COMPACT OPERATORS AND THE ORLICZ-PETTIS PROPERTY
IN $p$-ADIC ANALYSIS

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Report 9101
January 1991
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ABSTRACT. For a non-archimedean locally convex space $E$ we study in this paper
the property
1. "Every weakly convergent sequence in $E$ is convergent"
as related to
2. "Every continuous linear map from $\ell^\infty$ to $E$ is compact".
Also, we show that for a Banach space $E$ there is a duality between these properties
and the property
3. "Every $\sigma(E', E)$-convergent sequence in $E'$ is norm convergent in $E''$",
which has been studied by N. de Grande-de Kimpe in [1] and [2] and recently by T.
Kiyosawa in [4].

TERMINOLOGY

Throughout $K$ is a non-archimedean valued field that is complete under the metric
induced by the non-trivial valuation $|\cdot|$, and $E, F, \ldots$ are locally convex spaces over $K$.We always assume that $E, F, \ldots$ are Hausdorff.
$L(E, F)$ will be the $K$-vector space consisting of all continuous linear maps $E \to F$.
The topological dual space of $E$ is $E' := L(E, K)$. Also, the algebraic dual space of $E$will be denoted $E^*$. Observe that the weak topology $\sigma(E, E')$ of $E$ is Hausdorff if andonly if $E'$ separates the points of $E$. 

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By $E \cong F$ we mean that there is a linear homeomorphism from $E$ onto $F$.

A nonempty subset $A$ of $E$ is absolutely convex if $x, y \in A$, $\lambda, \mu \in K$, $|\lambda| \leq 1$, $|\mu| \leq 1$ implies $\lambda x + \mu y \in A$. The absolutely convex hull of $A$ is denoted by $\text{co}(A)$ and the $K$-linear hull of $A$ by $[A]$. We shall write $\overline{\text{co}}(A)$ instead of $\text{co}(A)$.

If $p$ is a continuous seminorm in $E$ we denote by $E_p$ the associated normed space $E/\text{Ker} p$.

For unexplained terms and background we refer to [6] (locally convex spaces) and [13] (normed spaces).
§1. (O.P.) - SPACES.

The classical Banach space $\ell^1$ (over $\mathbb{R}$ or $\mathbb{C}$) has the property that every weakly convergent sequence is norm convergent. This fact is known as the Orlicz - Pettis Theorem. Let us define in the non-archimedean theory:

**DEFINITION 1.1.** A locally convex space $E$ over $K$ is called an *Orlicz - Pettis space* (($\text{O.P.}$)-space) if every weakly convergent sequence is convergent (or equivalently, if for every sequence $x_1, x_2, \ldots$ in $E$, $\lim_{n \to \infty} x_n = 0$ weakly implies $\lim_{n \to \infty} x_n = 0$ in the original topology).

It is well-known that $c_0$ is an (O.P.)-space ([13], p. 158).

Also, it follows easily that if $E$ is an (O.P.)-space, then the dual $E'$ separates the points of $E$. However, $\ell^\infty$ is a space whose dual separates the points, but, if $K$ is not spherically complete, is not an (O.P.)-space (see [6], Remark following Proposition 4.11).

We now study some stability properties of the class of (O.P.)-spaces.

**PROPOSITION 1.2.**

a) A subspace of an (O.P.)-space is an (O.P.)-space.

b) The product of a family of (O.P.)-spaces is an (O.P.)-space.

*Proof.*

a) Let $D$ be a subspace of the (O.P.)-space $(E, \tau)$ and let $x_1, x_2, \ldots$ be a sequence in $D$ converging weakly to 0 in $\sigma(D, D')$. Then, certainly $x_n \to 0$ with respect to $\sigma(E, E')$. Hence, $x_n \to 0$ by assumption.

b) Let, for each $i$ belonging to some set $I$, $E_i$ be an (O.P.)-space and let $(x^1_i)_{i \in I}$, $(x^2_i)_{i \in I}, \ldots$ converge weakly to 0 in the product space $\prod_{i \in I} E_i$. By continuity of projections $x^1_i, x^2_i, \ldots$ converges weakly to 0 in $E_i$ for each $i \in I$, hence by assumption in the initial topology of $E_i$. But this is precisely convergence to 0 of $(x^1_i)_{i \in I}, (x^2_i)_{i \in I}, \ldots$ in the product topology.

Let us denote the locally convex direct sum of a family $\{E_i : i \in I\}$ of locally convex spaces by $\bigoplus_{i \in I} E_i$. Recall that each $e \in \bigoplus_{i \in I} E_i$ has a unique decomposition $e = \sum_{i \in I} e_i$, where $e_i \in E_i$ for each $i$ and where $\{i \in I : e_i \neq 0\}$ is finite.

**LEMMA 1.3.** Let $\{E_i : i \in I\}$ be a family of locally convex spaces such that the weak topology on each $E_i$ is Hausdorff. Then, for any weakly bounded set $X$ in $\bigoplus_{i \in I} E_i$,

$$J := \{i \in I : \text{there exists an } x \in X \text{ with } x_i \neq 0\}$$
Proof. Suppose $J$ is infinite. Then inductively one can find $i_1, i_2, \ldots \in I$ and $x^{j}_1, x^{j}_2, \ldots \in X$ such that $x^{j}_{i_m} = 0$ if $j < m$ and $x^{m}_{i_m} \neq 0$ for each $m \in \{1, 2, \ldots \}$. Then, again inductively, one can construct $f_{i_1} \in E'_{i_1}, f_{i_2} \in E'_{i_2}, \ldots$ such that for each $m \in \mathbb{N}$

$$|f_{i_m}(x^{m}_{i_m})| \geq m + \sum_{k < m} |f_{i_k}(x^{k}_{i_k})|.$$ 

If $i \in I \setminus \{i_1, i_2, \ldots \}$ we define $f_i \in E'_i$ to be 0. Then, the formula

$$f(\sum_{i \in I} e_i) = \sum_{i \in I} f_i(c_i)$$

defines an element $f \in \left( \bigoplus_{i \in I} E_i \right)'$. Also for each $m \in \mathbb{N}$ we have

$$|f(x^m)| = \left| \sum_{i \in I} f_i(x^m_i) \right| = \sum_{k \in \mathbb{N}} |f_{i_k}(x^{m}_{i_k})| = \sum_{k \leq m} |f_{i_k}(x^{m}_{i_k})| \geq |f_{i_m}(x^{m}_{i_m})| - \sum_{k < m} |f_{i_k}(x^{m}_{i_k})| \geq m.$$ 

It follows that $X$ is not weakly bounded, a contradiction.

**Proposition 1.4.** The locally convex direct sum of a family of (O.P.)-spaces is again an (O.P.)-space.

**Proof.** Any weakly convergent sequence in the direct sum $\bigoplus_{i \in I} E_i$ of the (O.P.)-spaces $E_i$ is, by weak boundedness and Lemma 1.3, contained in $\bigoplus_{i \in J} E_i \simeq \prod_{i \in J} E_i$ for some finite set $J \subset I$, and also weakly convergent in that space, hence, by Proposition 1.2.b), convergent in the restricted topology of $\bigoplus_{i \in J} E_i \subset \bigoplus_{i \in I} E_i$.

**Remark.** The class of (O.P.)-spaces is not closed for forming of quotients.

Indeed, let $K$ be not spherically complete. Then $\ell^\infty$ is not an (O.P.)-space. On the other hand, one can make a quotient map $c_0(I) \to \ell^\infty$ if $I$ has sufficiently large cardinal, and we shall see in Theorem 1.6 (vi) that $c_0(I)$ is an (O.P.)-space.

However, we do have the following

**Proposition 1.5.** Let $E$ be a locally convex space and let $D$ be a finite dimensional subspace.

(i) If $E$ is an (O.P.)-space then so is $E/D$. 

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(ii) If \( E' \) separates the points of \( E \) and if \( E/D \) is an (O.P.)-space, then so is \( E \).

Proof. An elementary reasoning shows that, if \( E' \) separates the points of \( E \), then \( D \) is complemented in \( E \), i.e., \( E = D \oplus H \) for some closed subspace \( H \) of \( E \). It follows that \( H \) is linearly homeomorphic to \( E/D \). Now apply Proposition 1.2.a) to find (i) and Proposition 1.4 to find (ii).

To obtain examples of (O.P.)-space (in Theorem 1.6) we first recall some definitions.

Following [6] a locally convex space \( E \) is of countable type if for every continuous seminorm \( p \) the normed space \( E_p \) is of countable type (a normed space is of countable type if there exists a countable set whose linear span is dense). Also, \( E \) is strongly polar if for every continuous seminorm \( p \) the formula \( p = \sup \{|f| : f \in E^*, |f| \leq p\} \) holds. Finally, following [12] we say that \( E \) has property \((*)\) if for each subspace \( D \) of countable type, each \( f \in D' \) has a continuous linear extension \( \tilde{f} \in E' \).

**THEOREM 1.6.** The following spaces are (O.P.)-spaces.

(i) Any locally convex space \( E \) such that for every continuous seminorm \( p \) on \( E \), the associated normed space \( E_p \) is an (O.P.)-space.

(ii) Every locally convex space of countable type.

(iii) Every locally convex space with the property \((*)\).

(iv) Every strongly polar space.

(v) Every locally convex space over a spherically complete field.

(vi) Every Banach space with a base.

(vii) Any vector space \( E \) equipped with the strongest locally convex topology \( \tau^* \).

Proof.

(i) We know that if \( \mathcal{P} \) is a family of seminorms determining the topology of \( E \), then \( E \) can be considered as a subspace of \( \prod_{p \in \mathcal{P}} E_p \). Now apply Proposition 1.2.

Property (ii) is a direct consequence of property (i).

Also, the proof of (iii) is just like the corresponding one given in [12], Theorem 5.2 for normed spaces.

Now (iv), (v) and (vi) are special cases of (iii) (see [6], Theorem 4.2 and [13], Corollary 3.18).

To prove (vii) just observe that \((E, \tau^*)\) is linearly homeomorphic to \( \bigoplus_{i \in I} K_i \) where \( I \) has the cardinality of an algebraic base of \( E \) and where \( K_i = K \) for each \( i \). Now apply Proposition 1.4.
§2. (O.P.)-LIKE SPACES.

In this section we consider the following variants of the Orlicz-Pettis property.

**DEFINITION 2.1.** Let $E$ be a locally convex space over $K$.

(a) $E$ is called a (B.O.P.)-space if every bounded weakly convergent sequence is convergent.

(b) $E$ is called a (C.O.P.)-space if every Cauchy sequence that is weakly convergent, is convergent.

Clearly we have

$E$ is an (O.P.)-space $\iff E$ is a (B.O.P.)-space $\iff E$ is a (C.O.P.)-space

and also that the dual of a (C.O.P.)-space separates points.

Further, it is not hard to see by looking at the proofs of 1.2 - 1.5 that the class of (B.O.P.)-spaces ((C.O.P.)-spaces) is stable with respect to subspaces, products, locally convex direct sums and quotients by finite dimensional subspaces.

The space $c_0$, over a nonspherically complete $K$, is a (C.O.P.)-space but, as we saw in §1, not a (B.O.P.)-space.

If $E$ is a normed space, we have that $E$ is a (B.O.P.)-space $\iff E$ is an (O.P.)-space. Indeed, suppose that $E$ is a (B.O.P.)-space and that $x_1, x_2, \ldots$ tends to zero weakly but $\|x_n\| \uparrow \infty$. Then there exist $\lambda_1, \lambda_2, \ldots \in K$ such that $\{\|\lambda_n x_n\| : n \in \{1, 2, \ldots\}\}$ is bounded away from 0 and $\lambda_n \to 0$. Then $\lambda_1 x_1, \lambda_2 x_2, \ldots$ is a bounded sequence such that $\lambda_n x_n \to 0$ weakly. Hence, $\|\lambda_n x_n\| \to 0$, a contradiction.

From this fact, the following question arises in a natural way:

**PROBLEM:** Is every (B.O.P.)-space an (O.P.)-space?

In 2.1, 2.2, 2.3 below we shall prove characterizations of (C.O.P.)-, (B.O.P.)-, (O.P.)-spaces respectively, yielding a comparison between these classes. Recall (3) that an absolutely convex subset $A$ of a locally convex space $E$ is said to be (a) compactoid if for each neighbourhood $U$ of 0 in $E$ there exists a finite set $H \subseteq E$ such that $A \subseteq U + \text{co}(H)$.

Then,

**THEOREM 2.2.** For a locally convex space $E$, the following properties are equivalent.

(i) $E$ is a (C.O.P.)-space.

(ii) On metrizable and compactoid sets in $E$, the weak topology $\sigma(E, E')$ and the original topology $\tau$ coincide.

(iii) For every metrizable and compactoid set $A \subseteq E$, $\overline{A}^{\sigma(E, E')} = \overline{A}^\tau$.

(iv) Every closed metrizable and compactoid subset of $E$ is weakly closed.
Proof.
(i) ⇒ (ii). Let \( A \subseteq E \) be a metrizable and compactoid subset of \( E \) and let \( \hat{E} \) denote the completion of \( E \). Set
\[
E^* := \{ x \in \hat{E} : \text{there is a sequence } (x_n) \text{ in } E \\
\text{such that } x_n \to x \text{ in } E \}
\]

By [8], Theorem 6.1 we have that \( A^{E^*} \) is also a metrizable and compactoid subset of \( E^* \) endowed with the restricted topology induced by \( \hat{E} \). Also, \( A^{E^*} \) is complete.

On the other hand, one proves very easily that if \( E \) is a (C.O.P.)-space then the weak topology \( \sigma(E^*, (E^*)') \) is Hausdorff.

Now apply Theorem 3.2 of [7] to conclude that \( \sigma(E, E')|A = \tau|A \).

The implications (ii) ⇒ (iii) and (iii) ⇒ (iv) are obvious.

Finally we prove (iv) ⇒ (i). Let \( (x_n) \) be a Cauchy sequence converging weakly to 0. By [8], Theorem 6.1 we have that \( A := \overline{co}(x_1, x_2, \ldots) = \overline{co}(x_1, x_2, x_2 - x_1, x_3, \ldots) \) is a metrizable compactoid subset of \( E \).

Metrizability of \( A \) implies the existence of a sequence \( U_1 \supset U_2 \supset \ldots \supset U_n \supset \ldots \) of clopen neighbourhoods of 0 such that \( \{ U_n \cap A : n \in \{ 1, 2, \ldots \} \} \) generate \( \tau|A \).

Take \( m \in \{ 1, 2, \ldots \} \). Since \( (x_n) \) is a Cauchy sequence we have that \( x_k - x_\ell \in U_m \cap A \) for sufficiently large \( k, \ell \). Also, \( U_m \cap A \) is a closed metrizable compactoid set and by (iv) it is also weakly closed. Further, \( (x_\ell) \) converges weakly to 0. Putting together these facts we conclude that \( x_k \in U_m \cap A \) for large \( k \). Thus, \( x_1, x_2, \ldots \) converges to 0 in the original topology \( \tau \), and we are done.

**Corollary 2.3.** For a locally convex space \( E \), the following are equivalent.
(i) \( E \) is a (B.O.P.)-space.
(ii) \( E \) is a (C.O.P.)-space and every absolutely convex bounded and \( \sigma(E, E') \)-metrizable set of \( E \) is metrizable and compactoid.
(iii) \( E' \) separates the points of \( E \) and on every absolutely convex bounded and \( \sigma(E, E') \)-metrizable set in \( E \), the weak topology \( \sigma(E, E') \) and the original topology \( \tau \) coincide.

Proof.
(i) ⇒ (ii). We already know that if \( E \) is a (B.O.P.)-space then \( E \) is a (C.O.P.)-space and \( E' \) separates the points. Now let \( A \subseteq E \) be an absolutely convex bounded and \( \sigma(E, E') \)-metrizable subset of \( E \). Let \( \lambda \in K, |\lambda| > 1 \) if the valuation is dense, \( \lambda = 1 \) if the valuation is discrete. By [6], Proposition 8.2 there exists a sequence \( e_1, e_2, \ldots \) in \( \lambda A \) (and hence it is a bounded sequence) with \( \lim_{n \to \infty} e_n = 0 \) in \( \sigma(E, E') \) and such that \( A \subseteq \overline{co}(e_1, e_2, \ldots) \). Since \( E \) is a (B.O.P.)-space we conclude that \( e_n \rightharpoonup 0 \), which implies that \( co(e_1, e_2, \ldots) \) is a metrizable compactoid set ([8], Theorem 6.1). By
Theorem 2.2 it follows that $\overline{\sigma(E,E')}(e_1, e_2, \ldots)$ (and hence also $A$) is a metrizable and compactoid set.

The implication (ii) $\Rightarrow$ (iii) is a direct consequence of Theorem 2.2.

(iii) $\Rightarrow$ (i). Let $(x_n)$ be a bounded sequence such that $x_n \rightharpoonup 0$ weakly. Then, $A := co(x_1, x_2, x_3, \ldots)$ is a bounded and $\sigma(E, E')$-metrizable subset of $E$ ([8], Theorem 6.1). By (iii), $\tau|A = \sigma(E, E')|A$. Hence, $x_n \rightharpoonup 0$.

**COROLLARY 2.4.** For a locally convex space $E$ the following are equivalent.

(i) $E$ is an (O.P.)-space.

(ii) $E$ is a (B.O.P.)-space and every weakly bounded set is bounded.

(iii) $E$ is a (C.O.P.)-space and every absolutely convex weakly bounded and $\sigma(E, E')$-metrizable subset of $E$ is metrizable and compactoid.

(iv) $E'$ separates the points of $E$ and for every absolutely convex weakly bounded and $\sigma(E, E')$-metrizable set in $E$, the weak topology $\sigma(E, E')$ and the original topology coincide.

**Proof.**

(i) $\Rightarrow$ (ii). We only have to prove that if $E$ is an (O.P.)-space then every weakly bounded set is bounded. Thus, let $X \subset E$ be weakly bounded but not $\tau$-bounded. Then, there is a $\tau$-continuous seminorm $p$ and a sequence $x_1, x_2, \ldots \in X$ such that $p(x_n) \geq n$ for each $n$. There is a $\rho > 0$ and there are $\lambda_1, \lambda_2, \ldots \in K$ such that $\rho \leq p(\lambda_n x_n) \leq 1$ for each $n$. Then $\lambda_n \rightharpoonup 0$ and, by the weak boundedness of $\{x_1, x_2, \ldots\}$, $\lim_{n \to \infty} \lambda_n x_n = 0$ weakly. By (i), $\lambda_n x_n \rightharpoonup 0$ which is impossible as $p(\lambda_n x_n) \geq \rho$ for all $n$.

The implication (ii) $\Rightarrow$ (iii) follows directly from Corollary 2.3 (i) $\Rightarrow$ (ii) and the implication (iii) $\Rightarrow$ (iv) follows from Theorem 2.2 (i) $\Rightarrow$ (ii). Finally the implication (iv) $\Rightarrow$ (i) can be proved just like the corresponding one in Corollary 2.3.

In the next section we shall study the (O.P.)-property from a different angle. With an eye on Theorem 1.6 (v) we shall assume that $K$ is not spherically complete.
§3. (∞)-SPACES.

FROM NOW ON IN THIS PAPER WE ASSUME THAT $K$ IS NOT SPHERICALLY COMPLETE

DEFINITION 3.1. A locally convex space $E$ is said to be an $(\infty)$-space if every continuous linear map $\ell^\infty \to E$ is compact (i.e. the image of the unit ball of $\ell^\infty$ is a compactoid set in $E$).

The next theorems reveal the connection with the previous sections.

THEOREM 3.2. For a locally convex space $(E, \tau)$ consider the following properties,

(a) $E$ is an $(O,P)$-space.
(b) $E$ is an $(\infty)$-space and every weakly bounded set is bounded.
(c) $E$ is an $(\infty)$-space, $E'$ separates the points of $E$ and every absolutely convex weakly bounded and $\sigma(E, E')$-metrizable set is bounded.

Then, (a) $\Rightarrow$ (b) $\iff$ (c).

If in addition $E$ is weakly sequentially complete, then properties (a) - (c) are equivalent.

Proof.

(a) $\Rightarrow$ (b). It suffices, by Corollary 2.4 (i) $\Rightarrow$ (ii), to show that $E$ is an $(\infty)$-space. So let $T \in L(\ell^\infty, E)$ and let $e_1 := (1,0,0,...)$, $e_2 = (0,1,0,...),\ldots$ be the unit vectors of $\ell^\infty$. $K$ is not spherically complete so $(\ell^\infty)' \simeq c_0$ ([13], Theorem 4.17) and therefore for each $y = (\eta_1, \eta_2,\ldots)$ in $\ell^\infty$ we have $y = \lim_{n \to \infty} \sum_{i=1}^{n} \eta_i e_i$ weakly. By weak continuity

$$Ty = \lim_{n \to \infty} \sum_{i=1}^{n} \eta_i Te_i \text{ weakly.}$$

By (a) the above limit is in the sense of $\tau$. Thus, the unit ball of $\ell^\infty$ is mapped into the $\tau$-closure of $co\{Te_1, Te_2,\ldots\}$ which is a compactoid since $Te_n \to 0$. We conclude that $T$ is compact and that $E$ is an $(\infty)$-space.

The proof of (b) $\Rightarrow$ (c) is elementary; we prove (c) $\Rightarrow$ (b). Suppose $X \subset E$ is weakly bounded but not bounded. Then there exist a continuous seminorm $p$ and a sequence $x_1, x_2,\ldots$ in $X$ such that $p(x_n) \to \infty$. There exist $\lambda_1, \lambda_2, \ldots \in K$ such that $\lim_{n \to \infty} \lambda_n = 0$ and $p(\lambda_n x_n) \to \infty$. As $\lambda_n \to 0$ and $\{x_1, x_2,\ldots\}$ is weakly bounded we have $\lambda_n x_n \to 0$ weakly. Then $co(\lambda_1 x_1, \lambda_2 x_2,\ldots)$ is weakly metrizable, weakly bounded, absolutely convex, hence bounded by (c). But this conflicts $p(\lambda_n x_n) \to \infty$.

Now assume that $E$ is weakly sequentially complete.
\[(\gamma) \Rightarrow (\alpha). \text{ Let } x_1, x_2, \ldots \text{ be a sequence in } E \text{ converging weakly to } 0. \text{ Then the formula} \]

\[(\eta_1, \eta_2, \ldots) \xrightarrow{T} \sigma(E, E') \lim_{n \to \infty} \sum_{i=1}^{n} \eta_i x_i \]

defines, by weak sequential completeness, a linear map \(T : \ell^\infty \to E\). It maps the unit ball of \(\ell^\infty\) into the set \(A := \overline{\sigma(E, E')} (x_1, x_2, \ldots)\). Since \(x_n \to 0\) weakly we have that \(A\) is a weakly bounded and \(\sigma(E, E')\)-metrizable set and from \((\gamma)\) we derive that \(A\) is bounded. Then \(T\) maps the unit ball into a bounded set so that \(T\) is continuous. By assumption \(T\) is compact. Then \(T(\ell^\infty)\) is a space of countable type ([7], Proposition 4.3). We have that \(x_n = Te_n \to 0\) in the topology \(\sigma(T\ell^\infty, (T\ell^\infty)'\). Theorem 1.6 implies that \(x_n \xrightarrow{} 0\).

**THEOREM 3.3.** For a locally convex space \((E, \tau)\) consider the following properties.

\((\alpha)\) \(E\) is a (B.O.P.)-space.

\((\beta)\) \(E\) is an \((\infty)\)-space, \(E'\) separates the points of \(E\) and for every absolutely convex bounded and \(\sigma(E, E')\)-metrizable set \(A \subset E, \overline{A}^{\sigma(E, E')}\) is bounded.

Then, \((\alpha) \Rightarrow (\beta)\). If in addition \(E\) is weakly sequentially complete, then also \((\beta) \Rightarrow (\alpha)\).

**Proof.**

\((\alpha) \Rightarrow (\beta)\). Observe that for each \(T \in L(\ell^\infty, E)\) and for each \(y := (\eta_1, \eta_2, \ldots)\) in \(\ell^\infty\), the sequence consisting of the partial sums \(\sum_{i=1}^{n} \eta_i e_i\) is bounded. As in \((\alpha) \Rightarrow (\beta)\) of Theorem 3.2 we conclude that \(E\) is an \((\infty)\)-space. The rest follows from Corollary 2.3.

Now, assume that \(E\) is weakly sequentially complete. Then \((\beta) \Rightarrow (\alpha)\) can be proved in a similar way as \((\gamma) \Rightarrow (\alpha)\) in Theorem 3.2.

**REMARKS.**

1. There exist (C.O.P.)-spaces which are not \((\infty)\)-spaces (e.g. \(\ell^\infty\)).

2. Of course, \((\alpha) \Rightarrow (\beta)\) in Theorems 3.2 and 3.3 is not true if the field is spherically complete.

3. There exist Banach \((\infty)\)-spaces which are not (B.O.P.)-spaces. To construct an example we need a preliminary concept. Let us say that an ultrametric group is an abelian group \(G\), together with an invariant ultrametric \(d\). A surjective homomorphism \(\varphi : (G_1, d_1) \to (G_2, d_2)\), where \((G_1, d_1), (G_2, d_2)\) are ultrametric groups, is called a quotient map if

\[d_2(\varphi(x), 0) = \inf\{d_1(y, 0) : y \in G_1, \varphi(x) = \varphi(y)\}\]

for each \(x \in G_1\).
The following result is crucial for our purpose.

**Theorem 3.4.** Let $E, F$ be Banach spaces over $K$. Let $\varphi : E \to F$ be a quotient map. Let $D := \ker \varphi$. Then, if $F$ is spherically complete we have for each $f \in E'$,

$$
\sup_{x \in B_E} |f(x)| = \sup_{x \in B_D} |f(x)|
$$

(1)

(where $B_E$ and $B_D$ denote the "closed" unit ball in $E$ and $D$ respectively).

**Proof.** Suppose there exists an $f \in E'$ such that $\|f\| = 1$ and $|f| \leq r$ on $B_D$ for some $r < 1$. Let $\rho : B_K \to B_K/\{\lambda \in K : |\lambda| \leq r\} =: k_r$ be the canonical surjection. With the natural quotient metric $k_r$ is an ultrametric group (which is easily seen to be not spherically complete) and $\rho$ is a quotient map in the sense of above. Then $\rho f : B_E \to k_r$ is a quotient map as well, which is zero on $B_D$. The restriction of $\varphi$ to $B_E$ (again called $\varphi$) is a quotient map $\varphi : B_E \to \varphi(B_E)$ whose kernel equals $B_D$.

There is a unique homomorphism of groups $\psi$ making

$$
\begin{array}{ccc}
B_E & \xrightarrow{\varphi} & \varphi(B_E) \\
\rho f & \searrow & \downarrow \psi \\
k_r & & B_D
\end{array}
$$

commute. Since $\varphi$ and $\rho \circ f$ are quotient maps, so is $\psi$. Also, as $\varphi$ is a quotient map, $\varphi(B_E)$ is spherically complete. Similarly, $k_r$ is spherically complete, a contradiction.

Now we are ready to provide examples of $(\infty)$-spaces which are not (O.P.)-spaces.

**Theorem 3.5.** Let $E$ be an $(\infty)$-Banach space and suppose there is a closed subspace $D \neq E$ satisfying property (1) of Theorem 3.4. (For example, take $E := c_0(X)$ where $X$ is the "closed" unit ball of $F := \ell^\infty/c_0$, $D := \ker \varphi$ where $\varphi : E \to F$ is given by $\varphi(\sum \lambda_x e_x) = \sum \lambda_x x$; now apply Theorem 3.4.)

For each $n \in \mathbb{N}$, let $E_n$ be the space $E$ endowed with the norm

$$
N_n(x) = \max(\|x\|, n \operatorname{dist}(x, D))
$$

Then

$$
G := \{(x_1, x_2, \ldots) \in \prod_{n \in \mathbb{N}} E_n : \lim_{n \to \infty} N_n(x_n) = 0\}
$$

with the norm given by

$$
N(x_1, x_2, \ldots) = \max_n N_n(x_n)
$$

is an $(\infty)$-Banach space which is not an (O.P.)-space.
Proof. Firstly observe that the norms $N_n$ and $\| \cdot \|$ are equivalent so that, as sets, $E'_n = E'$ for each $n \in \mathbb{N}$. Also, by property (I) it follows directly that the identity map from $E'$ onto $E'_n$ is an isometry.

Now we prove that $G$ is an $(\infty)$-space. Let $T \in L(\ell^\infty, G)$ and for each $n \in \mathbb{N}$ let $\pi_n : G \to E_n$ be the obvious continuous projection. Then $\pi_n \circ T \in L(\ell^\infty, E_n) = L(\ell^\infty, E)$ is a compact map, so $\pi_nT(\ell^\infty)$ is of countable type. Then $T\ell^\infty$ is in the $N$-closure of $\sum_n \pi_nT(\ell^\infty)$ and is therefore of countable type. Now, apply Theorem 3.9 (a) $\iff$ (γ) below.

Finally we show that $(G, N)$ is not an (O.P.)-space. For each $n \in \mathbb{N}$ choose $u_n \in E_n$ such that $\|u_n\| \leq \frac{1}{\sqrt{n}}$, and set $z_n = (0, 0, \ldots, u_n, 0, \ldots) \in G$.

We see that $N(z_n) = N_n(u_n) \geq n \text{ dist } (u_n, D) \geq \sqrt{n}$, so the sequence $\{z_1, z_2, \ldots\}$ is not bounded in $(G, N)$. On the other hand, let $f \in (G, N)'$. Then (see [13], Exercise 3.Q) there exist $f_n \in E'_n$ such that

$$f((x_1, x_2, \ldots)) = \sum_{n=1}^{\infty} f_n(x_n) \quad ((x_1, x_2, \ldots) \in G)$$

while

$$M := \sup_n N_n(f_n) < \infty.$$ 

We have:

$$|f(z_n)| = |f_n(u_n)| \leq \|f_n\| \|u_n\| = N_n(f_n)\|u_n\| \leq M\|u_n\| \leq M/\sqrt{n}.$$ 

so $z_n \to 0$ weakly. Thus, $(G, N)$ is not an (O.P.)-space.

Next, we shall see that for $(\infty)$-spaces we have the same stability properties as for (O.P.) ((B.O.P.) or (C.O.P.))-spaces. To see that we need the following lemma.

**Lemma 3.6.** Let $\{E_i : i \in I\}$ be a family of locally convex spaces. Then for any bounded set $X \subset \bigoplus_{i \in I} E_i$

$$J := \{i \in I : \text{ there is an } x \in X \text{ with } x_i \neq 0\}$$

is a finite set.

**Proof.** Suppose we have distinct $i_1, i_2, \ldots \in I$ such that for each $n$ there exists an $x^n \in X$ with $x^n_{i_n} \neq 0$. Then there are continuous seminorms $q_{i_1}$ on $E_{i_1}, q_{i_2}$ on $E_{i_2}, \ldots$ such that
q_i(x^n) \geq n for each n. For i \in \Lambda \{i_1, i_2, \ldots\} we define q_i to be zero on E_i. Then 
q(x) = \max_{i \in I} q_i(x_i) is a continuous seminorm on \( \bigoplus_{i \in I} E_i \). We have 
g(x^n) \geq q_i(x^n_i) \geq n 
for each n so that X is not bounded, a contradiction.

**PROPOSITION 3.7.**

a) A subspace of an \((\infty)\)-space is again an \((\infty)\)-space.

b) The product of a family of \((\infty)\)-spaces is again an \((\infty)\)-space.

c) The locally convex direct sum of a family of \((\infty)\)-spaces is again an \((\infty)\)-space.

d) If E is a locally convex space such that E' separates the points of E and D is a 
finite dimensional subspace of E, then 

d.i) If E is an \((\infty)\)-space then so is E/D.

d.ii) If E/D is an \((\infty)\)-space then so is E.

**Proof.** The proof of a) is direct verification and the proof of (d) is analogous to the 
proof of Proposition 1.5.

b) Let \{E_i : i \in I\} be a family of \((\infty)\)-spaces and let \( T \in L(\ell^\infty, \prod_{i \in I} E_i) \). Then 
\( \pi_i \circ T \) (where \( \pi_i \) is the projection onto \( E_i \)) is compact for each i, implying that 
\( TB_{\ell^\infty} \subset \prod_{i \in I} \pi_i(TB_{\ell^\infty}) \) (where \( B_{\ell^\infty} \) denotes the (closed) unit ball of \( \ell^\infty \)) is a compactoid, i.e., \( T \) 
is compact.

c) Let \{E_i : i \in I\} be a family of \((\infty)\)-spaces and let \( T \in L(\ell^\infty, \bigoplus_{i \in I} E_i) \). Then 
\( TB_{\ell^\infty} \) is bounded so it lies in \( \bigoplus_{i \in I} E_i \) for some finite \( J \subset I \) by Lemma 3.6. Now apply 
b) and the fact that \( \bigoplus_{i \in J} E_i \simeq \prod_{i \in J} E_i \) to arrive at the compactness of \( T \).

It turns out that we can sharpen statement d.ii) for \((\infty)\)-spaces (see Proposition 
3.11). As a stepping stone we prove several characterizations of \((\infty)\)-spaces (see Theorem 
3.9).

**LEMMA 3.8.** Let \((E, \tau)\) be a locally convex space and let \( T \in L(\ell^\infty, E) \). Then \( T \) is 
compact if and only if it has the form 

\[(\eta_1, \eta_2, \ldots) \mapsto \sum_{n=1}^{\infty} \eta_n x_n \quad (*)\]

where \( x_1, x_2, \ldots \in E, x_n \rightharpoonup 0 \). In particular, if \( T \) is compact then 
\( TB_{\ell^\infty} \) is a metrizable compactoid.

**Proof.** Suppose \( T \) is compact. Set \( x_n := T e_n \) where \( e_1, e_2, \ldots \) are the unit vectors of 
\( \ell^\infty \). Then \( e_n \to 0 \) weakly so \( x_n \to 0 \) weakly in \( F = T \ell^\infty \), a space of countable type,
hence an (O.P.)-space. So $x_n \to 0$ and

$$T(\eta_1, \eta_2, \ldots) = \sigma(E, E') - \lim_{n \to \infty} \sum_{i=1}^{n} \eta_i x_i =$$

$$= \tau - \lim_{n \to \infty} \sum_{i=1}^{n} \eta_i x_i = \sum_{i=1}^{\infty} \eta_i x_i$$

for each $(\eta_1, \eta_2, \ldots) \in \ell^\infty$.

Conversely, if $T$ has the form (*) then $TB_{t^\infty}$ is contained in the $\tau$-closure of $\text{co}\{x_1, x_2, \ldots\}$ where $x_n \to 0$. So $TB_{t^\infty}$ is a $\tau$-metrizable (see [8], Theorem 6.1) compactoid.

**Theorem 3.9.** For a locally convex space $E$ the following are equivalent:

1. $E$ is an $(\infty)$-space.
2. For every $T \in L(\ell^\infty, E)$ the image of the unit ball is a metrizable compactoid.
3. For every $T \in L(\ell^\infty, E)$ the range of $T$ is of countable type.
4. For every $T \in L(\ell^\infty, E)$ the range of $T$ is an (O.P.)-space.
5. For every $T \in L(\ell^\infty, E)$ the range of $T$ is an $(\infty)$-space.
6. For every $T \in L(\ell^\infty, E)$ and for every weakly convergent sequence $y_1, y_2, \ldots \in \ell^\infty$, the image $Ty_1, Ty_2, \ldots$ is a convergent sequence in $E$.

**Proof.** $(\alpha) \Rightarrow (\beta)$ follows from Lemma 3.8. The implications $(\beta) \Rightarrow (\delta) \Rightarrow (\varepsilon) \Rightarrow (\alpha)$ are obvious.

Clearly $(\delta) \Rightarrow (\eta)$. Also $(\eta) \Rightarrow (\alpha)$ follows easily by applying Lemma 3.8.

**Lemma 3.10.** Let $E$ be a locally convex space and let $D$ be a closed subspace of countable type. If $S$ is a subspace of countable type of $E/D$ then $\pi^{-1}(S)$ is also of countable type, where $\pi : E \to E/D$ is the quotient map.

**Proof.** Let $p$ be a continuous seminorm on $E$; we produce a countable set $X \subset \pi^{-1}(S)$ such that $[X]$ is $p$-dense in $\pi^{-1}(S)$. There exists a countable set $F \subset S$ such that $F$ is $\overline{p}$-dense in $S$ where $\overline{p}$ is the quotient seminorm of $p$ on $E/D$. Choose a countable set $Y \subset \pi^{-1}(S)$ with $\pi(Y) = F$ and a countable set $Z \subset D$ such that $[Z]$ is $p$-dense in $D$. Set $X := Z + Y$. To prove that $[X]$ is $p$-dense in $\pi^{-1}(S)$, let $x \in \pi^{-1}(S)$ and $\varepsilon > 0$. There is an $y \in [Y]$ with $\overline{p} \pi(x - y) < \varepsilon$, so there exists $d \in D$ with $p(x - y - d) < \varepsilon$. There is a $z \in [Z]$ with $p(d - z) < \varepsilon$. Then $p(x - (y + z)) < \varepsilon$ and we are done.

**Remark.** By taking $S = E/D$ in the above proof we obtain the following: If $D$ and $E/D$ are of countable type then so is $E$. 

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Now we state the announced strong version of Proposition 3.7. d.ii).

**Proposition 3.11.** Let $E$ be a locally convex space and let $D$ be a closed subspace of countable type. Then, if $E/D$ is an $(\infty)$-space then so is $E$.

**Proof.** Let $T \in L(\ell^\infty, E)$, let $\pi : E \to E/D$ be the quotient map. Then $\pi \circ T$ is compact so $(\pi \circ T) (\ell^\infty)$ is of countable type in $E/D$. By Lemma 3.10 $\pi^{-1} (\pi \circ T (\ell^\infty))$ is of countable type in $E$, hence so is $T (\ell^\infty)$. By Theorem 3.9 $\gamma \Rightarrow (\alpha)$, $E$ is an $(\infty)$-space.

Of similar nature is the following "3-space property" for Fréchet (i.e., complete and metrizable) spaces.

**Theorem 3.12.** Let $E$ be a Fréchet space and let $D$ be a closed linear subspace of $E$. If $D$ and $E/D$ are $(\infty)$-spaces then so is $E$.

**Proof.** Let $T \in L(\ell^\infty, E)$, let $\pi : E \to E/D$ be the quotient map. By assumption $\pi \circ T$ is compact so it has the form (Lemma 3.8)

$$
(\eta_1, \eta_2, \ldots) \overset{\pi \circ T}{\longrightarrow} \sum_{n=1}^{\infty} y_n x_n
$$

where $x_n \to 0$ in $E/D$. By metrizability we can find a sequence $y_1, y_2, \ldots$ in $E$ with $y_n \to 0$ and $\pi(y_n) = x_n$ for all $n$. By completeness, the formula

$$
V(\eta_1, \eta_2, \ldots) = \sum_{n=1}^{\infty} \eta_n y_n
$$

defines a compact map $V \in L(\ell^\infty, E)$. We have $\pi \circ (T - V) = 0$ so that $T - V \in L(\ell^\infty, D)$. By assumption $T - V$ is compact and so is $T = (T - V) + V$.  

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§4. POLARITY.

Recall ([6], Definition 3.5) that a locally convex space $E$ is polar if its topology is generated by a base of polar seminorms, where a seminorm $p$ is polar if $p = \sup\{|f| : f \in E^*, |f| \leq p\}$.

It is natural to ask whether each (O.P.)-space is automatically polar. Let us show that the answer, in general, is no.

**Proposition 4.1.** Let $(E, \tau)$ be an (O.P.)-space admitting a non-trivial continuous seminorm $p$ with

$$ f \in E^*, |f| \leq p \Rightarrow f = 0. $$

Then, there is a locally convex topology $\tau_1$ with $\sigma(E, E') \subset \tau_1 \subset \tau$ such that $(E, \tau_1)$ is nonpolar but an (O.P.)-space.

**Proof.** We may assume that $\tau$ is polar. (otherwise take $\tau_1 := \tau$). Let $\tau_1$ be the locally convex topology induced by $\sigma(E, E')$ and the single seminorm $p$. Then, clearly $\sigma(E, E') \subset \tau_1 \subset \tau$ and hence $(E, \tau_1)$ is an (O.P.)-space.

Suppose $(E, \tau_1)$ is polar; we arrive at a contradiction. There is a $\tau_1$-continuous polar seminorm $q$ such that $p \leq q$. As $\sigma(E, E')$ and $p$ generate the topology $\tau_1$ we have $q \leq \max(cp, r)$ for some $c > 0$ and some $\sigma(E, E')$-continuous seminorm $r$. Without loss, $r = \max(|f_1|, |f_2|, \ldots, |f_n|)$ where $f_1, \ldots, f_n \in E'$. Set $H := \bigcap_{i=1}^n \ker (f_i)$. Then on $H$ we have $p \leq q \leq cp$. We can find a non-trivial $g \in E'$ with $|g| \leq \frac{1}{2c}q$. Then $|g| \leq \frac{1}{2}p$ on $H$. Since $H$ has finite codimension we can extend $g$ to a $\tilde{g} \in E^*$ such that $|\tilde{g}| \leq p$ on $E$, a contradiction.

To find an example of an (O.P.)-space which is not polar, take a set $I$ with cardinality large enough to make possible a linear surjection $\pi : c_0(I) \to \ell^\infty / c_0$. Then $p : x \to \|\pi(x)\| \quad (x \in c_0(I))$ satisfies the requirement of Proposition 4.1 and $(c_0(I), \tau_1)$ is the wanted space where $\tau_1$ is the topology generated by the weak topology and the seminorm $p$.

The example we have above is not metrizable which is not accidental. In fact, we shall see that metrizable locally convex (O.P.)-spaces are automatically polar (see Corollary 4.4).

**Lemma 4.2.** Let $E$ be a $K$-vector space and let $\tau_1, \tau_2$ be locally convex topologies both induced by countably many seminorms. Suppose $\tau_1$-bounded = $\tau_2$-bounded for subsets of $E$. Then, $\tau_1 = \tau_2.$
Proof. Let \( p_1 \leq p_2 \leq \ldots \leq p_n \leq \ldots \) be a sequence of seminorms defining \( \tau_2 \) and let \( q_1 \leq q_2 \leq \ldots \leq q_n \leq \ldots \) be a sequence of seminorms defining \( \tau_1 \). It suffices to prove that \( \tau_2 \leq \tau_1 \).

First we show that \( p_1 \leq C q_n \) for some constant \( C > 0 \) and some \( n \in \mathbb{N} \).

Suppose not. Then \( p_1 \leq C q_1 \) is true for no \( C > 0 \) so there exists a sequence \( x_{11}, x_{12}, \ldots \) in \( E \) with \( q_1(x_{1n}) \leq 1 \) for each \( n \) while \( p_1(x_{1n}) \geq n \) for each \( n \). Also \( p_1 \leq C q_2 \) is true for no \( C > 0 \) so there exists a sequence \( x_{21}, x_{22}, \ldots \) in \( E \) with \( q_2(x_{2n}) \leq 1 \) and \( p_1(x_{2n}) \geq n \) for each \( n \), etc.

Inductively we find a double sequence

\[
\begin{align*}
x_{11}, & \quad x_{12}, \quad x_{13}, \ldots \\
x_{21}, & \quad x_{22}, \quad x_{23}, \ldots \\
\vdots & \quad \vdots \quad \vdots
\end{align*}
\]

in \( E \) such that \( q_k(x_{kn}) \leq 1 \) for each \( k, n \) while \( p_1(x_{kn}) \geq n \) for each \( k, n \). We see that the diagonal sequence \( x_{11}, x_{22}, \ldots \) is \( q_k \)-bounded for each \( k \) so it is \( \tau_1 \)-bounded. But \( p_1(x_{nn}) \geq n \), which implies that the sequence \( x_{11}, x_{22}, \ldots \) is not \( \tau_2 \)-bounded: a contradiction.

In a similar way as above we can prove that \( p_2, p_3, \ldots \) are majorized by positive multiples of some \( q_n \). It follows that \( \tau_2 \leq \tau_1 \).

**Theorem 4.3.** Let \((E, \tau)\) be a metrizable locally convex space such that every weakly bounded set is bounded. Then \( E \) is polar.

**Proof.** Let \( p_1 \leq p_2 \leq \ldots \leq p_n \leq \ldots \) be an increasing sequence of seminorms inducing \( \tau \). Let, for each \( n \), \( \bar{p}_n \) be the polar seminorm associated to \( p_n \) (i.e. \( \bar{p}_n = \sup \{|f| : f \in E^*, |f| \leq p_n\} \)). We have \( \bar{p}_1 \leq \bar{p}_2 \leq \ldots \), the topology induced by these \( \bar{p}_n \) is polar. It is easily seen that \( \bar{\tau} \) is the strongest polar topology that is \( \subseteq \tau \). Now let \( X \subset E \).

By [6] Theorem 7.5 we have \( X \) is weakly bounded \( \iff \) \( X \) is \( \bar{\tau} \)-bounded. Then, by assumption, \( \tau \)-bounded \( \Rightarrow \) \( \bar{\tau} \)-bounded. Then \( \tau = \bar{\tau} \) by Lemma 4.2, i.e., \( \tau \) is polar.

**Corollary 4.4.** A metrizable \((O.P.)\)-space is polar.

**Proof.** It is a direct consequence of Corollary 2.4 i) \( \Rightarrow \) ii) and Theorem 4.3.

**Remarks.**

1. There are metrizable \((C.O.P.)\)-spaces which are not polar.

   *Example:* Take \( E = \bigoplus_n \ell_1^n \) as in [13] Exercise 4.5. \( E \) is a Banach space whose dual \( E' \) separates the points of \( E \) and which is not polar.
On the other hand, by looking at the proof of Theorem 2.2 it is very easy to see that a Banach space is a (C.O.P.)-space if and only if its dual separates the points. Then $E = \bigoplus_n \ell^\infty_n$ as above is a (C.O.P.)-space which is not polar.

2. As we know (see p.5), every (B.O.P.) normed space $E$ is an (O.P.)-space and hence it is polar.

Corollary 4.4 allows us to formulate the following question:

**PROBLEM.** $E$ is a metrizable (B.O.P.)-space $\Rightarrow E$ is polar?

Now, we shall prove interesting characterizations of polar ($\infty$)-spaces. The heart of the matter is contained in the next lemma.

**LEMMA 4.5.** Let $(E, \tau)$ be a polar locally convex space. Suppose $E$ does not contain (a subspace linearly homeomorphic to) $\ell^\infty$. Then $E$ is an ($\infty$)-space.

**Proof.** Let $T \in L(\ell^\infty, E)$ be not compact; we derive a contradiction by showing that $\ell^\infty$ is a subspace of $E$. Let $e_1, e_2, \ldots$ be the unit vectors of $\ell^\infty$. Then $\{Te_1, Te_2, \ldots\}$ is not a compactoid (otherwise, $TB_{\ell^\infty}$, being a subset of the weak closure of $co\{Te_1, Te_2, \ldots\}$ would be a compactoid, so $T$ would be compact). So, there exists a continuous polar seminorm $p$ such that $\{Te_1, Te_2, \ldots\}$ is not $p$-compactoid. By [11], Theorem 2 there exists a $t \in (0, 1)$ and a subsequence $z_1, z_2, \ldots$ of $Te_1, Te_2, \ldots$ that is $t$-orthogonal with respect to $p$ and such that $\inf p(z_n) > 0$. Without loss, assume $p(z_n) \geq 1$ for each $n$.

Now, inductively we shall construct a subsequence $u_1, u_2, \ldots$ of $z_1, z_2, \ldots$ and $f_1, f_2, \ldots \in E'$ such that $|f_n| \leq 2t^{-1}p$ for all $n$ and

$$|f_m(u_n)| = \begin{cases} 0 & \text{if } m > n \\ 1 & \text{if } m = n \\ |f_m(u_n)| & \text{if } m < n \end{cases}$$

To do that, observe that the function $h_1 : \lambda z_1 \mapsto \lambda$ ($\lambda \in K$) satisfies $|h_1| \leq p$. By polarity it can be extended to an $f_1 \in E'$ such that $|f_1| \leq 2p$. Set $u_1 := z_1$. Suppose $f_1, \ldots, f_{m-1}$ and $u_1, \ldots, u_{m-1}$ are chosen with the required properties. Since $Te_n \rightharpoonup 0$ weakly we have $z_n \rightharpoonup 0$ weakly. So we can find a $k$ (larger than the indexes with respect to $z$ of $u_1, \ldots, u_{m-1}$) such that $|f_1(z_n)| \leq 1/2$, $|f_{m-1}(z_n)| \leq 1/2$ for $n \geq k$. Choose $u_m := z_k$. The function $h_m : \lambda_1 u_1 + \cdots + \lambda_m u_m \mapsto \lambda_m$ ($\lambda_1, \ldots, \lambda_m \in K$) satisfies $|h_m| \leq t^{-1}p$ so it can be extended to a function $f_m \in E'$ such that $|f_m| \leq 2t^{-1}p$. We see that $f_1, \ldots, f_m$ and $u_1, \ldots, u_m$ have the required properties.

Now, we have that $u_1, u_2, \ldots$ is a subsequence, say $Te_{i_1}, Te_{i_2}, \ldots$ of $Te_1, Te_2, \ldots$. Define a linear isometry $\Omega : \ell^\infty \to \ell^\infty$ by the formula

$$\Omega(y_1, y_2, \ldots)_n = \begin{cases} 0 & \text{if } n \notin \{i_1, i_2, \ldots\} \\ y_n & \text{if } n \in \{i_1, i_2, \ldots\} \end{cases}$$

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and set $S := T \circ \Omega$. Then obviously $S \in L(\ell^\infty, E)$ and $S$ is described by the formula

$$S(y_1, y_2, \ldots) = \sigma(E, E') - \sum_{n=1}^{\infty} y_n u_n.$$ 

Finally let $y = (y_1, y_2, \ldots) \in \ell^\infty$, $y \neq 0$. There is an $m \in \mathbb{N}$ such that $|y_m| > \frac{1}{2} \|y\|$. We have $p(Sy) \geq \frac{1}{2} t|f_m(Sy)| = \frac{1}{2} t \sum_{n \geq m} y_n f_m(u_n)$. If $n > m$ we have $|y_n f_m(u_n)| \leq \frac{1}{2} |y_n| \leq \frac{1}{2} \|y\|$ whereas $|y_m f_m(v_m)| = |y_m| > \frac{1}{2} \|y\|$ so $p(Sy) \geq \frac{1}{2} \|y\|$ implying that $S$ is a linear homeomorphism from $\ell^\infty$ onto $S(\ell^\infty) \subseteq E$ which gives the desired contradiction.

**Lemma 4.6.** Let $E$ be a polar locally convex space and let $i : \ell^\infty \to E$ be a linear homeomorphism of $\ell^\infty$ onto $i(\ell^\infty)$. Then there exists a continuous linear map $P : E \to \ell^\infty$ such that $P \circ i$ is the identity on $\ell^\infty$.

**Proof.** There exists a continuous polar seminorm $p$ on $E$ such that $x \mapsto p(i(x))$ ($x \in \ell^\infty$) is equivalent to the standard norm on $\ell^\infty$. Let $\overline{p} : E/\ker p \to [0, \infty)$ be the quotient norm of $p$ and let $\pi : E \to E/\ker p$ be the quotient map. The map

$$\pi \circ i : \ell^\infty \to E \to E/\ker p$$

is a linear homeomorphism of $\ell^\infty$ into the normed space $(E/\ker p, \overline{p})$. Then (see [10]) there exists a linear continuous map $Q : (E/\ker \overline{p}, \overline{p}) \to \ell^\infty$ such that $Q \circ \pi \circ i$ is the identity on $\ell^\infty$. Now set $P := Q \circ \pi$.

**Corollary 4.7.** For a polar locally convex space $(E, \tau)$ the following are equivalent.

(a) $E$ is an $(\infty)$-space.
(b) For every $T \in L(\ell^\infty, E)$, $Te_n \to 0$ in $E$ (where $e_1, e_2, \ldots$ are the unit vectors of $\ell^\infty$).
(c) For every $T \in L(\ell^\infty, E)$ the restriction $T|c_0$ is compact.
(d) $E$ does not contain a subspace linearly homeomorphic to $\ell^\infty$.
(e) $E$ does not contain a complemented subspace linearly homeomorphic to $\ell^\infty$.

If in addition $E$ is weakly sequentially complete, properties (a) – (e) are equivalent to

(f) $E$ is an (O.P.)-space.
(g) Every bounded absolutely convex and $\sigma(E, E')$-metrizable subset of $E$ is compactoid.
(h) If $F$ is a locally convex space and $T \in L(F, E)$ then $T$ maps weakly convergent sequences in $F$ into convergent sequences in $E$. 

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Proof.

(a) $\Rightarrow$ (b) follows by Theorem 3.9 ($\alpha$) $\Rightarrow$ ($\eta$).

(b) $\Rightarrow$ (c). Let $T \in L(\ell^\infty, E)$. Given $x = \sum_{n=1}^{\infty} x_n e_n$ in $c_0$ we have $Tx = \sum_{n=1}^{\infty} x_n T e_n$.

So $T$ maps the unit ball of $c_0$ into $\overline{\sigma}(T e_1, T e_2, \ldots)$ which is a compactoid set since $T e_n \to 0$.

(c) $\Rightarrow$ (d) is obvious; (d) $\Rightarrow$ (e) and (e) $\Rightarrow$ (a) follow from Lemmas 4.5 and 4.6.

Now assume that $E$ is weakly sequentially complete.

The equivalence (a) $\iff$ (f) is a direct consequence of Theorem 3.2 ($\alpha$) $\iff$ ($\beta$).

The implication (f) $\Rightarrow$ (g) follows from Corollary 2.4 i) $\Rightarrow$ iii).

(g) $\Rightarrow$ (h). Let $F$ be a locally convex space and let $T \in L(F, E)$. If $x_n \to 0$ weakly in $F$ clearly we have $Tx_n \to 0$ weakly in $E$, so ([8], Theorem 6.1) $\overline{\sigma}(E, E')\{Tx_1, Tx_2, \ldots\}$ is $\sigma(E, E')$-metrizable and by (g) it is a compactoid set. Thus, $Tx_n \to 0$ in $\tau$.

Finally, observe that by taking $F = \ell^\infty$ in property (h) we derive property ($\eta$) of Theorem 3.9 and hence $E$ is an $\infty$-space.
§5. METRIZABLE (O.P.)-SPACES.

**Proposition 5.1.** Let \((E, \tau)\) be a metrizable locally convex space and let \(D\) be a dense subspace of \(E\). If \(D\) is an (O.P.)-space then so is \(E\).

**Proof.** There is an invariant metric \(d\) on \(E\) inducing \(\tau\). Let \(x_1, x_2, \ldots\) be a sequence in \(E\) such that \(\lim_{n \to \infty} x_n = 0\) in \(\sigma(E, E')\). For each \(n\), choose a \(y_n \in D\) with \(d(x_n, y_n) \leq \frac{1}{n}\). Then \(x_n - y_n \to 0\) so \(x_n - y_n \to 0\) in \(\sigma(E, E')\) and also \(y_n = y_n - x_n + x_n \to 0\) in \(\sigma(E, E')\) hence in \(\sigma(D, D')\). Since \(D\) is an (O.P.)-space, \(y_n \to 0\). But then \(x_n = x_n - y_n + y_n \to 0\).

**Problem.** Does the conclusion hold if we drop the metrizability condition?

A standard application of the Closed Graph Theorem yields the following lemma.

**Lemma 5.2.** Let \((E, \tau_1), (F, \tau_2)\) be Fréchet spaces and let \(T : E \to F\) be a linear map. Suppose \((F, \tau_2)\) separates the points of \(F\) and that \(T : (E, \tau_1) \to (F, \sigma(F, F'))\) is continuous. Then \(T : (E, \tau_1) \to (F, \tau_2)\) is continuous.

The following results give some characterizations of metrizable (O.P.)-spaces.

**Theorem 5.3.** For a Fréchet space \(E\) the following are equivalent:

1. \(E\) is an (O.P.)-space.
2. \(E'\) separates the points of \(E\), \(E\) is weakly sequentially complete, \(E\) is an \((\infty)\)-space.

**Proof.**

(\(\alpha\)) \(\Rightarrow\) (\(\beta\)). That \(E\) is an \((\infty)\)-space with separating dual follows from Theorem 3.2

(\(\alpha\)) \(\Rightarrow\) (\(\beta\)).

Also if \(x_1, x_2, \ldots\) is weakly Cauchy then \(x_{n+1} - x_n \to 0\) weakly hence strongly. As \(E\) is Fréchet, \(x_n \to x\) strongly for some \(x \in E\), hence weakly. Then, \(E\) is weakly sequentially complete.

(\(\beta\)) \(\Rightarrow\) (\(\alpha\)). Let \(x_1, x_2, \ldots\) be a sequence in \(E\) tending weakly to 0. Then the formula

\[(\eta_1, \eta_2, \ldots) \mapsto T(\sigma(E, E')) - \sum_{i=1}^{\infty} \eta_i x_i\]

defines, by weakly sequential completeness a linear map \(T : \ell^\infty \to E\). It is easily seen that \(T\) is strong to weak continuous. By Lemma 5.2, \(T\) is continuous. By Theorem 3.9

(\(\alpha\)) \(\iff\) (\(\eta\)) we conclude that \(x_n \to 0\) in the initial topology of \(E\).
THEOREM 5.4. For a metrizable locally convex space $E$ the following are equivalent.

(a) $E$ is a complete (O.P.)-space.

(β) $E$ is a polar weakly sequentially complete ($\infty$)-space.

Proof. $(\alpha) \Rightarrow (\beta)$ follows from Theorem 5.3 and Corollary 4.4. To prove $(\beta) \Rightarrow (\alpha)$ observe that from Theorem 3.2 $(\beta) \Rightarrow (\alpha)$ it follows that $E$ is an (O.P.)-space. To prove completeness, let $x_1, x_2, \ldots$ be a Cauchy sequence in $E$. It is weakly Cauchy so that $x_n \to x$ weakly for some $x \in E$. Since $E$ is an (O.P.)-space we conclude that $x_n \to x$ in the initial topology of $E$.

For metrizable ($\infty$)-spaces we have the following extension of Corollary 4.7.

PROPOSITION 5.5. For a polar Fréchet space the following are equivalent.

(a) $E$ is an ($\infty$)-space.

(β) No continuous linear map $\ell^\infty \to E$ is semi-Fredholm (A continuous linear map $T$ is semi Fredholm if its Kernel, Ker($T$) is finite dimensional and its range, $T\ell^\infty$, is closed).

(γ) For every $T \in L(\ell^\infty, E)$ there exists a compact map $S \in L(\ell^\infty, E)$ such that Ker($T - S$) is infinite dimensional.

Proof.

$(\alpha) \Rightarrow (\beta)$. Assume there exists a semi-Fredholm $T : \ell^\infty \to E$. Then the corresponding bijection $\ell^\infty / \text{Ker}(T) \to T\ell^\infty$ is a linear isomorphism ([5], Corollary 2.75). Compactness of $T$ implies that the canonical quotient map $\ell^\infty \to \ell^\infty / \text{Ker}(T)$ is compact, a contradiction.

$(\beta) \Rightarrow (\alpha)$ follows directly from Corollary 4.7 (d) $\Rightarrow$ (a).

$(\alpha) \Rightarrow (\gamma)$ is obvious. (Choose $S := T$.)

$(\gamma) \Rightarrow (\alpha)$ is also a consequence of Corollary 4.7 (d) $\Rightarrow$ (a): Observe that if there is an injection $i : \ell^\infty \to E$ such that $i(\ell^\infty)$ is isomorphic to $\ell^\infty$, then by (γ) there is a compact map $S \in L(\ell^\infty, E)$ such that Ker($i - S$) is infinite dimensional. Since $i | \text{Ker}(i - S) = S | \text{Ker}(i - S)$ we derive that the restriction $i | \text{Ker}(i - S)$ is a compact map, which is impossible (see [13], Theorem 4.40).
§6. BANACH (O.P.)-SPACES.

The following definition is in a sense dual to the definition of \((\infty)\)-space (see 3.1).

**DEFINITION 6.1.** A locally convex space \(E\) is said to be a \((0)\)-space if every continuous linear map \(T : E \to c_0\) is compact (i.e., there exists a continuous seminorm \(p\) on \(E\) such that \(T\{x \in E : p(x) \leq 1\}\) is a compactoid in \(c_0\)).

In this section we study, for Banach spaces, the duality between \((0)\)-spaces on one hand and \((\infty)\)-spaces or (O.P.)-spaces on the other.

\((0)\)-SPACES have been studied by N. DE GRANDE - DE KIMPE in [1] and [2] (here the base field was spherically complete) and by T. KIYOSAWA in [4]. Putting together Theorem 8 of [2] (which also works for non-spherically complete fields) and Theorem 14 of [4] we obtain the following characterizations of Banach \((0)\)-spaces.

**THEOREM 6.2.** (see [2] and [4]). For a Banach space \(E\) the following are equivalent.

1. \(E\) is a \((0)\)-space.
2. \(E\) does not contain a complemented subspace linearly homeomorphic to \(c_0\) (Recall that every infinite dimensional Banach space contains a subspace which is isomorphic to \(c_0\)).
3. No quotient of \(E\) is isomorphic to \(c_0\).
4. In \(E'\) is every \(\sigma(E', E)\)-convergent sequence also norm convergent.
5. Let \((T_n)\) be a sequence of compact continuous linear maps from \(E\) to a Banach space \(F\), converging pointwise to \(T\). Then \(T\) is compact.
6. The space \(C(E, c_0)\) of compact continuous linear maps from \(E\) to \(c_0\) is complemented in \(L(E, c_0)\).

From this result we derive

**COROLLARY 6.3.** For a polar Banach space \(E\) we have the following.

1. \(E\) is a \((0)\)-space \(\iff\) \(E'\) is an (O.P.)-space \(\iff\) \(E'\) is an \((\infty)\)-space.
2. \(E'\) is a \((0)\)-space \(\Rightarrow\) \(E\) is an \((\infty)\)-space.

If in addition there exists a closed subspace \(D\) of \(E''\) (the bidual of \(E\)) such that \(D\) is an \((\infty)\)-space and \(E''/D\) is isomorphic to \(E\) (e.g., when \(E\) is reflexive), then

3. \(E\) is an \((\infty)\)-space \(\Rightarrow\) \(E'\) is a \((0)\)-space.
Proof.

(i) Assume $E$ is a $(0)$-space and let $f_1, f_2, \ldots$ be a sequence in $E'$ such that $f_n \to 0$ in $\sigma(E', E'')$. Then $f_n \to 0$ in $\sigma(E', E)$ and from Theorem 6.2 $(\alpha) \Rightarrow (\delta)$ we obtain that $f_1, f_2, \ldots$ is norm convergent in $E'$, i.e. $E'$ is an $(O.P.)$-space.

Clearly, if $E'$ is an $(O.P.)$-space then $E'$ is an $(\infty)$-space (see Theorem 3.2).

Now assume $E'$ is an $(\infty)$-space and let $T \in L(E, c_0)$. Since $T' \in L(\ell_\infty, E')$ is compact we derive that $T$ is also compact ([9], Proposition 5.8), i.e. $E$ is a $(0)$-space.

Property (ii) follows directly from the definition of $(\infty)$-and $(0)$-spaces and from

[9], Proposition 5.8.

(iii) Theorem 3.12 implies that under the assumptions of (iii), $E''$ is an $(\infty)$-space. Now apply (i) to conclude that $E'$ is a $(0)$-space.

**Problem.** Let $E$ be a Banach space. If $E$ is an $(\infty)$-space, does it imply that $E'$ is a $(0)$-space?

**Remarks.**

1. Let $I$ be a small set (i.e. the cardinality of $I$ is nonmeasurable). By [13] Theorem 4.21, $c_0(I)$ is a reflexive Banach space and by Theorem 1.6 vi) it is also a $(\infty)$-space. Applying Corollary 6.3 (iii) we deduce that $\ell_\infty(I)$ is a $(0)$-space. Also we have (we like to thank Arnoud van Rooij for the proof)

**Proposition 6.4.** If $K$ is small then for every index set $I$, $\ell_\infty(I)$ is a $(0)$-space.

**Proof.**

Let $T \in L(\ell_\infty(I), c_0)$. Since $K$ is small, $c_0$ is also small. So, there exists a small set $A \subset \ell_\infty(I)$ such that $TA = T\ell_\infty(I)$.

Define in $I$ the following equivalence relation

$$i_1 \sim i_2 \iff a(i_1) = a(i_2) \quad \forall a \in A$$

and let $[i]$ denote the class of $i \in I$. Let $J$ be the collection of these classes. Let $\pi : I \to J$ be the canonical surjection. Then $\pi$ induces a continuous linear map $P : \ell_\infty(J) \to \ell_\infty(I)$. It is easily seen that $A \subset P\ell_\infty(J)$.

There is also an injection $J \overset{\cong}{\longrightarrow} K^A$ given by:

If $[i] \in J$, then $\varphi([i])$ is the map $a \mapsto a(i)$.

Then $J$ is also small.

Thus by the above remark, $T \circ P : \ell_\infty(J) \to c_0$ is compact. But $(T \circ P)(\ell_\infty(J)) \supset T(A) = T\ell_\infty(I)$. Hence, $T$ is compact ([13], Theorem 4.40).
2. The "duals" of the properties (α) − (δ) in Theorem 6.2 have been studied in this paper (see Corollaries 4.7 and 6.3). Property (ε) has also a counterpart. In fact, we have

**Proposition 6.5.** Let $E$ be a weakly sequentially complete Banach space such that $E'$ separates the points of $E$. Then the following are equivalent.

(α) $E$ is an (O.P.)-space.

(β) Let $F$ be a Banach (0)-space and let $(T_n)$ be a sequence of compact continuous linear maps from $F$ to $E$ such that $T_n(y) \to T(y)$ in $σ(E, E')$ for each $y \in F$. Then $T$ is compact.

**Proof.**

(α) ⇒ (β). Let $F$ and $(T_n)$ be as in (β). By (O.P.) we have that $T_n(y) \to T(y)$ in norm for each $y \in F$. Now apply Theorem 6.2 (α) ⇒ (ε) to conclude that $T$ is compact.

(β) ⇒ (α). By Theorem 5.3 it suffices to prove that $E$ is an (∞)-space. So let $T \in L(ℓ^∞, E)$ and for each $m \in \mathbb{N}$ let $T_m : ℓ^∞ \to E$ be given by

$$(η_1, η_2, \ldots) \mapsto \sum_{n=1}^{m} η_n T e_n.$$  

It follows easily that $T_m(x) \to T(x)$ in $σ(E, E')$ for each $x \in ℓ^∞$. Also, every $T_m$ is a continuous linear map of finite rank, so it is compact. By (β) we conclude that $T$ is compact.

Observe that in property (β) of Proposition 6.5 the condition of $F$ being a (0)-space cannot be dropped. In fact, take $E = F = c_0$ and let $T_n : c_0 \to c_0$ be given by

$$T_n(x_1, x_2, \ldots) = (x_1, \ldots, x_n, 0, 0, \ldots) \quad (n \in \mathbb{N}).$$  

Then $(T_n)$ is a sequence of finite rank (and hence compact) continuous linear maps converging pointwise to the identity map on $c_0$, which is not compact.

3. By considering the "dual counterpart" of property (η) in Theorem 6.2, we obtain - for Banach spaces - the following improvement of Theorem 3.2.

**Proposition 6.6.** For a Banach space $E$ we consider the following properties:

(α) $E$ is an (O.P.)-space.

(β) The space $C(ℓ^∞, E)$ of all compact continuous linear maps from $ℓ^∞$ to $E$ is complemented in $L(ℓ^∞, E)$ and every weakly bounded set in $E$ is bounded.

Then (α) ⇒ (β). If, in addition, $E$ is weakly sequentially complete we have also (β) ⇒ (α).
Proof.

(a) ⇒ (β) follows from Theorem 3.2.

(β) ⇒ (α) can be proved similarly to Theorem 14 of [4]. In fact, observe that if $E$ is not an (O.P.)-space there exists a sequence $(x_n)$ in $E$ such that $x_n \to 0$ in $\sigma(E, E')$ and $1 \leq \|x_n\| \leq 1/|\pi|$ ∀n (where $\pi \in K, |\pi| > 1$ is fixed). For $\lambda = (\lambda_n) \in \ell^\infty$ we define $H_\lambda \in L(\ell^\infty, E)$ by $H_\lambda(\alpha_1, \alpha_2, \ldots) = \sigma(E, E') - \sum_{n=1}^{\infty} \alpha_n \lambda_n x_n \quad ((\alpha_1, \alpha_2, \ldots) \in \ell^\infty)$. From now we can follow the proof as in Theorem 14 of [4].
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