The $p$-adic bounded weak topologies

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INTRODUCTION

Throughout, let $K$ be a nonarchimedean nontrivially valued field that is complete under the metric induced by the valuation $| |$. For a normed space $E$ over $K$ one has the well known weak topology $w = \sigma(E, E')$ which is the weakest topology on $E$ making all $f \in E'$ (the dual space of $E$) continuous. Similarly, the so-called 'weak star' topology $w' = \sigma(E', E)$ is by definition the weakest topology on $E'$ making all evaluations $f \mapsto f(x)$ continuous. (The expression 'weak star' goes back to the old days when the topological dual space of $E$ was denoted $E^*$ rather than $E'$.) In this paper we study the following related less known topologies. The bounded weak topology $bw$ is the strongest locally convex topology on $E$ that coincides with $w$ on (norm) bounded subsets of $E$. The bounded weak star topology $bw'$ is the strongest locally convex topology on $E'$ that coincides with $w'$ on (norm) bounded subsets of $E'$. These two topologies have appeared incidentally in [8], [9], but in this paper we shall consider them in their own right.

PRELIMINARIES

We shall use notations and terminology of [3]. Throughout $E$ is a normed space over $K$ (where the norm $|| \cdot ||$ is assumed to satisfy the strong triangle inequality $||x+y|| \leq \max(||x||,||y||)$ $(x, y \in E)$). The 'closed' ball with center 0 and radius $\varepsilon > 0$, $\{x \in E : ||x|| \leq \varepsilon\}$ is written $B(0, \varepsilon)$. Sometimes we write $BE$ in place of $\{x \in E : ||x|| \leq 1\}$. The canonical map $E \to E''$ is denoted $j_E$. WARNING. $j_E$ need not be injective; the topologies $w$ and $bw$ introduced above need not be Hausdorff!

A subset $A$ of a $K$-vector space is absolutely convex if $x, y \in A$, $\lambda, \mu \in K$, $|\lambda|, |\mu| \leq 1$ implies $\lambda x + \mu y \in A$. If $A$ is absolutely convex we define $A^e := A$ if the valuation of $K$ is discrete, $A^e := \bigcap_{|\lambda|>1} \lambda A$ if the valuation of $K$ is dense. $A$ is edged if $A = A^e$. The smallest absolutely
convex set containing $X$ is denoted by $\text{co}X$, its linear span by $[X]$. Let $Z$ be a locally convex space over $K$, let $X \subset Z$ and $Y \subset Z'$. We write

$$X^0 := \{ f \in Z' : |f(x)| \leq 1 \text{ for all } x \in X \}$$

$$Y_0 := \{ z \in Z : |f(z)| \leq 1 \text{ for all } f \in Y \}.$$  

$X$ is a polar set if $(X^0)_0 = X$.

### §1. THE BOUNDED WEAK TOPOLOGY

We first collect some elementary consequences of the definitions.

**Proposition 1.1.** For an absolutely convex set $U \subset E$ the following are equivalent.

-(a) $U$ is bw-open.

-(b) $U \cap B$ is weakly open in $B$ for each bounded set $B \subset E$.

-(c) $U \cap \lambda B_E$ is weakly open in $\lambda B_E$ for each $\lambda \in K$, $\lambda \neq 0$.

**Proof.** Straightforward.

**Proposition 1.2.** For a seminorm $p$ on $E$ the following are equivalent.

-(a) $p$ is bw-continuous.

-(b) $p|B$ is weakly continuous for each bounded set $B \subset E$.

-(c) $p|B_E$ is weakly continuous.

**Proof.** Direct consequence of Proposition 1.1. (Observe that from (c) it follows that $p|\lambda B_E$ is weakly continuous for each $\lambda \in K$, $\lambda \neq 0$.)

**Proposition 1.3.** Let $(X, \tau)$ be a locally convex space over $K$, let $A : E \to X$ be a linear map. Then $A$ is bw to $\tau$ continuous if and only if $A|B$ is weakly continuous for each bounded set $B \subset E$.

**Proof.** If $A$ is bw to $\tau$ continuous then so is $A|B$ and $A|B$ is weakly continuous. If $A|B$ is weakly continuous for each bounded $B \subset E$ then for each $\tau$-continuous seminorm $q$ on $X$ we have that $q \circ A$ satisfies (b) of Proposition 1.2.

The next Proposition is deeper and gives a concrete description of the bw-continuous seminorms ((δ) and (ε) below).

**Proposition 1.4.** For a seminorm $p$ on $E$ the following are equivalent.

-(a) $p$ is bw-continuous.

-(b) $B_E$ is a $p$-compactoid.

-(γ) $p$ is polar, norm continuous, $\{ f \in E' : |f| \leq p \}$ is a norm compactoid in $E'$.

-(δ) There exist $f_1, f_2, \ldots$ in $E'$ with $\lim_{n \to \infty} \|f_n\| = 0$ such that $p$ is equivalent to $\sup |f_n|$.

-(ε) There exist $f_1, f_2, \ldots$ in $E'$ with $\lim_{n \to \infty} \|f_n\| = 0$ such that $p \leq \sup |f_n|$.
Proof. $(\alpha) \implies (\beta)$. $B_{E}$ is a compactoid for the weak topology; the topologies $bw$ and $w$ coincide on $B_{E}$.

$(\beta) \implies (\gamma)$. As $B_{E}$ is a $p$-compactoid we have $(E,p) = ([B_{E}],p)$ is of countable type so $p$ is a polar seminorm. Also $p$ is bounded on $B_{E}$ hence norm continuous. To show that $\{f \in E' : |f| \leq p\}$ is a normcompactoid consider the identity map $(E,||\ ||) \rightarrow (E,p)$ which is compact by $(\beta)$. Then so is its adjoint which is the inclusion $(E,p)' \hookrightarrow (E,||\ ||)'$. Hence, the unit ball $\{f \in E' : |f| \leq p\}$ of $(E,p)'$ is a compactoid in $(E,||\ ||)'$.

$(\gamma) \implies (\delta)$. Set $S := \{f \in E' : |f| \leq p\}$. By polarity and norm continuity

\[ p = \sup\{|f| : f \in E' : |f| \leq p\} = \sup_{f \in S}|f|. \]

The normcompactoid $S$ is easily seen to be absolutely convex and norm closed. So, by [6], Lemma 1.3, if $\lambda \in K$, $|\lambda| > 1$ there exist $f_1, f_2, \ldots \in S$ with $\lim_{n \to \infty}||f_n|| = 0$ such that

\[ \overline{\mathfrak{C}}\{f_1, f_2, \ldots\} \subset S \subset |\lambda| \overline{\mathfrak{C}}\{f_1, f_2, \ldots\} \]

We obtain

\[ \sup_{n} |f_n| = \sup\{|f| : f \in \overline{\mathfrak{C}}\{f_1, f_2, \ldots\}\} \leq \sup_{S}|f| \leq \sup\{|f| : f \in |\lambda| \overline{\mathfrak{C}}\{f_1, f_2, \ldots\}\} = |\lambda| \sup_{n}|f_n|. \]

Together with $(\ast)$ this yields

\[ \sup_{n} |f_n| \leq p \leq |\lambda| \sup_{n}|f_n| \]

and $(\delta)$ follows. The implication $(\delta) \implies (\varepsilon)$ is obvious. Finally, assume $(\varepsilon)$. To arrive at $(\alpha)$ we prove (see Proposition 1.2(\gamma)) that for a net $i \mapsto x_i$ in $B_{E}$ converging weakly to $x \in B_{E}$ it follows that $p(x-x_i) \rightarrow 0$. In fact, let $N$ be such that $||f_n|| \leq \varepsilon$ for $n > N$. We have $|f_j(x-x_i)| \leq \varepsilon$ for all $j \in \{1, \ldots, N\}$ and sufficiently large $i$. For these $i$

\[ p(x-x_i) \leq \max_{1 \leq j \leq N} |f_j(x-x_i)| \vee \sup_{j > N}|f_j(x-x_i)| \leq \varepsilon \vee \sup_{j > N}||f_j|| \|(x-x_i)\| \]

\[ \leq \varepsilon \vee \sup_{j > N}||f_j|| \leq \varepsilon. \]

Several corollaries obtain.

**Corollary 1.5.** The topology $bw$ is of countable type. The dual of $(E,bw)$ is $E'$. An absolutely convex edged set in $E$ is $bw$-closed if and only if it is weakly closed.

**Proof.** In the proof of $(\beta) \implies (\gamma)$ it is observed that $bw$ is of countable type. The identity maps $(E,||\ ||) \rightarrow (E,bw) \rightarrow (E,w)$ are continuous (Proposition 1.4(\gamma)). Since $(E,||\ ||)' = (E,w)' = E'$ the dual of $(E,bw)$ also must be $E'$. As a consequence the topologies $bw$ and

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w have the same collection of polar sets and, by strong polarness, have the same collection of absolutely convex edged sets ([5], Theorem 4.7).

Remarks. (See [5], [9] for proofs.)
1. If $K$ is spherically complete we may extend the last statement of Corollary 1.5. as follows. An absolutely convex set in $E$ is $bw$-closed if and only if it is weakly closed if and only if it is norm closed.
2. If $K$ is not spherically complete but $E$ is strongly polar we may draw the same conclusion as in 1. but only for edged absolutely convex sets.
3. If $K$ is not spherically complete and $E'$ is infinite dimensional there exists a $bw$-closed absolutely convex set in $E$ that is not weakly closed.

Corollary 1.6. Let $X$ be a locally convex space. A linear map $A : E \to X$ is $bw$-continuous if and only if it is compact for the norm topology on $E$.

Proof. If $A$ is $bw$-continuous then (Proposition 1.4) $B_E$ is a $bw$-compactoid so $AB_E$ is a compactoid in $X$. If, conversely, $A$ is norm compact then by [7], 1.2, for each net $i \to x_i \in B_E$ converging weakly to 0 we have $Ax_i \to 0$ in $X$. By Proposition 1.3 $A$ is $bw$-continuous.

The weak topology is defined as the weakest topology for which all $f \in E'$ are continuous. For $bw$ we have a similar description.

Corollary 1.7. The bounded weak topology on $E$ is the weakest topology on $E$ for which all compact operators $E \to c_0$ are continuous.

Proof. Let $\tau$ be the weakest topology on $E$ for which all compact operators $E \to c_0$ are continuous. By Corollary 1.6, $bw \geq \tau$. To prove $bw \leq \tau$, let $p$ be a $bw$-continuous seminorm, say (Proposition 1.4.)

$$p = \sup_n |f_n|$$

where $f_1, f_2, \ldots \in E'$, $\lim_{n \to \infty} ||f_n|| = 0$. The operator $A : E \to c_0$, given by

$$Ax = (f_1(x), f_2(x), \ldots)$$

is easily seen to be compact and therefore $\tau$-continuous. Hence, so is $x \mapsto ||Ax||$, which is $p$.

Corollary 1.8.

(i) $bw$ is the strongest locally convex topology on $E$ for which each norm bounded set is a compactoid.

(ii) $bw$ is nuclear. In fact it is the strongest nuclear topology, weaker than the norm topology. (See also [2], Example 5.2.)
Proof. (i) Follows from Proposition 1.4, \((\alpha) \iff (\beta)\). To prove nuclearity observe that \((E, bw)\) is of countable type (Corollary 1.5) so \(\{1\} \iff \{2\}\) it suffices to prove that every continuous linear map \(A : (E, bw) \to c_0\) is compact. Such an \(A\) has the form

\[
z \mapsto Az = (f_1(z), f_2(z), \ldots)
\]

for certain \(f_1, f_2 \in E'\). By Corollary 1.6 \(A\) is compact for the norm topology on \(E\) so \(\lim_{n \to \infty} \|f_n\| = 0\). There exist \(\lambda_1, \lambda_2 \in K\) and \(g_1, g_2, \ldots \in E'\) such that \(f_n = \lambda_n g_n\) for each \(n\), \(\lim \lambda_n = 0\), \(\lim_{n \to \infty} \|g_n\| = 0\). Then \(\{z : |g_n(z)| \leq 1\text{ for each } n\}\) is a bw-neighbourhood of 0 whose image under \(A\) lies in \(\{(a_1, a_2, \ldots) \in c_0 : |a_n| \leq |\lambda_n|\text{ for each } n\}\). It follows that \(A : (E, bw) \to c_0\) is compact.

Now let \(\tau\) be a nuclear topology on \(E\), weaker than the norm topology. Then, in particular, each norm bounded set is \(\tau\)-bounded and is, by nuclearity, a \(\tau\)-compactoid. We conclude that, by (i), \(\tau \leq bw\).

We have seen in Proposition 1.3 that in order to check \(bw\)-continuity of \(A : E \to X\) it suffices to consider the behaviour of \(A\) on \textit{bounded} \(bw\)-convergent nets. Yet we shall prove (Proposition 1.10) that there do exist 'essentially' unbounded \(bw\)-convergent nets.

**Lemma 1.9.** Let \(E'\) be infinite dimensional. If \(X \subset E'\) is a normcompactoid then \(X_0 = \{z \in E : |f(z)| \leq 1\text{ for each } f \in X\}\) is unbounded.

**Proof.** Without loss assume \(X\) is absolutely convex, edged and normcomplete. As \(X\) is also metrizable, by [4], Theorem 10(ii) the norm topology and the \(w'\)-topology coincide on \(X\). It follows that \(X\) is \(w'\)-closed and edged. Thus, ([5], Theorem 4.7) \(X\) is a polar set in \((E', \sigma(E', E))\) so that \(X = (X_0)^0 := \{f \in E' : |f| \leq 1\text{ on } X_0\}\). Now, if \(X_0\) were bounded then \(X = (X_0)^0\) is a norm neighbourhood of 0 which is in conflict to compactoidity.

**Proposition 1.10.** Let \(E'\) be infinite dimensional. Then there exists a net \(i \mapsto z_i\) in \(E\), converging to 0 in \(bw\) but such that \(\{z_i : i \geq i_0\}\) is unbounded for each \(i_0\).

**Proof.** For each normcompactoid \(X \subset E'\) and \(\varepsilon > 0\) we choose an \(x_{X, \varepsilon} \in E\) such that \(|f(x_{X, \varepsilon})| \leq \varepsilon\) for each \(f \in X\) and \(\|x_{X, \varepsilon}\| \geq \varepsilon^{-1}\) (Lemma 1.9). The set \(I := \{(X, \varepsilon) : X\text{ is a compactoid in } E', \varepsilon > 0\}\) is a directed set under: \((X_1, \varepsilon_1) \geq (X_2, \varepsilon_2)\) iff \(X_1 \supset X_2\) and \(\varepsilon_1 \leq \varepsilon_2\). It is not hard to see that the net \(i \mapsto z_i\) we just defined satisfies the requirements.

Let us consider the following related question (compare Proposition 1.1 (\(\beta\))). Let \(A\) be an absolutely convex subset of \(E\) such that \(A \cap B\) is weakly closed in \(B\) for each bounded set \(B \subset E\). Does it follow that \(A\) is \(bw\)-closed? Does it follow that \(A\) is \(w\)-closed?

If \(K\) is spherically complete (and also if \(E\) is a normed space over \(R\) or \(C\)) the answers are yes \((A \cap B\) is norm closed in \(B\) for each bounded set \(B \subset E\) so \(A\) is norm closed hence \(w\)-closed (and \(bw\)-closed) by Remark 1 following Corollary 1.5). But, if \(K\) is not spherically complete, say, \(K = C_p\) the answers are no for \(E = \ell^\infty(m)\) where \(m \geq \#K\) (see [9], Corollary 3.6).
§2. THE BOUNDED WEAK STAR TOPOLOGY

It can be expected (for the definitions, see the Introduction) that, to some extent, the topologies $bw$ and $bw'$ behave in a similar way. Indeed, the obvious $bw'$-versions of Propositions 1.1, 1.2, 1.3 are easily seen to be true. Also, $B_{E'}$ is a $bw'$-compactoid, $bw'$ is of countable type, $bw'$ is weaker than the norm topology on $E'$, etc.; we leave the details to the reader. The counterpart (Theorem 2.3) of Proposition 1.4 (a) $\iff$ (b) is less innocent. First recall that the dual of $(E', w')$ is equal to $j(E)$. ([5], Lemma 7.1). We prove

**Proposition 2.1.** The dual of $(E', bw')$ is the norm closure of $j(E)$ in $E''$.

*Proof.* Every $\theta \in j(E)$ is, on $B_{E'}$, the uniform limit of a sequence in $j(E)$ so $\theta|B_{E'}$ is $w'$-continuous i.e. $\theta$ is $bw'$-continuous. Thus, we have $j(E) \subset (E', bw')$. Conversely, let $\theta \in (E', bw')$. Then $\theta$ is norm continuous i.e. $\theta \in E''$. Let $\varepsilon > 0$; we shall find an $x \in E$ such that $||\theta - j(x)|| < \varepsilon$. Choose $\alpha \in K$, $0 < |\alpha| < \varepsilon$. The $w'$-continuity of $\theta|B_{E'}$ yields a finite set $F \subset E$ such that $f \in F^0 \cap B_{E'}$ implies $|\theta(f)| \leq |\alpha|$, in other words

$$f \in j(E(F)) \cap (B_{E'})_0 \implies |(\alpha^{-1}\theta)(f)| \leq 1.$$  

So we see that $\alpha^{-1}\theta \in (j(E(F)) \cap (B_{E'})_0)^0 = (A + B_{E'})^0$ where $A = j(E(coF))$. Now $B_{E'}$ is $w'$-closed and $A$ is finite dimensional so by [6], 1.4, $(A + B_{E'})^0 = (A + B_{E'})^*$. For any $\lambda \in K$ such that $|\lambda| > 1$ and $|\lambda\alpha| < \varepsilon$ we have $\alpha^{-1}\theta \in \lambda A + \lambda B_{E'}$ hence $\theta \in j(E) + \alpha \lambda B_{E'}$ and there is an $x \in E$ with $\theta - j(x) \in \alpha \lambda B_{E'}$ i.e. $||\theta - j(x)|| < \varepsilon$.

To avoid needless complications we shall from now on restrict ourselves to polar spaces i.e. spaces $E$ such that $j(E)$ is an isometry. (Most 'natural' spaces are polar.)

**Corollary 2.2.** Let $E$ be a polar Banach space.

(i) The dual of $(E', bw')$ is $j(E)$.

(ii) An edged absolutely convex subset of $E'$ is $bw'$-closed if and only if it is $w'$-closed.

*Proof.* (i) is immediate. (ii) follows from (i) as $(E', w')$ and $(E', w')$ have the same dual space.

**Theorem 2.3.** Let $E$ be a polar $K$-Banach space. Then the topology $bw'$ on $E'$ is the topology $c$ of compact convergence i.e. it is generated by the seminorms $f \mapsto \max_X |f|$ where $X$ runs through the family of all compact subsets of $E$.

*Proof.* A routine proof shows that $c$, restricted to bounded sets, coincides with $w'$ so, by definition, $bw'$ is stronger than $c$. Let us prove that $bw'$ is weaker than $c$. Let $p$ be a $bw'$-continuous seminorm on $E'$. Then $p$ is certainly $bw$-continuous so by Proposition 1.4 there exist $\theta_1, \theta_2, \ldots \in E''$ with $\lim_{n \to \infty} ||\theta_n|| = 0$ such that $p$ is equivalent to $f \mapsto \max_n |\theta_n(f)|$. Hence,
\(\theta_1, \theta_2, \ldots\) are bw'-continuous so by Corollary 2.2 (i) there exist \(z_1, z_2, \ldots\) with \(\theta_n = j_E(z_n)\) for each \(n\). We have \(\lim_{n \to \infty} \|z_n\| = 0\), as \(j_E\) is an isometry. We see that \(p\) is equivalent to
\[
f \mapsto \max_{x \in X} |f(x)|
\]
where \(X\) is the compact set \(\{0, x_1, x_2, \ldots\}\) and we are done.

The proof of Theorem 2.3 depends on the theory of §1. We now show a way to arrive at the same result avoiding the use of the bw'-theory. The key is the following 'convexification' of the Ascoli Theorem.

**Theorem 2.4.** Let \(X\) be a subset of a polar \(K\)-Banach space \(E\). The following are equivalent.

1. \((\alpha)\) \(X\) is compactoid.
2. \((\beta)\) If \(i \mapsto f_i\) is a (norm) bounded net in \(E'\) converging w' to \(f \in E'\) then \(f_i \to f\) uniformly on \(X\).

**Proof.** \((\alpha) \implies (\beta)\). To prove \((\beta)\) we may assume \(f = 0\), \(\|f_i\| \leq 1\) for all \(i\). Let \(\varepsilon > 0\). There exist \(x_1, x_2, \ldots, x_n \in E\) such that \(X \subset B(0, \varepsilon) + \text{co}\{x_1, \ldots, x_n\}\). There is an \(i_0\) such that \(|f_i(x_j)| \leq \varepsilon\) for all \(i \geq i_0\) and all \(j \in \{1, \ldots, n\}\). As \(|f_i| \leq \varepsilon\) on \(B(0, \varepsilon)\) for all \(i\) we find \(|f_i| \leq \varepsilon\) on \(X\) if \(i \geq i_0\).

**Remark.** In this part of the proof the polarity assumption was not needed.

\((\beta) \implies (\alpha)\). It is an easy exercise to show that \(E/D\) is a polar \(K\)-Banach space for every finite dimensional subspace \(D\) of \(E\). Now assume \(X\) is not a compactoid. Then, almost from the definition, there is an \(\alpha > 0\) such that \(\text{diam } \pi_D(X) > \alpha\) for each \(D \in \mathcal{D}\), where \(\mathcal{D}\) is the directed set of all finite dimensional subspaces of \(E\) and where \(\pi_D : E \to E/D\) is the quotient map.

Now \(E/D\) is polar so we can find a \(g_D \in (E/D)'\) with \(\|g_D\| \leq 1\) and \(|g_D(\pi_D(x))| \geq \alpha\) for some \(x \in X\). Set \(f_D := g_D \circ \pi_D\). Then \(f_D \to 0\) in w', \(\|f_D\| \leq 1\) for all \(D\) but not \(f_D \to 0\) uniformly on \(X\), conflicting \((\beta)\).

**Second proof of Theorem 2.3.** Let \(p\) be a bw'-continuous seminorm on \(E'\). Then \(p\) is polar so \(p(f) = \sup\{\theta(f) : \theta \leq p, \theta \in (E', \text{bw}')\}\). By Corollary 2.2(i) there is a subset \(X \subset E\) for which
\[
p(f) = \sup\{|f(x)| : x \in X\}.
\]
To prove that \(X\) is a compactoid, let \(f_i \in E', \|f_i\| \leq 1\), \(f_i \to 0\) in w' then, by definition, \(f_i \to 0\) in bw' so that \(p(f_i) \to 0\) implying \(f_i \to 0\) uniformly on \(X\). The compactoidity now follows from Theorem 2.4, \((\beta) \implies (\alpha)\).

Finally we consider briefly the bw'-version of the last problem of §1 (we keep assuming \(E\) to be a polar Banach space). Let \(A \subset E'\) be absolutely convex. Assume that \(A \cap B\) is w'-closed in \(B\) for each bounded \(B \subset E'\). Does it follow that \(A\) is bw'-closed? Does it follow that \(A\) is w'-closed?
If $K$ is spherically complete the answers are yes. Its classical counterpart is known as the Krein-Šmulian Theorem. The proof is by no means trivial. See [8]. If $K$ is not spherically complete a counterexample to both questions is given by [9], Corollary 3.7.

REFERENCES