The $p$-adic bounded weak topologies

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INTRODUCTION

Throughout, let $K$ be a nonarchimedean nontrivially valued field that is complete under the metric induced by the valuation $|\cdot|$. For a normed space $E$ over $K$ one has the well known weak topology $w = \sigma(E, E')$ which is the weakest topology on $E$ making all $f \in E'$ (the dual space of $E$) continuous. Similarly, the so-called 'weak star' topology $w' = \sigma(E', E)$ is by definition the weakest topology on $E'$ making all evaluations $f \mapsto f(x)$ continuous. (The expression 'weak star' goes back to the old days when the topological dual space of $E$ was denoted $E^*$ rather than $E'$.) In this paper we study the following related less known topologies. The bounded weak topology $bw$ is the strongest locally convex topology on $E$ that coincides with $w$ on (norm) bounded subsets of $E$. The bounded weak star topology $bw'$ is the strongest locally convex topology on $E'$ that coincides with $w'$ on (norm) bounded subsets of $E'$. These two topologies have appeared incidentally in [8], [9], but in this paper we shall consider them in their own right.

PRELIMINARIES

We shall use notations and terminology of [3]. Throughout $E$ is a normed space over $K$ (where the norm $\| \cdot \|$ is assumed to satisfy the strong triangle inequality $\|x+y\| \leq \max(\|x\|,\|y\|)$). The 'closed' ball with center $0$ and radius $\varepsilon > 0$, $\{x \in E : \|x\| \leq \varepsilon\}$ is written $B(0,\varepsilon)$. Sometimes we write $B_E$ in place of $\{x \in E : \|x\| \leq 1\}$. The canonical map $E \to E''$ is denoted $j_E$. WARNING. $j_E$ need not be injective; the topologies $w$ and $bw$ introduced above need not be Hausdorff!

A subset $A$ of a $K$-vector space is absolutely convex if $x, y \in A$, $\lambda, \mu \in K$, $|\lambda|, |\mu| \leq 1$ implies $\lambda x + \mu y \in A$. If $A$ is absolutely convex we define $A^e := A$ if the valuation of $K$ is discrete, $A^e := \bigcap_{|\lambda| > 1} \lambda A$ if the valuation of $K$ is dense. $A$ is edged if $A = A^e$. The smallest absolutely
convex set containing $X$ is denoted by $\text{co}X$, its linear span by $[X]$. Let $Z$ be a locally convex space over $K$, let $X \subset Z$ and $Y \subset Z'$. We write

$$X^0 := \{f \in Z' : |f(x)| \leq 1 \text{ for all } x \in X\}$$

$$Y_0 := \{x \in Z : |f(x)| \leq 1 \text{ for all } f \in Y\}.$$  

$X$ is a polar set if $(X^0)_0 = X$.

§1. THE BOUNDED WEAK TOPOLOGY

We first collect some elementary consequences of the definitions.

**Proposition 1.1.** For an absolutely convex set $U \subset E$ the following are equivalent.

(a) $U$ is bw-open.

(b) $U \cap B$ is weakly open in $B$ for each bounded set $B \subset E$.

(c) $U \cup \lambda B_E$ is weakly open in $\lambda B_E$ for each $\lambda \in K$, $\lambda \neq 0$.

**Proof.** Straightforward.

**Proposition 1.2.** For a seminorm $p$ on $E$ the following are equivalent.

(a) $p$ is bw-continuous.

(b) $p|B$ is weakly continuous for each bounded set $B \subset E$.

(c) $p|B_E$ is weakly continuous.

**Proof.** Direct consequence of Proposition 1.1. (Observe that from (c) it follows that $p|\lambda B_E$ is weakly continuous for each $\lambda \in K$, $\lambda \neq 0$.)

**Proposition 1.3.** Let $(X, \tau)$ be a locally convex space over $K$, let $A : E \to X$ be a linear map. Then $A$ is bw to $\tau$ continuous if and only if $A|B$ is weakly continuous for each bounded set $B \subset E$.

**Proof.** If $A$ is bw to $\tau$ continuous then so is $A|B$ and $A|B$ is weakly continuous. If $A|B$ is weakly continuous for each bounded $B \subset E$ then for each $\tau$-continuous seminorm $q$ on $X$ we have that $q \circ A$ satisfies $b$ of Proposition 1.2.

The next Proposition is deeper and gives a concrete description of the bw-continuous seminorms $(\delta)$ and $(\varepsilon)$ below.

**Proposition 1.4.** For a seminorm $p$ on $E$ the following are equivalent.

(a) $p$ is bw-continuous.

(b) $B_E$ is a $p$-compactoid.

(c) $p$ is polar, norm continuous, $\{f \in E' : |f| \leq p\}$ is a norm compactoid in $E'$.

(d) There exist $f_1, f_2, \ldots$ in $E'$ with $\lim_{n \to \infty} \|f_n\| = 0$ such that $p$ is equivalent to $\sup_n |f_n|$.

(e) There exist $f_1, f_2, \ldots$ in $E'$ with $\lim_{n \to \infty} \|f_n\| = 0$ such that $p \leq \sup_n |f_n|$.
Proof. $(a) \implies (\beta)$. $B_\mathcal{E}$ is a compactoid for the weak topology; the topologies $bw$ and $w$ coincide on $B_\mathcal{E}$.

$(\beta) \implies (\gamma)$. As $B_\mathcal{E}$ is a $p$-compactoid we have $(E,p) = ([B_\mathcal{E}],p)$ is of countable type so $p$ is a polar seminorm. Also $p$ is bounded on $B_\mathcal{E}$ hence norm continuous. To show that $\{f \in E' : |f| \leq p\}$ is a normcompactoid consider the identity map $(E,\|\|) \rightarrow (E,p)$ which is compact by $(\beta)$. Then so is its adjoint which is the inclusion $(E,p)' \hookrightarrow (E,\|\|)'$. Hence, the unit ball $\{f \in E' : |f| \leq p\}$ of $(E,p)'$ is a compactoid in $(E,\|\|)'$.

$(\gamma) \implies (\delta)$. Set $S := \{f \in E' : |f| \leq p\}$. By polarity and norm continuity

\[ p = \sup\{|f| : f \in E' : |f| \leq p\} = \sup_{f \in S} |f|. \]

The normcompactoid $S$ is easily seen to be absolutely convex and norm closed. So, by [6], Lemma 1.3, if $\lambda \in K, |\lambda| > 1$ there exist $f_1, f_2, \ldots \in S$ with $\lim_{n \to \infty} \|f_n\| = 0$ such that

\[ \mathcal{V}\{f_1, f_2, \ldots\} \subset S \subset |\lambda| \mathcal{V}\{f_1, f_2, \ldots\} \]

We obtain

\[ \sup_n |f_n| = \sup\{|f| : f \in \mathcal{V}\{f_1, f_2, \ldots\}\} \leq \sup_S |f| \leq \sup\{|f| : f \in |\lambda| \mathcal{V}\{f_1, f_2, \ldots\}\} = |\lambda| \sup_n |f_n|. \]

Together with $(*)$ this yields

\[ \sup_n |f_n| \leq p \leq |\lambda| \sup_n |f_n| \]

and $(\delta)$ follows. The implication $(\delta) \implies (\varepsilon)$ is obvious. Finally, assume $(\varepsilon)$. To arrive at $(a)$ we prove (see Proposition 1.2$(\gamma)$) that for a net $i \mapsto x_i$ in $B_\mathcal{E}$ converging weakly to $x \in B_\mathcal{E}$ it follows that $p(x-x_i) \rightarrow 0$. In fact, let $N$ be such that $|f_n| \leq \varepsilon$ for $n > N$. We have $|f_j(x-x_i)| \leq \varepsilon$ for all $j \in \{1, \ldots, N\}$ and sufficiently large $i$. For these $i$

\[ p(x-x_i) \leq \max_{1 \leq j \leq N} |f_j(x-x_i)| = \varepsilon \sup_{j > N} |f_j||x-x_i| \leq \varepsilon \sup_{j > N} \|f_j\||x-x_i| \leq \varepsilon \sup_{j > N} \|f_j\| \leq \varepsilon. \]

Several corollaries obtain.

Corollary 1.5. The topology $bw$ is of countable type. The dual of $(E,bw)$ is $E'$. An absolutely convex edged set in $E$ is $bw$-closed if and only if it is weakly closed.

Proof. In the proof of $(\beta) \implies (\gamma)$ it is observed that $bw$ is of countable type. The identity maps $(E,\|\|) \rightarrow (E,bw) \rightarrow (E,w)$ are continuous (Proposition 1.4$(\gamma)$). Since $(E,\|\|)' = (E,w)' = E'$ the dual of $(E,bw)$ also must be $E'$. As a consequence the topologies $bw$ and
have the same collection of polar sets and, by strong polarness, have the same collection of absolutely convex edged sets ([5], Theorem 4.7.).

Remarks. (See [5], [9] for proofs.)
1. If \( K \) is spherically complete we may extend the last statement of Corollary 1.5. as follows.
An absolutely convex set in \( E \) is \( bw \)-closed if and only if it is weakly closed if and only if it is norm closed.
2. If \( K \) is not spherically complete but \( E \) is strongly polar we may draw the same conclusion as in 1. but only for edged absolutely convex sets.
3. If \( K \) is not spherically complete and \( E' \) is infinite dimensional there exists a \( bw \)-closed absolutely convex set in \( E \) that is not weakly closed.

Corollary 1.6. Let \( X \) be a locally convex space. A linear map \( A : E \to X \) is \( bw \)-continuous if and only if it is compact for the norm topology on \( E \).

Proof. If \( A \) is \( bw \)-continuous then (Proposition 1.4) \( B_E \) is a \( bw \)-compactoid so \( AB_E \) is a compactoid in \( X \). If, conversely, \( A \) is norm compact then by [7], 1.2, for each net \( i \to x, x \in B_E \) converging weakly to 0 we have \( Ax_i \to 0 \) in \( X \). By Proposition 1.3 \( A \) is \( bw \)-continuous.

The weak topology is defined as the weakest topology for which all \( f \in E' \) are continuous. For \( bw \) we have a similar description.

Corollary 1.7. The bounded weak topology on \( E \) is the weakest topology on \( E \) for which all compact operators \( E \to c_0 \) are continuous.

Proof. Let \( \tau \) be the weakest topology on \( E \) for which all compact operators \( E \to c_0 \) are continuous. By Corollary 1.6, \( bw \geq \tau \). To prove \( bw \leq \tau \), let \( p \) be a \( bw \)-continuous seminorm, say (Proposition 1.4.)

\[
p = \sup_n |f_n|
\]

where \( f_1, f_2, \ldots \in E' \), \( \lim_{n \to \infty} \|f_n\| = 0 \). The operator \( A : E \to c_0 \), given by

\[
Ax = (f_1(x), f_2(x), \ldots)
\]

is easily seen to be compact and therefore \( \tau \)-continuous. Hence, so is \( z \mapsto \|Ax\| \), which is \( p \).

Corollary 1.8.

(i) \( bw \) is the strongest locally convex topology on \( E \) for which each norm bounded set is a compactoid.

(ii) \( bw \) is nuclear. In fact it is the strongest nuclear topology, weaker than the norm topology.

(See also [2], Example 5.2.)
Proof. (i) Follows from Proposition 1.4, ($\alpha \iff \beta$). To prove nuclearity observe that $(E, bw)$ is of countable type (Corollary 1.5) so (1), §4, Proposition 2) it suffices to prove that every continuous linear map $A : (E, bw) \to c_0$ is compact. Such an $A$ has the form

$$z \mapsto Az = (f_1(z), f_2(z), \ldots)$$

for certain $f_1, f_2 \in E'$. By Corollary 1.6 $A$ is compact for the norm topology on $E$ so $\lim_{n \to \infty} \|f_n\| = 0$. There exist $\lambda_1, \lambda_2 \in K$ and $g_1, g_2, \ldots \in E'$ such that $f_n = \lambda_n g_n$ for each $n$, $\lim \lambda_n = 0$, $\lim_{n \to \infty} \|g_n\| = 0$. Then $\{z : |g_n(z)| \leq 1 \text{ for each } n\}$ is a $bw$-neighbourhood of 0 whose image under $A$ lies in $\{(a_1, a_2, \ldots) \in c_0 : |a_n| \leq |\lambda_n| \text{ for each } n\}$. It follows that $A : (E, bw) \to c_0$ is compact. Now let $\tau$ be a nuclear topology on $E$, weaker than the norm topology. Then, in particular, each norm bounded set is $\tau$-bounded and is, by nuclearity, a $\tau$-compactoid. We conclude that, by (i), $\tau \leq bw$.

We have seen in Proposition 1.3 that in order to check $bw$-continuity of $A : E \to X$ it suffices to consider the behaviour of $A$ on bounded $bw$-convergent nets. Yet we shall prove (Proposition 1.10) that there do exist 'essentially' unbounded $bw$-convergent nets.

Lemma 1.9. Let $E'$ be infinite dimensional. If $X \subset E'$ is a normcompactoid then $X_0 = \{x \in E : |f(x)| \leq 1 \text{ for each } f \in X\}$ is unbounded.

Proof. Without loss assume $X$ is absolutely convex, edged and normcomplete. As $X$ is also metrizable, by [4], Theorem 10(ii) the norm topology and the $w'$-topology coincide on $X$. It follows that $X$ is $w'$-closed and edged. Thus, ([5], Theorem 4.7) $X$ is a polar set in $(E', \sigma(E', E))$ so that $X = (X_0)^0 := \{f \in E' : |f| \leq 1 \text{ on } X_0\}$. Now, if $X_0$ were bounded then $X = (X_0)^0$ is a norm neighbourhood of 0 which is in conflict to compactoidity.

Proposition 1.10. Let $E'$ be infinite dimensional. Then there exists a net $i \mapsto x_i$ in $E$, converging to 0 in $bw$ but such that $\{x_i : i \geq i_0\}$ is unbounded for each $i_0$.

Proof. For each normcompactoid $X \subset E'$ and $\varepsilon > 0$ we choose an $z_{x,\varepsilon} \in E$ such that $|f(z_{x,\varepsilon})| \leq \varepsilon$ for each $f \in X$ and $\|z_{x,\varepsilon}\| \geq \varepsilon^{-1}$ (Lemma 1.9). The set $I := \{(X, \varepsilon') : X \text{ is a compactoid in } E', \varepsilon' > 0\}$ is a directed set under: $(X, \varepsilon_1) \geq (X, \varepsilon_2)$ iff $X_1 \supset X_2$ and $\varepsilon_1 \leq \varepsilon_2$. It is not hard to see that the net $i \mapsto x_i$ we just defined satisfies the requirements.

Let us consider the following related question (compare Proposition 1.1 ($\beta$)). Let $A$ be an absolutely convex subset of $E$ such that $A \cap B$ is weakly closed in $B$ for each bounded set $B \subset E$. Does it follow that $A$ is $bw$-closed? Does it follow that $A$ is $w$-closed? If $K$ is spherically complete (and also if $E$ is a normed space over $R$ or $C$) the answers are yes ($A \cap B$ is norm closed in $B$ for each bounded set $B \subset E$ so $A$ is norm closed hence $w$-closed (and $bw$-closed) by Remark 1 following Corollary 1.5). But, if $K$ is not spherically complete, say, $K = C_p$ the answers are no for $E = \ell^m(m)$ where $m \geq \#K$ (see [9], Corollary 3.6).
§2. THE BOUNDED WEAK STAR TOPOLOGY

It can be expected (for the definitions, see the Introduction) that, to some extent, the topologies $bw$ and $bw'$ behave in a similar way. Indeed, the obvious $bw'$-versions of Propositions 1.1, 1.2, 1.3 are easily seen to be true. Also, $B_{E'}$ is a $bw'$-compactoid, $bw'$ is of countable type, $bw'$ is weaker than the norm topology on $E'$, etc.; we leave the details to the reader. The counterpart (Theorem 2.3) of Proposition 1.4 (α) $\iff$ (β) is less innocent. First recall that the dual of $(E', w')$ is equal to $j_E(E)$. ([5], Lemma 7.1). We prove

Proposition 2.1. The dual of $(E', bw')$ is the norm closure of $j_E(E)$ in $E''$.

Proof. Every $\theta \in j_E(E)$ is, on $B_{E'}$, the uniform limit of a sequence in $j_E(E)$ so $\theta|B_{E'}$ is $w'$-continuous i.e. $\theta$ is $bw'$-continuous. Thus, we have $j_E(E) \subset (E', bw')$. Conversely, let $\theta \in (E', bw')$. Then $\theta$ is norm continuous i.e. $\theta \in E''$. Let $\varepsilon > 0$; we shall find an $x \in E$ such that $||\theta - j_E(x)|| < \varepsilon$. Choose $x \in E$, $0 < ||x|| < \varepsilon$. The $w'$-continuity of $\theta|B_{E'}$ yields a finite set $F \subset E$ such that $f \in F \cap B_{E'}$ implies $|\theta(f)| \leq ||x||$, in other words

$$f \in j_E(F) \cap (B_{E'})_0 \implies |(\alpha^{-1}\theta)(f)| \leq 1.$$ 

So we see that $\alpha^{-1}\theta \in (j_E(F)_0 \cap (B_{E'})_0)^0 = (A + B_{E''})^0$ where $A = j_E(coF)$. Now $B_{E''}$ is $w'$-closed and $A$ is finite dimensional so by [6], 1.4, $(A + B_{E''})^0 = (A + B_{E''})^e$. For any $\lambda \in K$ such that $|\lambda| > 1$ and $|\lambda x| < \varepsilon$ we have $\alpha^{-1}\theta \in \lambda A + \lambda B_{E''}$ hence $\theta \in j_E(E) + \alpha \lambda B_{E''}$ and there is an $x \in E$ with $\theta - j_E(x) \in \alpha \lambda B_{E''}$ i.e. $||\theta - j_E(x)|| < \varepsilon$.

To avoid needless complications we shall from now on restrict ourselves to polar spaces i.e. spaces $E$ such that $j_E$ is an isometry. ('Most' 'natural' spaces are polar.)

Corollary 2.2. Let $E$ be a polar Banach space.

(i) The dual of $(E', bw')$ is $j_E(E)$.

(ii) An edged absolutely convex subset of $E'$ is $bw'$-closed if and only if it is $w'$-closed.

Proof. (i) is immediate. (ii) follows from (i) as $(E', w')$ and $(E', w)$ have the same dual space.

Theorem 2.3. Let $E$ be a polar $K$-Banach space. Then the topology $bw'$ on $E'$ is the topology $c$ of compact convergence i.e. it is generated by the seminorms $f \mapsto \max_X |f|$ where $X$ runs through the family of all compact subsets of $E$.

Proof. A routine proof shows that $c$, restricted to bounded sets, coincides with $w'$ so, by definition, $bw'$ is stronger than $c$. Let us prove that $bw'$ is weaker than $c$. Let $p$ be a $bw'$-continuous seminorm on $E'$. Then $p$ is certainly $bw$-continuous so by Proposition 1.4 there exist $\theta_1, \theta_2, \ldots \in E''$ with $\lim_{n \to \infty} ||\theta_n|| = 0$ such that $p$ is equivalent to $f \mapsto \max_n |\theta_n(f)|$. Hence,
\( \theta_1, \theta_2, \ldots \) are \( bw' \)-continuous so by Corollary 2.2 (i) there exist \( z_1, z_2, \ldots \) with \( \theta_n = j_E(x_n) \) for each \( n \). We have \( \lim_{n \to \infty} \|x_n\| = 0 \), as \( j_E \) is an isometry. We see that \( p \) is equivalent to

\[
\int_{x \in X} \max_{z \in X} |f(z)|
\]

where \( X \) is the compact set \( \{0, x_1, x_2, \ldots \} \) and we are done.

The proof of Theorem 2.3 depends on the theory of \( \S 1 \). We now show a way to arrive at the same result avoiding the use of the \( bw' \)-theory. The key is the following 'convexification' of the Ascoli Theorem.

**Theorem 2.4.** Let \( X \) be a subset of a polar \( K \)-Banach space \( E \). The following are equivalent.

1. \( X \) is compactoid.
2. If \( i \mapsto f_i \) is a (norm) bounded net in \( E' \) converging \( w' \) to \( f \in E' \) then \( f_i \to f \) uniformly on \( X \).

**Proof.** \((a) \Rightarrow (b)\). To prove \((b)\) we may assume \( f = 0, \|f_i\| \leq 1 \) for all \( i \). Let \( \varepsilon > 0 \). There exist \( x_1, x_2, \ldots, x_n \in E \) such that \( X \subseteq B(0, \varepsilon) + co\{x_1, \ldots, x_n\} \). There is an \( i_0 \) such that \( |f_i(x_j)| \leq \varepsilon \) for all \( i \geq i_0 \) and all \( j \in \{1, \ldots, n\} \). As \( |f_i| \leq \varepsilon \) on \( B(0, \varepsilon) \) for all \( i \) we find \( |f_i| \leq \varepsilon \) on \( X \) if \( i \geq i_0 \).

(Remark. In this part of the proof the polarity assumption was not needed.)

\((b) \Rightarrow (a)\). It is an easy exercise to show that \( E/D \) is a polar \( K \)-Banach space for every finite dimensional subspace \( D \) of \( E \). Now assume \( X \) is not a compactoid. Then, almost from the definition, there is a \( \alpha > 0 \) such that \( \operatorname{diam} \pi_D(X) > \alpha \) for each \( D \in \mathcal{D} \), where \( \mathcal{D} \) is the directed set of all finite dimensional subspaces of \( E \) and where \( \pi_D : E \to E/D \) is the quotient map.

Now \( E/D \) is polar so we can find a \( g_D \in (E/D)' \) with \( \|g_D\| \leq 1 \) and \( |g_D(\pi_D(x))| \geq \alpha \) for some \( x \in X \). Set \( f_D := g_D \circ \pi_D \) \((D \in \mathcal{D})\). Then \( f_D \to 0 \) in \( w' \), \( \|f_D\| \leq 1 \) for all \( D \) but not \( f_D \to 0 \) uniformly on \( X \), conflicting \((b)\).

**Second proof of Theorem 2.3.** Let \( p \) be a \( bw' \)-continuous seminorm on \( E' \). Then \( p \) is polar so \( p(f) = \sup \{ |f(\theta)| : |\theta| \leq p, \theta \in (E', bw')' \} \). By Corollary 2.2(i) there is a subset \( X \subseteq E \) for which

\[
p(f) = \sup \{ |f(z)| : z \in X \}.
\]

To prove that \( X \) is a compactoid, let \( f_i \in E', \|f_i\| \leq 1, f_i \to 0 \) in \( w' \) then, by definition, \( f_i \to 0 \) in \( bw' \) so that \( p(f_i) \to 0 \) implying \( f_i \to 0 \) uniformly on \( X \). The compactoidity now follows from Theorem 2.4, \((b) \Rightarrow (a)\).

Finally we consider briefly the \( bw' \)-version of the last problem of \( \S 1 \) (we keep assuming \( E \) to be a polar Banach space). Let \( A \subseteq E' \) be absolutely convex. Assume that \( A \cap B \) is \( w' \)-closed in \( B \) for each bounded \( B \subseteq E' \). Does it follow that \( A \) is \( bw' \)-closed? Does it follow that \( A \) is \( w' \)-closed?
If $K$ is spherically complete the answers are yes. Its classical counterpart is known as the Krein-Šmulian Theorem. The proof is by no means trivial. See [8]. If $K$ is not spherically complete a counterexample to both questions is given by [9], Corollary 3.7.

REFERENCES