in some scalar multiple of $A$ with $\lim_{n \to \infty} x_n = 0$ and $Tx_n = y_n$ for each $n$?

(It suffices to consider the case where $T$ is a quotient map. If in the above we require the $x_1, x_2, \ldots$ to be in $A$ the problem has a negative answer. In fact, a positive answer would imply that $TA$ is closed whenever $A$ is closed, which is not true if $K$ is not spherically complete; see [6], Theorem 6.28.)

7. STRONG POLARITY

(See [4] for definitions and for related concepts and problems.) It is well known that $E$ is strongly polar (SP) if and only if each closed edged absolutely convex subset of $E$ is polar. Let us define:

$E$ is **boundedly strongly polar** (BSP) if every bounded closed edged absolutely convex subset of $E$ is polar or, equivalently, if each norm inducing the topology is polar.

$E$ has the **almost orthogonal complementation property** (AOC) if, for each $t \in (0, 1)$, every closed linear subspace of $E$ has a $t$-orthogonal complement.

Obviously we have

$E$ is of countable type $\implies E$ has AOC $\implies E$ is SP $\implies E$ is BSP.

**Problem.** Let $K$ be not spherically complete. Which ones of the opposite implications are true?

8. REFLEXIVITY

**Problem.** Let $a \in E$, $a \neq 0$. If $E/[a]$ is reflexive, must $E$ itself be reflexive?

(If $E$ is reflexive, then so is $E/[a]$. Proof. Let $\varepsilon > 0$; we are done if we can find a reflexive space $D$ and a linear bijection $T : D \to E/[a]$ with $\|T\| \leq 1$, $\|T^{-1}\| \leq 1 + \varepsilon$. There exists an $f \in E'$ such that $f(a) = 1$, $\|f\| \|a\| \leq 1 + \varepsilon$. Take $D = f^{-1}(0)$ and let $T$ be the restriction of the quotient map $E \to E/[a]$.)
A Banach space $F$ is called pseudoreflexive (or polar) if the natural map

$$j_F : F \to F''$$

is isometric. The following variation on the above problem is just as meaningful: if $E/\{a\}$ is pseudoreflexive, must $E$ be pseudoreflexive too?

However, there exist a Banach space $E$ and an $a \in E$, $a \neq 0$ such that $j_{E/\{a\}}$ is injective but $j_E$ is not! Such $E$ and $a$ can be made as follows. For every $n \in \mathbb{N}$ let $F_n$ be a Banach space such that $F_n'$ separates the points of $F_n$ and there exists an $a_n \in F_n$ with

$$\|a_n\| = 1, \quad |f(a_n)| \leq n^{-1}\|f\| \quad (f \in F_n').$$

(See [5], 3.1 and 4.N; take $a_n = (1, 1, 1, \ldots)$.) Let $F = \oplus F_n$, $D = \{(\lambda_1 a_1, \lambda_2 a_2, \ldots) : \lambda_1, \lambda_2, \ldots \in K, \sum \lambda_n = 0\}$, $E = F/D$ and let $a \in E$ be the element corresponding to $(a_1, 0, 0, \ldots) \in F$.

A related problem is:

**Problem.** Let $E$ be a Banach space. Let $A, B$ be closed linear subspaces with $A + B = E$, $A \cap B = \{0\}$ and suppose both are (pseudo)reflexive. Does it follow that $E$ is (pseudo)reflexive?

### 9. WEAKLY CLOSED CONVEX SUBSETS OF $c_0$

If $K$ is spherically complete each closed absolutely convex subset of $c_0$ is weakly closed. In general, each closed *edged* absolutely convex subset of $c_0$ is weakly closed. Also, closed compactoids are weakly closed. However, if $K$ is not spherically complete, one can always find a closed absolutely convex subset of $c_0$ which is not weakly closed ([8], Theorem 1.1).

**Problem.** Characterize the weakly closed absolutely convex subsets of $c_0$.

See also Problem 5.
10. CLOSED CONVEX SETS DETERMINING TOPOLOGY

Let $K$ be spherically complete and $\dim E = \infty$. Then the norm topology and the weak topology differ, yet these topologies have the same collection of closed convex sets. On the other hand, if $K$ is not spherically complete and $\dim E = \infty$ it follows from [8], Theorem 1.1 that one can find a norm closed absolutely convex set that is not weakly closed. This leads naturally to the following general

**Problem.** Let $\tau_1, \tau_2$ be two locally convex topologies on a vector space over a non-spherically complete $K$. Suppose $\tau_1$-closed = $\tau_2$-closed for absolutely convex sets. Does it follow that $\tau_1 = \tau_2$?

11. HYPERISOMORPHIC SPACES

Let us say that $E$ is *hyperisomorphic* if there exists a closed hyperplane which is linearly homeomorphic to $E$.

**Problem.** Is, for spherically complete $K$, every infinite dimensional Banach space hyperisomorphic? More generally, is every polar infinite dimensional Banach space hyperisomorphic?

12. COMPLETELY CONTINUOUS OPERATORS

Let us say that an $A \in \mathcal{L}(E, F)$ is *completely continuous* if for every sequence $x_1, x_2, \ldots$ in $E$

$$\lim_{n \to \infty} x_n = 0 \text{ weakly } \implies \lim_{n \to \infty} \|Ax_n\| = 0.$$ 

Clearly every compact operator is completely continuous. The converse is not true as for spherically complete $K$ every $A \in \mathcal{L}(E, F)$ is completely continuous.

**Problem.** For nonspherically complete $K$, characterize the completely continuous operators in $\mathcal{L}(E, F)$.
13. THE INVARIANT SUBSPACE PROBLEM

If \( T \in \mathcal{L}(E) \), must \( E \) have a nontrivial closed linear subspace that is invariant under \( T \)?

If \( K \) is not algebraically closed, the answer clearly is negative: Let \( E \neq K \) be an algebraic field extension of \( K \) generated by an element \( a \) and let \( Tx = ax \ (x \in E) \).

The answer is also negative if \( K \) is not spherically complete. To see this, provide the spherical completion \( \bar{K} \) of \( K \) with the structure of a valued field ([5], Th. 4.49), take \( a \in \bar{K}\setminus K \), let \( E \) be the closed linear hull of \( \{1, a, a^2, \ldots\} \) in \( \bar{K} \) and \( Tx = ax \ (x \in E) \). Then \( E \) is a subalgebra of \( \bar{K} \). Actually, \( E \) is a field: If \( x \in E\setminus\{0\} \), there exists a \( \xi \in K \) with \( |x - \xi| < |\xi| \); then

\[
x^{-1} = \xi^{-1} \sum_{n=0}^{\infty} (1 - \xi^{-1}x)^n \in E.
\]

Every closed linear subspace of \( E \) that is invariant under \( T \) is an ideal in \( E \), hence must be either \( E \) or \( \{0\} \).

There remains the following

**Problem.** Assume that \( K \) is algebraically closed and spherically complete. Let \( T \in \mathcal{L}(E) \). Does \( E \) necessarily have a nontrivial closed linear subspace that is invariant under \( T \)?

If we restrict ourselves to infinite dimensional Banach spaces, the reasoning we gave for non-algebraically closed \( K \) falls through and the one for non-spherically complete \( K \) works only if \( \bar{K} \) contains an element that is not algebraic over \( K \). This brings us to a question that in itself has nothing to do with invariant subspaces:

**Problem.** If \( K \) is not spherically complete, can \( \bar{K} \) consist only of elements that are algebraic over \( K \)?

14. ARCHIMEDEAN NORMS

An \( A \)-norm ([5]) on a vector space \( D \) is a function \( q : D \to [0, \infty) \) satisfying

\[
q(\lambda x) = |\lambda|q(x) \quad (\lambda \in K, \ x \in D), \quad q(x + y) \leq q(x) + q(y) \quad (x, y \in D), \quad q(x) = 0 \implies x = 0 \quad (x \in D).
\]
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Such an $A$-norm induces a vector space topology.

**Problem.** (See [4]) Let $q : D \rightarrow [0, \infty)$ be an $A$-norm and assume that for every linear subspace $D_0$ of $D$, every continuous linear function $D_0 \rightarrow K$ extends to a continuous linear function $D \rightarrow K$. Must there exist a (non-Archimedean) norm $\| \|_\cdot$ on $E$ that is equivalent to $q$?

(There is a natural candidate for $\| \|_\cdot$:

On the space $D'$ of all $q$-continuous linear functions $D \rightarrow K$ we impose a (non-Archimedean) norm $q'$ by

$$q'(f) = \inf \{ c \in [0, \infty) : |f(x)| \leq cq(x) \text{ for all } x \in D \}.$$ 

$q'$ determines a non-Archimedean norm $\bar{q}$ on $D$:

$$\bar{q}(x) = \inf \{ c \in [0, \infty) : |f(x)| \leq cq'(f) \text{ for all } f \in D' \}.$$ 

Then $\bar{q} \leq q$ and $D'$ is just the space of all $\bar{q}$-continuous linear functions on $D$.

For $x_1, x_2, \ldots \in D$ write

$$x_n \rightarrow 0$$

if $f(x_n) \rightarrow 0$ for all $f \in D'$. Trivially,

$$q(x_n) \rightarrow 0 \implies x_n \rightarrow 0.$$ 

By 5.2 of [6],

$$x_n \rightarrow 0 \iff \bar{q}(x_n) \rightarrow 0$$

and

$$x_n \rightarrow 0 \iff \|x_n\| \rightarrow 0$$

if $\| \|_\cdot$ is a norm providing a positive answer to our Problem. It follows that, if such a norm $\| \|_\cdot$ exists at all, then $\bar{q}$ is one. Also, this is the case if and only if

$$x_n \rightarrow 0 \implies q(x_n) \rightarrow 0.$$ 

Consequently, to solve the Problem one may restrict oneself to spaces $D$ of countable type.
Van Gisbergen ([3]) gives an example of an A-norm that does not have the extension property mentioned in the Problem although $D'$ separates the points of $D$.

### 15. ULTRAMETRIZABILITY

Let $\ell^1$ be the space of all sequences $x = (x_1, x_2, \ldots)$ where $x_i \in K$ for each $i$ such that $\|x\| := \sum |x_i| < \infty$. Then $\ell^1$ is a $K$-Banach space with respect to the $A$-norm $\| \|$ (see Problem 14).

**Problem.** Is $\ell^1$ ultrametrizable?

(One can show that the dense subspace $c_{00} := \{(x_1, x_2, \ldots) : x_i = 0 \text{ for large } i\}$ is ultrametrizable. For separable $K$ this is easily seen as follows. One verifies that $B(0, r^-)$ is closed in $c_{00}$. Then $c_{00}$ is zero-dimensional, separable and metrizable, hence ultrametrizable.)

An affirmative answer would solve the more general

**Problem.** Does there exist a complete ultrametrizable topological vector space over $K$ which is not locally convex?

### REFERENCES


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Open problems


Katholieke Universiteit
Mathematisch Instituut
Toernooiveld
6525 ED Nijmegen, The Netherlands