SEVEN PAPERS ON $p$-ADIC FUNCTIONAL ANALYSIS

by

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Report 9125
November 1991
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The Axiom of Choice in $p$-adic Functional Analysis

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Functional analysts generally are little concerned with the set-theoretic background of their theories. This note is an attempt to evoke interest in the role of the Axiom of Choice (AC) in $p$-adic Functional Analysis.

It is a deplorably wide-spread attitude with mathematicians in general and with functional analysts in particular, blindly to accept the Axiom of Choice without thinking of its cost. The Axiom enables one to claim the existence of certain objects but only if one manages not to think about what "existence" means. It is a magic key to open a door that is closed to constructivists, but the door leads to a phantom world of things one cannot touch.

Let me hasten to add that the Countable Axiom of Choice is a quite different matter. It can be understood, it can be made constructive and, best of all, it seems to be everything one needs, at least in Functional Analysis over $\mathbb{R}$ and $\mathbb{C}$. For instance, it implies the Hahn-Banach Theorem for separable normed spaces. Admittedly it does not yield the Hahn-Banach Theorem for $l^\infty$, but does anyone ever use that?

The dubious value of the full Axiom of Choice becomes apparent if one considers alternative axioms, just as valid, that lead to a theory in which all subsets of $\mathbb{R}$ are Lebesgue measurable, all linear functionals on Banach spaces are continuous and all finitely additive measures on a $\sigma$-algebra are $\sigma$-additive. Such a theory is in no sense inferior; it is just unfashionable. Next year may be different.

It is not my purpose to abolish AC. (After all, it may well be "true", if that means anything.) I do wish that mathematicians were more aware of it and did not take it for granted.

As I said before, the Countable Axiom of Choice seems to suffice for Functional Analysis over $\mathbb{R}$ and $\mathbb{C}$. It may be more crucial in the $p$-adic theory. Indeed, it looks
as if from the point of view of the set-theoretic foundations, there is a big difference between classical and $p$-adic Functional Analysis. As a case in point I would like to discuss the Hahn-Banach Theorem and its $p$-adic counterpart, Ingleton’s Theorem.

The standard proofs of the Hahn-Banach Theorem (HBT) one encounters in literature rely on the Axiom of Choice (AC) and, indeed, it is well known that the theorem does not follow from the Zermelo-Fraenkl axioms for Set Theory. On the other hand it does not require the full strength of the Axiom of Choice, either. A weaker axiom implying HBT is the Ultrafilter Theorem (UT), stated farther on. Assuming the Zermelo-Fraenkl axioms one has

$$\text{AC} \Rightarrow \text{UT} \Rightarrow \text{HBT}.$$ 

It can be proved that the reverse implications are false:

$$\text{AC} \nRightarrow \text{UT} \nRightarrow \text{HBT}.$$ 

We propose to show how for locally compact $K$, Ingleton’s Theorem can be deduced from UT. Our proof is only a trivial adaption of Luxemburg’s proof of HBT. (See [1].)

Throughout, we base ourselves on the Zermelo-Fraenkl axioms. We do not use other set-theoretic axioms without explicit mention.

As usual, $K$ is a field provided with a nontrivial non-Archimedean valuation. In addition we assume that $K$ is locally compact

\section*{§1. First, the Ultrafilter Theorem itself.}

Let $T$ be a nonempty set. A nonempty collection $C$ of subsets of $T$ is said to have the finite intersection property if

$$N \in \mathbb{N}, \quad X_1, \ldots, X_N \in C \implies X_1 \cap \ldots \cap X_N \neq \emptyset.$$ 

(Then in particular $\emptyset \notin C$.)

An ultrafilter in $T$ is a collection of subsets of $T$ that is maximal among the collections that have the finite intersection property. E.g., for every $\tau_0 \in T$, $C := \{X : \tau_0 \in X\}$ is an ultrafilter.

The Ultrafilter Theorem states:

$$\text{UT} \begin{cases} \text{Let } T \text{ be a nonempty set, } C \text{ a} \\ \text{nonempty collection of subsets of } T \\ \text{having the finite intersection property.} \\ \text{Then } C \text{ is contained in an ultrafilter.} \end{cases}$$
Clearly, UT is a special case of Zorn's Lemma, hence follows from the Axiom of Choice. The reverse is false:

\[
\text{AC} \not\Rightarrow \text{UT}.
\]

Various theorems in mathematics that commonly are obtained from AC actually follow already from UT. An example of this is the classical Gelfand-Naimark-Segal representation theorem for commutative \(C^*\)-algebras. Another is the Alaoglu Theorem: AC is known to be equivalent to the Tychonoff Theorem, but UT is enough to imply compactness of \([0,1]^S\) for every \(S\), and thereby the Alaoglu Theorem (but not the Krein-Milman Theorem).

\section{2}
Let \(\mathcal{U}\) be an ultrafilter in a set \(T\). Then:

1. \(T \in \mathcal{U}\). More generally: if \(X \in \mathcal{U}\) and \(X \subseteq Y \subseteq T\), then \(Y \in \mathcal{U}\).
2. If \(X \in \mathcal{U}\) and \(Y \in \mathcal{U}\), then \(X \cap Y \in \mathcal{U}\).
3. If \(X \cup Y \in \mathcal{U}\), then \(X \in \mathcal{U}\) or \(Y \in \mathcal{U}\).

The proofs of these statements are elementary.

The following terminology will be convenient for our proof of Ingleton's Theorem from UT. Again, \(\mathcal{U}\) is an ultrafilter in \(T\).

Suppose for every \(\tau \in T\) we are given a proposition \(P(\tau)\). We say

\[
P(\tau) \text{ for almost every } \tau
\]

if \(\{\tau : P(\tau)\} \in \mathcal{U}\). Instead of "almost every" we also use the abbreviation "a.e."

Assume that for every \(\tau\) we have propositions \(P(\tau)\) and \(Q(\tau)\). Then from (1)-(3) one easily obtains:

\[
\begin{align*}
(1') \quad & P(\tau) \text{ for a.e. } \tau \\
\Rightarrow & Q(\tau) \text{ for a.e. } \tau \\
(2') \quad & P(\tau) \text{ for a.e. } \tau \\
\Rightarrow & P(\tau) \land Q(\tau) \text{ for a.e. } \tau \\
(3') \quad & P(\tau) \lor Q(\tau) \text{ for a.e. } \tau \\
\Rightarrow & Q(\tau) \text{ for a.e. } \tau.
\end{align*}
\]

\section{3}
Now we turn to Ingleton's Theorem.

\(K\) is locally compact, \(E\) is a normed vector space (norm: \(||\cdot||\)), \(D\) a linear subspace of \(E\), \(f\) a continuous linear function \(D \rightarrow K\). Our problem is to extend \(f\) to a linear
function $\bar{f} : E \to K$ with

$$|\bar{f}(x)| \leq \|f\| \|x\| \quad (x \in E).$$

$$D \subset E$$

$\quad f \quad \bar{f} \quad K$

If there exist $a_1, \ldots, a_N$ in $E$ such that $D + Ka_1 + \cdots + Ka_N = E$, there is no problem: $K$ being spherically complete one can use the standard technique to extend $f$ successively to the spaces

$$D + Ka_1, \; D + Ka_1 + Ka_2, \ldots, \; D + Ka_1 + \cdots + Ka_N.$$

In other words, for any finite dimensional linear subspace $F$ of $E$ there is a linear $g : D + F \to K$ extending $f$ and with $\|g\| \leq \|f\|$. The entire difficulty is to get from here to an extension defined on all of $E$, and this is, of course, where UT comes in.

§4. Let $T$ be the set of all pairs $(F, g)$ where $F$ is a finite dimensional linear subspace of $E$ and $g$ is a linear function $D + F \to K$ for which

$$|g(x)| \leq \|f\| \|x\| \quad (x \in E + F).$$

For $\tau = (F, g) \in T$ we write $F_{\tau} := F$, $g_{\tau} := g$.

As we have already observed, for any finite dimensional linear subspace $F$ of $E$ there exists a "Hahn-Banach extension" of $f$ to $E + F$, i.e. there exists a $\tau \in T$ with $F_{\tau} = F$.

For $x \in E$, set $T_x := \{\tau \in T : x \in F_{\tau}\}$.

By the above, the collection of sets $\{T_x : x \in E\}$ has the finite intersection property. The Ultrafilter Theorem claims the existence of an ultrafilter $U$ in $T$ such that

$$T_x \in U \quad (x \in E).$$

Using the phrase "almost every" as in §2 we obtain:
Therefore, 
(2) \( x \in E \implies g_\tau(x) \) exists for a.e. \( \tau \).

If \( x, y \in E \) then for almost every \( \tau \) we have \( x \in F_\tau, y \in F_\tau \) and \( x + y \in F_\tau \). Hence,

(3) \( x, y \in E \implies g_\tau(x) + g_\tau(y) = g_\tau(x + y) \) for a.e. \( \tau \).

Similarly,

(4) \( x \in E, \lambda \in K \implies \lambda g_\tau(x) = g_\tau(\lambda x) \) for a.e. \( \tau \).

Take \( x \in E \).

Let \( \varepsilon > 0 \). \( K \) being locally compact, the (bounded) set \( \{ g_\tau(x) : \tau \in T_\varepsilon \} \) is covered by finitely many balls of radius \( \varepsilon \). It follows from (3') of §2 that one of these balls must contain \( g_\tau(x) \) for a.e. \( \tau \).

Thus, for every \( \varepsilon > 0 \) there is a (unique) ball \( B_\varepsilon \) of radii \( \varepsilon \) with \( g_\tau(x) \in B_\varepsilon \) for a.e. \( \tau \). Consequently, we have a unique element \( \overline{f}(x) \) of \( K \) such that for every \( \varepsilon > 0 \)

\[
|g_\tau(x) - \overline{f}(x)| < \varepsilon \quad \text{for a.e. } \tau.
\]

In this way we have obtained a function \( \overline{f} \) on \( E \). It is a simple matter to show that \( \overline{f} \) is linear, that

\[
|\overline{f}(x)| \leq \|f\| \|x\| \quad (x \in E)
\]

and that \( \overline{f} \) coincides with \( f \) on \( D \).

§5. So far, no problem. All we have done is to prove a special case of Ingleton’s Theorem under slightly weaker set-theoretical assumptions than is customary. Now the questions.

A. Is it possible to derive the full Ingleton Theorem from UT? Owing to the great diversity of the admissible scalar fields \( K \), Ingleton’s Theorem seems to have a much wider scope than the Hahn-Banach Theorem and may well imply AC. By the same token, Ingleton’s Theorem for locally compact scalar fields may well imply UT.

B. Is there any implication between Ingleton’s Theorem and the Hahn-Banach Theorem itself?

C. There are other places in \( p \)-adic Functional Analysis where AC is used, such as the construction of the spherical completion. It is really needed there?

D. Van der Put’s theorem stating that for every compact space \( X \), \( C(X) \) has an orthonormal base relies on the existence of a Hamel base in a vector space over the residue
class field. It is known that, if every vector space has a Hamel base, then AC holds (Th. 6.33 in [2]). Does van der Put's theorem imply AC?

E. Is it possible to obtain the spherical completion of a normed vector space without esoteric tools such as AC or UT?

REFERENCES


Open Problems

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This is a selection of 19 problems all centered around Banach space theory. We do not claim that they reflect the state of the art in $p$-adic Functional Analysis. We also wish to point out that - as far as we know - none of the problems stated in [5] has been solved yet!

Note added in proof. The problem of item 6 has a positive solution that appears elsewhere in this Report (p.59). A negative answer to the first problem of item 15 was found recently by J.M. Bayod and will be published in the Proceedings mentioned in [4] and [8].

NOTATIONS & TERMINOLOGY

We follow the conventions of [5]. Throughout $K$ is a nonarchimedean nontrivially valued field that is complete under the metric induced by the valuation $|\ |$. Norms and seminorms are nonarchimedean. $E, F$ are $K$-Banach spaces; $E \oplus F$ is the orthogonal direct sum of $E$ and $F$; the formula $E \sim F$ means that $E, F$ are isometrically isomorphic. The ‘open’ (‘closed’) ball with radius $r$ and center $a$ is denoted $B(a, r^−) (B(a, r))$. For $a \in E$ we write $[a]$ for the linear subspace generated by $a$. An absolutely convex subset $A$ of $E$ is edged if for each $a \in E$ the set $\{|\lambda| : \lambda \in K, \lambda a \in A\}$ is closed in $|K|$.

1. FINITE DIMENSIONAL PRIME DECOMPOSITION

Let us call an $E \neq \{0\}$ prime if there do not exist non-zero Banach spaces $E_1, E_2$ such
that $E \sim E_1 \oplus E_2$. For a finite dimensional Banach space $E$, $E \neq \{0\}$, clearly there exist prime spaces $E_1, \ldots, E_N$ such that $E \sim E_1 \oplus \ldots \oplus E_N$.

**Problem.** Is the prime decomposition of a finite dimensional Banach space in the natural sense unique?

(One can show: if $E, F$ are finite dimensional and $K \oplus E \sim K \oplus F$ then $E \sim F$. Also, if $E_1, \ldots, E_N$ are prime, $D$ is a Banach space and $E_1 \oplus \ldots \oplus E_N \sim K \oplus D$ then $E_n \sim K$ for some $n$.)

**Problem.** If $E$ is prime, must $\{\|x\| : x \in E, x \neq 0\}$ be just one coset of the value group of $K$ in $(0, \infty)$?

2. NORMS ON $K^2$

Let $K$ not be spherically complete. Let $\mathcal{N}$ be the set of all norms $\nu$ on $K^2$ for which

1. $\nu(x) \in |K|$ ($x \in K^2$);
2. $K^2$ has no base that is orthogonal relative to $\nu$.

For $\mu, \nu \in \mathcal{N}$ write

$$\mu \sim \nu$$

if there is a linear bijection $A : K^2 \to K^2$ such that $\mu = \nu \circ A$.

**Problem.** Describe the set of all $\sim$-equivalence classes. E.g., can it be finite?

The following consideration may be of use.

By a hole we mean a maximal chain of balls of $K$ with empty intersection. From a hole we can make other holes by translation and by multiplication with a nonzero scalar. Let us call two holes, $\Omega$ and $\Omega'$ equivalent,

$$\Omega \sim \Omega'$$

if there exist $\alpha \in K$ and $\beta \in K \setminus \{0\}$ for which $\Omega' = \alpha + \beta \Omega$.

A hole $\Omega$ determines a norm $\nu_{\Omega}$ on $K^2$ in the following fashion. Let $|\Omega|$ be the infimum of the radii of the balls belonging to $\Omega$. For $r \in (|\Omega|, \infty)$ let $\Omega_r$ be the element of $\Omega$ whose radius is $r$. If $(\alpha, \beta) \in K^2$, then for sufficiently small $r \in (|\Omega|, \infty)$ the function $\lambda \mapsto |\alpha - \beta \lambda|$ is constant on $\Omega_r$; let $\nu_{\Omega}(\alpha, \beta)$ be its value.
It turns out that the map $\Omega \mapsto \nu_{\Omega}$ establishes a bijective correspondence between the equivalence classes of norms and the equivalence classes of holes: every $\nu_{\Omega}$ lies in $\mathcal{N}$, for every $\nu \in \mathcal{N}$ there is a hole $\Omega$ with $\nu \sim \nu_{\Omega}$, and for any two holes, $\Omega$ and $\Omega'$, one has

$$\nu_{\Omega} \sim \nu_{\Omega'} \iff \Omega \sim \Omega'. $$

Thus, our problem may also be formulated as: describe the set of all equivalence classes of the holes.

3. MULTI-ORTHOGONAL BASES

The following observation was made already in 1964 by Carpentier [2]. (It also follows easily from [6], Theorem 1.11).

Let $\| \|_1, \| \|_2$ be norms on a finite-dimensional space $E$ such that $(E, \| \|_1), (E, \| \|_2)$ both have orthogonal bases. Then there exists a base of $E$ which is orthogonal to both $\| \|_1$ and $\| \|_2$.

**Problem.** Do we have a similar result for three (finitely many) norms? For which $K$ is the ‘natural’ infinite dimensional version of the above true?

4. CARTESIAN SPACES

In the terminology of S. Bosch et al. [1], a normed vector space is *Cartesian* if every finite dimensional subspace has an orthogonal base.

Let us call a normed space *Hilbertian* of every 1-dimensional linear subspace has an orthogonal complement. Every finite dimensional linear subspace of a Hilbertian space has an orthogonal complement; it follows that a Hilbertian space is Cartesian.

**Problem.** Is every Cartesian normed space Hilbertian?

(If $K$ is spherically complete, then every normed space is Hilbertian and Cartesian. In general, every normed space that has an orthogonal base is Hilbertian and Cartesian. Conversely, a Cartesian space of countable type has an orthogonal base, hence is Hilbertian.)
However, if the valuation of $K$ is dense, there exists a Cartesian space without orthogonal base. We sketch a proof of this statement. Let $\mathcal{C}_0$ be the spherical completion of $c_0$. By Zorn's Lemma, there is a maximal Cartesian subspace $E$ of $\mathcal{C}_0$ containing $c_0$. Suppose $E$ has an orthogonal base. Then $E$ has a countable orthonormal base \{${a_1, a_2, \ldots}$\}. Choose $\varepsilon_1, \varepsilon_2, \ldots \in K$ with $|\varepsilon_1| > |\varepsilon_2| > \ldots \to 1$. Take $u \in \mathcal{C}_0$ such that

$$
\|u - \sum_{n=1}^{N} \varepsilon_n a_n\| \leq |\varepsilon_{N+1}| \quad (N = 1, 2, \ldots).
$$

Then $u \notin E$. Set $F = E + [u]$. Putting

$$
b_N = u - \sum_{n=1}^{N} \varepsilon_n a_n \quad (N = 1, 2, \ldots)
$$

one can show that \{${u, b_1, b_2, \ldots}$\} is an orthogonal base for $F$, so that $F$ is Cartesian. This contradicts the maximality of $E$.)

5. ORTHOGONAL ALMOST COMPLEMENTS

A closed subspace $S$ of $E$ is said to be an orthogonal almost complement (o.a.c.) of a closed subspace $T$ if $S \perp T$ and $S + T$ has finite codimension in $E$. (To show that this concept does not appear out of the blue consider the following easily proved Proposition. The ‘open’ balls $B(0, r^-)$ are weakly closed for each $r > 0$ if and only if each onedimensional subspace has an o.a.c.)

**Proposition.** (Compare [5], Lemma 4.35 (iii).) *If each onedimensional subspace has an o.a.c. then does it follow that each finite dimensional subspace has an o.a.c.?*

6. IMAGES OF COMPACTOIDS (see the note at the beginning of this paper)

If $A \subset E$ is a compactoid and $T \in \mathcal{L}(E, F)$ is injective then $T$ maps $A$ homeomorphically onto $TA$. This leads to the

**Problem.** *Let $A \subset E$ be an absolutely convex compactoid and let $T \in \mathcal{L}(E, F)$. If $y_1, y_2, \ldots$ is a sequence in $TA$ with $\lim_{n \to \infty} y_n = 0$ does there exist a sequence $x_1, x_2, \ldots$ in some scalar multiple of $A$ with $\lim_{n \to \infty} x_n = 0$ and $Tx_n = y_n$ for each $n$?*
(It suffices to consider the case where $T$ is a quotient map. If in the above we require the $x_1, x_2, \ldots$ to be in $A$ the problem has a negative answer. In fact, a positive answer would imply that $TA$ is closed whenever $A$ is closed, which is not true if $K$ is not spherically complete; see [6], Theorem 6.28.)

7. STRONG POLARITY

(See [4] for definitions and for related concepts and problems.) It is well known that $E$ is strongly polar (SP) if and only if each closed edged absolutely convex subset of $E$ is polar. Let us define:

$E$ is **boundedly strongly polar** (BSP) if every bounded closed edged absolutely convex subset of $E$ is polar or, equivalently, if each norm inducing the topology is polar.

$E$ has the **almost orthogonal complementation property** (AOCP) if, for each $t \in (0,1)$, every closed linear subspace of $E$ has a $t$-orthogonal complement.

Obviously we have

$E$ is of countable type $\implies E$ has AOCP $\implies E$ is SP $\implies E$ is BSP.

**Problem.** Let $K$ be not spherically complete. Which ones of the opposite implications are true?

8. REFLEXIVITY

**Problem.** Let $a \in E$, $a \neq 0$. If $E/\langle a \rangle$ is reflexive, must $E$ itself be reflexive?

(If $E$ is reflexive, then so is $E/\langle a \rangle$. Proof. Let $\varepsilon > 0$; we are done if we can find a reflexive space $D$ and a linear bijection $T : D \to E/\langle a \rangle$ with $\|T\| \leq 1$, $\|T^{-1}\| \leq 1 + \varepsilon$. There exists an $f \in E'$ such that $f(a) = 1$, $\|f\| \|a\| \leq 1 + \varepsilon$. Take $D = f^{-1}(0)$ and let $T$ be the restriction of the quotient map $E \to E/\langle a \rangle$.)

A Banach space $F$ is called **pseudoreflexive** (or **polar**) if the natural map

$$j_F : F \to F''$$

is isometric. The following variation on the above problem is just as meaningful: if $E/\langle a \rangle$ is pseudoreflexive, must $E$ be pseudoreflexive too?
However, there exist a Banach space $E$ and an $a \in E$, $a \neq 0$ such that $f_{E/[a]}$ is injective but $f_E$ is not! Such $E$ and $a$ can be made as follows. For every $n \in \mathbb{N}$ let $F_n$ be a Banach space such that $F_n'$ separates the points of $F_n$ and there exists an $a_n \in F_n$ with

$$\|a_n\| = 1, \quad |f(a_n)| \leq n^{-1}\|f\| \quad (f \in E_n').$$

(See [5], 3.1 and 4.N; take $a_n = (1,1,1,\ldots)$.) Let $F = \oplus F_n$, $D = \{(\lambda_1 a_1, \lambda_2 a_2, \ldots) : \lambda_1, \lambda_2, \ldots \in K, \sum \lambda_n = 0\}$, $E = F/D$ and let $a \in E$ be the element corresponding to $(a_1, 0, 0, \ldots) \in F$.

A related problem is:

**Problem.** Let $E$ be a Banach space. Let $A, B$ be closed linear subspaces with $A + B = E$, $A \cap B = \{0\}$ and suppose both are (pseudo)reflexive. Does it follow that $E$ is (pseudo)reflexive?

9. **WEAKLY CLOSED CONVEX SUBSETS OF $c_0$**

If $K$ is spherically complete each closed absolutely convex subset of $c_0$ is weakly closed. In general, each closed *edge* absolutely convex subset of $c_0$ is weakly closed. Also, closed compactoids are weakly closed. However, if $K$ is not spherically complete, one can always find a closed absolutely convex subset of $c_0$ which is not weakly closed ([8], Theorem 1.1, see also p.19 of this Report).

**Problem.** Characterize the weakly closed absolutely convex subsets of $c_0$.

See also Problem 5.

10. **CLOSED CONVEX SETS DETERMINING TOPOLOGY**

Let $K$ be spherically complete and $\dim E = \infty$. Then the norm topology and the weak topology differ, yet these topologies have the same collection of closed convex sets. On the other hand, if $K$ is not spherically complete and $\dim E = \infty$ it follows from [8], Theorem 1.1 that one can find a norm closed absolutely convex set that is not weakly closed. This leads naturally to the following general

**Problem.** Let $\tau_1, \tau_2$ be two locally convex topologies on a vector space over a non-spherically complete $K$. Suppose $\tau_1$-closed = $\tau_2$-closed for absolutely convex sets. Does it follow that $\tau_1 = \tau_2$?
11. HYPERISOMORPHIC SPACES

Let us say that $E$ is hyperisomorphic if there exists a closed hyperplane which is linearly homeomorphic to $E$.

**Problem.** Is, for spherically complete $K$, every infinite dimensional Banach space hyperisomorphic? More generally, is every polar infinite dimensional Banach space hyperisomorphic?

12. COMPLETELY CONTINUOUS OPERATORS

Let us say that an $A \in \mathcal{L}(E, F)$ is completely continuous if for every sequence $x_1, x_2, \ldots$ in $E$

$$\lim_{n \to \infty} x_n = 0 \text{ weakly } \implies \lim_{n \to \infty} \|Ax_n\| = 0.$$ 

Clearly every compact operator is completely continuous. The converse is not true as for spherically complete $K$ every $A \in \mathcal{L}(E, F)$ is completely continuous.

**Problem.** For non-spherically complete $K$, characterize the completely continuous operators in $\mathcal{L}(E, F)$.

13. THE INVARIANT SUBSPACE PROBLEM

If $T \in \mathcal{L}(E)$, must $E$ have a nontrivial closed linear subspace that is invariant under $T$?

If $K$ is not algebraically closed, the answer clearly is negative: Let $E \neq K$ be an algebraic field extension of $K$ generated by an element $a$ and let $Tx = ax \ (x \in E)$.

The answer is also negative if $K$ is not spherically complete. To see this, provide the spherical completion $\bar{K}$ of $K$ with the structure of a valued field ([5], Th. 4.49), take $a \in \bar{K}\setminus K$, let $E$ be the closed linear hull of $\{1, a, a^2, \ldots\}$ in $\bar{K}$ and $Tx = ax \ (x \in E)$. Then $E$ is a subalgebra of $\bar{K}$. Actually, $E$ is a field: If $x \in E\setminus\{0\}$, there exists a $\xi \in K$ with $|x - \xi| < |\xi|$; then

$$x^{-1} = \xi^{-1} \sum_{n=0}^{\infty} (1 - \xi^{-1} x)^n \in E.$$ 

Every closed linear subspace of $E$ that is invariant under $T$ is an ideal in $E$, hence must be either $E$ or $\{0\}$.

There remains the following
Problem. Assume that $K$ is algebraically closed and spherically complete. Let $T \in \mathcal{L}(E)$. Does $E$ necessarily have a nontrivial closed linear subspace that is invariant under $T$?

If we restrict ourselves to infinite dimensional Banach spaces, the reasoning we gave for non-algebraically closed $K$ falls through and the one for non-spherically complete $K$ works only if $K$ contains an element that is not algebraic over $K$. This brings us to a question that in itself has nothing to do with invariant subspaces:

Problem. If $K$ is not spherically complete, can $K$ consist only of elements that are algebraic over $K$?

14. ARCHIMEDEAN NORMS

An $A$-norm ([5]) on a vector space $D$ is a function $q : D \to [0, \infty)$ satisfying

$$q(\lambda x) = |\lambda|q(x) \quad (\lambda \in K, \ x \in D),$$
$$q(x + y) \leq q(x) + q(y) \quad (x, y \in D),$$
$$q(x) = 0 \implies x = 0 \quad (x \in D).$$

Such an $A$-norm induces a vector space topology.

Problem. (See [4]) Let $q : D \to [0, \infty)$ be an $A$-norm and assume that for every linear subspace $D_0$ of $D$, every continuous linear function $D_0 \to K$ extends to a continuous linear function $D \to K$. Must there exist a (nonarchimedean) norm $\| \|$ on $E$ that is equivalent to $q$?

(There is a natural candidate for $\| \|$):

On the space $D'$ of all $q$-continuous linear functions $D \to K$ we impose a (nonarchimedean) norm $q'$ by

$$q'(f) = \inf\{c \in [0, \infty) : |f(x)| \leq cq(x) \text{ for all } x \in D\}.$$ 

$q'$ determines a nonarchimedean norm $\bar{q}$ on $D$:

$$\bar{q}(x) = \inf\{c \in [0, \infty) : |f(x)| \leq cq'(f) \text{ for all } f \in D'\}.$$ 

Then $\bar{q} \leq q$ and $D'$ is just the space of all $\bar{q}$-continuous linear functions on $D$.

For $x_1, x_2, \ldots \in D$ write

$$x_n \to 0$$
if \( f(x_n) \to 0 \) for all \( f \in D' \). Trivially,

\[
q(x_n) \to 0 \implies x_n \rightharpoonup 0.
\]

By 5.2 of [6],

\[
x_n \rightharpoonup 0 \iff \bar{q}(x_n) \to 0
\]

and

\[
x_n \rightharpoonup 0 \iff \|x_n\| \to 0
\]

if \( \| \cdot \| \) is a norm providing a positive answer to our Problem. It follows that, if such a norm \( \| \cdot \| \) exists at all, then \( \bar{q} \) is one. Also, this is the case if and only if

\[
x_n \rightharpoonup 0 \implies q(x_n) \to 0.
\]

Consequently, to solve the Problem one may restrict oneself to spaces \( D \) of countable type.

Van Gisbergen ([3]) gives an example of an \( A \)-norm that does not have the extension property mentioned in the Problem although \( D' \) separates the points of \( D \).)

15. ULTRAMETRIZABILITY (see the note at the beginning of this paper)

Let \( \ell^1 \) be the space of all sequences \( x = (x_1, x_2, \ldots) \) where \( x_i \in K \) for each \( i \) such that

\[
\|x\| := \sum_i |x_i| < \infty.
\]

Then \( \ell^1 \) is a \( K \)-Banach space with respect to the \( A \)-norm \( \| \cdot \| \) (see Problem 14).

**Problem.** Is \( \ell^1 \) ultrametrizable?

(One can show that the dense subspace \( c_{00} := \{(x_1, x_2, \ldots) : x_i = 0 \text{ for large } i\} \) is ultrametrizable. For separable \( K \) this is easily seen as follows. One verifies that \( B(0, r^-) \) is closed in \( c_{00} \). Then \( c_{00} \) is zero-dimensional, separable and metrizable, hence ultrametrizable.)

An affirmative answer would solve the more general

**Problem.** Does there exist a complete ultrametrizable topological vector space over \( K \) which is not locally convex?
REFERENCES


The $p$-adic Krein-Šmulian theorem

by

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ABSTRACT. The natural nonarchimedean version of the Krein-Šmulian Theorem holds essentially only when the base field is spherically complete (Corollary 1.6). For Banach spaces over non-spherically complete scalar fields two restricted versions of the Krein-Šmulian Theorem (Theorems 2.2 and 3.1) are proved.

INTRODUCTION. (For unexplained terms see below.) Consider the following statement $(*)$.

\[
(*) \quad \begin{cases}
\text{Let } E \text{ be a } K\text{-Banach space and let } A \subset E' \text{ be convex. If } A \cap B \\
\text{is } w'\text{-closed in } B \text{ for each bounded set } B \subset E' \text{ then } A \text{ is } w'\text{-closed.}
\end{cases}
\]

If $K = \mathbb{R}$ or $\mathbb{C}$ then $(*)$ is known as the Krein-Šmulian Theorem. By modifying classical techniques (in particular by using c-compactness arguments) $(*)$ can also be proved if $K$ is a spherically complete nonarchimedean valued field ([4], Theorem 5.1). Now see the Abstract.

PRELIMINARIES. (For terms still unexplained see [2].) Throughout $K$ is a nonarchimedean nontrivially valued field that is complete under the metric induced by the valuation $| \cdot |$. We always assume that $K$ is not spherically complete, so $|K| := \{ |\lambda| : \lambda \in K \}$ is dense. If $X \subset K$ is bounded, nonempty, we set $\text{diam } X = \sup\{ |x - y| : x, y \in X \}$.
Let $E$ be a $K$-vector space. A nonempty subset $A$ of $E$ is *absolutely convex* if $x,y \in A$, $\lambda, \mu \in K$, $|\lambda| \leq 1$, $|\mu| \leq 1$ implies $\lambda x + \mu y \in A$. For such $A$ we set $A^e := \cap \{\lambda A : \lambda \in K, |\lambda| > 1\}$. $A$ is *edged* if $A = A^e$. The smallest absolutely convex set containing $X \subset E$ is denoted $\overline{co} X$. A nonempty set in $E$ is *convex* (edged convex) if it is an additive coset of an absolutely convex (edged absolutely convex) set. By definition, the empty set is convex. The *algebraic dual* of $E$ is the vector space $E^*$ consisting of all linear functions $E \to K$. The weakest topology on $E$ for which all $f \in E^*$ are continuous is denoted $\sigma(E,E^*)$.

A *seminorm* on $E$ is a map $p : E \to [0,\infty)$ such that $p(x) \geq 0$, $p(\lambda x) = |\lambda| p(x)$, $p(x+y) \leq \max(p(x),p(y))$ for all $x,y \in E$, $\lambda \in K$. We shall use expressions such as ‘$p$-convergence’, ‘$p$-closure’, ‘$p$-compactoid’, ‘$p$-orthogonal’ without further explanation. A seminorm $p$ is *of finite type* if Ker $p$ has finite codimension, *of countable type* if $E/$Ker $p$ is finite dimensional and by the norm induced by $p$ is of countable type. A seminorm $p$ is *polar* if $p = \sup \{|f| : f \in E^*, |f| \leq p\}$. A seminorm $p$ is a *norm* if $p(x) = 0$ implies $x = 0$. Norms are usually denoted $\| \cdot \|$ rather than $p$.

Let $(E, \| \cdot \|)$ be a normed space over $K$. Let $a \in E$, $r > 0$. We write $B_E(a,r) := \{x \in E : \|x-a\| \leq r\}$ and $B_E := B_E(0,1)$. The *dual space* $E'$ is the Banach space consisting of all continuous linear functions $E \to K$, normed by $f \mapsto \|f\| := \sup_B |f|$. The natural map $j_E : E \to E''$ is continuous. $E$ is *pseudoreflexive* if $j_E$ is an isometry (which is equivalent to polarity of the norm on $E$). A linear map $T$ from a $K$-Banach space $E$ to a $K$-Banach space $F$ is a *quotient map* if $T$ maps $\{x \in E : \|x\| < 1\}$ onto $\{x \in F : \|x\| < 1\}$.

Let $(E, \tau)$ be a locally convex space over $K$. It is called *of finite (countable) type* if every continuous seminorm is of finite (countable) type. $(E, \tau)$ is *strongly polar* if each continuous seminorm is polar, *polar* if there exists a base of polar continuous seminorms. Let $E' = (E, \tau')$ be the space of all continuous linear functions $E \to K$. The *weak topology* $\omega = \sigma(E,E')$ is the weakest topology on $E$ such that all $f \in E'$ are continuous. Similarly, the *weak-star topology* $\omega' = \sigma(E',E)$ is the weakest topology on $E'$ such that for each $x \in E$ the evaluation $f \mapsto f(x)$ ($f \in E'$) is continuous. It is well known (see [5]) that the natural map $E \to \sigma(E',E)$ is surjective. Let $X \subset E$, $Y \subset E'$. We set $X^0 := \{f \in E' : |f(x)| \leq 1 \text{ for all } x \in X\}$ and $Y_0 := \{x \in E : |f(x)| \leq 1 \text{ for all } f \in Y\}$. $X$ is a *polar set* if $X^0 = X$. For a ball $B_E(0,r)$ in a normed space $E$ we have $B_E(0,r)^0 = B_E(0,1/r)$. If $E$ is pseudoreflexive, $B_E(0,r)_0 = B_E(0,1/r)$. The closure of a set $X \subset E$ is $\overline{X}$. Instead of $\overline{co} X$ we write $\overline{co} X$. Let $E,F$ be locally convex spaces over $K$. The *adjoint* of a continuous linear map $T : E \to F$ is the map
$T' : F' \to E'$ defined by $f \mapsto f \circ T$. Following [1] we say that a subspace $D$ of $E$ has the Weak Extension Property (WEP) if the adjoint $E' \to D'$ of the inclusion map $D \hookrightarrow E$ is surjective.

§1. FAILURE OF THE KREIN-ŠMULIAN THEOREM

The key theorem of this section is the following. Recall that $K$ is not spherically complete.

**Theorem 1.1.** Let $\tau_1, \tau_2$ be locally convex topologies on a $K$-vector space $E$ such that $\tau_2$ is of finite type while $\tau_1$ is not. Then there exists a $\tau_1$-closed absolutely convex set in $E$ that is not $\tau_2$-closed.

We signal the following corollary which is in sharp contrast to the theory over spherically complete base fields.

**Corollary 1.2.** Let $E$ be a locally convex space over $K$ whose topology is not the weak topology. Then there exists a closed absolutely convex set in $E$ that is not weakly closed.

The proof of Theorem 1.1 runs in a few steps. Let us say that a seminorm $q$ on a $K$-vector space is special if $q(x) \in |K|$ for each $x \in E$ and if for all $x, y \in E$

$$x \perp y \text{ in the sense of } q \implies q(x) = 0 \text{ or } q(y) = 0$$

**Lemma 1.3.** On a normed space of countable type over $K$ there exists an equivalent special norm.

*Proof.* Let $(\tilde{K}, | |)$ be the spherical completion of $(K, | |)$ in the sense of [2], Theorem 4.49. Then $| |$, considered as a norm on the $K$-vector space $\tilde{K}$ is special. (Indeed, we have $|\tilde{K}| = |K|$. If $x, y \in \tilde{K}$, $x \perp y$, $y \neq 0$ then $xy^{-1} \perp 1$ so $xy^{-1} \perp K$. But $\tilde{K}$ is an immediate extension of $K$ so $xy^{-1} = 0$ i.e. $x = 0$.) As $\tilde{K}$ is infinite dimensional over $K$ we can, for a given normed space $E$ of countable type over $K$, make a $K$-linear homeomorphism $T$ of $E$ into $\tilde{K}$. Then $x \mapsto |Tx|$ is the required norm.

**Lemma 1.4.** Let $E$ be a strongly polar locally convex space over $K$. If $E$ is not of finite type then there exists a continuous special seminorm $q$ on $E$, $q$ not of finite type.

*Proof.* There is a continuous seminorm of infinite type $p$ on $E$. The $p$-continuous linear functions form an infinite dimensional space so we can find linearly independent
\(f_1, f_2, \ldots \in E'\) such that \(n|f_n| \leq p\) for each \(n \in \mathbb{N}\). The formula \(\tilde{p}(x) = \max |f_n(x)|\) defines a continuous seminorm \(\tilde{p}\) on \(E\), of infinite countable type. Now Lemma 1.3 (applied to \(E/\text{Ker } \tilde{p}\)) leads to a special seminorm \(q\) equivalent to \(\tilde{p}\).

**Remark.** The conclusion of Lemma 1.4 holds for any polar space \(E\) that is not of finite type.

**Lemma 1.5.** \(\text{Let } q \text{ be a special seminorm on a } K\text{-vector space } E. \text{ If } q \text{ is not of finite type then } \{x \in E : q(x) < 1\} \text{ is not } \sigma(E, E^*)\text{-closed.}\)

**Proof.** Let \(x \in E, \ q(x) = 1\) (such \(x\) exist!). We shall prove that \(x\) is in the \(\sigma(E, E^*)\)-closure of \(A := \{x \in E : q(x) < 1\}\) by producing, for given \(f_1, \ldots, f_n \in E^*\), a point \(a \in A\) such that \(f_i(x-a) = 0\) for \(i \in \{1, \ldots, n\}\).

(i) Suppose \(f_1(x-a) \neq 0\) for all \(a \in A\). Then \(f_1(x) \notin f_1(A)\) so, by convexity, \(f_1(A)\) is bounded and \(f_1\) is \(q\)-continuous. We have

\[|f_1(x)| \geq \text{diam } f_1(A) = \sup\{|f_1(a)| : q(a) < 1\} = \|f_1\|\]

where \(\|f_1\|\) is the operator seminorm of \(f_1\) with respect to \(q\). For each \(y \in \text{Ker } f_1\)

\[\|f_1\|q(x-y) \geq |f_1(x-y)| = |f_1(x)| \geq \|f_1\|\]

and we find \(x \perp y\) in the sense of \(q\). As \(q\) is special and \(q(x) = 1\) we must have \(q = 0\) on \(\text{Ker } f_1\) implying that \(q\) is of finite type, a contradiction. Thus, we may conclude that there exists an \(a_1 \in A\) with \(f_1(x-a_1) = 0\).

(ii) Now we repeat the argument in (i) where \(E\) is replaced by \(\text{Ker } f_1, q\) by \(q|\text{Ker } f_1, x\) by \(x-a_1, A\) by \(A \cap \text{Ker } f_1\) and \(f_1\) by \(f_2|\text{Ker } f_1\). (Indeed, \(q|\text{Ker } f_1\) is special, of infinite type and \(q(x-a_1) = 1\). So there exists an \(a_2 \in A \cap \text{Ker } f_1\) such that \(f_2(x-a_1-a_2) = 0\). Observe that also \(f_1(x-a_1-a_2) = 0\). In this spirit we arrive inductively at points \(a_1, a_2, \ldots, a_n \in A\) such that \(f_i(x-a) = 0\) \((i \in \{1, \ldots, n\})\) where \(a := \sum_{i=1}^{n} a_i \in A\).

**Proof of Theorem 1.1.** If \((E, \tau_1)\) is not strongly polar, choose any nonpolar continuous seminorm \(q\) and set \(A := \{x \in E : q(x) \leq 1\}\). \(A\) is \(\tau_1\)-closed but, as \(q\) is not polar and \(A\) is edged, \(A\) is not \(\sigma(E, E^*)\)-closed so certainly \(A\) is not \(\tau_2\)-closed. If \((E, \tau_1)\) is strongly polar, let \(q\) be as in Lemma 1.4. By Lemma 1.5 the set \(A := \{x \in E : q(x) < 1\}\) is not \(\sigma(E, E^*)\)-closed, so not \(\tau_2\)-closed.

Part (i) of the next corollary demonstrates the failure of the Krein-Šmulian Theorem for non-spherically complete base fields.
COROLLARY 1.6. Let $E$ be a normed space over $K$ such that $E'$ is infinite dimensional.

(i) There exists an absolutely convex set $A \subset E'$ such that $A \cap B$ is $w'$-closed in $B$ for each bounded set $B \subset E'$ while $A$ is not $w'$-closed.

(ii) There exists an absolutely convex set $A \subset E$ such that $A \cap B$ is $w$-closed in $B$ for each bounded set $B \subset E$ while $A$ is not $w$-closed.

Proof. (i) $(E', w')$ is an infinite dimensional Hausdorff space of countable type so its dual (which is $j_B(E)$) is infinite dimensional. Thus, we can choose $x_1, x_2, \ldots$ in $E$ such that $j_E(x_1), j_E(x_2), \ldots$ are linearly independent and $\lim_{n \to \infty} \|x_n\| = 0$. The seminorm $p$ on $E'$ defined by

$$p(f) = \max_n |f(x_n)| = \max_n |j_E(x_n)(f)|$$

is therefore not of finite type. By Theorem 1.1 there exists an absolutely convex set $A \subset E'$ which is $p$-closed but not $w'$-closed. But it is easily seen that, on any bounded set $B \subset E'$, $w'$-convergence implies $p$-convergence. Thus, the $p$-closedness of $A$ implies that $A \cap B$ is $w'$-closed in $B$.

(ii) Similar to the above proof but now with the seminorm $x \mapsto \max_n |f_n(x)|$ ($x \in E$), where $f_1, f_2, \ldots$ is a linearly independent sequence in $E'$ for which $\lim_{n \to \infty} \|f_n\| = 0$. We leave the details to the reader.

§2. SAVE THE KREIN-ŠMULIAN THEOREM! (PART ONE)

To save the Krein-Šmulian Theorem we shall concentrate on *edged* convex sets. As such sets are translates of edged absolutely convex sets no harm is done by considering only the latter. Thus, we arrive at

**DEFINITION 2.1.** A normed space $E$ over $K$ is a *Krein-Šmulian space* if the following holds. If $A \subset E'$ is absolutely convex and edged and if $A \cap B$ is $w'$-closed in $B$ for each bounded set $B \subset E'$ then $A$ is $w'$-closed.

Observe that, for an absolutely convex $A \subset E'$, the expression ‘$A \cap B$ is $w'$-closed in $B$ for each bounded set $B \subset E$’ is equivalent to ‘for each $n \in \mathbb{N}$ the set $A \cap B_{E'}(0, n)$ is $w'$-closed’ and, if $A$ is a subspace, to ‘$A \cap B_{E'}$ is $w'$-closed’.

The main result of this section is

**THEOREM 2.2.** A strongly polar Banach space is a Krein-Šmulian space.
For the proof we need first a lemma on Banach spaces. Let us call a sequence \( X_1 \supset X_2 \supset \ldots \) of closed absolutely convex subsets of a \( K \)-Banach space \( E \) quasi Cauchy if for each \( \lambda \in K, |\lambda| > 1 \) and \( N \in \mathbb{N} \)

\[
X_n \subset \lambda(X_m + BE(0, \frac{1}{N})) \quad (m, n \geq N)
\]

**Lemma 2.3.** Let \( X_1 \supset X_2 \supset \ldots \) be a quasi Cauchy sequence in a \( K \)-Banach space. Set \( X := \bigcap X_n \). Then, for each \( n \in \mathbb{N} \) and \( x_n \in X_n \), and each \( \lambda \in K, |\lambda| > 1 \) there is an \( x \in \lambda X \) such that \( \|x_n - x\| \leq \frac{|\lambda|}{n} \).

**Proof.** Choose \( \lambda_1, \lambda_2, \ldots \in K \) with \( |\lambda_i| > 1 \) for each \( i \), \( \prod_i |\lambda_i| = |\lambda| \). We have

\[
X_n \subset \lambda_1(X_{n+1} + BE(0, \frac{1}{n})) \quad \text{whence} \quad X_n \subset \lambda_1X_{n+1} + BE(0, \frac{|\lambda|}{n})
\]

\[
X_{n+1} \subset \lambda_2(X_{n+1} + B(0, \frac{1}{n+1})) \quad \text{whence} \quad \lambda_1X_{n+1} \subset \lambda_1\lambda_2X_{n+2} + B(0, \frac{|\lambda|}{n+1})
\]

etc.

So, given \( x_n \in X_n \), we can find a sequence \( x_{n+1}, x_{n+2}, \ldots \) where \( x_{n+1} \in \lambda_1X_{n+1}, x_{n+2} \in \lambda_1\lambda_2X_{n+2}, \ldots \) such that for all \( k \in \{0, 1, 2, \ldots \} \)

\[
\|x_{n+k} - x_{n+k+1}\| \leq \frac{|\lambda|}{n+k}.
\]

By completeness \( x := \lim_{k \to \infty} x_{n+k} \) exists. We have \( \lambda^{-1}x_{n+1} \in \lambda^{-1}\lambda_1X_{n+1} \subset X_{n+1}; \lambda^{-1}x_{n+2} \in \lambda^{-1}\lambda_1\lambda_2X_{n+2} \subset X_{n+2}, \ldots \), so \( \lambda^{-1}x = \lim_{k \to \infty} \lambda^{-1}x_{n+k} \in \bigcap_{i \geq n+1} X_i = X \) and it follows that \( x \in \lambda X \). Further, we have

\[
\|x_n - x\| \leq \max\{\|x_n - x_{n+1}\|, \|x_{n+1} - x_{n+2}\|, \ldots\} \leq \max\left(\frac{|\lambda|}{n}, \frac{|\lambda|}{n+1}, \ldots\right) \leq \frac{|\lambda|}{n}.
\]

**Proof of Theorem 2.2.** Let \( A \subset E' \) be absolutely convex, edged and assume that \( A \cap B_{E'}(0, n) \) is \( w' \)-closed for each \( n \in \mathbb{N} \). Then \( (A \cap B_{E'}(0, n) \) is also edged) \( A \cap B_{E'}(0, n) \) is a polar set. Setting

\[
X_n := (A \cap B_{E'}(0, n))_0 \quad (n \in \mathbb{N})
\]

\[
X := \bigcap_n X_n
\]

one verifies immediately (i), (ii), (iii), (iv) below.
(i) Each $X_n$ is a polar subset of $E$.
(ii) $X_n^0 = A \cap B_E'(0, n)$ for each $n \in \mathbb{N}$.
(iii) $X_1 \supset X_2 \supset \ldots$
(iv) $X = A_0$.
(v) For each $N \in \mathbb{N}$ and $m, n \geq N$

$$X_n \subset (X_m + B_E(0, \frac{1}{N}))_0^0.$$ 

(Proof: $(X_m + B_E(0, \frac{1}{N}))_0^0 = X_m^0 \cap B_E(0, \frac{1}{N})_0^0 = A \cap B_E'(0, m) \cap B_E'(0, N) = A \cap B_E'(0, N)$, so $X_n \subset X_N = (A \cap B_E'(0, N))_0^0 = (X_m + B_E(0, \frac{1}{N}))_0^0$.)

(vi) $X_1, X_2, \ldots$ is quasi Cauchy. (Proof. Let $\lambda \in K, |\lambda| > 1, N \in \mathbb{N}, m, n \geq N$.
The set $X_m + B_E(0, \frac{1}{N})$ is norm open hence norm closed. So $(X_m + B_E(0, \frac{1}{N}))^e$ is norm closed and edged, hence polar (as $E$ is strongly polar). It follows via (v), that $X_n \subset (X_m + B_E(0, \frac{1}{N}))_0^0 \subset \lambda(X_m + B_E(0, \frac{1}{N})).$

(vii) $X^0 \subset A$. (Proof. Let $f \in X^0, \lambda \in K, |\lambda| > 1$. It suffices to prove that $f \in \lambda A$.
Let $n \in \mathbb{N}$ be such that $\|f\| \leq n$. Choose any $x \in X_n$. By Lemma 2.3 there is a $y \in \lambda X$ with $\|x - y\| \leq \frac{|\lambda|}{n}$. We have

$$|f(x)| \leq |f(x-y)| \vee |f(y)| \leq \|f\| \|x-y\| \vee |\lambda| \leq n \cdot \frac{|\lambda|}{n} \vee |\lambda| = |\lambda|$$

and we see that $|\lambda^{-1} f| \leq 1$ on $X_n$, so $\lambda^{-1} f \in X_n^0 = A \cap B_E'(0, n) \subset A$ i.e. $f \in \lambda A$.)

Now (iv) combined with (vii) yields $A = X^0$ is $w'$-closed.

**COROLLARY 2.4.** A subspace of the dual of a strongly polar Banach space is $w'$-closed as soon as its intersection with the closed unit ball is $w'$-closed.

**COROLLARY 2.5.** An edged absolutely convex subset $A$ of $\ell^\infty$ is $\sigma(\ell^\infty, c_0)$-closed as soon as $A \cap B_{\ell^\infty}(0, n)$ is $\sigma(\ell^\infty, c_0)$-closed for each $n \in \mathbb{N}$.

*Proof.* $c_0$ is a (reflexive) strongly polar space.

We also have:

**THEOREM 2.6.** If $E$ is a Krein-Šmulian space and $D \subset E$ is a closed subspace then $E/D$ is a Krein-Šmulian space.

*Proof.* Let $i : (E/D)' \to E'$ be the adjoint of the quotient map $E \to E/D$. It is easily seen that $i$ is an isometry, that $\text{Im} i$ is $w'$-closed in $E'$ and that $i$ is a $w'$ to $w'$ homeomorphism $(E/D)' \to \text{Im} i$. 23
Now let $A$ be an edged absolutely convex subset of $(E/D)'$ such that $A \cap B$ is $w'$-closed in $B$ for each bounded set $B$ in $(E/D)'$. Then $i(A)$ is edged. If $X \subseteq E'$ is bounded then $i(A) \cap X$ is $w'$-closed in $X$. (Proof. Let $j \mapsto a_j$ be a net in $A$ such that $i(a_j) \in X$ for all $j$ and let $w' - \lim_j i(a_j) = b \in X$. As $\text{Im } i$ is $w'$-closed $b = i(a)$ for some $a \in i^{-1}(X)$. Then $w' - \lim_j a_j = a$. Now $a_j \in A \cap i^{-1}(X)$ for all $j$, $a \in i^{-1}(X)$ and $i^{-1}(X)$ is bounded, so by assumption on $A$ we have $a \in A$, so $b \in i(A) \cap X$.) Since $E$ is a Krein-Šmulian space, $i(A)$ is $w'$-closed in $E'$ so that $A = i^{-1}(i(A))$ is $w'$-closed in $(E/D)'$.

**Theorem 2.7.** If $E$ is a Krein-Šmulian space and if $D \subseteq E$ is a weakly closed subspace having the WEP then $D$ is a Krein-Šmulian space.

Proof. Let $\pi : E' \rightarrow D'$ be the adjoint of the inclusion map $D \hookrightarrow E$. Then $\pi$ is surjective and $w'$ to $w'$ continuous. If $A$ is an edged absolutely convex set in $D'$ and $\pi^{-1}(A)$ is $w'$-closed then $A$ is $w'$-closed. (Proof. Let $g \in D'$, $g \notin A$. There is an $f \in E'$ with $\pi(f) = g$. Then $f \notin \pi^{-1}(A)$. Now $\pi^{-1}(A)$ is $w'$-closed and edged so there exists an $x \in E$ such that $f(x) = 1$ and $|h(x)| < 1$ for all $h \in \pi^{-1}(A)$. In particular, $|h(x)| < 1$ for all $h \in \text{Ker } \pi = D^0$ i.e. $h(x) = 0$ for all $h \in D^0$ so $x \in D_0^0 = D$. Then $g(x) = f(x) = 1$ and $|h(x)| < 1$ for all $h \in A$.)

Now let $A$ be an absolutely convex edged subset of $D'$ such that $A \cap B$ is $\sigma(D', D)$-closed in $B$ for each bounded set $B \subseteq D'$. Then for such $B$, $\pi^{-1}(A) \cap \pi^{-1}(B)$ is $\sigma(E', E)$-closed in $\pi^{-1}(B)$. If $X \subseteq E'$ is bounded then $\pi(X)$ is bounded and $X \subseteq \pi^{-1}(\pi(X))$ so it follows that $\pi^{-1}(A) \cap X$ is $w'$-closed in $X$ for each bounded set $X \subseteq E'$. Since $E$ is Krein-Šmulian we have that $\pi^{-1}(A)$ is $w'$-closed, so by the remark above, $A$ is $w'$-closed.

**Remark.** Not every Krein-Šmulian polar space is strongly polar; $\ell^\infty$ is an easy example. In §3 we will see that, if $I$ is large enough, $c_0(I)$ is not Krein-Šmulian. This leads to the

**Problem.** Characterize the class of Krein-Šmulian spaces.

A concrete help would be the answer to the following two questions.
- Is $c_0 \times \ell^\infty$ a Krein-Šmulian space? (More generally, if $E_1$ and $E_2$ are Krein-Šmulian spaces then does it follow that $E_1 \times E_2$ is Krein-Šmulian?)
- Is the subspace of $D$ of $\ell^\infty$ constructed in [2], Ex. 4.J Krein-Šmulian?

§3. **Save the Krein-Šmulian Theorem! (Part Two)**

In this section we shall prove the following version of the Krein-Šmulian Theorem. Observe that $(\alpha)$ holds for any polar $K$-Banach space.

24
THEOREM 3.1. For a normed space $E$ over $K$ the following are equivalent.

(α) $j_E(E)$ is norm closed in $E''$.

(β) If $H \subset E'$ is a subspace of finite codimension and if $H \cap B_{E'}$ is $w'$-closed then so is $H$.

For a normed space $E$ over $K$ the $bw'$-topology (the 'bounded-weak-star topology') is by definition the strongest locally convex topology on $E'$ that coincides with $w'$ on bounded subsets of $E'$.

PROPOSITION 3.2. Let $E$ be a normed space over $K$.

(i) $bw'$ is stronger than $w'$ but weaker than the norm topology on $E'$.

(ii) $(E', bw')$ is of countable type.

(iii) A seminorm $p$ on $E'$ is $bw'$-continuous if and only if $p|B_{E'}$ is $w'$-continuous.

(iv) For any locally convex space $(X, \tau)$ and any linear map $T : E' \to X$ we have that $T$ is $bw'$ to $\tau$ continuous if and only if $T|B_{E'}^*$ is $w'$ to $\tau$ continuous.

Proof. $E' = [B_{E'}]$ and $B_{E'}$ is a $w'$-compactoid, hence a $bw'$-compactoid. This implies (ii). The other proofs are straightforward.

We know that $(E', w')' = j_E(E)$ ([5])). We now prove

PROPOSITION 3.3. For a normed space $E$ over $K$ the dual of $(E', bw')$ is the norm closure of $j_E(E)$ in $E''$.

Proof. Every $\theta \in \overline{j_E(E)}$ is, on $B_{E'}$, the uniform limit of a sequence in $j_E(E)$ so $\theta|B_{E'}$ is $w'$-continuous and $\theta$ is $bw'$-continuous (Proposition 3.2 (iv)). Thus $\overline{j_E(E)} \subset (E', bw')$.

Conversely, let $\theta \in (E', bw')$. Then (Proposition 3.2 (i)) $\theta \in E''$. Let $\varepsilon > 0$; we shall find an $x \in E$ such that $\|\theta - j_E(x)\| < \varepsilon$. Let $\alpha \in K$, $0 < |\alpha| < \varepsilon$. The $w'$-continuity of $\theta|B_{E'}$ yields a finite set $F \subset E$ such that $f \in F^0 \cap B_{E'}$ implies $|\theta(f)| \leq |\alpha|$, in other words

$$f \in j_E(F)_0 \cap (B_{E'})_0 \implies |(\alpha^{-1}\theta)(f)| \leq 1.$$ 

So we see that $\alpha^{-1}\theta \in (j_E(F)_0 \cap (B_{E'})_0)^0 = (A + B_{E'})_0^0$, where $A = j_E(\text{co } F)$. Now $B_{E'}$ is $w'$-closed and $A$ is finite dimensional so by [3], 1.4, $(A + B_{E'})_0^0 = (A + B_{E'})^0$. For any $\lambda \in K$ such that $|\lambda| > 1$ and $|\lambda| < \varepsilon$ we have $\alpha^{-1}\theta \in \lambda A + \lambda B_{E'}$ hence $\theta \in j_E(E) + \alpha \lambda B_{E'}$ and there is an $x \in E$ with $\theta - j_E(x) \in \alpha \lambda B_{E'}$ i.e. $\|\theta - j_E(x)\| < \varepsilon$.

COROLLARY 3.4. Let $j_E(E)$ be closed in $E''$, let $A \subset E'$ be absolutely convex and edged. Then $A$ is $w'$-closed if and only if $A$ is $bw'$-closed.
Proof. Let $A$ be $bw'$ closed. As $A$ is also edged and $(E', bw')$ is strongly polar (Proposition 3.2 (ii)), $A$ is a polar set i.e. $A = S_0$ for some $S \subset (E', bw')$. But by Proposition 3.3 $S \subset (E', w')'$ so that $A$ is $w'$-closed.

Further, we need the following general lemma.

**Lemma 3.5.** Let $A$ be a closed absolutely convex subset of a Hausdorff locally convex space over $\mathbb{F}$; let $D$ be a finite dimensional subspace such that $A + D = \{ 0 \}$. Then $A + D$ is closed and the addition map is a homeomorphism $A \times D \to A + D$.

**Proof.** (i) If addition is homeomorphic then $A + D$ is closed. In fact, let $i \to a_i + d_i$ be a net in $A + D$ (where $a_i \in A$, $d_i \in D$ for each $i$), converging to some $z$. Then $(i, j) \mapsto a_i - a_j + d_i - d_j$ converges to 0. By homeomorphism, $d_i - d_j \to 0$, by completeness of $D$, $d_i \to d$ for some $d \in D$. Then $a_i \to z - d$ and, by closedness of $A$, $z - d \in A$. We see that $z \in A + D$.

(ii) Assume $n := \dim D = 1$, say $D = Kx$ for some nonzero $x$. Let $i \mapsto a_i + \lambda_i x$ ($a_i \in A$, $\lambda_i \in K$) be a net in $A + D$ converging to 0. If not $\lambda_i \to 0$ we may assume $|\lambda_i| \geq |\alpha| > 0$ for all $i$ and some $\alpha \in K$. Then $\alpha \lambda_i^{-1} (a_i + \lambda_i x) \to 0$ so $\alpha x = -\lim \alpha \lambda_i^{-1} a_i \in A = x$ conflicting $Kx \cap A = \{ 0 \}$. Thus, addition is homeomorphic and via (i) the lemma is proved if $n = 1$.

(iii) The proof of the induction step $n - 1 \to n$ is now standard and left to the reader.

**Proof of Theorem 3.1.**

(i) Suppose $(\alpha)$, and let $H \subset E'$ be a subspace of finite codimension such that $H \cap B_{E'}$ is $w'$-closed. Then $H \cap B_{E'}$ is norm closed, hence so is $H$. For some $t \in (0, 1)$ $H$ has a $t$-orthogonal complement $D$. Let $P : E' \to D$ be the obvious projection. For $\lambda \in K$, $|\lambda| \geq t^{-1}$ we have

$$B_{E'} \subset \lambda(H \cap B_{E'}) + \lambda(D \cap B_{E'}) \subset \lambda(H \cap B_{E'}) + D.$$  

Let $i \mapsto f_i$ be a net in $B_{E'}$, $w' - \lim f_i = 0$. Then, by Lemma 3.5, $\lim_i P \circ f_i = 0$. We see that $P|_{B_{E'}}$ is continuous, so (Proposition 3.2 (iv)) $P$ is $bw'$ to norm continuous and $\text{Ker} \ P = H$ is $bw'$-closed, hence $w'$-closed by Corollary 3.4, and $(\beta)$ is proved.

(ii) Suppose $\alpha$ is not true. Choose $\theta \in \mathfrak{f}_E(\mathcal{E}) \setminus \mathfrak{f}_E(\mathcal{E})$. Then $\theta$ is not $w'$-continuous so $H := \text{Ker} \ \theta$ is not $w'$-closed. But $\theta$ is $bw'$-continuous by Proposition 3.3. so $H \cap B_{E'}$ is $w'$-closed.

The results of this section yield the existence of polar non-Krein-Šmulian spaces (see §2).
**COROLLARY 3.6.** If $m$ is a cardinality $\geq \#K$ then $c_0(m)$ is not a Krein-Šmulian space.

*Proof.* In [2], Exercise 4.N a Banach space $E$ is constructed such that $j_E(E)$ is a proper dense subset of $E''$. From this construction it is easily seen that $\#E = \#\ell^\infty \leq \#K^N = \#K$. Now let $I$ be a set with cardinality $\geq \#K$ and let $\{e_i : i \in I\}$ be the natural orthonormal base of $c_0(I)$. There is a surjection $\{e_i : i \in I\} \to B_E$, it extends to a quotient map $c_0(I) \to E$. Now $E$ is not a Krein-Šmulian space by Theorem 3.1, neither is $c_0(I)$ by Theorem 2.6. (It is not hard to see by looking at the proof of Theorem 2.6 that one can even find a subspace $D \subset \ell^\infty(I)$ that is not $w'$-closed while $D \cap B_{\ell^\infty(I)}$ is.)

**COROLLARY 3.7.** If $m$ is a nonmeasurable cardinality $\geq \#K^K$ then $\ell^\infty(m)$ is not a Krein-Šmulian space.

*Proof.* In the spirit of the previous proof one constructs a quotient map $\pi : c_0(m) \to \ell^\infty(n)$ where $n = \#K$. By reflexivity ([2], Theorem 4.21) the adjoint $\pi' : c_0(n) \to \ell^\infty(m)$ is an isometry and $\pi'(c_0(n))$ has the WEP in $\ell^\infty(m)$. From [4], Lemma 2.2 we obtain that $\pi'(c_0(n))$ is also weakly closed in $\ell^\infty(m)$. By the previous corollary $c_0(n)$ is not a Krein-Šmulian space, neither is $\ell^\infty(m)$ by Theorem 2.7.

**PROBLEM.** Determine the smallest cardinality $m$ for which $c_0(m)$ ($\ell^\infty(m)$ if $\#K$ is nonmeasurable) is not a Krein-Šmulian space.

As a further application we now prove a nonarchimedean version of a classical reflexivity criterion (Theorem 3.8). First we ‘dualize’ the notion of a polar seminorm as follows. A seminorm $p$ on the dual $E'$ of a locally $K$-convex space $E$ is a *dual seminorm* if there exists an $X \subset E$ such that $p(f) = \sup\{|f(x)| : x \in X\}$ for all $f \in E'$. An easy exercise shows that $p$ is dual if and only if $\{f \in E' : p(f) \leq 1\}$ is $\sigma(E', E)$-closed. Dual seminorms are automatically polar.

**THEOREM 3.8.** Let $E$ be a pseudoreflexive $K$-Banach space. Then $E$ is reflexive if and only if each polar norm on $E'$ inducing the topology is dual.

*Proof.* Let $E$ be reflexive and let $\nu$ be a polar norm on $E'$ inducing the topology. Then $\{f \in E' : \nu(f) \leq 1\}$ is weakly closed so, by reflexivity, $w'$-closed. Hence $\nu$ is dual by the above remark. Conversely, suppose each polar norm on $E'$ inducing the topology is dual. To prove reflexivity of $E$ it suffices (by pseudoreflexivity) to show that any $\theta \in E''$ is $w'$-continuous. For each $n \in \mathbb{N}$ the norm $f \mapsto n|\theta(f)| \vee \|f\|$ (where $\|\|$ is the ‘natural’ norm on $E'$) is easily seen to be polar and it is obviously equivalent to $\|\|$. 27
By assumption its closed unit ball

\[ B_n := \{ f \in E' : |\theta(f)| \leq \frac{1}{n}, \|f\| \leq 1 \} \]

is \( w' \)-closed. Hence so is \( \bigcap_n B_n \) which is \( \operatorname{Ker} \theta \cap B_{E'} \). By Theorem 3.1 \( \theta \) is \( w' \)-closed implying that \( \theta \) is \( w' \)-continuous.

**Remark.** One also may consider a 'predual form' of the Krein-Šmulian property (compare Definition 1.1, see also Corollary 1.6 (ii)) as follows. A normed space \( E \) is PKŠ space if for each absolutely convex edged \( A \subset E \):

\[ A \cap B \text{ is } w\text{-closed in } B \text{ for each bounded } B \subset E \implies A \text{ is } w\text{-closed.} \]

(Of obviously this notion is of no use in classical Banach space theory.) The reader will not have difficulties in proving results about PKŠ spaces similar to the one of KS-spaces of this paper. More precisely, we have

(i) A strongly polar normed space is PKŠ.
(ii) Let \( E \) be a normed PKŠ space, let \( D \) be a closed subspace. Then \( E/D \) is PKŠ. If \( D \), in addition, is weakly closed and has the WEP then \( D \) has PKŠ.
(iii) Let \( E \) be a normed space. If \( H \subset E \) is a subspace with finite codimension and \( H \cap B_E \) is weakly closed then \( H \) is weakly closed.

**REFERENCES**

The $p$-adic bounded weak topologies

W.H. Schikhof

INTRODUCTION

Throughout, let $K$ be a nonarchimedean nontrivially valued field that is complete under the metric induced by the valuation $| |$. For a normed space $E$ over $K$ one has the well known weak topology $w = \sigma(E, E')$ which is the weakest topology on $E$ making all $f \in E'$ (the dual space of $E$) are continuous. Similarly, the so-called 'weak star' topology $w' = \sigma(E', E)$ is by definition the weakest topology on $E'$ making all evaluations $f \mapsto f(x)$ continuous. (The expression 'weak star' goes back to the old days when the topological dual space of $E$ was denoted $E^*$ rather than $E'$.) In this paper we study the following related less known topologies. The bounded weak topology $bw$ is the strongest locally convex topology on $E$ that coincides with $w$ on (norm) bounded subsets of $E$. The bounded weak star topology $bw'$ is the strongest locally convex topology on $E'$ that coincides with $w'$ on (norm) bounded subsets of $E'$. These two topologies have appeared incidentally in [8], [9], but in this paper we shall consider them in their own right.

PRELIMINARIES

We shall use notations and terminology of [3]. Throughout $E$ is a normed space over $K$ (where the norm $|| |$ is assumed to satisfy the strong triangle inequality $\|x+y\| \leq \max(\|x\|, \|y\|)$ $(x, y \in E)$). The 'closed' ball with center 0 and radius $\varepsilon > 0$, $\{x \in E : \|x\| \leq \varepsilon\}$ is written $B(0, \varepsilon)$. Sometimes we write $B_E$ in place of $\{x \in E : \|x\| \leq 1\}$. The canonical map $E \to E''$ is denoted $j_E$. WARNING. $j_E$ need not be injective; the topologies $w$ and $bw$ introduced above need not be Hausdorff!

A subset $A$ of a $K$-vector space is absolutely convex if $x, y \in A$, $\lambda, \mu \in K$, $|\lambda|, |\mu| \leq 1$...
implies $\lambda x + \mu y \in A$. If $A$ is absolutely convex we define $A^e := A$ if the valuation of $K$ is discrete, $A^e := \bigcap_{|\lambda| > 1} \lambda A$ if the valuation of $K$ is dense. $A$ is *edged* if $A = A^e$. The smallest absolutely convex set containing $X$ is denoted by $\text{co}X$, its linear span by $[X]$. Let $Z$ be a locally convex space over $K$, let $X \subset Z$ and $Y \subset Z'$. We write

$$X^0 := \{ f \in Z' : |f(x)| \leq 1 \text{ for all } x \in X \}$$

$$Y_0 := \{ x \in Z : |f(x)| \leq 1 \text{ for all } f \in Y \}.$$  

$X$ is a *polar set* if $(X^0)_0 = X$.

§1. THE BOUNDED WEAK TOPOLOGY

We first collect some elementary consequences of the definitions.

**PROPOSITION 1.1.** For an absolutely convex set $U \subset E$ the following are equivalent.

(a) $U$ is bw-open.

(b) $U \cap B$ is weakly open in $B$ for each bounded set $B \subset E$.

(c) $U \cap \lambda B_E$ is weakly open in $\lambda B_E$ for each $\lambda \in K$, $\lambda \neq 0$.

*Proof.* Straightforward.

**PROPOSITION 1.2.** For a seminorm $p$ on $E$ the following are equivalent.

(a) $p$ is bw-continuous.

(b) $p|B$ is weakly continuous for each bounded set $B \subset E$.

(c) $p|B_E$ is weakly continuous.

*Proof.* Direct consequence of Proposition 1.1. (Observe that from (c) it follows that $p|\lambda B_E$ is weakly continuous for each $\lambda \in K$, $\lambda \neq 0$.)

**PROPOSITION 1.3.** Let $(X, \tau)$ be a locally convex space over $K$, let $A : E \to X$ be a linear map. Then $A$ is bw to $\tau$ continuous if and only if $A|B$ is weakly continuous for each bounded set $B \subset E$.

*Proof.* If $A$ is bw to $\tau$ continuous then so is $A|B$ and $A|B$ is weakly continuous. If $A|B$ is weakly continuous for each bounded $B \subset E$ then for each $\tau$-continuous seminorm $q$ on $X$ we have that $q \circ A$ satisfies (b) of Proposition 1.2.

The next Proposition is deeper and gives a concrete description of the bw-continuous seminorms ((d) and (e) below).

**PROPOSITION 1.4.** For a seminorm $p$ on $E$ the following are equivalent.

(a) $p$ is bw-continuous.

(b) $B_E$ is a $p$-compactoid.
(γ) \( p \) is polar, norm continuous, \( \{ f \in E' : |f| \leq p \} \) is a norm compactoid in \( E' \).

(δ) There exist \( f_1, f_2, \ldots \) in \( E' \) with \( \lim_{n \to \infty} \|f_n\| = 0 \) such that \( p \) is equivalent to \( \sup_n |f_n| \).

(ε) There exist \( f_1, f_2, \ldots \) in \( E' \) with \( \lim_{n \to \infty} \|f_n\| = 0 \) such that \( p \leq \sup_n |f_n| \).

Proof. \( (\alpha) \implies (\beta) \). \( B_E \) is a compactoid for the weak topology; the topologies \( bw \) and \( w \) coincide on \( B_E \).

(β) \( \implies (\gamma) \). As \( B_E \) is a \( p \)-compactoid we have \( (E, p) = ([B_E], p) \) is of countable type so \( p \) is a polar seminorm. Also \( p \) is bounded on \( B_E \) hence norm continuous. To show that \( \{ f \in E' : |f| \leq p \} \) is a normcompactoid consider the identity map \( (E, \| \|) \rightarrow (E, p) \) which is compact by (β). Then so is its adjoint which is the inclusion \( (E, p)' \rightarrow (E, \| \|)' \). Hence, the unit ball \( \{ f \in E' : |f| \leq p \} \) of \( (E, p)' \) is a compactoid in \( (E, \| \|)' \).

(γ) \( \implies (\delta) \). Set \( S := \{ f \in E' : |f| \leq p \} \). By polarity and norm continuity

\[
(*) \quad p = \sup_{f \in S} \{|f| : f \in E' : |f| \leq p\} = \sup_{f \in S} |f|.
\]

The normcompactoid \( S \) is easily seen to be absolutely convex and norm closed. So, by [6], Lemma 1.3, if \( \lambda \in K, |\lambda| > 1 \) there exist \( f_1, f_2, \ldots \in S \) with \( \lim_{n \to \infty} \|f_n\| = 0 \) such that

\[
\overline{c} \sigma \{ f_1, f_2, \ldots \} \subset S \subset |\lambda| \overline{c} \sigma \{ f_1, f_2, \ldots \}
\]

We obtain

\[
\sup_n |f_n| = \sup \{|f| : f \in \overline{c} \sigma \{ f_1, f_2, \ldots \}\} \leq \sup_{f \in S} |f| \leq \sup \{|f| : f \in \lambda \overline{c} \sigma \{ f_1, f_2, \ldots \}\} = |\lambda| \sup_n |f_n|.
\]

Together with (\*) this yields

\[
\sup_n |f_n| \leq p \leq |\lambda| \sup_n |f_n|
\]

and (δ) follows. The implication (δ) \( \implies (ε) \) is obvious. Finally, assume (ε). To arrive at (α) we prove (see Proposition 1.2(γ)) that for a net \( i \mapsto x_i \) in \( B_E \) converging weakly to \( x \in B_E \) it follows that \( p(x-x_i) \to 0 \). In fact, let \( N \) be such that \( \|f_n\| \leq \varepsilon \) for \( n > N \). We have \( |f_j(x-x_i)| \leq \varepsilon \) for all \( j \in \{1, \ldots, N\} \) and sufficiently large \( i \). For these \( i \)

\[
p(x-x_i) \leq \max_{1 \leq j \leq N} |f_j(x-x_i)| \vee \sup_{j > N} |f_j(x-x_i)| \leq \varepsilon \vee \sup_{j > N} \|f_j\| \|x-x_i\|
\]

\[
\leq \varepsilon \vee \sup_{j > N} \|f_j\| \leq \varepsilon.
\]

Several corollaries obtain.
COROLLARY 1.5. The topology $bw$ is of countable type. The dual of $(E, bw)$ is $E'$. An absolutely convex edged set in $E$ is $bw$-closed if and only if it is weakly closed.

Proof. In the proof of $(\beta) \implies (\gamma)$ it is observed that $bw$ is of countable type. The identity maps $(E,||)|| \to (E,bw) \to (E,w)$ are continuous (Proposition 1.4(\gamma)). Since $(E,||)||' = (E,w)' = E'$ the dual of $(E,bw)$ also must be $E'$. As a consequence the topologies $bw$ and $w$ have the same collection of polar sets and, by strong polarness, have the same collection of absolutely convex edged sets ([5], Theorem 4.7.).

Remarks. (See [5], [9] for proofs.)

1. If $K$ is spherically complete we may extend the last statement of Corollary 1.5. as follows. An absolutely convex set in $E$ is $bw$-closed if and only if it is weakly closed if and only if it is norm closed.

2. If $K$ is not spherically complete but $E$ is strongly polar we may draw the same conclusion as in 1. but only for edged absolutely convex sets.

3. If $K$ is not spherically complete and $E'$ is infinite dimensional there exists a $bw$-closed absolutely convex set in $E$ that is not weakly closed.

COROLLARY 1.6. Let $X$ be a locally convex space. A linear map $A : E \to X$ is $bw$-continuous if and only if it is compact for the norm topology on $E$.

Proof. If $A$ is $bw$-continuous then (Proposition 1.4) $B_E$ is a $bw$-compactoid so $AB_E$ is a compactoid in $X$. If, conversely, $A$ is norm compact then by [7], 1.2, for each net $i \mapsto x_i \in B_E$ converging weakly to 0 we have $Ax_i \to 0$ in $X$. By Proposition 1.3 $A$ is $bw$-continuous.

The weak topology is defined as the weakest topology for which all $f \in E'$ are continuous. For $bw$ we have a similar description.

COROLLARY 1.7. The bounded weak topology on $E$ is the weakest topology on $E$ for which all compact operators $E \to c_0$ are continuous.

Proof. Let $\tau$ be the weakest topology on $E$ for which all compact operators $E \to c_0$ are continuous. By Corollary 1.6, $bw \geq \tau$. To prove $bw \leq \tau$, let $p$ be a $bw$-continuous seminorm, say (Proposition 1.4.)

$$p = \sup_n |f_n|$$

where $f_1, f_2, \ldots \in E'$, $\lim_{n \to \infty} ||f_n|| = 0$. The operator $A : E \to c_0$, given by

$$Ax = (f_1(x), f_2(x), \ldots)$$

32
is easily seen to be compact and therefore $\tau$-continuous. Hence, so is $x \mapsto \|Ax\|$, which is $p$.

**COROLLARY 1.8.**

(i) $bw$ is the strongest locally convex topology on $E$ for which each norm bounded set is a compactoid.

(ii) $bw$ is nuclear. In fact it is the strongest nuclear topology, weaker than the norm topology. (See also [2], Example 5.2.)

Proof. (i) Follows from Proposition 1.4, ($\alpha \iff \beta$). To prove nuclearity observe that $(E, bw)$ is of countable type (Corollary 1.5) so ([1], §4, Proposition 2) it suffices to prove that every continuous linear map $A : (E, bw) \to c_0$ is compact. Such an $A$ has the form

$$x \mapsto Ax = (f_1(x), f_2(x), \ldots)$$

for certain $f_1, f_2 \in E'$. By Corollary 1.6 $A$ is compact for the norm topology on $E$ so

$$\lim_{n \to \infty} \|f_n\| = 0.$$ 

There exist $\lambda_1, \lambda_2 \in K$ and $g_1, g_2, \ldots \in E'$ such that $f_n = \lambda_n g_n$ for each $n$, $\lim_{n \to \infty} \lambda_n = 0$, $\lim_{n \to \infty} \|g_n\| = 0$. Then $\{x : |g_n(x)| \leq 1 \text{ for each } n\}$ is a $bw$-neighbourhood of 0 whose image under $A$ lies in $\{ (a_1, a_2, \ldots) \in c_0 : |a_n| \leq |\lambda_n| \text{ for each } n\}$. It follows that $A : (E, bw) \to c_0$ is compact. Now let $\tau$ be a nuclear topology on $E$, weaker than the norm topology. Then, in particular, each norm bounded set is $\tau$-bounded and is, by nuclearity, a $\tau$-compactoid. We conclude that, by (i), $\tau \leq bw$.

We have seen in Proposition 1.3 that in order to check $bw$-continuity of $A : E \to X$ it suffices to consider the behaviour of $A$ on bounded $bw$-convergent nets. Yet we shall prove (Proposition 1.10) that there do exist 'essentially' unbounded $bw$-convergent nets.

**LEMMA 1.9.** Let $E'$ be infinite dimensional. If $X \subset E'$ is a normcompactoid then $X_0 = \{x \in E : |f(x)| \leq 1 \text{ for each } f \in X\}$ is unbounded.

Proof. Without loss assume $X$ is absolutely convex, edged and normcomplete. As $X$ is also metrizable, by [4], Theorem 10(ii) the norm topology and the $w'$-topology coincide on $X$. It follows that $X$ is $w'$-closed and edged. Thus, ([5], Theorem 4.7) $X$ is a polar set in $(E', \sigma(E', E))$ so that $X = (X_0)^0 := \{f \in E' : |f| \leq 1 \text{ on } X_0\}$. Now, if $X_0$ were bounded then $X = (X_0)^0$ is a norm neighbourhood of 0 which is in conflict to compactoidity.

**PROPOSITION 1.10.** Let $E'$ be infinite dimensional. Then there exists a net $i \mapsto x_i$ in $E$, converging to 0 in $bw$ but such that $\{x_i : i \geq i_0\}$ is unbounded for each $i_0$.

33
Proof. For each normcompactoid \( X \subset E' \) and \( \epsilon > 0 \) we choose an \( x_{X,\epsilon} \in E \) such that
\[
|f(x_{X,\epsilon})| \leq \epsilon \quad \text{for each } f \in X \text{ and } \|x_{X,\epsilon}\| \geq \epsilon^{-1} \quad \text{(Lemma 1.9)}.
\]
The set \( I := \{(X,\epsilon) : X \text{ is a compactoid in } E', \epsilon > 0\} \) is a directed set under: \( (X_1,\epsilon_1) \geq (X_2,\epsilon_2) \) iff \( X_1 \supset X_2 \) and \( \epsilon_1 \leq \epsilon_2 \). It is not hard to see that the net \( i \mapsto x_i \) we just defined satisfies the requirements.

Let us consider the following related question (compare Proposition 1.1 \((\beta)\)). Let \( A \) be an absolutely convex subset of \( E \) such that \( A \cap B \) is weakly closed in \( B \) for each bounded set \( B \subset E \). Does it follow that \( A \) is \( w \)-closed? Does it follow that \( A \) is \( w \)-closed?
If \( K \) is spherically complete (and also if \( E \) is a normed space over \( \mathbb{R} \) or \( \mathbb{C} \)) the answers are yes \( (A \cap B \) is norm closed in \( B \) for each bounded set \( B \subset E \) so \( A \) is norm closed hence \( w \)-closed (and \( bw \)-closed) by Remark 1 following Corollary 1.5). But, if \( K \) is not spherically complete, say, \( K = \mathbb{C}_p \) the answers are no for \( E = \ell^\infty(m) \) where \( m \geq \#K \) (see [9], Corollary 3.6).

§2. THE BOUNDED WEAK STAR TOPOLOGY

It can be expected (for the definitions, see the Introduction) that, to some extent, the topologies \( bw \) and \( bw' \) behave in a similar way. Indeed, the obvious \( bw' \)-versions of Propositions 1.1, 1.2, 1.3 are easily seen to be true. Also, \( B_{E'} \) is a \( bw' \)-compactoid, \( bw' \) is of countable type, \( bw' \) is weaker than the norm topology on \( E' \), etc.; we leave the details to the reader. The counterpart (Theorem 2.3) of Proposition 1.4 \((\alpha) \iff (\beta)\) is less innocent. First recall that the dual of \( (E',w') \) is equal to \( j_E(E) \). ([5], Lemma 7.1). We prove

**Proposition 2.1.** The dual of \( (E',bw') \) is the norm closure of \( j_E(E) \) in \( E'' \).

*Proof.*** Every \( \theta \in j_E(E) \) is, on \( B_{E'} \), the uniform limit of a sequence in \( j_E(E) \) so \( \theta|B_{E'} \) is \( w' \)-continuous i.e. \( \theta \) is \( bw' \)-continuous. Thus, we have \( j_E(E) \subset (E',bw') \). Conversely, let \( \theta \in j_E(E) \). Then \( \theta \) is norm continuous i.e. \( \theta \in E'' \). Let \( \epsilon > 0 \); we shall find an \( x \in E \) such that \( \|\theta - j_E(x)\| < \epsilon \). Choose \( \alpha \in K, \ 0 < |\alpha| < \epsilon \). The \( w' \)-continuity of \( \theta|B_{E'} \) yields a finite set \( F \subset E \) such that \( f \in F \cap B_{E'} \) implies \( |\theta(f)| \leq |\alpha|, \) in other words
\[
f \in j_E(F)_0 \cap (B_{E'})_0 \implies |(\alpha^{-1}\theta)(f)| \leq 1.
\]
So we see that \( \alpha^{-1}\theta \in (j_E(F)_0 \cap (B_{E'})_0)_0 = (A + B_{E''})_0 \) where \( A = j_E(coF) \). Now \( B_{E''} \) is \( w' \)-closed and \( A \) is finite dimensional so by [6], 1.4, \( (A + B_{E''})_0 = (A + B_{E''})^e \). For any \( \lambda \in K \) such that \( |\lambda| > 1 \) and \( |\lambda\alpha| < \epsilon \) we have \( \alpha^{-1}\theta \in \lambda A + \lambda B_{E''} \) hence \( \theta \in j_E(E) + \alpha \lambda B_{E''} \) and there is an \( x \in E \) with \( \theta - j_E(x) \in \alpha \lambda B_{E''} \) i.e. \( \|\theta - j_E(x)\| < \epsilon. \)
To avoid needless complications we shall from now on restrict ourselves to polar spaces i.e. spaces $E$ such that $j_E$ is an isometry. ('Most' 'natural' spaces are polar.)

**COROLLARY 2.2.** Let $E$ be a polar Banach space.

(i) The dual of $(E', bw')$ is $j_E(E)$.

(ii) An edged absolutely convex subset of $E'$ is $bw'$-closed if and only if it is $w'$-closed.

*Proof.* (i) is immediate. (ii) follows from (i) as $(E', w')$ and $(E', w')$ have the same dual space.

**THEOREM 2.3.** Let $E$ be a polar $K$-Banach space. Then the topology $bw'$ on $E'$ is the topology $c$ of compact convergence i.e. it is generated by the seminorms $f \mapsto \max_{X} |f|$ where $X$ runs through the family of all compact subsets of $E$.

*Proof.* A routine proof shows that $c$, restricted to bounded sets, coincides with $w'$ so, by definition, $bw'$ is stronger than $c$. Let us prove that $bw'$ is weaker than $c$. Let $p$ be a $bw'$-continuous seminorm on $E'$. Then $p$ is certainly $bw$-continuous so by Proposition 1.4 there exist $\theta_1, \theta_2, \ldots \in E''$ with $\lim_{n \to \infty} \|\theta_n\| = 0$ such that $p$ is equivalent to $f \mapsto \max_{X} |\theta_n(f)|$. Hence, $\theta_1, \theta_2, \ldots$ are $bw'$-continuous so by Corollary 2.2 (i) there exist $x_1, x_2, \ldots$ with $\theta_n = j_E(x_n)$ for each $n$. We have $\lim_{n \to \infty} \|x_n\| = 0$, as $j_E$ is an isometry. We see that $p$ is equivalent to

$$f \mapsto \max_{x \in X} |f(x)|$$

where $X$ is the compact set $\{0, x_1, x_2, \ldots\}$ and we are done.

The proof of Theorem 2.3 depends on the theory of §1. We now show a way to arrive at the same result avoiding the use of the $bw$-theory. The key is the following 'convexification' of the Ascoli Theorem.

**THEOREM 2.4.** Let $X$ be a subset of a polar $K$-Banach space $E$. The following are equivalent.

(a) $X$ is compactoid.

(b) If $i \mapsto f_i$ is a (norm) bounded net in $E'$ converging $w'$ to $f \in E'$ then $f_i \to f$ uniformly on $X$.

*Proof.* (a) $\Longrightarrow$ (b). To prove (b) we may assume $f = 0$, $\|f_i\| \leq 1$ for all $i$. Let $\varepsilon > 0$. There exist $x_1, x_2, \ldots, x_n \in E$ such that $X \subset B(0, \varepsilon) + \text{co}\{x_1, \ldots, x_n\}$. There is an $i_0$ such that $|f_i(x_j)| \leq \varepsilon$ for all $i \geq i_0$ and all $j \in \{1, \ldots, n\}$. As $|f_i| \leq \varepsilon$ on $B(0, \varepsilon)$ for all $i$ we find $|f_i| \leq \varepsilon$ on $X$ if $i \geq i_0$. (Remark. In this part of the proof the polarity assumption was not needed.)
It is an easy exercise to show that $E/D$ is a polar $K$-Banach space for every finite dimensional subspace $D$ of $E$. Now assume $X$ is not a compactoid. Then, almost from the definition, there is an $\alpha > 0$ such that $\text{diam } \pi_D(X) > \alpha$ for each $D \in \mathcal{D}$, where $\mathcal{D}$ is the directed set of all finite dimensional subspaces of $E$ and where $\pi_D : E \to E/D$ is the quotient map. Now $E/D$ is polar so we can find a $g_D \in (E/D)'$ with $\|g_D\| \leq 1$ and $|g_D(\pi_D(x))| \geq \alpha$ for some $x \in X$. Set $f_D := g_D \circ \pi_D (D \in \mathcal{D})$. Then $f_D \to 0$ in $w'$, $\|f_D\| \leq 1$ for all $D$ but not $f_D \to 0$ uniformly on $X$, conflicting $(\beta)$.

Second proof of Theorem 2.3. Let $p$ be a $bw'$-continuous seminorm on $E'$. Then $p$ is polar so $p(f) = \sup\{|\theta(f)| : |\theta| \leq p, \theta \in (E',bw')'\}$. By Corollary 2.2(i) there is a subset $X \subset E$ for which

$$p(f) = \sup\{|f(x)| : x \in X\}.$$

To prove that $X$ is a compactoid, let $f_i \in E'$, $\|f_i\| \leq 1$, $f_i \to 0$ in $w'$ then, by definition, $f_i \to 0$ in $bw'$ so that $p(f_i) \to 0$ implying $f_i \to 0$ uniformly on $X$. The compactoidity now follows from Theorem 2.4, $(\beta) \implies (\alpha)$.

Finally we consider briefly the $bw'$-version of the last problem of §1 (we keep assuming $E$ to be a polar Banach space). Let $A \subset E'$ be absolutely convex. Assume that $A \cap B$ is $w'$-closed in $B$ for each bounded $B \subset E'$. Does it follow that $A$ is $bw'$-closed? Does it follow that $A$ is $w'$-closed?

If $K$ is spherically complete the answers are yes which is also true for the analogous classical questions (the Krein-Šmulian Theorem). The proof is by no means trivial. See [8]. If $K$ is not spherically complete a counterexample to both questions is given by [9], Corollary 3.7.

REFERENCES


A determinant for almost-isometric linear maps

A. van Rooij

$K$ is a field with a non-trivial non-Archimedean valuation relative to which $K$ is complete.

In [2], for any two normed vector spaces $E$ and $F$ of the same finite dimension we have introduced a real-valued function $\Delta$ on the space $\mathcal{L}(E, F)$ of all linear maps $E \to F$. This function reduces to the absolute value of the determinant function in case $E = F$. It is our purpose in this paper to extend $\Delta$ to the set of all contractive linear operators between infinite dimensional normed spaces.

For notations and definitions we refer to [1] and [2], §1.

Throughout, $E$, $F$ and $G$ are normed vector spaces over $K$. Their norms are all indicated by: $\| \|$.

$\mathcal{L}(E, F)$ is the space of all continuous linear maps $E \to F$.

The restriction of a map $T$ to a subset $D$ of its domain is denoted: $T|D$.

§1. We start with a synopsis of [2], §1 as far as is relevant at this stage. First some notations.

For elements $a_1, \ldots, a_N$ of a vector space, by

$$[a_1, \ldots, a_N]$$

we denote the linear hull of $\{a_1, \ldots, a_N\}$, with the understanding that

$$[a_1, \ldots, a_0] := [\ ] := \{0\}.$$  

If $E$ is a normed vector space, $a \in E$ and $D$ is a linear subspace of $E$, we set

$$\text{dist}(a, D) := \inf_{x \in D} \|a - x\|.$$
For \( x_1, \ldots, x_N \in E \) we define

\[
\text{Vol}_E(x_1, \ldots, x_N) := \prod_{n=1}^{N} \text{dist}(x_n, [x_1, \ldots, x_{n-1}]).
\]

1.2-1.6 are essentially quotes from [2], §1.

1.2 LEMMA ([2], 1.2). Let \( x_1, \ldots, x_N \in E \). Then

(i) \( \text{Vol}_E(x_1, \ldots, x_N) \leq \Pi \|x_n\| \).

(ii) \( \text{Vol}_E(x_1, \ldots, x_N) = \Pi \|x_n\| \) if and only if either the \( x_n \) form an orthogonal sequence or one of them is 0.

(iii) \( \text{Vol}_E(x_1, \ldots, x_N) = 0 \) if and only if the \( x_n \) are linearly dependent. \( \square \)

1.3 THEOREM ([2], 1.3). If \( E \) is \( N \)-dimensional \((N \in \mathbb{N})\) and \( T : E \to E \) is linear, then for all \( x_1, \ldots, x_N \in E \)

\[
\text{Vol}_E(Tx_1, \ldots, Tx_N) = \det T \cdot \text{Vol}_E(x_1, \ldots, x_N).
\]

\( \square \)

1.4 COROLLARY ([2], 1.3(A)). If \( x_1, \ldots, x_N \in E \) and if \( \pi \) is a permutation of \( \{1, \ldots, N\} \), then

\[
\text{Vol}_E(x_{\pi(1)}, \ldots, x_{\pi(N)}) = \text{Vol}_E(x_1, \ldots, x_N).
\]

\( \square \)

1.5 COROLLARY ([2], 1.5). Let \( E \) be \( N \)-dimensional \((N \in \mathbb{N})\). Then for every \( T \in \mathcal{L}(E,F) \) there exists a unique real number \( \Delta(T) \) such that

\[
\text{Vol}_F(Tx_1, \ldots, Tx_N) = \Delta(T) \cdot \text{Vol}_E(x_1, \ldots, x_N) \quad (x_1, \ldots, x_N \in E).
\]

\( \square \)

1.6 THEOREM ([2], 1.8). Let \( E \) be \( N \)-dimensional. If \( x_1, \ldots, x_N \in E \) and \( f_1, \ldots, f_N \in E' \), then

\[
\text{Vol}_E(x_1, \ldots, x_N)\text{Vol}_{E'}(f_1, \ldots, f_N) = |\det f(x)|
\]

where \( f(x) \) is the \( N \times N \)-matrix defined by

\[
f(x)_{nm} := f_n(x_m) \quad (n, m \in \{1, \ldots, N\}).
\]

\( \square \)

In the above we have taken some liberties. Properly speaking, [2] deals only with the "volume" of a sequence of vectors whose length is equal to the dimension of the surrounding vector space. Thus, [2] produces only the particular cases of 1.2-1.6 in which \( \dim E = \dim F = N \). This condition is retained in 1.3 and 1.6, partly in 1.5 and 40
not at all in 1.2 and 1.4. One easily obtains the general versions from the special ones if one observes that the definition of $\text{Vol}_E(x_1, \ldots, x_N)$ requires no connection between $N$ and $\dim E$, and that

$$\text{Vol}_E(x_1, \ldots, x_N) = \text{Vol}_D(x_1, \ldots, x_N)$$

whenever $D$ is a linear subspace of $E$ containing $x_1, \ldots, x_N$.

Accordingly, from here on we drop the subscript and write “Vol” instead of “Vol$_E$” or “Vol$_D$”.

Corollary 1.5 is to be viewed as the definition of a function $\Delta$ on $L(E, F)$.

We need a few simple additions to 1.2-1.6. From the definitions one obtains immediately:

1.7 COROLLARY. If $E, F, G$ are $N$-dimensional normed spaces ($N \in \mathbb{N}$) and if $T : E \to F$ and $S : F \to G$ are linear, then

$$\Delta(ST) = \Delta(S)\Delta(T). \quad \Box$$

1.8 LEMMA. Let $0 < t \leq 1$; let $x_1, \ldots, x_N$ be a $t$-orthogonal sequence in $E$. Then

$$\text{Vol}(x_1, \ldots, x_N) \geq t^N \prod_n \|x_n\| \quad \Box$$

In every $N$-dimensional normed vector space, for $0 < t < 1$ there is a $t$-orthogonal base. Lemma 1.8 therefore implies the first part of Theorem 1.9. For the second part, apply 1.7 with $S = T^{-1}$.

1.9 THEOREM. Let $E$ be $N$-dimensional; let $T : E \to F$ be linear.

(i) Then $\Delta(T) \leq \|T\|^N$.

(ii) If $c \in (0, \infty)$ and $\|Tx\| \geq c\|x\|$ ($x \in E$), then $\Delta(T) \geq c^N$. \quad \Box

1.10 COROLLARY. Let $E, F$ be $N$-dimensional normed spaces. Let $(x_1, \ldots, x_N)$ be a base for $E$, $(f_1, \ldots, f_N)$ a base for $F'$. For every linear $T : E \to F$,

$$\Delta(T) = \frac{|\det f(Tx)|}{\text{Vol}(x_1, \ldots, x_N)\text{Vol}(f_1, \ldots, f_N)}$$

where $f(Tx)$ is the $N \times N$-matrix defined by

$$(f(Tx))_{nm} = f_n(Tx_m).$$
Proof. Apply 1.6 to $Tx_1, \ldots, Tx_N$ and $f_1, \ldots, f_N$. \hfill $\Box$

1.11 COROLLARY. If $E, F$ are $N$-dimensional normed spaces and $T : E \to F$ is linear, then

$$\Delta(T') = \Delta(T)$$

where $T' : F' \to E'$ is the adjoint of $T$.

Proof. Use 1.10 and the reflexivity of $E$. \hfill $\Box$

1.12 MONOTONICITY LEMMA. Let $E$ be finite dimensional, let $T \in \mathcal{L}(E,F)$ and let $D$ be a linear subspace of $E$. Then

$$\Delta(T) \leq \Delta(T|D)\|T\|^{\dim E/D}.$$ 

Proof. Without restriction, let $\dim E/D = 1$. Take $0 < t < 1$. There is an $e \in E, e \notin D$ such that $\text{dist}(e, D) \geq t\|e\|$ ([1], 3.2). If $(d_1, \ldots, d_N)$ is a base of $D$, then

$$\Delta(T) = \frac{\text{Vol}(Td_1, \ldots, Td_N, Te)}{\text{Vol}(d_1, \ldots, d_N, e)} = \Delta(T|D) \frac{\text{dist}(Te, T(D))}{\text{dist}(e, D)} \leq \Delta(T|D) \frac{\|T\|\|e\|}{\text{dist}(e, D)} \leq t\Delta(T|D)\|T\|.$$ 

\hfill $\Box$

1.13 CONTINUITY LEMMA. For every $N$ the function $\text{Vol}$ is continuous on $E^N$. Explicitly: Let $s > 0, \varepsilon > 0$. If $a_1, \ldots, a_N, b_1, \ldots, b_N \in E$ are such that

$$\|a_n\| \leq s, \|b_n\| \leq s, \|a_n - b_n\| \leq \varepsilon$$

for every $n$, then

$$|\text{Vol}(a_1, \ldots, a_N) - \text{Vol}(b_1, \ldots, b_N)| \leq N s^{N-1} \varepsilon.$$ 

Proof. Setting $A := [a_1, \ldots, a_{N-1}]$ we have

$$|\text{Vol}(a_1, \ldots, a_{N-1}, a_N) - \text{Vol}(a_1, \ldots, a_{N-1}, b_N)| = \text{Vol}(a_1, \ldots, a_{N-1}) \cdot |\text{dist}(a_N, A) - \text{dist}(b_N, A)| \leq \text{Vol}(a_1, \ldots, a_{N-1}) \cdot \|a_N - b_N\| \leq s^{N-1} \varepsilon.$$
By symmetry (1.4) we see that for all $n$
\[ |\text{Vol}(a_1, \ldots, a_{n-1}, a_n, b_{n+1}, \ldots, b_N) - \text{Vol}(a_1, \ldots, a_{n-1}, b_n, b_{n+1}, \ldots, b_N)| \leq s^{N-1}\varepsilon. \]

The announced inequality follows. □

1.14 COROLLARY. If $E$ is finite dimensional, the function $\Delta$ is continuous on $\mathcal{L}(E, F)$. □

§2. Now we drop the condition that $E$ be finite dimensional. In exchange, we have to restrict our class of operators: we can define $\Delta(T)$ only if $\|T\| \leq 1$.

By
\[ \mathcal{L}_1(E, F) \]
we indicate the set $\{T \in \mathcal{L}(E, F) : \|T\| \leq 1\}$.

If $T \in \mathcal{L}_1(E, F)$ and if $D_1, D_2$ are finite dimensional linear subspaces of $E$ with $D_1 \subset D_2$, then by our Monotonicity Lemma,
\[ \Delta(T|D_1) \geq \Delta(T|D_2). \]

Therefore, it is not unreasonable to define
\begin{align*}
2.1 \quad \Delta(T) &:= \inf\{\Delta(T|D) : D \text{ is a finite dimensional linear subspace of } E\} = \\
&= \inf\left\{ \frac{\text{Vol}(Te_1, \ldots, Te_N)}{\text{Vol}(e_1, \ldots, e_N)} : e_1, \ldots, e_N \in E \text{ are linearly independent}\right\}.
\end{align*}

with the added convention that $\Delta(T) = 1$ in case $E = \{0\}$.

Obviously, for finite dimensional $E$ this definition ties in with the older one.

It is clear that
\begin{align*}
(2.2) \quad \Delta(T) &\leq 1 \quad (T \in \mathcal{L}_1(E, F)).
\end{align*}

Also, if $T \in \mathcal{L}_1(E, F)$ and $a \in E$, $a \neq 0$, then
\[ \Delta(T) \leq \Delta(T|[a]) = \frac{\|Ta\|}{\|a\|}. \]

Thus,
\begin{align*}
(2.3) \quad \Delta(T) = 1 &\iff T \text{ is an isometry.}
\end{align*}
It also follows that; if $\Delta(T) > 0$, then $T$ is a homeomorphism of $E$ onto $T(E)$, so that $T(E)$ is closed. This means that in proofs we may often assume $T$ to be surjective $E \to F$.

For a space $E$ of countable type we have a more straightforward description of $\Delta(T)$:

**2.4 Theorem** Let $e_1, e_2, \ldots \in E$ be linearly independent and such that their linear hull is dense in $E$. Then

$$\Delta(T) = \lim_{N \to \infty} \frac{\text{Vol}(Te_1, \ldots, Te_N)}{\text{Vol}(e_1, \ldots, e_N)} \quad (T \in \mathcal{L}_1(E, F)).$$

*Proof.* Let $D_N := [e_1, \ldots, e_N], D := \bigcup_N D_N$. Take $T \in \mathcal{L}_1(E, F)$ and set

$$\delta_N := \frac{\text{Vol}(Te_1, \ldots, Te_N)}{\text{Vol}(e_1, \ldots, e_N)} \quad (N \in \mathbb{N}).$$

Then $\delta_N = \Delta(T|D_N)$. By the Monotonicity Lemma $\delta_1 \geq \delta_2 \geq \ldots$, so $\lim \delta_N$ exists. Every finite dimensional subspace of $D$ is contained in some $D_N$. Therefore,

$$\Delta(T|D) = \lim \delta_N.$$

But by continuity (1.13), $\Delta(T) = \Delta(T|D)$ since $D$ is dense in $E$. 

**2.5 Example.** Let $\alpha_1, \alpha_2, \ldots \in K, |\alpha_n| \leq 1$, and define $T : c_0 \to c_0$ by

$$Tx := (\alpha_1x_1, \alpha_2x_2, \ldots) \quad (x = (x_1, x_2, \ldots) \in c_0).$$

Then $\Delta(T) = \prod_n |\alpha_n|.$

**2.6 Theorem.** Let $T \in \mathcal{L}_1(E, F), S \in \mathcal{L}_1(F, G)$. Then

$$\Delta(ST) = \Delta(T)\Delta(S|T(E)).$$

*Proof.* We may assume $T$ to be injective. (Otherwise $\Delta(T) = 0$ and $\Delta(ST) = 0$.)

For every finite dimensional $D \subset E$, using 1.7 we get

$$\Delta((ST)|D) = \Delta(T|D)\Delta(S|T(D)) \geq \Delta(T)\Delta(S|T(E)), \quad \text{so} \quad \Delta(ST) \geq \Delta(T)\Delta(S|T(E)).$$

Conversely, for any two finite dimensional spaces $E_1 \subset E$ and $F_1 \subset T(E)$ make $D := E_1 + T^{-1}(F_1)$; then $D$ is finite dimensional and

$$\Delta(T|E_1)\Delta(S|F_1) \geq \Delta(T|D)\Delta(S|T(D)) = \Delta((ST)|D) \geq \Delta(ST).$$

Hence, $\Delta(T)\Delta(S|T(D)) \geq \Delta(ST)$.
2.7 COROLLARY. Let $T \in \mathcal{L}_1(E, F)$, $S \in \mathcal{L}_1(E, G)$ and suppose

$$\|Sx\| \leq \|Tx\| \quad (x \in E).$$

Then $\Delta(S) \leq \Delta(T)$.

Proof. There is a $U$ in $\mathcal{L}_1(T(E), G)$ with $S = UT$. □

§ 3. Let $T \in \mathcal{L}_1(E, F)$ and let $A$ be a closed linear subspace of $E$. Then $T$ induces a linear map

$$x \mod A \mapsto Tx \mod \overline{T(A)} \quad (x \in E)$$

of $E/A$ into $F/\overline{T(A)}$, $\overline{T(A)}$ being the closure of $T(A)$ in $F$. We denote this map by $T \mod A$.

It is easy to see that

$$T \mod A \in \mathcal{L}_1(E/A, F/\overline{T(A)}),$$

and we have a commutative diagram

$$\begin{array}{ccc}
E & \xrightarrow{T} & F \\
\downarrow & & \downarrow \\
E/A & \xrightarrow{T \mod A} & F/\overline{T(A)}
\end{array}$$

There is a simple connection between $\Delta(T)$, $\Delta(T|A)$ and $\Delta(T \mod A)$:

3.1 THEOREM. Let $T \in \mathcal{L}_1(E, F)$ and let $A$ be a closed linear subspace of $E$. Then

$$\Delta(T) = \Delta(T|A)\Delta(T \mod A).$$

Proof. Let $P$ and $Q$ be the quotient maps $E \to E/A$ and $F \to F/\overline{T(A)}$, and set $T_A := T \mod A$:

$$\begin{array}{ccc}
E & \xrightarrow{T} & F \\
P\downarrow & & \downarrow Q \\
E/A & \xrightarrow{T_A} & F/\overline{T(A)}
\end{array}$$
(I) We first show that \( \Delta(T) \geq \Delta(T|A)\Delta(T_A) \). To this end, choose a finite dimensional linear subspace \( D \) of \( E \) and let \( 0 < t < 1 \). We are done if we can find finite dimensional \( D_1 \subset A \) and \( D_2 \subset E/A \) with

\[
\Delta(T|D) \geq t^2 \Delta(T|D_1)\Delta(T_A|D_2).
\]

Take \( D_2 := P(D) \). Let \( N := \dim D_2 \), \( s := t^{1/N} \) and choose an \( s \)-orthogonal base \((u_1, \ldots, u_N)\) in \( D_2 \). For each \( n \), choose \( d_n \in D \) with \( Pd_n = u_n \), and \( d'_n \in A \) with \( \|d_n - d'_n\| \leq s^{-1}\text{dist}(d_n, A) \). Set \( D_1 := D \cap A + [d'_1, \ldots, d'_N] \), \( \tilde{D} := D_1 + [d_1, \ldots, d_N] \). Then \( P(\tilde{D}) = D_2 \), \( \tilde{D} \cap A = D_1 \) and \( \tilde{D} \supset D \). For every \( n \),

\[
\text{dist}(d_n, D_1) \leq \|d_n - d'_n\| \leq s^{-1}\text{dist}(d_n, A) = s^{-1}\|Pd_n\| = s^{-1}|u_n|.
\]

If \((e_1, \ldots, e_M)\) is a base for \( D_1 \), then \((e_1, \ldots, e_M, d_1, \ldots, d_N)\) is a base for \( \tilde{D} \). Therefore

\[
\Delta(T|D) \leq \Delta(T|\tilde{D}) = \Delta(T|D_1) \frac{\prod \text{dist}(Td_n, [Te_1, \ldots, Te_M, Td_1, \ldots, Td_{n-1}])}{\prod \text{dist}(d_n, [e_1, \ldots, e_M, d_1, \ldots, d_{n-1}])}.
\]

Now for every \( n \),

\[
\text{dist}(Td_n, [Te_1, \ldots, Td_{n-1}]) \leq \text{dist}(Td_n, T(A) + [Td_1, \ldots, Td_{n-1}]) = \\
= \text{dist}(QTd_n, [QTd_1, \ldots, QTd_{n-1}]) = \\
= \text{dist}(T_Au_n, [T_Au_1, \ldots, T_Au_{n-1}])
\]

whereas, by (2),

\[
\text{dist}(d_n, [e_1, \ldots, d_{n-1}]) \leq \text{dist}(d_n, D_1) \leq s^{-1}\|u_n\|.
\]

Combining (4) and (5) with (3) and observing that

\[
\text{Vol}(u_1, \ldots, u_N) \geq s^N \Pi \|u_n\|
\]

one obtains (1).

(II) The proof of the converse inequality, \( \Delta(T) \leq \Delta(T|A)\Delta(T_A) \), follows similar principles.

Let \( D_1 \) and \( D_2 \) be finite dimensional subspaces of \( A \) and \( E/A \), respectively; let \( 0 < t < 1 \). It suffices to construct a finite dimensional \( D \subset E \) for which

\[
\Delta(T|D) \leq t^{-2}\Delta(T|D_1)\Delta(T_A|D_2).
\]
Let $N := \dim T_A(D_2)$, $s := t^{1/N}$. Take $v_1, \ldots, v_N \in D_2 \subset E/A$ such that $(T_A v_1, \ldots, T_A v_N)$ is an $s$-orthogonal base in $T_A(D_2)$. Choose $d_n \in E$ with $Pd_n = v_n$ and $a_n \in A$ such that

$$\dist(Td_n, T(A)) \geq s\|Td_n - Ta_n\|.$$  

Then

$$\|Td_n - Ta_n\| \leq s^{-1}\|Qd_n\| = s^{-1}\|T_A v_n\|.$$  

Put

$$\tilde{D}_1 := D_1 + [a_1, \ldots, a_N], \quad D := \tilde{D}_1 + [d_1, \ldots, d_N].$$

Then $D \cap A = \tilde{D}_1$. If $(e_1, \ldots, e_M)$ is a base of $\tilde{D}_1$, then $(e_1, \ldots, e_M, d_1, \ldots, d_N)$ is a base of $D$. Hence,

$$\Delta(T|D) = \Delta(T|\tilde{D}_1) \frac{\prod dist(Td_n, [Te_1, \ldots, Td_{n-1}])}{\prod dist(d_n, [e_1, \ldots, d_{n-1}])}.$$

For every $n$,

$$\dist(Td_n, [Te_1, \ldots, Td_{n-1}]) \leq \|Td_n - Ta_n\| \leq s^{-1}\|T_A v_n\|$$

and

$$\dist(d_n, [e_1, \ldots, d_{n-1}]) \geq \dist(d_n, A + [d_1, \ldots, d_{n-1}]) = \dist(Pd_n, [Pd_1, \ldots, Pd_{n-1}]) = \dist(v_n, [v_1, \ldots, v_{n-1}]).$$

Considering the inclusion $D_1 \subset \tilde{D}_1$ and the $s$-orthogonality of $(T_A v_1, \ldots, T_A v_N)$ we find

$$\Delta(T|D) \leq \Delta(T|D_1) \frac{s^{-N}\|T_A v_n\|}{\prod dist(v_n, [v_1, \ldots, v_{n-1}])} \leq \frac{t^{-2}\Delta(T|D_1)}{\Vol(T_A v_1, \ldots, T_A v_N)} = t^{-2}\Delta(T|D_1)\Delta(T_A|D_2).$$

\[ \blacksquare \]

3.2 COROLLARY. If $T \in \mathcal{L}(E, F)$ and $\Delta(T) \neq 0$, then there exists a closed linear subspace $A$ of $E$ that is of countable type and such that $T|A$ is isometric, i.e.,

$$\dist(x, A) = \dist(Tx, T(A)) \quad (x \in E). \blacksquare$$

3.3 COROLLARY. Let $E$ be a Banach space, $E_1$, and $E_2$ closed linear subspaces of $E$ such that $E_1 + E_2 = E$, $E_1 \cap E_2 = (0)$. Consider the natural maps

$$T : E_1 \oplus E_2 \rightarrow E \quad (x, y) \mapsto x + y$$

$$P_1 : E_1 \rightarrow E/E_2 \quad x \mapsto x \mod E_2$$

$$P_2 : E_2 \rightarrow E/E_1 \quad y \mapsto y \mod E_1$$

47
Then
\[ \Delta(T) = \Delta(P_1) = \Delta(P_2). \]

**Proof.**
\[ \Delta(T) = \Delta(T \mid E_1 \oplus \{0\})\Delta(T \mod E_1 \oplus \{0\}) = 1 \cdot \Delta(P_2) = \Delta(P_2). \]

§4. We consider the infinite dimensional analogue of 1.11.

**4.1 Lemma.** Let \( A \) be an \( N \)-dimensional linear subspace of \( E' \) and \( 0 < t < 1 \). Then there exists an \( N \)-dimensional linear subspace \( B \) of \( E \) such that
\[ t\|f\| \leq \|f|B\| \leq \|f\| \quad (f \in A). \]

**Proof.** Let \( s > 0, s^{N+1} = t \). Choose \( f_1, b_1, f_2, b_2, \ldots, b_N \) such that
\[ f_n \in A, f_n \neq 0; f_n(b_i) = 0 \text{ for } i < n; \]
\[ b_n \in E, b_n \neq 0; |f_n(b_n)| \geq s\|f_n\|\|b_n\|. \]
Take \( B := [b_1, \ldots, b_N] \). Trivially, \( \|f|B\| \leq \|f\| \) for \( f \in A \).

Set \( g_n := f_n|B \) \( n = 1, \ldots, N \); then \( \|g_n\| \geq s\|f_n\| \). If \( \lambda_1, \ldots, \lambda_N \in K \), then
\[ \| \sum_{M}^{N} \lambda_n g_n \| \geq s\|\lambda_M g_M\| \quad (M = 1, \ldots, N); \]
(evaluate at \( b_M \)). It follows that \( (g_1, \ldots, g_N) \) is \( s^N \)-orthogonal. Then for \( \lambda_1, \ldots, \lambda_N \in K \),
\[ \| \sum_{M}^{N} \lambda_n g_n \| \geq s^N \max\|\lambda_n g_n\| \geq s^{N+1} \max\|\lambda_n f_n\| \geq s^{N+1} \| \sum_{M}^{N} \lambda_n f_n \|, \]
i.e. \( \|f|B\| \geq t\|f\| \) for all \( f \in A \).

**4.2 Theorem.** Let \( T \in \mathcal{L}_1(E, F) \) be bijective. Then
\[ \Delta(T') \geq \Delta(T). \]
(The bijectivity is no luxury. Consider what happens if \( E = K, F = K^2 \).)

**Proof.** Let \( A \) be an \( N \)-dimensional subspace of \( F' \). By the lemma there exists an \( N \)-dimensional subspace \( D \) of \( E \) such that
\[ t\|f\| \leq \|f|T(D)\| \quad (f \in A). \]
Let \( P : E' \to D' \) and \( Q : F' \to T(D)' \) be the restriction maps; choose a base \( e_1, \ldots, e_N \) in \( D \) and a base \( f_1, \ldots, f_N \) in \( A \). Applying Cor. 1.10 we find

\[
\Vol(T' f_1, \ldots, T' f_N) \Vol(e_1, \ldots, e_N) \geq \Vol(P T' f_1, \ldots, P T' f_N) \Vol(e_1, \ldots, e_N) =
\]

\[
= |\det (P T')(e)| = |\det (T')(e)| =
\]

\[
= |\det f(Te)| = |\det (Q f)(Te)| =
\]

\[
= \Vol(Q f_1, \ldots, Q f_N) \Vol(T e_1, \ldots, T e_N).
\]

Now we have

\[
t\|f\| \leq \|Q f\| \quad (f \in A).
\]

It follows from 1.9(ii) that \( \Delta(Q|A) \geq t^N \), so that

\[
\Vol(Q f_1, \ldots, Q f_N) \geq t^N \Vol(f_1, \ldots, f_N).
\]

Hence,

\[
\Vol(T' f_1, \ldots, T' f_N) \Vol(e_1, \ldots, e_N) \geq t^N \Vol(f_1, \ldots, f_N) \Vol(T e_1, \ldots, T e_N).
\]

Then

\[
\Delta(T'|A) \geq t^n \Delta(T|D) \geq t^n \Delta(T).
\]

It is clear how the proof is completed. \( \square \)

Recall that a normed vector space \( E \) is said to be \textit{pseudoreflexive} if the natural map

\[
j_E : E \to E''
\]

is isometric.

\textbf{4.3 Corollary.} Let \( T \in \mathcal{L}_1(E, F) \) be bijective and assume \( E \) to be pseudoreflexive. Then

\[
\Delta(T') = \Delta(T).
\]

Moreover, if \( \Delta(T) \neq 0 \), then \( F \) is pseudoreflexive.

Proof.

\[
\begin{array}{ccc}
E & \xrightarrow{T} & F \\
j_E & \downarrow & j_F \\
E'' & \xrightarrow{T''} & F''
\end{array}
\]
$j_E$ is an isometry and $\|j_F\| \leq 1$. From 2.6 we infer that
\[
\Delta(T''j_E) = \Delta(T''|j_E(E)) \geq \Delta(T''), \\
\Delta(j_FT) = \Delta(j_F)\Delta(T) \leq \Delta(T).
\]

But $T''j_E = j_FT$, so, by 4.2
\[
\Delta(T'') \leq \Delta(j_F)\Delta(T) \leq \Delta(T) \leq \Delta(T') \leq \Delta(T'').
\]

The corollary follows. \qed

The final part of the corollary is surprising as the mere existence of a linear homeomorphism $T \in \mathcal{L}_1(E, F)$ does not guarantee pseudoreflexivity of $F$.

4.4 EXAMPLE. Let $\alpha_1, \alpha_2, \ldots \in K$, $|\alpha_n| \leq 1$ for each $n$, and define $S : l^\infty \to l^\infty$ by
\[Sx := (\alpha_1x_1, \alpha_2x_2, \ldots) \quad (x = (x_1, x_2, \ldots) \in l^\infty).
\]
Then $\Delta(S) = \prod_n |\alpha_n|$, since $S = T'$ where $T$ is as in 2.5.

REFERENCES


\documentclass{article}
\usepackage{amsmath}
\usepackage{amssymb}
\usepackage{amsthm}
\usepackage{enumerate}

\title{\textit{p}-Adic frames}
\author{W.H. Schikhof}
\date{ }

\begin{document}
\maketitle
\abstract{Throughout $K$ is a nonarchimedean valued complete field whose valuation $|\cdot|$ is not trivial. For a $K$-Banach space we introduce the concept of a $t$-frame (Definition 1.1), weaker than 'base'. Applying the emerging theory we find alternative proofs of [2], Lemma 3.2 and Corollary 3.5 concerning the type of a Banach space avoiding the use of Gruson's Theorem (Corollary 2.7). Also $t$-frames enable us to prove that a metrizable Montel space over $K$ is of countable type (Theorem 3.1).}

\section*{Terminology}
We shall use the conventions and notations of [1] and [2]. Throughout this note $E$ is a $K$-Banach space with (nonarchimedean) norm $\|\cdot\|$. Following [2] we define the Volume Function as follows. For $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in E$ we set
\[ \text{Vol}(x_1, \ldots, x_n) := \prod_{i=1}^{n} \text{dist}(x_i, \sum_{j>1} Kx_j). \]
See [2] for properties of this function, in particular its symmetry and its connection with the determinant. The algebraic $K$-linear span of a set $X \subset E$ is $[X]$. The closure of a set $Y \subset E$ is denoted $\overline{Y}$, its cardinality $\#Y$. The closed ball with center $a \in E$ and radius $r > 0$ is $B(a, r) = \{x \in E : \|x - a\| \leq r\}$.

\section{Definition and First Properties}
\begin{definition} \label{def:1.1}
Let $t \in (0, 1)$, $X \subset E$, $0 \notin X$. We say that $X$ is a $t$-frame if for any $n \in \mathbb{N}$ and distinct $x_1, \ldots, x_n \in X$ we have
\[ \text{Vol}(x_1, \ldots, x_n) \geq t^{n-1}\|x_1\| \ldots \|x_n\|. \]
\end{definition}

51
\end{document}
The proofs of the following observations are either trivial or straightforward. Let \( t \in (0, 1) \).

1. For each element \( x \) of a \( t \)-frame \( X \), let \( \lambda_x \in K, \lambda_x \neq 0 \). Then \( \{\lambda_x \cdot x : x \in X\} \) is also a \( t \)-frame. Thus, on many occasions we shall assume that \( \inf\{\|x\| : x \in X\} > 0 \) i.e. ‘\( X \) is away from zero’.

2. Every \( t \)-frame is a linearly independent set.

3. Every subset of a \( t \)-frame is a \( t \)-frame.

4. Let \( t \leq s \leq 1 \). Every \( s \)-frame is a \( t \)-frame.

5. Every singleton set \( \neq \{0\} \) is a 1-frame.

6. Let \( X \subset E, 0 \notin X \). Then \( X \) is a 1-frame if and only if \( X \) is an orthogonal set.

7. Any \( t \)-orthogonal set not containing 0 is a \( t \)-frame.

8. Let \( X \subset E \) be a \( t \)-frame, let \( X \subset Y \subset E \). Then, among all \( t \)-frames \( Z \) satisfying \( X \subset Z \subset Y \) there exist maximal ones.

9. Let \( \| \cdot \|_1 \) be a norm on \( E \) such that for some positive constants \( c, d \)

\[
c\|x\| \leq \|x\|_1 \leq d\|x\| \quad (x \in E)
\]

and let \( X \) be a \( t \)-frame in \( (E, \| \cdot \|_1) \). Then \( X \) is a \( cd^{-1}t \)-frame in \( (E, \| \cdot \|) \).

For a partial converse of 7. see §4, last proposition.

Frames are somehow opposite to compactoids:

**Lemma 1.2.** Let \( 0 < t \leq 1 \), let \( X \subset E \) be a \( t \)-frame, away from zero. Then every compactoid in \( X \) is finite.

*Proof.* We have \( s := \inf\{\|x\| : x \in X\} > 0 \). Let \( x_1, x_2, \ldots \) be a sequence in \( X \) where \( x_n \neq x_m \) as soon as \( n \neq m \). Then for each \( n \)

\[
\text{Vol}(x_1, \ldots, x_n) \geq t^{n-1}\|x_1\| \ldots \|x_n\| \geq t^{n-1}s^n \geq t^n s^n
\]

so that

\[
\liminf_{n \to \infty} \sqrt[n]{\text{Vol}(x_1, \ldots, x_n)} \geq ts > 0
\]

implying that \( \{x_1, x_2, \ldots\} \) is not a compactoid ([2], Corollary 6.10).

The converse of Lemma 1.2 does not hold. In fact, let \( e_1, e_2, \ldots \) be the canonical base of \( c_0 \), let \( \lambda \in K, 0 < |
\lambda| < 1 \) and set

\[
X := \{e_1, e_2, \ldots\} \cup \{(1+\lambda)e_1, (1+\lambda^2)e_2, \ldots\}.
\]
Each compactoid in $X$ is finite but from

$$\text{Vol}(e_n, (1+\lambda^n)e_n) = |\lambda|^n \quad (n \in \mathbb{N})$$

one derives easily that $X$ is a $t$-frame for no $t \in (0,1]$. Thus we are led to the following.

**PROPOSITION 1.3.** For a bounded subset $X$ of $E$ the following are equivalent.

(a) Every compactoid subset of $X$ is finite.

(β) Every infinite subset of $X$ contains, for some $t \in (0,1]$, an infinite $t$-orthogonal subset away from zero.

(γ) Every infinite subset of $X$ contains, for some $t \in (0,1]$, an infinite $t$-frame away from zero.

**Proof.** (α) $\implies$ (β). An infinite subset of $X$ is not a compactoid and therefore contains by [3], Theorem 2 a $t$-orthogonal sequence not tending to 0. (γ) $\implies$ (α). Let $Y$ be an infinite subset of $X$. By assumption it contains an infinite $t$-frame $Z$, away from zero. Hence every compactoid in $Z$ is finite (Lemma 1.2) and so $Z$ itself is not a compactoid, neither is $Y$. ■

**Remark.** One might call a bounded subset $X$ of $E$ (an) anticom pactoid if every compactoid subset of $X$ is finite. One proves easily that $Y \subset E$ is a compactoid if and only if every anticom pactoid subset of $Y$ is finite, making the two concepts indeed 'opposite' to one another.

§2. FRAMES AND THE TYPE OF A BANACH SPACE

Following [2] p.36 we define the type $m(E)$ of a $K$-Banach space $E$ to be the smallest among the cardinalities of the subsets $X$ of $E$ for which $[X] = E$. This notion links up with the well-known concept ‘of countable type’ which, in our terminology, reads ‘of type $\leq \aleph_0$’.

**PROPOSITION 2.1.** Let $E$ be infinite dimensional, let $X \subset E$ have cardinality strictly greater than $m(E)$. Then $X$ contains an infinite compactoid.

**Proof.** If, for each $n \in \mathbb{N}$, the set $X \cap B(0,n)$ had cardinality $\leq m(E)$ then so would $X = \bigcup_{n} X \cap B(0,n)$, a contradiction. Thus, we may assume that $X$ is bounded.

Choose a subset $Y$ of $E$ with $\#Y = m(E)$, $[Y] = E$. Let $\mathcal{F}$ be the collection of finite subsets of $Y$. Then $\#\mathcal{F} = \#Y$ and $[Y] = \bigcup_{F \in \mathcal{F}} [F]$ so that $E = \bigcup_{F \in \mathcal{F}} (B(0,1) + [F])$.
and \( X = \bigcup_{F \in \mathcal{F}} X_F \) where \( X_F := (B(0,1) + [F]) \cap X \). If \( \#X_F \leq m(E) \) for all \( F \in \mathcal{F} \) then \( \#X \leq (\#\mathcal{F}) \cdot m(E) = m(E)^2 = m(E) \), a contradiction. So there exists an \( F_1 \in \mathcal{F} \) such that

\[
X_1 := (B(0,1) + [F_1]) \cap X
\]

has cardinality \( > m(E) \). From

\[
X_1 = \bigcup_{F \in \mathcal{F}} (B(0,\frac{1}{2}) + [F]) \cap X_1
\]

we obtain by a similar argument an \( F_2 \in \mathcal{F} \) such that

\[
X_2 := (B(0,\frac{1}{2}) \cap [F_2]) \cap X_1
\]

has cardinality \( > m(E) \), etc.. Inductively we arrive at infinite sets \( X \supset X_1 \supset X_2 \ldots \) and finite subsets \( F_1, F_2, \ldots \) of \( Y \) such that \( X_n \subset B(0,\frac{1}{n}) + [F_n] \) for each \( n \in \mathbb{N} \). Now choose mutually distinct \( x_1 \in X_1, x_2 \in X_2, \ldots \) Then \( Z := \{x_1, x_2, \ldots \} \) is infinite, bounded, and for each \( n \in \mathbb{N} \) we have \( Z \subset B(0,\frac{1}{n}) + [F_n] + [x_1, \ldots, x_{n-1}] = B(0,\frac{1}{n}) + [\tilde{F}_n] \) where \( \tilde{F}_n \) is a finite set. We see that \( Z \) is a compactoid.

**COROLLARY 2.2.** Let \( 0 < t \leq 1 \). For every \( t \)-frame \( X \subset E \) we have \( \#X \leq m(E) \).

*Proof.* We may assume \( X \) to be away from zero. Now combine Lemma 1.2 and Proposition 2.1. ■

In addition to this we have (notice the clause \( t \neq 1 \)):

**PROPOSITION 2.3.** Let \( 0 < t < 1 \). If \( X \) is a maximal \( t \)-frame in \( E \) then \( [X] = E \).

*Proof.* Set \( D := [X] \). If \( D \neq E \) then we could find a nonzero \( y \in E \) with \( \text{dist}(y,D) \geq t\|y\| \). It therefore suffices to prove that for \( x \in E \setminus X \) we have \( \text{dist}(x,D) < t\|x\| \). By maximality \( X \cup \{x\} \) is no longer a \( t \)-frame. So there is an \( n \in \mathbb{N} \) and there are \( x_1, \ldots, x_n \in X \) such that \( \text{Vol}(x, x_1, \ldots, x_n) < t^n \|x\| \|x_1\| \cdots \|x_n\| \). On the other hand \( \text{Vol}(x, x_1, \ldots, x_n) \geq \text{dist}(x, [x_1, \ldots, x_m]) \text{Vol}(x_1, \ldots, x_n) \geq \text{dist}(x, D) t^{n-1} \|x_1\| \cdots \|x_n\| \). It follows that \( \text{dist}(x,D) < t\|x\| \). ■

Combining these results we obtain

**THEOREM 2.4.** Let \( 0 < t < 1 \). If \( X \) is a maximal \( t \)-frame in \( E \) then \( \#X = m(E) \) and \( [X] = E \). ■
COROLLARY 2.5. For each \( s, t \in (0, 1) \), maximal \( s \)-frames and maximal \( t \)-frames in \( E \) have the same cardinality. ■

COROLLARY 2.6. Let \( 0 < t < 1 \). \( E \) is of countable type if and only if each \( t \)-frame is at most countable. ■

COROLLARY 2.7. ([2], 3.2, 3.5.) If \( E \) has a base \( X \) then \( \#X = m(E) \). If \( D \) is a closed subspace of \( E \) then \( m(D) \leq m(E) \).

Proof. If \( X \) is a base of \( E \) then \( [X] = E \) so, by definition, \( \#X \geq m(E) \). But \( X \) is also a \( t \)-frame for some \( t \in (0, 1) \) so that, by Corollary 2.2, \( \#X \leq m(E) \). To prove the second statement, let \( Y \subset D \) be a maximal \( \frac{1}{2} \)-frame in \( D \). Extend it to a maximal \( \frac{1}{2} \)-frame \( X \) in \( E \). Then we have \( m(E) = \#X \geq \#Y = m(D) \). ■

§3. METRIZABLE MONTEL SPACES

Frames have been invented to prove the following theorem.

THEOREM 3.1. A metrizable locally convex space \( F \) over \( K \) in which each bounded set is a compactoid is of countable type.

Proof. Let \( p_1 \leq p_2 \leq \ldots \) be a collection of seminorms defining the topology of \( F \). It suffices to show that \( (F,p_1) \) (or, \( F/\text{Ker} \ p_1 \) with the norm induced by \( p_1 \)) is of countable type. Suppose the latter is not true. By Proposition 2.3 (the proof of which being also valid for noncomplete normed spaces) we can choose an uncountable \( \frac{1}{2} \)-frame \( Y \) in \( F/\text{Ker} \ p_1 \), \( Y \) away from zero. Let \( \pi : F \to F/\text{Ker} \ p_1 \) be the quotient map, let \( i : F/\text{Ker} \ p_1 \to F \) be a map such that \( \pi \circ i \) is the identity, set \( X := i(Y) \). \( X \) is covered by the balls \( \{ x \in F : p_1(x) \leq n \} \) \((n \in \mathbb{N})\) so \( X \) contains an uncountable \( p_1 \)-bounded set \( X_1 \). By the same token there is an \( X_2 \subset X_1 \) that is \( p_2 \)-bounded and uncountable. Inductively we obtain infinite sets \( X \supset X_1 \supset X_2 \ldots \). Choose mutually distinct \( x_1 \in X_1 \), \( x_2 \in X_2 \), … Then \( \{ x_1, x_2, \ldots \} \) is bounded hence a compactoid by assumption. But then \( \{ \pi(x_1), \pi(x_2), \ldots \} \) is bounded, infinite, and a subset of \( Y \) which is a \( \frac{1}{2} \)-frame away from 0. From Lemma 1.2 it follows that the set \( \{ \pi(x_1), \pi(x_2), \ldots \} \) is finite, a contradiction. ■

§4. NOTES

1. If \( X \subset E \) is such that \( [X] = E \), then does there exist a subset \( Y \) of \( X \) such that \( [X] = E \) and \( Y \) is a \( t \)-frame for some \( t \in (0, 1) \)? The following example shows that the
answer is no. Let \( \lambda \in \mathbb{K}, 0 < |\lambda| < 1 \) and set \( X := \{ e_1, e_1 + \lambda e_2, e_1 + \lambda^2 e_3, \ldots \} \subset c_0 \). This \( X \) generates \( c_0 \) but is compact so \( t \)-frames in \( X \) must be finite. However we do have the following.

2. Let \( X \subset E \) be such that \( \overline{[X]} = E \). Choose, for each \( n \in \mathbb{N} \), a maximal \( \frac{1}{n} \)-frame \( X_n \) in \( X \). Then \( \bigcup X_n \) is compact so \( t \)-frames in \( X \) must be finite. However we do have the following.

3. Let \( \lambda \in \mathbb{K}, 0 < |\lambda| < 1 \). If \( t \in (0,1) \), \( t < |\lambda| \) then the canonical base \( e_1, e_2, \ldots \) of \( c_0 \) is not a maximal \( t \)-frame.

4. Observe that the set \( X \) we just constructed is a \( t \)-frame in \( c_0 \) but is \( s \)-orthogonal for no \( s \in (0,1] \).

5. Let \( 0 < t < 1 \), let \( D \) be a closed subspace of \( E \), let \( Z \) be a \( t \)-frame in \( E/D \). Let \( 0 < s < 1 \) and let \( i : E/D \to E \) be a map such that \( \pi \circ i \) is the identity and \( s ||i(y)|| \leq ||y|| \) for all \( y \in E/D \). (Here \( \pi : E \to E/D \) is the quotient map.) Then \( X := i(Y) \) is an \( s^t \)-frame in \( E \).

6. Let \( t \in (0,1], n \in \mathbb{N} \) and let \( \{ x_1, \ldots, x_n \} \) be a \( t \)-frame. Suppose that \( D := [x_1, \ldots, x_n] \) has an orthonormal base. Then \( \{ x_1, \ldots, x_n \} \) is \( t^{n-1} \)-orthogonal.

Proof. Let \( e_1, e_2, \ldots \) be an orthonormal base of \( D \). We may assume that \( ||x_i|| = 1 \) for all \( i \in \{1, \ldots, n\} \). Define a linear map \( A : D \to D \) by the formula \( A e_i = x_i \) (\( i \in \{1, \ldots, n\} \)), and let \( (a_{ij}) \) be the matrix of \( A \) with respect to the base \( e_1, \ldots, e_n \). Then \( |a_{ij}| \leq 1 \) for all \( i, j \in \{1, \ldots, n\} \). By [2], Theorem 1.3 we have

\[
\text{Vol}(x_1, \ldots, x_n) = |\det A| \text{ Vol}(e_1, \ldots, e_n) = |\det A|,
\]
while, by assumption,
\[ \text{Vol}(x_1, \ldots, x_n) \geq t^{n-1}\|x_1\| \cdots \|x_n\| = t^{n-1} \]

so that we conclude that \(|\det A| \geq t^{n-1}\).

Now let \((b_{ij})\) be the matrix of \(A^{-1}\) with respect to \(e_1, \ldots, e_n\). By Cramer's rule we have \(|b_{ij}| \leq |\det A|^{-1}\) for all \(i, j\) so that \(\|A^{-1}\| = \max |b_{ij}| \leq |\det A|^{-1} \leq t^{-n+1}\).

Finally, let \(\lambda_1, \ldots, \lambda_n \in K\). Then \(\|A^{-1}\| \|\lambda_1 x_1 + \cdots + \lambda_n x_n\| = \|A^{-1}\| \|A(\lambda_1 e_1 + \cdots + \lambda_n e_n)\| \geq \|\lambda_1 e_1 + \cdots + \lambda_n e_n\| = \max_{1 \leq i \leq n} |\lambda_i| \|x_i\|\). We see that the \(x_1, \ldots, x_n\) are \(\|A^{-1}\|^{-1}\)-orthogonal, hence \(t^{n-1}\)-orthogonal. \(\blacksquare\)

**7. Proposition.** Suppose the valuation of \(K\) is dense, let \(t \in (0, 1]\). If \(n \in \mathbb{N}\) and \(\{x_1, \ldots, x_n\}\) is a \(t\)-frame in \(E\) then \(\{x_1, \ldots, x_n\}\) is \(t^{n-1}\)-orthogonal.

**Proof.** Set \(D := [x_1, \ldots, x_n]\), let \(0 < s < 1\). Then \(D\) has an \(s\)-orthogonal base \(f_1, \ldots, f_n\) with \(s \leq \|f_i\| \leq 1\) for each \(i \in \{1, \ldots, n\}\). The norm

\[ N : \sum_{i=1}^{n} \lambda_i f_i \mapsto \max |\lambda_i| \quad (\lambda_1, \ldots, \lambda_n \in K) \]

satisfies (with \(x = \sum_{i=1}^{n} \lambda_i f_i \in D\) \(N(x) \geq \|x\| \geq s \max \|\lambda_i f_i\| \geq s^2 \max |\lambda_i| = s^2 N(x)\) and also \((D, N)\) has an orthonormal base. Now, \(\{x_1, \ldots, x_n\}\) is a \(t\)-frame in \((D, \| \|)\) hence an \(s^2 t\)-frame in \((D, N)\) so by the previous statement it is \((s^2 t)^{n-1}\)-orthogonal in \((D, N)\) implying, in its turn, \(s^2 (s^2 t)^{n-1}\)-orthogonality in \((D, \| \|)\). Now the statement follows after taking the limit for \(s\) tending to 1. \(\blacksquare\)

**REFERENCES**


Zero sequences in $p$-adic compactoids

W.H. Schikhof

ABSTRACT. We prove the following theorem, solving Problem 6 of [3]. Let $E, F$ be Banach spaces over a nonarchimedean valued field $K$, let $A$ be an absolutely convex closed compactoid in $E$ and $T \in \mathcal{L}(E, F)$. Then, if $y_1, y_2, \ldots$ is a sequence in $TA$ tending to 0 then there is a sequence $x_1, x_2, \ldots$ in some scalar multiple of $A$ tending to 0 such that $Tx_n = y_n$ for each $n$. (See Theorem 2.7).

INTRODUCTION. For a proof we decompose $T$:

$$
\begin{array}{c}
A \\
\downarrow \pi
\end{array}
\xrightarrow{T}
\begin{array}{c}
TA \\
\downarrow i
\end{array}
\rightarrow
\begin{array}{c}
A/A \cap \text{Ker } T
\end{array}
$$

Here, the topology of $TA$ is inherited from the norm on $F$ and on $A/A \cap \text{Ker } T$ we take the quotient topology induced by the quotient map $\pi$. Then $i$ (which is the unique map making the diagram commute) is continuous. The object $A/A \cap \text{Ker } T$ is in a natural way a topological module over $\{\lambda \in K : |\lambda| \leq 1\}$ whose topology is induced by the metric $(\pi(x), \pi(y)) \mapsto \|\pi(x) - \pi(y)\|$ where

$$
\|\pi(z)\| = \text{dist}(z, A \cap \text{Ker } T) \quad (z \in A)
$$

This “norm” $\| \|$ on $A/A \cap \text{Ker } T$ satisfies the strong triangle inequality and

$$(*) \quad \|\lambda \pi(z)\| \leq |\lambda| \|\pi(z)\| \quad (z \in A, \lambda \in K, |\lambda| \leq 1)$$

but one does not always have equality in $(*)$. Thus, we shall study a class of topological modules over the valuation ring which properly contains the class of absolutely convex
subsets of locally convex spaces. This study involves a careful modification of the theory given in [4] for absolutely convex sets. We then will be able to conclude that although \( i \) (see diagram) is not always a homeomorphism it does have the property that \( i(t_n) \to 0 \) implies \( \|\lambda t_n\| \to 0 \) for each \( \lambda \in K, |\lambda| < 1 \); the statement announced in the Abstract follows easily.

**Terminology.** (For unexplained terms we refer to [2])

1. Throughout \( K \) is a nonarchimedean nontrivially valued field, that is complete under the metric induced by the valuation \( |\cdot| \). We set

\[
B_K := \{ \lambda \in K : |\lambda| \leq 1 \}
\]

\[
B_K^\circ := \{ \lambda \in K : |\lambda| < 1 \}
\]

2. Let \( A \) be a module over the ring \( B_K \), let \( B \) be a submodule of \( A \), let \( \lambda, \mu \in K \). We set

\[
\lambda B := \begin{cases} \{ \lambda b : b \in B \} & \text{if } |\lambda| \leq 1 \\ \{ x \in A : \lambda^{-1} x \in B \} & \text{if } |\lambda| \geq 1 \end{cases}
\]

(observe that this causes no ambiguity if \( |\lambda| = 1 \)). We have the following obvious consequences.

(i) \( \lambda B \) is a submodule of \( A \), \( 1 \cdot B = B \).

(ii) If \( |\lambda| \leq |\mu| \) then \( \lambda B \subset \mu B \). In particular, \( |\lambda| = |\mu| \) implies \( \lambda B = \mu B \).

(iii) If either \( |\lambda|, |\mu| \leq 1 \) or \( |\lambda|, |\mu| \geq 1 \) then \( (\lambda \mu)B = \lambda(\mu B) = \mu(\lambda B) \).

We shall say that \( B \) absorbs a subset \( X \) of \( A \) if \( X \subset \bigcup_{\lambda \in K} \lambda B \). The module generated by \( X \subset A \) is denoted \( \text{co} X \).

3. A locally convex topology on a \( B_K \)-module \( A \) is a topology \( \tau \) on \( A \) such that

(i) \( (A, \tau) \) is a topological \( B_K \)-module (of course, the topology on \( B_K \) is the valuation topology),

(ii) there is a neighbourhood base of \( 0 \) consisting of \( (\tau\text{-open}) B_K \)-submodules of \( A \).

Then we call \( A = (A, \tau) \) a locally convex module.

Clearly, absolutely convex subsets of locally convex spaces over \( K \) are examples of locally convex modules. Any submodule of a locally convex module is, with the restriction topology, a locally convex module. If \( B \) is a (closed) submodule of a locally convex module \( A \) the quotient topology on \( A/B \) is (Hausdorff) and locally convex.
Let $A$ be a locally convex module. The closure of a subset $X$ of $A$ is denoted $\overline{X}$. Instead of $\operatorname{co}X$ we write $\operatorname{co}X$. Let $X_1, X_2, \ldots$ be subsets of $A$. We shall write $\lim_{n \to \infty} X_n = 0$ if for each neighbourhood $U$ of 0 in $A$ we have $X_n \subset U$ for large $n$. The following is not hard to see. If $X_1, X_2, \ldots$ are submodules, $\lim_{n \to \infty} X_n = 0$ then

$$\bigcap_{n=1}^{\infty} (B + X_n) = B$$

for every submodule $B$ of $A$.

Let $i \mapsto x_i$ be a net in a locally convex module converging to 0. Then, for any net $i \mapsto \lambda_i$ in $B_K$, the net $i \mapsto \lambda_i x_i$ converges to 0. This is a direct consequence of local convexity.

**THROUGHOUT §1 AND §2 WE ASSUME THAT THE VALUATION OF $K$ IS DENSE.**

### §1. LOCALLY CONVEX $B_K$-MODULES

In this section $A$ is a locally convex module over $B_K$.

**LEMMA 1.1.** Let $B$ be a closed submodule of $A$ let $a \in A$, let $\lambda \in B_K$. If

$$i \mapsto x_i = b_i + c_i a \quad (b_i \in B, \ c_i \in B_K)$$

is a net in $B + \operatorname{co}(a)$ converging to 0 then $\lim_{i} c_i = 0$ or $\lambda x_i \in B$ eventually, where the latter case occurs if $B$ absorbs $\{a\}$.

**Proof.** Set $C := \{\xi \in B_K : \xi a \in B\}$. Then $C$ is absolutely convex; let $r$ be its diameter. Suppose $\lim_{i} |c_i| > r$. Then there exists a $\mu \in K$ and a cofinal $J \subset I$ such that

$$|c_j| \geq |\mu| > r \quad (j \in J)$$

Then $\mu c_j^{-1} \in B_K$ so that

$$\mu c_j^{-1} b_j + \mu a = \mu c_j^{-1} x_j \to 0$$

We see that $\mu a = -\lim_{j} \mu c_j^{-1} b_j \in B = B$ so that $\mu \in C$ conflicting $|\mu| > r$. Thus, we have proved that $\lim_{i} |c_i| \leq r$. If $r = 0$ then $\lim_{i} c_i = 0$. If $r > 0$ then, eventually, $|\lambda c_i| < r$, i.e. $\lambda c_i \in C$ i.e. $\lambda c_i a \in B$ i.e. $\lambda x_i = \lambda b_i + \lambda c_i a \in \lambda B + B \subset B$.  

61
To prove the second assertion, let $B$ absorb $\{a\}$ and let $\lim c_i = 0$. We have $\mu a \in B$ for some nonzero $\mu \in B_K$. Then, eventually, $|c_i| \leq |\mu|$ yielding $c_i a \in B$ implying $x_i \in B$.

For a submodule $B$ of $A$ we define

$$B^e := \bigcap \{\lambda B : \lambda \in K, |\lambda| > 1\}$$

The following elementary facts are easily verified. $B^e$ is a submodule, $B \subset B^e$, $B^{ee} = B^e$, $B^e = \{x \in A : \lambda x \in B \text{ for all } \lambda \in B_K^\times\}$. $\overline{B^e} \subset \overline{B}$. Further, we have: $B^e$ is closed $\iff \overline{B} \subset B^e$ $\iff \lambda B \subset B$ for each $\lambda \in B_K^\times$.

**Lemma 1.2.** Let $B$ be a closed submodule of $A$, let $a \in A$. Then $(B + \text{co}\{a\})^e$ is closed.

*Proof.* Let $i \mapsto x_i = b_i + c_i a$ ($b_i \in B$, $c_i \in B_K$) be a net in $B + \text{co}\{a\}$ converging to some $x \in A$. Let $\lambda \in B_K^\times$; we shall prove that $\lambda x \in B + \text{co}\{a\}$. In fact, the net $(i, j) \mapsto x_i - x_j$ converges to 0. So by Lemma 1.1 we have either $\lim_{i,j}(c_i - c_j) = 0$ (then $\lim c_i = c$ for some $c \in B_K$ and it follows easily that even $x \in B + \text{co}\{a\}$) or $\lambda(x_i - x_j) \in B$ for, say, all $i, j \geq s$. Then $\lambda x_i \in \lambda x_s + B$ ($i \geq s$) so $\lambda x = \lim_{i} \lambda x_i \in \overline{\lambda x_s + B} = \lambda x_s + B \in \lambda(B + \text{co}\{a\}) + B \subset B + \text{co}\{a\}$.

**Lemma 1.3.** Let $B$ be a closed submodule of $A$, let $a_1, \ldots, a_n \in A$. Then $(B + \text{co}\{a_1, \ldots, a_n\})^e$ is closed.

*Proof.* Lemma 1.2 covers the case $n = 1$. Suppose, for some $m$, $Z := (B + \text{co}\{a_1, \ldots, a_{m-1}\})^e$ is closed. Then, again by Lemma 1.2, $B + \text{co}\{a_1, \ldots, a_m\} \subset Z + \text{co}\{a_m\} \subset (Z + \text{co}\{a_m\})^e = (B + \text{co}\{a_1, \ldots, a_m\})^e$, so $(B + \text{co}\{a_1, \ldots, a_m\})^e$ is closed.

**Lemma 1.4.** Let $B$ be a closed submodule of $A$, let $a_1, \ldots, a_n \in A$, and suppose that $B$ absorbs $\{a_1, \ldots, a_n\}$. If $i \mapsto x_i$ is a net in $B + \text{co}\{a_1, \ldots, a_n\}$ converging to 0 then $\lambda x_i \in B$ eventually, for each $\lambda \in B_K^\times$.

*Proof.* Choose $\mu_1, \ldots, \mu_n \in B_K^\times$ such that $|\prod_{i=1}^n \mu_i|^2 \geq |\lambda|$. We have $x_i \in (B + \text{co}\{a_1, \ldots, a_{n-1}\}) + \text{co}\{a_n\}$. By Lemma 1.1 we have, eventually,

$$\mu_1 x_i \in \overline{B + \text{co}\{a_1, \ldots, a_{n-1}\}} \subset (B + \text{co}\{a_1, \ldots, a_{n-1}\})^e \quad (\text{Lemma 1.3}),$$

so that, eventually,

$$\mu_1^2 x_i \in B + \text{co}\{a_1, \ldots, a_{n-1}\}$$

Inductively we find

$$\mu_1^2 \cdots \mu_n^2 x_i \in B \quad \text{eventually}$$
implying $\lambda x_i \in B$ eventually.

§2. COMPACTOID MODULES

DEFINITION 2.1. A locally convex module $A$ is a compactoid module if for each $\lambda \in B_K^{-}$ and each neighbourhood $U$ of 0 in $A$ there exists a finite set $F \subset A$ such that

$$\lambda A \subset U + \text{co} F$$

Remark. An absolutely convex subset of a locally convex space over $K$ is a compactoid module if and only if it is a compactoid in the usual sense. So Definition 2.1 generalizes the notion of compactoid to a larger class of objects, and we will see that it suits the purpose of the paper. (Yet we must warn the reader that, in general, compactoid modules are no longer ‘compact-like’: For any Banach space $E$ the module $\{x \in E : \|x\| \leq 1\} / \{x \in E : \|x\| < 1\}$ has the discrete topology but is a compactoid module.)

We need the following algebraic lemma.

**Lemma 2.2.** Let $B,U$ be submodules of a $B_K$-module $A$, let $x_1, \ldots, x_n \in A$ and suppose

$$B \subset U + \text{co}\{x_1, \ldots, x_n\}.$$

Then, for each $\lambda \in B_K^{-}$ there exist $b_1, \ldots, b_n \in B$ such that

$$\lambda B \subset U + \text{co}\{b_1, \ldots, b_n\}.$$

**Proof.** The proofs of Lemmas 1.1 and 1.2 of [1] apply in this more general situation.

**Proposition 2.3.** Every submodule of a compactoid module is compactoid.

**Proof.** Let $B$ be a submodule of a compactoid module $A$, let $V$ be a zero neighbourhood in $B$, let $\lambda \in B_K^{-}$. Choose $\mu \in B_K^{-}$ with $|\lambda| \leq |\mu|^2$, choose an open submodule $U$ of $A$ with $U \cap B \subset V$. There is a finite set $F_1 \subset A$ such that

$$\mu B \subset \mu A \subset U + \text{co} F_1$$

By Lemma 2.2 there exists a finite set $F_2 \subset B$ such that

$$\mu^2 B \subset U + \text{co} F_2$$

63
It follows that
\[ \lambda B \subset \mu^2 B \subset U \cap B + \co F_2 \subset V + \co F_2 \]
which proves that \( B \) is a compactoid module.

**PROPOSITION 2.4.** Let \( A \) be a metrizable compactoid module, let \( \lambda \in B_K^- \). Then there exists a sequence \( x_1, x_2, \ldots \) with \( \lim_{n \to \infty} x_n = 0 \) such that
\[ \overline{\lambda A} \subset \overline{\{x_1, x_2, \ldots\}} \subset A \]

**Proof.** Choose \( \lambda_1, \lambda_2, \ldots \in B_K^- \) such that \( |\Pi \lambda_n| \geq |\lambda| \). By metrizability and local convexity there exist open submodules
\[ A = U_1 \supset U_2 \supset \cdots \]

of \( A \) forming a neighbourhood base at 0. By compactoidity there is a finite set \( F_1 \subset U_1 \) such that
\[ \lambda_1 U_1 \subset U_2 + \co F_1. \]

By Proposition 2.3 \( U_2 \) is a compactoid so
\[ \lambda_2 U_2 \subset U_3 + \co F_2 \]

for some finite set \( F_2 \subset U_2 \), etc. Inductively we find finite sets \( F_1, F_2, \ldots \) such that
(i) \( F_n \subset U_n \) for each \( n \). Hence \( \lim_{n \to \infty} F_n = 0 \) (see TERMINOLOGY, 3)
(ii) \( \lambda_n U_n \subset U_{n+1} + \co F_n \) for each \( n \).

Repeated application of (ii) yields for \( n \in \mathbb{N} \)
\[
\lambda A \subset \lambda_1 \lambda_2 \ldots \lambda_n A \subset \lambda_2 \ldots \lambda_n (\lambda_1 U_1) \subset \lambda_2 \ldots \lambda_n (U_2 + \co F_1) \\
\subset \lambda_2 \ldots \lambda_n U_2 + \co F_1 \subset \lambda_3 \ldots \lambda_n U_3 + \co F_2 + \co F_1 \subset \cdots \\
\subset U_{n+1} + \co (F_1 \cup F_2 \cup \ldots \cup F_n). 
\]

We see that for each \( n \)
\[ \overline{\lambda A} \subset U_{n+1} + \co \{F_1 \cup F_2 \cup \ldots\} \]
so that
\[ \overline{\lambda A} \subset \co \{F_1 \cup F_2 \cup \ldots\} \]

64
By enumerating $F_1, F_2, \ldots$ successively we obtain a sequence $x_1, x_2, \ldots$ in $A$ tending to 0 (as $\lim F_n = 0$) such that

$$\overline{\lambda A} \subset \overline{\{x_1, x_2, \ldots\}}.$$

**Theorem 2.5.** Let $E, F$ be locally convex $B_K$-modules, let $A$ be a complete metrizable compactoid submodule of $E$.

(i) If $F$ is Hausdorff and $T : E \rightarrow F$ is a continuous module homomorphism then $(TA)^e$ is closed.

(ii) If $B \subset E$ is a closed submodule of $E$ then $(A + B)^e$ is closed.

**Proof.** We show that for a closed submodule $B$ of $F$ the set $(TA + B)^e$ is closed (then (i) follows by taking $B = \{0\}$, (ii) is the case $E = F$, $T$ is the identity). Thus, let $\lambda \in B_K$; we prove that $\lambda(TA + B) \subset TA + B$.

Let $\lambda_1, \mu_1, \mu_2, \ldots \in B_K$ such that $|\lambda_1| \prod \mu_n \geq |\lambda|$. By Proposition 2.4 there exist $x_1, x_2, \ldots \in A$ such that $\lim_{n \rightarrow \infty} x_n = 0$ and

$$(*) \quad \overline{\lambda_1 A} \subset \overline{\{x_1, x_2, \ldots\}} \subset A$$

Write $X_n := \overline{\{x_n\}}$ and $A_n := \sum_{i \geq n} X_i$. Then $\lim_{n \rightarrow \infty} X_n = 0$, $\lim_{n \rightarrow \infty} A_n = 0$ and $\lim_{n \rightarrow \infty} TA_n = 0$. First observe that by $(*)$

$$\lambda_1(TA + B) \subset TA_1 + \overline{B}$$

Also we have by Lemma 1.2

$$\mu_1(TA_1 + B) = \mu_1(TX_1 + TA_2 + B) \subset TX_1 + TA_2 + \overline{B}$$

$$\mu_2(TA_2 + B) = \mu_2(TX_2 + TA_3 + B) \subset TX_2 + TA_3 + \overline{B}, \text{ etc.}$$

Let $x \in \lambda(TA + B)$. Then $x \in (\prod \mu_n)\lambda_1(TA + B) \subset TX_1 + (\prod \mu_n)(TA_2 + \overline{B})$.

There is a $y_1 \in X_1$ such that

$$x - Ty_1 \in \prod_{n \geq 2} \mu_n(TA_2 + \overline{B}) \subset TX_2 + (\prod_{n \geq 3} \mu_n)(TA_3 + \overline{B}).$$

So there is a $y_2 \in X_2$ such that

$$x - Ty_1 - Ty_2 \in (\prod_{n \geq 3} \mu_n)(TA_3 + \overline{B}), \text{ etc.}$$

Inductively we find $y_1 \in X_1, y_2 \in X_2, \ldots$ such that for each $n$

$$x = T(y_1 + \cdots + y_n) + z_{n+1}$$

65
when \( z_{n+1} \in TA_{n+1} + B \). Now \( \lim_{n \to \infty} y_n = 0 \), so, by completeness of \( A \), \( y := \sum_{n=1}^{\infty} y_n \), exists, and lies in \( A \). Then \( z : \lim_{n \to \infty} z_{n+1} \) also must exist and it lies in \( \bigcap_{n} (TA_n + B) \) which is \( B \) since \( \lim_{n \to \infty} TA_n = 0 \). We see that \( x = Ty + z \in TA + B \).

As a corollary we obtain

**THEOREM 2.6.** Let \( A_1, A_2 \) be locally convex modules, let \( A_1 \) be metrizable, complete and compactoid and let \( T : A_1 \rightarrow A_2 \) be a continuous injective module homomorphism. Let \( i \mapsto a_i \) be a net in \( A_1 \) such that \( Ta_i \rightarrow 0 \). Then, for each \( \lambda \in B_K, \lambda a_i \rightarrow 0 \).

**Proof.** Let \( U \) be an open submodule of \( A_1 \), let \( \lambda \in B_K^- \). Choose \( \mu \in B_K^- \), \( |\mu|^3 \geq |\lambda| \). There exists a finite set \( F \subset A_1 \) such that \( \mu A \subset U + \text{co } F \). Hence

\[
\mu TA_1 \subset TU + \text{co } TF \subset (TU)^e + \text{co } TF.
\]

The net \( i \mapsto \mu Ta_i \) lies in \( (TU)^e + \text{co } TF \) and converges to 0. Further, \( (TU)^e \) is closed (Proposition 2.3 and Theorem 2.5) and it absorbs \( TF \). Thus, by Lemma 1.4 we have, eventually: \( \mu^2 Ta_i \in (TU)^e \) which implies \( \mu^3 Ta_i \in TU \). As \( |\lambda| \leq |\mu|^3 \) we have \( T(\lambda a_i) \in TU \). By injectivity of \( T \), \( \lambda a_i \in U \). If follows that \( \lambda a_i \rightarrow 0 \).

Now we can prove the result announced in the Abstract.

**THEOREM 2.7.** Let \( E, F \) be \( K \)-Banach spaces, let \( T \in \mathcal{L}(E, F) \) and let \( A \subset E \) be an absolutely convex and closed compactoid. If \( y_1, y_2, \ldots \) is a sequence in \( TA \) tending to 0 then for each \( \lambda \in K, |\lambda| > 1 \) there is a sequence \( x_1, x_2, \ldots \) in \( \lambda A \) tending to 0 such that \( Tx_n = y_n \) for each \( n \).

**Proof.** Decompose \( T \) (see the Introduction):

\[
A \xrightarrow{T} TA \xrightarrow{i} A/A \cap \text{Ker } T
\]

Let \( z_n := i^{-1}(y_n) \) for each \( n \), let \( \lambda \in K, |\lambda| > 1 \). By Theorem 2.6 \( \lim_{n \to \infty} \lambda^{-1} z_n = 0 \) in the quotient topology of \( A/A \cap \text{Ker } T \). By metrizability there exist \( a_1, a_2, \ldots \in A \) such that \( \lim_{n \to \infty} a_n = 0 \) and \( \pi(a_n) = \lambda^{-1} z_n \) for each \( n \). Set \( x_n := \lambda a_n (n \in \mathbb{N}) \). Then \( x_n \in \lambda A \), \( \lim_{n \to \infty} x_n = 0 \) and \( Tx_n = y_n \) for each \( n \).

The following is a more general version of Theorem 2.7.

**THEOREM 2.8.** Let \( A_1, A_2 \) be locally convex modules, let \( A_1 \) be metrizable, complete and compactoid and let \( T : A_1 \rightarrow A_2 \) be a continuous surjective module homomorphism.
If $U \subset A_1$ is an open submodule then for every $\lambda \in K$, $|\lambda| > 1$ the module $\lambda TU$ is open in $A_2$.

Proof. We may assume that $T$ is bijective (decompose $T$ in the spirit of above). If $0$ is not in the interior of $\lambda TU$ then there exists a net $i \mapsto z_i$ in $A_2$ converging to 0 but $z_i \notin \lambda TU$ for each $i$. Let $x_i := T^{-1}(z_i)$. By Theorem 2.6 we have $\lambda^{-1}x_i \to 0$ so eventually $\lambda^{-1}x_i \in U$ so $\lambda^{-1}z_i = T(\lambda^{-1}x_i) \in TU$ i.e. $z_i \in \lambda TU$, a contradiction.

§3. THE CASE OF A DISCRETE VALUATION

We extend the definition of a compactoid module to arbitrary ground fields $K$ as follows. For $r \in [0, 1]$ and a $B_K$-module $A$ set

$$rA := \bigcap \{\lambda A : \lambda \in K, |\lambda| \geq r\}$$

A locally convex module $A$ is a compactoid if for each $r \in [0, 1)$ and each neighbourhood $U$ of 0 in $A$ there exists a finite set $F \subset A$ such that $rA \subset U + \text{co } F$. If the valuation of $K$ is dense, this definition is equivalent to 2.1 whereas for discretely valued $K$ one obtains Definition 2.1 but where "$\lambda \in B_K^-$" is replaced by "$\lambda = 1$".

It is not hard to see that all lemmas, propositions and theorems of §1, §2 remain true for discretely valued $K$ even when we take $\lambda := 1$ everywhere and replace $Z^e$ by $Z$ for every occurring module $Z$.

Problem. Is this last conclusion also true if $K$ is spherically complete?

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67