The $p$-adic bounded weak topologies

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INTRODUCTION

Throughout, let $K$ be a nonarchimedean nontrivially valued field that is complete under the metric induced by the valuation $|\cdot|$. For a normed space $E$ over $K$ one has the well known weak topology $w = \sigma(E, E')$ which is the weakest topology on $E$ making all $f \in E'$ (the dual space of $E$) continuous. Similarly, the so-called 'weak star' topology $w' = \sigma(E', E)$ is by definition the weakest topology on $E'$ making all evaluations $f \mapsto f(x)$ continuous. (The expression 'weak star' goes back to the old days when the topological dual space of $E$ was denoted $E^*$ rather than $E'$.) In this paper we study the following related less known topologies. The bounded weak topology $bw$ is the strongest locally convex topology on $E$ that coincides with $w$ on (norm) bounded subsets of $E$. The bounded weak star topology $bw'$ is the strongest locally convex topology on $E'$ that coincides with $w'$ on (norm) bounded subsets of $E'$. These two topologies have appeared incidentally in [8], [9], but in this paper we shall consider them in their own right.

PRELIMINARIES

We shall use notations and terminology of [3]. Throughout $E$ is a normed space over $K$ (where the norm $\|\cdot\|$ is assumed to satisfy the strong triangle inequality $\|x+y\| \leq \max(\|x\|, \|y\|)$ $(x, y \in E)$). The 'closed' ball with center 0 and radius $\varepsilon > 0$, $\{x \in E : \|x\| \leq \varepsilon\}$ is written $B(0, \varepsilon)$. Sometimes we write $B_E$ in place of $\{x \in E : \|x\| \leq 1\}$. The canonical map $E \to E''$ is denoted $j_E$. WARNING. $j_E$ need not be injective; the topologies $w$ and $bw$ introduced above need not be Hausdorff!

A subset $A$ of a $K$-vector space is absolutely convex if $x, y \in A, \lambda, \mu \in K, |\lambda|, |\mu| \leq 1$ implies $\lambda x + \mu y \in A$. If $A$ is absolutely convex we define $A^e := A$ if the valuation of $K$ is discrete, $A^e := \bigcap_{|\lambda| > 1} \lambda A$ if the valuation of $K$ is dense. $A$ is edged if $A = A^e$. The smallest absolutely
convex set containing $X$ is denoted by $\text{co}X$, its linear span by $[X]$. Let $Z$ be a locally convex space over $K$, let $X \subset Z$ and $Y \subset Z'$. We write

\[ X^0 := \{ f \in Z' : |f(x)| \leq 1 \text{ for all } x \in X \} \]
\[ Y_0 := \{ z \in Z : |f(x)| \leq 1 \text{ for all } f \in Y \}. \]

$X$ is a polar set if $(X^0)_0 = X$.

§1. THE BOUNDED WEAK TOPOLOGY

We first collect some elementary consequences of the definitions.

**Proposition 1.1.** For an absolutely convex set $U \subset E$ the following are equivalent.

1. $U$ is bw-open.
2. $U \cap B$ is weakly open in $B$ for each bounded set $B \subset E$.
3. $U \cap \lambda B_E$ is weakly open in $\lambda B_E$ for each $\lambda \in K$, $\lambda \neq 0$.

**Proof.** Straightforward.

**Proposition 1.2.** For a seminorm $p$ on $E$ the following are equivalent.

1. $p$ is bw-continuous.
2. $p|B$ is weakly continuous for each bounded set $B \subset E$.
3. $p|B_E$ is weakly continuous.

**Proof.** Direct consequence of Proposition 1.1. (Observe that from (3) it follows that $p|\lambda B_E$ is weakly continuous for each $\lambda \in K$, $\lambda \neq 0$.)

**Proposition 1.3.** Let $(X, \tau)$ be a locally convex space over $K$, let $A : E \to X$ be a linear map. Then $A$ is bw to $\tau$ continuous if and only if $A|B$ is weakly continuous for each bounded set $B \subset E$.

**Proof.** If $A$ is bw to $\tau$ continuous then so is $A|B$ and $A|B$ is weakly continuous. If $A|B$ is weakly continuous for each bounded $B \subset E$ then for each $\tau$-continuous seminorm $q$ on $X$ we have that $q \circ A$ satisfies (3) of Proposition 1.2.

The next Proposition is deeper and gives a concrete description of the bw-continuous seminorms ((δ) and (ε) below).

**Proposition 1.4.** For a seminorm $p$ on $E$ the following are equivalent.

1. $p$ is bw-continuous.
2. $B_E$ is a $p$-compactoid.
3. $p$ is polar, norm continuous, $\{ f \in E' : |f| \leq p \}$ is a norm compactoid in $E'$.
4. There exist $f_1, f_2, \ldots$ in $E'$ with $\lim_{n \to \infty} \|f_n\| = 0$ such that $p$ is equivalent to $\sup |f_n|$.
5. There exist $f_1, f_2, \ldots$ in $E'$ with $\lim_{n \to \infty} \|f_n\| = 0$ such that $p \leq \sup |f_n|$. 

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Proof. \((\alpha) \implies (\beta)\). \(B_E\) is a compactoid for the weak topology; the topologies \(bw\) and \(w\) coincide on \(B_E\).

\((\beta) \implies (\gamma)\). As \(B_E\) is a \(p\)-compactoid we have \((E,p) = ([B_E],p)\) is of countable type so \(p\) is a polar seminorm. Also \(p\) is bounded on \(B_E\) hence norm continuous. To show that \(\{f \in E' : |f| \leq p\}\) is a normcompactoid consider the identity map \((E,|| ||) \to (E,p)\) which is compact by \((\beta)\). Then so is its adjoint which is the inclusion \((E,p') \hookrightarrow (E,|| ||)'\). Hence, the unit ball \(\{f \in E' : |f| \leq p\}\) of \((E,p')\) is a compactoid in \((E,|| ||)'\).

\((\gamma) \implies (\delta)\). Set \(S := \{f \in E' : |f| \leq p\}\). By polarity and norm continuity

\[(*) \quad p = \sup\{|f| : f \in E' : |f| \leq p\} = \sup_{f \in S} |f|.

The normcompactoid \(S\) is easily seen to be absolutely convex and norm closed. So, by [6], Lemma 1.3, if \(\lambda \in K, |\lambda| > 1\) there exist \(f_1, f_2, \ldots \in S\) with \(\lim_{n \to \infty} \|f_n\| = 0\) such that

\[\bar{c}(f_1, f_2, \ldots) \subseteq S \subseteq |\lambda| \bar{c}(f_1, f_2, \ldots)\]

We obtain

\[\sup_n |f_n| = \sup\{|f| : f \in \bar{c}\{f_1, f_2, \ldots\}\} \leq \sup_{S} |f| \leq \sup\{|f| : f \in |\lambda| \bar{c}\{f_1, f_2, \ldots\}\} = |\lambda| \sup_n |f_n|.

Together with \((*)\) this yields

\[\sup_n |f_n| \leq p \leq |\lambda| \sup_n |f_n|\]

and \((\delta)\) follows. The implication \((\delta) \implies (\varepsilon)\) is obvious. Finally, assume \((\varepsilon)\). To arrive at \((\alpha)\) we prove (see Proposition 1.2(\(\gamma\))) that for a net \(i \mapsto x_i\) in \(B_E\) converging weakly to \(x \in B_E\) it follows that \(p(x-x_i) \to 0\). In fact, let \(N\) be such that \(\|f_n\| \leq \varepsilon\) for \(n > N\). We have \(|f_j(x-x_i)| \leq \varepsilon\) for all \(j \in \{1, \ldots, N\}\) and sufficiently large \(i\). For these \(i\)

\[p(x-x_i) \leq \max_{1 \leq j \leq N} |f_j(x-x_i)| \lor \sup_{j > N} |f_j(x-x_i)| \leq \varepsilon \lor \sup_{j > N} \|f_j\| \|x-x_i\| \leq \varepsilon \lor \sup_{j > N} \|f_j\| \leq \varepsilon.

Several corollaries obtain.

Corollary 1.5. The topology \(bw\) is of countable type. The dual of \((E,bw)\) is \(E'\). An absolutely convex edged set in \(E\) is \(bw\)-closed if and only if it is weakly closed.

Proof. In the proof of \((\beta) \implies (\gamma)\) it is observed that \(bw\) is of countable type. The identity maps \((E,|| ||) \to (E,bw) \to (E,w)\) are continuous (Proposition 1.4(\(\gamma\))). Since \((E,|| ||)' = (E,w)' = E'\) the dual of \((E,bw)\) also must be \(E'\). As a consequence the topologies \(bw\) and
have the same collection of polar sets and, by strong polarness, have the same collection of absolutely convex edged sets ([5], Theorem 4.7).

**Remarks.** (See [5], [9] for proofs.)

1. If $K$ is spherically complete we may extend the last statement of Corollary 1.5. as follows. An absolutely convex set in $E$ is $bw$-closed if and only if it is weakly closed if and only if it is norm closed.

2. If $K$ is not spherically complete but $E$ is strongly polar we may draw the same conclusion as in 1. but only for *edged* absolutely convex sets.

3. If $K$ is not spherically complete and $E'$ is infinite dimensional there exists a $bw$-closed absolutely convex set in $E$ that is not weakly closed.

**Corollary 1.6.** Let $X$ be a locally convex space. A linear map $A : E \to X$ is $bw$-continuous if and only if it is compact for the norm topology on $E$.

**Proof.** If $A$ is $bw$-continuous then (Proposition 1.4) $B_E$ is a $bw$-compactoid so $AB_E$ is a compactoid in $X$. If, conversely, $A$ is norm compact then by [7], 1.2, for each net $i \mapsto x_i \in B_E$ converging weakly to 0 we have $Ax_i \to 0$ in $X$. By Proposition 1.3 $A$ is $bw$-continuous.

The weak topology is defined as the weakest topology for which all $f \in E'$ are continuous. For $bw$ we have a similar description.

**Corollary 1.7.** The bounded weak topology on $E$ is the weakest topology on $E$ for which all compact operators $E \to c_0$ are continuous.

**Proof.** Let $\tau$ be the weakest topology on $E$ for which all compact operators $E \to c_0$ are continuous. By Corollary 1.6, $bw \geq \tau$. To prove $bw \leq \tau$, let $p$ be a $bw$-continuous seminorm, say (Proposition 1.4.)

$$p = \sup_n |f_n|$$

where $f_1, f_2, \ldots \in E'$, $\lim_{n \to \infty} \|f_n\| = 0$. The operator $A : E \to c_0$, given by

$$Ax = (f_1(x), f_2(x), \ldots)$$

is easily seen to be compact and therefore $\tau$-continuous. Hence, so is $z \mapsto \|Ax\|$, which is $p$.

**Corollary 1.8.**

(i) $bw$ is the strongest locally convex topology on $E$ for which each norm bounded set is a compactoid.

(ii) $bw$ is nuclear. In fact it is the strongest nuclear topology, weaker than the norm topology. (See also [2], Example 5.2.)
Proof. (i) Follows from Proposition 1.4, $(\alpha) \iff (\beta)$. To prove nuclearity observe that $(E,bw)$ is of countable type (Corollary 1.5) so $(1, \S 4, \text{Proposition 2})$ it suffices to prove that every continuous linear map $A : (E,bw) \to c_0$ is compact. Such an $A$ has the form

$$z \mapsto Az = (f_1(z), f_2(z), \ldots)$$

for certain $f_1, f_2 \in E'$. By Corollary 1.6 $A$ is compact for the norm topology on $E$ so $\lim_{n \to \infty} \|f_n\| = 0$. There exist $\lambda_1, \lambda_2 \in K$ and $g_1, g_2, \ldots \in E'$ such that $f_n = \lambda_n g_n$ for each $n$, $\lim \lambda_n = 0$, $\lim_{n \to \infty} \|g_n\| = 0$. Then $\{x : |g_n(x)| \leq 1 \text{ for each } n\}$ is a $bw$-neighbourhood of $0$ whose image under $A$ lies in $\{(a_1, a_2, \ldots) \in c_0 : |a_n| \leq |\lambda_n| \text{ for each } n\}$. It follows that $A : (E,bw) \to c_0$ is compact. Now let $\tau$ be a nuclear topology on $E$, weaker than the norm topology. Then, in particular, each norm bounded set is $\tau$-bounded and is, by nuclearity, a $\tau$-compactoid. We conclude that, by (i), $\tau \leq bw$.

We have seen in Proposition 1.3 that in order to check $bw$-continuity of $A : E \to X$ it suffices to consider the behaviour of $A$ on bounded $bw$-convergent nets. Yet we shall prove (Proposition 1.10) that there do exist 'essentially' unbounded $bw$-convergent nets.

**Lemma 1.9.** Let $E'$ be infinite dimensional. If $X \subset E'$ is a normcompactoid then $X_0 = \{x \in E : |f(x)| \leq 1 \text{ for each } f \in X\}$ is unbounded.

**Proof.** Without loss assume $X$ is absolutely convex, edged and normcomplete. As $X$ is also metrizable, by [4], Theorem 10(ii) the norm topology and the $\omega'$-topology coincide on $X$. It follows that $X$ is $\omega'$-closed and edged. Thus, ([5], Theorem 4.7) $X$ is a polar set in $(E', \sigma(E', E))$ so that $X = (X_0)^0 := \{f \in E' : |f| \leq 1 \text{ on } X_0\}$. Now, if $X_0$ were bounded then $X = (X_0)^0$ is a norm neighbourhood of $0$ which is in conflict to compactoidity. We conclude that, by (i), $\tau \leq bw$.

**Proposition 1.10.** Let $E'$ be infinite dimensional. Then there exists a net $i \mapsto x_i$ in $E$, converging to $0$ in $bw$ but such that $\{x_i : i \geq 0\}$ is unbounded for each $i_0$.

**Proof.** For each normcompactoid $X \subset E'$ and $\varepsilon > 0$ we choose an $x_{X, \varepsilon} \in E$ such that $|f(x_{X, \varepsilon})| \leq \varepsilon$ for each $f \in X$ and $\|x_{X, \varepsilon}\| \geq \varepsilon^{-1}$ (Lemma 1.9). The set $I := \{(X, \varepsilon) : X \text{ is a compactoid in } E', \varepsilon > 0\}$ is a directed set under: $(X_1, \varepsilon_1) \geq (X_2, \varepsilon_2)$ iff $X_1 \supset X_2$ and $\varepsilon_1 \leq \varepsilon_2$. It is not hard to see that the net $i \mapsto x_i$ we just defined satisfies the requirements.

Let us consider the following related question (compare Proposition 1.1 ($\beta$)). Let $A$ be an absolutely convex subset of $E$ such that $A \cap B$ is weakly closed in $B$ for each bounded set $B \subset E$. Does it follow that $A$ is $bw$-closed? Does it follow that $A$ is $\omega$-closed? If $K$ is spherically complete (and also if $E$ is a normed space over $\mathbb{R}$ or $\mathbb{C}$) the answers are yes ($A \cap B$ is norm closed in $B$ for each bounded set $B \subset E$ so $A$ is norm closed hence $\omega$-closed (and $bw$-closed) by Remark 1 following Corollary 1.5). But, if $K$ is not spherically complete, say, $K = C_p$ the answers are no for $E = \ell^m(m)$ where $m \geq \#K$ (see [9], Corollary 3.6).
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It can be expected (for the definitions, see the Introduction) that, to some extent, the topologies $bw$ and $bw'$ behave in a similar way. Indeed, the obvious $bw'$-versions of Propositions 1.1, 1.2, 1.3 are easily seen to be true. Also, $B_{E'}$ is a $bw'$-compactoid, $bw'$ is of countable type, $bw'$ is weaker than the norm topology on $E'$, etc.; we leave the details to the reader. The counterpart (Theorem 2.3) of Proposition 1.4 (α) $\Leftrightarrow$ (β) is less innocent. First recall that the dual of $(E', w')$ is equal to $j_E(E)$. ([5], Lemma 7.1). We prove

**Proposition 2.1.** The dual of $(E', bw')$ is the norm closure of $j_E(E)$ in $E''$.

**Proof.** Every $\theta \in \overline{j_E(E)}$ is, on $B_{E'}$, the uniform limit of a sequence in $j_E(E)$ so $\theta|B_{E'}$ is $w'$-continuous i.e. $\theta$ is $bw'$-continuous. Thus, we have $\overline{j_E(E)} \subset (E', bw')$. Conversely, let $\theta \in (E', bw')$. Then $\theta$ is norm continuous i.e. $\theta \in E''$. Let $\varepsilon > 0$; we shall find an $x \in E$ such that $||\theta - j_E(x)|| < \varepsilon$. Choose $\alpha \in K$, $0 < |\alpha| < \varepsilon$. The $w'$-continuity of $\theta|B_{E'}$ yields a finite set $F \subset E$ such that $f \in F^0 \cap B_{E'}$ implies $|\theta(f)| \leq |\alpha|$, in other words

$$f \in j_E(F)_0 \cap (B_{E'})_0 \implies |(\alpha^{-1}\theta)(f)| \leq 1.$$ 

So we see that $\alpha^{-1}\theta \in (j_E(F)_0 \cap (B_{E'})_0)^0 = (A + B_{E'})^0$ where $A = j_E(coF)$. Now $B_{E'}$ is $w'$-closed and $A$ is finite dimensional so by [6], 1.4, $(A + B_{E'})^0 = (A + B_{E'})^e$. For any $\lambda \in K$ such that $|\lambda| > 1$ and $|\lambda\alpha| < \varepsilon$ we have $\alpha^{-1}\theta \in \lambda A + \lambda B_{E'}$ hence $\theta \in j_E(E) + \alpha\lambda B_{E'}$ and there is an $x \in E$ with $\theta - j_E(x) \in \alpha\lambda B_{E'}$ i.e. $||\theta - j_E(x)|| < \varepsilon$.

To avoid needless complications we shall from now on restrict ourselves to polar spaces i.e. spaces $E$ such that $j_E$ is an isometry. ('Most' 'natural' spaces are polar.)

**Corollary 2.2.** Let $E$ be a polar Banach space.

(i) The dual of $(E', bw')$ is $j_E(E)$.

(ii) An edged absolutely convex subset of $E'$ is $bw'$-closed if and only if it is $w'$-closed.

**Proof.** (i) is immediate. (ii) follows from (i) as $(E', w')$ and $(E', w')$ have the same dual space.

**Theorem 2.3.** Let $E$ be a polar $K$-Banach space. Then the topology $bw'$ on $E'$ is the topology $c$ of compact convergence i.e. it is generated by the seminorms $f \mapsto \max_X |f|$ where $X$ runs through the family of all compact subsets of $E$.

**Proof.** A routine proof shows that $c$, restricted to bounded sets, coincides with $w'$ so, by definition, $bw'$ is stronger than $c$. Let us prove that $bw'$ is weaker than $c$. Let $p$ be a $bw'$-continuous seminorm on $E'$. Then $p$ is certainly $bw$-continuous so by Proposition 1.4 there exist $\theta_1, \theta_2, \ldots \in E''$ with $\lim_{n \to \infty} ||\theta_n|| = 0$ such that $p$ is equivalent to $f \mapsto \max_n |\theta_n(f)|$. Hence,
\(\theta_1, \theta_2, \ldots\) are \(bw'-\)continuous so by Corollary 2.2 (i) there exist \(x_1, x_2, \ldots\) with \(\theta_n = j_{E}(x_n)\) for each \(n\). We have \(\lim_{n \to \infty} \|x_n\| = 0\), as \(j_E\) is an isometry. We see that \(p\) is equivalent to

\[ f \mapsto \max_{x \in X} |f(x)| \]

where \(X\) is the compact set \(\{0, x_1, x_2, \ldots\}\) and we are done.

The proof of Theorem 2.3 depends on the theory of §1. We now show a way to arrive at the same result avoiding the use of the \(bw\)-theory. The key is the following ‘convexification’ of the Ascoli Theorem.

**Theorem 2.4.** Let \(X\) be a subset of a polar \(K\)-Banach space \(E\). The following are equivalent.

1. \(X\) is compactoid.
2. If \(i \mapsto f_i\) is a (norm) bounded net in \(E'\) converging \(w'\) to \(f \in E'\) then \(f_i \to f\) uniformly on \(X\).

**Proof.** (i) \(\Rightarrow\) (ii). To prove (ii) we may assume \(f = 0, \|f_i\| \leq 1\) for all \(i\). Let \(\varepsilon > 0\). There exist \(x_1, x_2, \ldots, x_n \in E\) such that \(X \subset B(0, \varepsilon) + c_0 \{x_1, \ldots, x_n\}\). There is an \(i_0\) such that \(|f_i(x_j)| \leq \varepsilon\) for all \(i \geq i_0\) and all \(j \in \{1, \ldots, n\}\). As \(|f_i| \leq \varepsilon\) on \(B(0, \varepsilon)\) for all \(i\) we find \(|f_i| \leq \varepsilon\) on \(X\) if \(i \geq i_0\). (Remark. In this part of the proof the polarity assumption was not needed.)

(ii) \(\Rightarrow\) (i). It is an easy exercise to show that \(E/D\) is a polar \(K\)-Banach space for every finite dimensional subspace \(D\) of \(E\). Now assume \(X\) is not a compactoid. Then, almost from the definition, there is an \(\alpha > 0\) such that \(\text{diam } \pi_D(X) > \alpha\) for each \(D \in \mathcal{D}\), where \(\mathcal{D}\) is the directed set of all finite dimensional subspaces of \(E\) and where \(\pi_D : E \to E/D\) is the quotient map. Now \(E/D\) is polar so we can find a \(g_D \in (E/D)'\) with \(\|g_D\| \leq 1\) and \(|g_D(\pi_D(z))| \geq \alpha\) for some \(z \in X\). Set \(f_D := g_D \circ \pi_D (D \in \mathcal{D})\). Then \(f_D \to 0\) in \(w'\), \(\|f_D\| \leq 1\) for all \(D\) but not \(f_D \to 0\) uniformly on \(X\), conflicting (ii).

**Second proof of Theorem 2.3.** Let \(p\) be a \(bw'-\)continuous seminorm on \(E'\). Then \(p\) is polar so \(p(f) = \sup \{|\theta(f)| : |\theta| \leq p, \theta \in (E', bw')'\}\). By Corollary 2.2(i) there is a subset \(X \subset E\) for which

\[ p(f) = \sup \{|f(x)| : x \in X\}. \]

To prove that \(X\) is a compactoid, let \(f_i \in E', \|f_i\| \leq 1, f_i \to 0\) in \(w'\) then, by definition, \(f_i \to 0\) in \(bw'\) so that \(p(f_i) \to 0\) implying \(f_i \to 0\) uniformly on \(X\). The compactoidity now follows from Theorem 2.4, (ii) \(\Rightarrow\) (i).

Finally we consider briefly the \(bw'\)-version of the last problem of §1 (we keep assuming \(E\) to be a polar Banach space). Let \(A \subset E'\) be absolutely convex. Assume that \(A \cap B\) is \(w'\)-closed in \(B\) for each bounded \(B \subset E'\). Does it follow that \(A\) is \(bw'\)-closed? Does it follow that \(A\) is \(w'\)-closed?
If $K$ is spherically complete the answers are yes. Its classical counterpart is known as the Krein-Šmulian Theorem. The proof is by no means trivial. See [8]. If $K$ is not spherically complete a counterexample to both questions is given by [9], Corollary 3.7.

REFERENCES