THE EQUALIZATION OF $p$-ADIC BANACH SPACES AND COMPACTOIDS

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ABSTRACT. In the set up of Non-Archimedean Functional Analysis, Banach spaces as well as compactoids play a fundamental role, although casted for different parts. The more surprising it is, that yet both concepts turn out to be closely related as will be revealed in Part One. In fact we shall prove that (roughly speaking) to any mathematical statement about compactoids there exists an equivalent dual statement formulated in terms of Banach spaces, and conversely. This is done by establishing an anti-equivalence between categories (Theorem 4.6). In Part Two we shall apply this result by translating both ways obtaining new theorems (6.1, 6.2, 7.1, 7.2(ii), 8.1, 8.2, 9.3, 9.4, 9.5, 10.1), new proofs of known theorems (6.3, 7.2(i), 7.4, 11.2) and equivalent formulations of open problems (Problems following 7.2, 7.3).

Remark. The Report [7] can be viewed as a forerunner of the present paper whereas in a succeeding note [11] some remaining details will be worked out.

Part One

1. BANACH SPACES VERSUS COMPACTOIDS, AN ANTI-EQUIVALENCE

1.1. Prologue. (For notations and terminology see the next section.) Recall [12] that a convex subset $C$ of a Hausdorff locally convex space over a (spherically complete) non-archimedean valued field $K$ is said to be c-compact if every collection of relatively closed convex subsets of $C$ with the finite intersection property has a non-void intersection. This notion suitably replaces the ordinary concept 'convex-compact' in case $K$ is not locally compact; in fact, certain convexified versions of compactness properties hold for c-compact sets $C$ as well. For example, $C$ is complete; if $T$ is a continuous linear map then $TC$ is c-compact, if in addition $T$ is injective then $T$ is a homeomorphism $C \to TC$. Within this context it therefore is natural to ask whether $TC$ always carries the quotient topology induced by $T$. As a test case consider the following situation. Let $K$ be spherically complete, let $A$ be an absolutely convex bounded c-compact (= an absolutely convex closed compactoid) set in $c_0$, let $T \in L(c_0)$. It is not hard to see that, by metrizability and the fact that $A$ is an additive topological group, the following assertions are equivalent.

(a) The (norm) topology on $TA$ equals the quotient topology induced by $T$.

(b) $T : A \to TA$ is an open mapping.

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For each sequence \( y_1, y_2, \ldots \) in \( TA \) tending to 0 there exists a sequence \( x_1, x_2, \ldots \) in \( A \) tending to 0 such that \( Tx_n = y_n \) for each \( n \).

It is known that \((\alpha)-(\gamma)\) hold if the valuation of \( K \) is discrete ([10] §3) or if \( T \) is injective ([1], [4], [5]). The following startling example shows that \((\alpha)-(\gamma)\) are false in general thus providing a negative answer to the problem stated in [10] §3. It was first obtained as a by-product of the main theory of this paper (see Corollary 9.4) but here it is considered to be shorn of these ornaments.

**Example 1.1.** Let \( K \) be spherically complete and let the valuation be dense. Then there exist a closed absolutely convex compactoid \( A \) in \( c_0 \) and a \( T \in L(c_0) \) such that its restriction \( A \to TA \) is not an open mapping.

**Proof.** Choose \( b_1, b_2, \ldots \in K \) with \( 0 < |b_n| < 1 \) for each \( n \) and \( \prod n|b_n| > 0 \), choose \( \rho \in K \), \( 0 < |ho| < 1 \). The formula

\[
T \left( \sum_{n=1}^{\infty} \lambda_n e_n \right) = \sum_{n=1}^{\infty} (\lambda_n - b_n \rho \lambda_{n+1}^{-1}) e_n
\]

where \( e_1, e_2, \ldots \) is the standard base of \( c_0 \) and \( \lambda_n \in K \), \( \lambda_n \to 0 \), defines a map \( T \in L(c_0) \). Setting \( y_n := \rho^n e_n \) we have \( y_n \to 0 \) so that

\[
A := \overline{co}\{y_1, y_2, \ldots \}
\]

is a closed compactoid. We first show that \( y_1, y_2, \ldots \) are in \( TA \). In fact, we have \( y_n = Tz_n \) where

\[
z_n := y_n + b_{n-1}y_{n-1} + b_{n-2}b_{n-1}y_{n-1} + \cdots + b_1b_2 \ldots b_{n-1}y_1 \in A.
\]

Also observe that \( \text{Ker} \ T = Ka \) where \( a := \sum_{n=1}^{\infty} a_n y_n \) and \( a_n := \prod_{i\geq n} b_i \). From \( \lim_{n\to\infty} |a_n| = 1 \) one obtains that \( A \cap \text{Ker} \ T = \text{co}\{a\} \).

Next, let \( x_1, x_2, \ldots \) be any sequence in \( A \) with \( Tx_n = y_n \) for each \( n \). Then by the above \( x_n \) has the form \( z_n - \lambda_n a \) where \( \lambda_n \in K \), \( |\lambda_n| \leq 1 \). The first coefficient of \( x_n \) in the expansion with respect to \( y_1, y_2, \ldots \) equals \( b_1b_2 \ldots b_{n-1} - \lambda_n a_1 = a_1(a_1^{-1} - \lambda_n) \) which does not tend to 0 since \( |a_1^{-1}| > 1 \). Then the sequence \( x_1, x_2, \ldots \) cannot tend to 0 and we conclude that \((\gamma)\) above is not true. \( \square \)

**Remark 1.** The above proof works also when \( K \) is not spherically complete although in this case the result is less spectacular as continuous linear images of \( A \) need not even be closed! ([2], 6.28).

**Remark 2.** Observe that \( A \) of above is edged in the sense of 2.2.
2. Terminology

In this section we collect definitions, notations, conventions, ... needed in this paper. For terms that remain unexplained we refer to [3], [6].

2.1. Throughout $K$ is a non-archimedean valued field that is complete with respect to its non-trivial valuation $|\cdot|$. We set $B_K := \{ \lambda \in K : |\lambda| \leq 1 \}$ and $B_K^* := \{ \lambda \in K : |\lambda| < 1 \}$.

2.2. A subset $A$ of a $K$-vector space $E$ is absolutely convex if it is a $B_K$-submodule of $E$. For a subset $X$ of $E$ we denote by $\text{co}X$ its absolutely convex hull, by $[X]$ its linear span. For an absolutely convex $A \subset E$ the formula

$$p_A(x) = \inf\{ |\lambda| : \lambda \in K, x \in \lambda A \}$$

defines a (non-archimedean) seminorm $p_A$ on $[A]$ called the Minkowski function of $A$. We define $A^e := A$ if the valuation of $K$ is discrete and $A^e := \bigcap \{ \lambda A : \lambda \in K, |\lambda| > 1 \}$ if the valuation of $K$ is dense. $A$ is edged if $A = A^e$ or, equivalently, if $A = \{ x \in [A] : p_A(x) \leq 1 \}$.

A seminorm $p$ on $E$ is polar if $p = \sup\{ |f| : f \in E^*, |f| \leq p \}$ when $E^*$ is the space of all linear functions $E \rightarrow K$.

2.3. Let $E = (E, \|\cdot\|)$ be a normed space over $K$. (Throughout norms are assumed to be non-archimedean.) For $a \in E$, $r > 0$ we write $B_E(a, r) := \{ x \in E : \|x-a\| < r \}$ (the ‘open’ ball) and $B_E^*(a, r) := \{ x \in E : \|x-a\| < r \}$ (the ‘closed’ ball). The ‘closed’ unit ball $B_E^*(0, 1)$ is sometimes denoted $B^*_E$; similarly we write $B_E^-(0, 1)$. For $a \in E$, $X \subset E$ we set $\text{dist}(a, X) := \inf\{ \|a-x\| : x \in X \}$.

Let $E, F$ be $K$-Banach spaces. Then $\mathcal{L}(E, F) := \{ T : E \rightarrow F/T \text{ linear and continuous} \}$ is a $K$-Banach space under the norm

$$\|T\| := \inf\{ M \geq 0 : \|Tx\| \leq M\|x\| \text{ for all } x \in E \}.$$ 

As usual we write $\mathcal{L}(E) := \mathcal{L}(E, E)$ and $E' := \mathcal{L}(E, K)$.

Let $D$ be a closed subspace of $E$, let $\pi : E \rightarrow E/D$ be the canonical map. The quotient norm on $E/D$ is defined by the formula $\|\pi(x)\| = \text{dist}(x, D)$. Then $\pi$ maps $B_E^-(0, r)$ onto $B_{E/D}^-(0, r)$ for each $r > 0$ and is called a quotient map. If, in addition, for each $r > 0$, $\pi$ maps $B_E(0, r)$ onto $B_{E/D}(0, r)$, or, equivalently, if $\|\pi\| \leq 1$ and for each $z \in E/D$ there is an $x \in E$ with $\pi(x) = z$ and $\|x\| = \|z\|$, we call $\pi$ a strict quotient map.

For each $T \in \mathcal{L}(E, F)$ its adjoint $T' \in \mathcal{L}(F', E')$ is defined by $T'(f) = f \circ T$ ($f \in F'$). It is easily seen that $\|T'\| \leq \|T\|$ and that the diagram

$$
\begin{array}{ccc}
E & \xrightarrow{T} & F \\
\downarrow{j_E} & & \downarrow{j_F} \\
E'' & \xrightarrow{T''} & F''
\end{array}
$$

holds.
commutes. Here, of course, \( j_E, j_F \) are the natural maps.

\( E \) is said to have the property \( HB \) (Hahn-Banach) if for each linear subspace \( D \), for each \( f \in D' \) there is an extension \( \tilde{f} \in E' \) of \( f \) with \( \| \tilde{f} \| = \| f \| \). If \( K \) is spherically complete each normed space over \( K \) has \( HB \); if \( K \) is not spherically complete then the only spaces with \( HB \) are the finite dimensional spaces with an orthogonal base ([3], 4.8, 4.54, 5.15).

\( E \) is said to have the property \( HB^+ \) if, for each linear subspace \( D \), for each \( s > 0 \), each \( f \in D' \) there is an extension \( \tilde{f} \in E' \) of \( f \) with \( \| \tilde{f} \| < (1+\varepsilon)\| f \| \). Every space \( E \) of countable type has \( HB^+ \) ([3], 3.16); it is an open problem as to whether there exist spaces \( E \) over a non-spherically complete \( K \) not of countable type but having \( HB^+ \).

A normed space \( (E, \| \|) \) is normpolar if \( \| \| \) is polar (see 2.1). If the valuation of \( K \) is discrete \( E \) is normpolar if and only if \( \| E \| \subset |K| \). If the valuation of \( K \) is dense \( E \) has \( HB^+ \) if and only if \( E \) has \( HB^+ \) (see 2.1). Then \( E \) is normpolar if \( \| \| \) is polar (see 2.1).

\( (E, \| \|) \) is called strongly normpolar if for each closed subspace \( D \) the quotient \( E/D \) is normpolar. If the valuation of \( K \) is discrete \( E \) is strongly normpolar if and only if \( \| E \| \subset |K| \). If the valuation of \( K \) is dense \( E \) is strongly normpolar if and only if \( E \) has \( HB^+ \). Thus, if in addition \( K \) is spherically complete, every \( K \)-normed space is strongly normpolar.

2.4. A locally convex space \( E \) over \( K \) is strongly polar if each continuous seminorm is polar, polar if there exists a base of continuous polar seminorms. Set \( E' = \{ f \in E^*: f \text{ is continuous} \} \). The weak topology \( w = \sigma(E, E') \) is the weakest topology on \( E \) for which all \( f \in E' \) are continuous. Similarly, the weak star topology \( w' = \sigma(E', E) \) is the weakest topology on \( E' \) for which all evaluations \( f \mapsto f(x) \) \( (x \in E) \) are continuous. For a normed space \( E \) the set \( B_E \), equipped with the weak topology will sometimes be called the weak unit ball of \( E \). In the same spirit we have the weak star unit ball (or \( w' \)-unit ball) of \( E' \). A subset \( A \) of a locally convex space \( E \) over \( K \) is (a) polar (set) if for each \( x \in E \setminus A \) there exists an \( f \in E' \) such that \( |f(x)| > 1 \) and \( |f(A)| \leq 1 \).

A subset \( X \) of a locally convex space \( E \) is (a) compactoid if for every zero neighbourhood \( U \) in \( E \) there exists a finite set \( F \subset E \) such that \( X \subset U + \infty F \). Then the space \( [X] \) is of countable type ([6], 4.3). Recall that, on compactoids in a polar locally convex space, the weak topology and the initial topology coincide ([6], 5.12). The closure of a set \( Y \subset E \) is \( \overline{Y} \). Instead of \( \text{co}\ Y \) we write \( \overline{\text{co}}\ Y \).

2.5. We shall adopt the convention to say that a map \( f : X \to Y \), where \( X, Y \) are topological spaces, is a homeomorphism into if \( f \) maps \( X \) homeomorphically onto \( f(X) \), equipped with the relative topology.
3. THE WEAK STAR UNIT BALL OF $E'$

Proposition 3.1 ($p$-adic Alaoglu Theorem). Let $E$ be a $K$-Banach space. Then the $w'$-unit ball $B_{E'}$ is an absolutely convex, edged, complete compactoid.

Proof. The formula $\phi(f) = (f(x))_{x \in B_E}$ defines a $B_K$-module homomorphism $\phi : B_{E'} \rightarrow B_{B_E}^K$ which is a homeomorphism into. Thus, $B_{E'}$ is isomorphic to a subset of a compactoid hence is one itself. The proofs of $w'$-completeness, absolute convexity and edgedness are straightforward. □

The bounded weak star topology $bw'$ on a $K$-Banach space $E$ is the strongest locally convex topology on $E'$ coinciding with $w'$ on bounded subsets of $E'$. In other words, a (non-archimedean) seminorm $p$ on $E'$ is $bw'$-continuous if and only if $p|B_{E'}$ is $w'$-continuous.

Proposition 3.2. Let $E$ be a $K$-Banach space, let $j_E : E \rightarrow E''$ be the canonical map. Then we have

(i) The dual of $(E',w')$ is $j_E(E)$.

(ii) The dual of $(E',bw')$ is the norm closure of $j_E(E)$ in $E''$.

Proof. See [9], 3.3.

Corollary 1. For a normpolar space $E$ the dual of $(E',bw')$ is $j_E(E)$.

Proposition 3.3 ($p$-adic Golstine Theorem). Let $E$ be a normpolar $K$-Banach space. Then the map $j_E$ is a homeomorphism of $(E,w)$ into $(E'',w')$ whose image is dense. We even have $(j_E(B_E)''')^e = B_{E''}$. 

Proof. We only prove the last statement. Since $j_E$ is an isometry we have $j_E(B_E) \subset B_{E''}$ and the $p$-adic Alaoglu Theorem yields that even $A := (j_E(B_E)''')^e \subset B_{E''}$. $A$ is closed and edged hence ([6], 4.8) a polar set in the topology $w' = \sigma(E'',E')$. So if $\theta \in B_{E''}$, $\theta \notin A$ we could find an element $\Omega$ in the dual of $(E'',w')$ such that $|\Omega(\theta)| > 1$ and $|\Omega| \leq 1$ on $A$. But by Proposition 3.2(i) $\Omega$ has the form $\theta \mapsto \theta(f)$ for some $f \in E'$. We then have $|\theta(f)| > 1$ and $|j_E(B_E)(f)| \leq 1$ i.e. $|f| \leq 1$ on $B_E$ which, by polarity, just means $\|f\| \leq 1$. But then $|\theta(f)| \leq \|\theta\| \|f\| \leq 1$, a contradiction. □

Proposition 3.4. Let $E, F$ be normpolar $K$-Banach spaces, let $S \in \mathcal{L}(F', E')$, $\|S\| \leq 1$. Then the following are equivalent.

(a) $S$ is the adjoint of a $T \in \mathcal{L}(E, F')$, $\|T\| \leq 1$.

(b) $S$ is continuous $(F', w') \rightarrow (E', w')$.

(c) $S|B_{F'}$ is continuous $(B_{F'}, w') \rightarrow (B_{E'}, w')$.

(d) $S$ is continuous $(F', bw') \rightarrow (E', bw')$.
Proof. The implications \((\alpha) \implies (\beta) \implies (\gamma)\) are obvious. Suppose \((\gamma)\). If \(p\) is a \(bw'\)-continuous seminorm on \(E'\) then \(p \circ S\) is \(w'\)-continuous on \(B_{F'}\) so by definition \(p \circ S\) is \(bw'\)-continuous and \((\delta)\) follows. Finally we prove \((\delta) \implies (\alpha)\). For each \(x \in E\) we have \(S'(j_E(x)) = j_E(x) \circ S \in (F'', bw')'\) which equals \(j_F(F')\) by Corollary 3.3. We see that \(S'\) maps \(j_E(E)\) into \(j_F(F')\) so by polarity there exists a unique map \(T : E \to F\) (which is linear and continuous) making the diagram

\[
\begin{array}{ccc}
E & \xrightarrow{T} & F \\
\downarrow{j_E} & & \downarrow{j_F} \\
E'' & \xrightarrow{S''} & F''
\end{array}
\]

commute. We see that \(T'' = S'\) on \(j_E(E)\), hence on \(E''\) by \(w'\)-continuity and Proposition 3.4. Hence \((T' - S)' = 0\), so \(T' = S\) and we have \((\alpha)\) as also \(\|T\| = \|T''\| = \|S\| \leq 1\). □

4. The anti-equivalence

In this section we prove the core of this paper, namely that the theory of normpolar Banach spaces is equivalent to the theory of complete edged compactoids in locally convex spaces. First let us describe in 4.1 and 4.2 the categories involved more precisely.

4.1. The category \(B_K\) of Banach spaces. We shall denote by \(B_K\) the category of the normpolar \(K\)-Banach spaces where for \(E, F \in B_K\) we set

\[
\text{Hom}(E, F) := \{T \in \mathcal{L}(E, F) : \|T\| \leq 1\}.
\]

4.2. The category \(C_K\) of compactoids. This takes a little bit of preparation. To define the objects of \(C_K\) properly we shall ‘free’ an absolutely convex set from the space it is embedded in, which can be done as follows. Let \(A\) be a module over the ring \(B_K\), let \(\tau\) be a topology on \(A\). We call \(A = (A, \tau)\) a topological module if the module operations are continuous. Any absolutely convex subset of a topological vector space over \(K\) is, with the inherited topology, a topological module. Conversely, we will say that a topological module \((A, \tau)\) is embeddable if there exists a Hausdorff locally convex space \(E\) over \(K\) and a \(B_K\)-module homomorphism \(i : A \to E\) which is also a homeomorphism into. We say that \(A\) is \((a)\) compactoid if \(i(A)\) is \((a)\) compactoid, edged if \(i(A)\) is edged. (It is not hard to see that these definitions do not depend on the particular choices of \(E\) and \(i\); one even can formulate intrinsic equivalent definitions as follows. A topological \(B_K\)-module \((A, \tau)\) is a compactoid if for each \(\tau\)-neighbourhood \(U\) of 0 and each \(\lambda \in B_K^\times\) there exists a finitely generated submodule \(F\) of \(A\) such that \(\lambda A \subset U + F\). A \(B_K\)-module \(A\) is edged if each homomorphism \(B_K^\times \to A\) can be extended to a homomorphism \(B_K \to A\).) Now we are able to introduce the second category.
We shall denote by $\mathcal{C}_K$ the category of the embeddable, edged, complete compactoid topological $B_K$-modules where for $A, B \in \mathcal{C}_K$ we set

$$\text{Hom}(A, B) := \{ \varphi : A \to B : \varphi \text{ is a continuous homomorphism} \}.$$ 

**Note.** For a more sophisticated definition of $\mathcal{C}_K$ avoiding embeddings in locally convex spaces, see [11]. For practical reasons in this paper we prefer the introduction of $\mathcal{C}_K$ as given above.

**4.3. The functor $B_K \to \mathcal{C}_K$.** To each $E \in B_K$ we assign the weak star unit ball $(B_{E'}, w')$. We shall often write $B_{E'}$ rather than $(B_{E'}, w')$. By the $p$-adic Alaoglu Theorem 3.1 this $B_{E'}$ is in $\mathcal{C}_K$.

Now let $E, F \in B_K$ and $T \in \text{Hom}(E, F)$. By normpolarity for the adjoint $T' : F' \to E'$ we have $\|T'\| = \|T\|$ so that $T' \in \text{Hom}(F', E')$. The restriction $T^d$ of $T'$ to the unit ball $B_{F'}$ maps into $B_{E'}$. It is easily seen (Proposition 3.5 if you wish) that $T^d : B_{F'} \to B_{E'}$ is an element of $\text{Hom}(B_{F'}, B_{E'})$. One verifies without pain that

$$T \mapsto T^d \quad (T \in \text{Hom}(E, F), \ E, F \in B_K)$$

defines a contravariant functor $B_K \to \mathcal{C}_K$.

Now let's get busy.

**Proposition 4.1.** Let $E, F \in B_K$. The map $T \mapsto T^d$ is a bijection $\text{Hom}(E, F) \to \text{Hom}(B_{F'}, B_{E'})$.

**Proof.** We only need to inspect surjectivity. If $\varphi \in \text{Hom}(B_{F'}, B_{E'})$ then its unique linear extension $S : F' \to E'$ has norm $\leq 1$ and satisfies $(\gamma)$, hence $(\alpha)$ of Proposition 3.5. Thus $\varphi = T^d$ for some $T \in \text{Hom}(E, F)$.

**Proposition 4.2.** Every $A \in \mathcal{C}_K$ is isomorphic to $B_{E'}$ for some suitably chosen $E \in B_K$.

**Proof.** 1. Let $\tau$ be the topology of $A$. We may assume that $A$ is an absolutely convex edged complete compactoid subset of some Hausdorff locally convex space $(X, \nu)$. Define a locally convex topology $\tau_1$ on $Y := [A]$ by declaring a seminorm $p$ on $Y$ to be $\tau_1$-continuous as soon as $p|A$ is $\tau$-continuous. Then $\tau_1$ is stronger than $\nu|Y$ but $\tau_1 = \nu = \tau$ on $A$. We also know that $(Y, \tau_1)$ is of countable type (2.4).

2. Now let $E$ be the linear space $(Y, \tau_1)'$ equipped with the norm $f \mapsto \|f\| := \sup\{|f(x)| : x \in A\}$. Then, indeed, $\| \cdot \|$ is a polar norm. To prove completeness, let $f_1, f_2, \ldots$ be a Cauchy sequence in $E$. Then for each $y \in Y$

$$f(y) := \lim_{n \to \infty} f_n(y)$$
exists and is a linear function on $Y$. As $f_n \to f$ uniformly on $A$ we have that $f|A$ is continuous and so is its absolute value. By the very construction of $\tau_1$ the seminorm $|f|$ is $\tau_1$-continuous. It follows that $f \in E$ and that $\|f - f_n\| \to 0$. We conclude that $E \in B_K$. 

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3. Finally we show that \((A, \tau)\) is isomorphic to \((B_E', w')\). Consider the map 
\(j : Y \to E'\) defined by the formula 
\[ j(y)(f) = f(y) \quad (y \in Y, f \in E). \]
This \(j\) is clearly a homeomorphism of \((Y, w)\) into \((E', w')\). Since \(Y\) is a polar space we have \(w = \tau\) on \(A\) (2.4). Thus \(j\) maps \((A, \tau)\) homeomorphically into \((B_E', w')\) so that \(j(A)\) is edged and \(w'\)-complete hence \(w'\)-closed. If we had an \(\Omega \in B_{E'} \setminus j(A)\) there would exist, by strong polarity of the space \((E', w')\), a \(w'\)-continuous linear function on \(E'\) separating \(j(A)\) and \(\{\Omega\}\) i.e. we would have an \(f \in E\) with \(|f| \leq 1\) on \(A\) (which precisely means \(\|f\| \leq 1\) and \(|\Omega(f)| > 1\). But then also \(|\Omega(f)| \leq \|\Omega\| \|f\| \leq 1\), a contradiction. It follows that \(j|A\) is surjective. Thus, \(j|A\) is an isomorphism between the topological \(B_K\)-modules \(A\) and \(B_E'\). 

\(\square\)

**Remark 3.** Part 3 of the above proof can be shortened by using the \(p\)-adic Mackey Theorem ([6], 7.4).

**Remark 4.** We have the following immediate corollary to Proposition 4.5.

Let \(A\) be a complete absolutely convex compactoid in a Hausdorff locally convex space over \(K\). Then \((|A|, p_A)\) is the dual of some \(K\)-Banach space. (For \(p^A\) see 2.2.)

Combination of the previous two Propositions yields the following.

**Theorem 1.** The categories \(B_K\) (of all normpolar \(K\)-Banach spaces) and \(C_K\) (of all embeddable absolutely convex complete edged compactoids) are anti-equivalent by means of the functor \(B_K \to C_K\) given by

\[
\begin{align*}
E &\mapsto B_{E'} \quad (E \in B_K) \\
T &\mapsto T^d \quad (T \in \text{Hom}(E, F), E, F \in B_K)
\end{align*}
\]

where \(B_{E'}\) carries the restriction of the \(w'\)-topology of \(E'\) and where \(T^d := T'|B_{E'} \in \text{Hom}(B_{F'}, B_{E'})\).

**Notation.** Let \(E, F \in B_K\) and let \(\varphi \in \text{Hom}(B_{F'}, B_{E'})\). The unique \(T \in \text{Hom}(E, F)\) for which \(T^d = \varphi\) is denoted \(\varphi^d\). Then obviously \(T^{dd} = T\) and \(\varphi^{dd} = \varphi\) for all \(T \in \text{Hom}(E, F)\) and \(\varphi \in \text{Hom}(B_{F'}, B_{E'})\).

**Remark 5.** Given an \(A \in C_K\) we can quickly construct its corresponding object \(E\) in \(B_K\) as follows. Set \(E := \text{Hom}(A, K)\) (with that we mean, of course, the set of all continuous \(B_K\)-module homomorphisms \(A \to K\); it is in a natural way a linear space and \(f \mapsto \|f\|_A := \sup\{|f(x)| : x \in A\}\) makes \(E\) into a normpolar Banach space. From the proof of Proposition 4.5 it follows directly that the \(w'\)-unit ball of \(E'\) is isomorphic to \(A\).
Part Two

5. BANACH SPACES VERSUS COMPACTOIDS Applications

For compactoids, convexified forms of compact-like stability properties (see §1) are interesting. After translating we will find them to be equivalent to certain Hahn-Banach properties in Banach spaces (6.1, 7.1, 8.1, 9.3, 9.5) which will shed new light on the theory of compactoids. Also we will characterize those $B_{E'}$ for which $E$ is of countable type (6.2), or has an orthogonal base (11.1), or is reflexive (10.1).

6. DECOMPOSITIONS IN $B_K$ AND $C_K$

For $E,F \in B_K$ and $T \in \text{Hom}(E,F)$ we have an obvious decomposition

$$T : E \xrightarrow{T} F \quad \quad T_1 \vee \quad T_2$$

into a map $T_1 \in \text{Hom}(E,TE)$ whose image is dense and an isometry $T_2 \in \text{Hom}(TE,F)$. (We have avoided the usual further decomposition of $T_1$ into a quotient map $E \rightarrow E/\ker T$ and a dense injection $E/\ker T \rightarrow TE$ because $E/\ker T$ is, in general, not in $B_K$.) We now look into its dual in $C_K$. Recall that $T^d$ is the restriction of $T^* : F'^* \rightarrow E'^*$ to the weak star unit ball $B_{E'}$.

**Theorem 2.** Let $E,F \in B_K$, let $T \in \text{Hom}(E,F)$. Then

(i) $TE$ is norm dense in $F$ if and only if $T^d$ is a ($w'$-) homeomorphism of $B_{E'}$ into $B_{E'}$,

(ii) $T$ is an isometry if and only if $(T^d(B_{E'}))^e = B_{E'}$, (the bar denoting the $w'$-closure).

**Proof.** (i) Suppose $TE$ is norm dense in $F$ and let $i \mapsto g_i$ be a net in $B_{E'}$ such that $T^d g_i \xrightarrow{w'} 0$; we prove that $g_i \xrightarrow{w'} 0$. We have $g_i \circ T \xrightarrow{w'} 0$ i.e., $g_i \rightarrow 0$ pointwise on $TE$. Let $y \in F$. By assumption there exist $y_1, y_2, \ldots \in TE$ such that $\|y-y_n\| \rightarrow 0$. From $|g_i(y)| \leq \|g_i\| \|y-y_n\| \vee |g_i(y_n)| \leq \|y-y_n\| \vee |g_i(y_n)|$ for all $i,n$ one infers easily that $g_i(y) \rightarrow 0$. Hence $g_i \xrightarrow{w'} 0$.

Conversely, let $T^d$ be a homeomorphism into and suppose we had an $y \in F \setminus TE$. Then $y$ is $s$-orthogonal to $TE$ for some $s > 0$. Choose a $\xi \in K$, $0 < |\xi| \leq \frac{1}{2}s\|a\|$. For each finite dimensional subspace $D$ of $TE$ the map

$$\lambda y + d \mapsto \lambda \xi \quad (\lambda \in K, \ d \in D)$$

has norm $\leq \frac{1}{2}$ on $Ky+D$ and, by normpolarity, can be extended to a $g_D \in B_{E'}$. These $g_D$ form a net in a natural way. We have $T^d(g_D) = g_D \circ T \xrightarrow{w'} 0$, so, by assumption, $g_D \xrightarrow{w'} 0$ which is a contradiction as $|g_D(y)| = |\xi| > 0$ for all $D$. 

(ii) Suppose $T$ is an isometry. It is not hard to see that $(T^d(B_{F'}))^e \subset B_{E'}$. If this inclusion were strict we would have an $f \in B_{F'}$ that can be separated from $(T^d(B_{F'}))^e$ by a $w'$-continuous linear function i.e. there is an $x \in E$ with $|f(x)| > 1$ and $|T^d(B_{F'})(x)| \leq 1$. The first inequality entails $1 < |f(x)| \leq \|f\| \|x\| \leq \|x\|$ whereas the second one yields $|g(Tx)| \leq 1$ for all $g \in B_{F'}$ i.e., $\|Tx\| \leq 1$. Hence $\|Tx\| < \|a\|$, a contradiction.

Conversely, suppose $(T^d(B_{F'}))^e = B_{E'}$. Let $x \in E$; we prove that $\|Tx\| \geq \|x\|$. Let $f \in T^d(B_{F'})$. Then, with $g \in B_{F'}$ such that $f = Tdg$,

$$|f(x)| = |g(Tx)| \leq \|Tx\|.$$  

This inequality then also holds for all $f \in T^d(B_{F'})$, all $f \in (T^d(B_{F'}))^e = B_{E'}$. Then, by normpolarity,

$$\|x\| = \text{sup}\{|f(x)| : f \in B_{E'}\} \leq \|Tx\|. \quad \square$$

Remark 6. We see that the decomposition $T = T_2 \circ T_1$ ($T_1$ dense, $T_2$ isometry) leads to a similar decomposition $T^d = T_1^d \circ T_2^d$ in $C_K$ because $T_2^d$ is "dense" in the sense that the smallest closed edged submodule of $B_{(TE)'y}$ containing $T_2^d(B_{F'})$ is $B_{(TE)'y}$, and $T_1^d$ is an embedding.

7. Stability of compactoids under injections

Let us say that a compactoid $A \in C_K$ is monocompact if, for each $B \in C_K$, every injective $\varphi \in \text{Hom}(A, B)$ is a homeomorphism into. In other words (Theorem 4.6), $A$ is monocompact if for every Hausdorff locally convex ‘overspace’ $E \supset A$, for each locally convex space $X$, for each injective continuous linear map $S : E \to X$ its restriction $A \to SA$ is a homeomorphism.

It is known ([1], [4]) that, if $K$ is spherically complete, every $A \in C_K$ is monocompact. This also follows from the next general criterion for monocompactness.

Proposition 7.1. A compactoid in $C_K$ is monocompact if and only if it is isomorphic to the weak star unit ball of the dual of a Banach space $E \in B_K$ satisfying one of the following equivalent conditions.

(a) Each weakly dense subspace of $E$ is norm dense.

(b) For each (norm) closed proper subspace $D$ we have $(E/D)' \neq \{0\}$.

Proof. (a) implies monocompactness of $B_{F'}$: Let $\varphi \in \text{Hom}(B_{F'}, X)$ be injective where $X \in C_K$. Then $X$ has the form $B_{F'}$ for some $F' \in B_K$; let us denote $\varphi^d$ (see 4.7) by $T : F \to E$. Injectivity of $\varphi$ implies that $T\varphi^w = E$ (if $f \in B_{F'}$, $f = 0$ on $TF$ then $\varphi(f) = T^d(f) = f \circ T = 0$, hence $f = 0$). By assumption $TF$ is norm dense so that $\varphi$ is a homeomorphism into by Theorem 5.1.

Monocompactness of $B_{F'}$ implies (a): Suppose $F$ is a weakly dense but norm closed subspace of $E$ with inclusion map $T : F \to E$. It follows easily that $T'$,
hence $T^d : B_E \to B_{E'}$ is injective. By assumption $T^d$ is a homeomorphism into, by
Theorem 5.1 $F$ is norm dense in $E$ i.e., $F = E$.

$(\alpha) \implies (\beta)$. A norm closed proper subspace $D$ is not weakly dense so $D^w \neq E$.
Then there exists a nonzero $f \in E'$ that is zero on $D^w$ hence on $D$. We see that
$(E/D)' \neq \{0\}$.

$(\beta) \implies (\alpha)$. If $D$ is weakly dense, with norm closure $\overline{D}$ then $f \in E'$, $f = 0$ on $\overline{D}$ implies $f = 0$ so that $(E/\overline{D})' = \{0\}$. By $(\beta)$ $\overline{D}$ is not proper i.e. $\overline{D} = E$ so $D$ is
norm dense. □

Remark 7. Let $K$ be not spherically complete. Although the properties $(\alpha)$, $(\beta)$
seem to be considerably weaker than normpolarity or strong normpolarity or even
strong polarity of $E$ (see 2.3) we do not know if they are equivalent, neither do we
know if one of them implies that $E$ is of countable type. The last property is reflected
in $C_K$ in the following way.

Proposition 7.2. (Compare [7], 5.1.) For a normpolar $K$-Banach space $E$ the fol­
lowing are equivalent.

$(\alpha)$ $E$ is of countable type.

$(\beta)$ The weak star unit ball of $E'$ is metrizable.

Proof. $(\alpha) \implies (\beta)$. Choose $e_1, e_2, \ldots$ in $B_E$ whose linear span is dense in $E$.
The formula $\phi(f) = \left(f(e_1), f(e_2), \ldots\right)$ defines a homomorphism $B_E \to B_K^N$ which
is easily seen to be a homeomorphism of $B_E$ into the product space $B_K^N$ which is
metrizable hence so are $\varphi(B_E)$ and $B_E$.

$(\beta) \implies (\alpha)$. Let $\lambda \in K$, $|\lambda| > 1$. By metrizability there exist ([6], 8.2)
$f_1, f_2, \ldots \in \lambda B_E^*$ with $w' - \lim_{n \to 0} f_n = 0$ such that

$(*)$ $B_E \subset \overline{\varnothing\{f_1, f_2, \ldots\}} \subset \lambda B_E^*$.

The formula $\phi(x) = \left(f_1(x), f_2(x), \ldots\right)$ defines a $K$-linear map $E \to c_0$. For each
$x \in E$ we have

$$||\phi(x)|| = \sup_n |f_n(x)| = \sup\{|g(x)| : g \in \varnothing\{f_1, f_2, \ldots\}\}$$

so that by normpolarity and $(*)$

$$||x|| \leq ||\phi(x)|| \leq |\lambda||x||.$$ 

Thus, $E$ is linearly homeomorphic to a subspace of $c_0$ and therefore of countable
type. □

This leads to the following corollary already proved in [4] in a different way.

Corollary 2. A metrizable compactoid in $C_K$ is monocompact.
Proof. Spaces of countable type are strongly polar ([6], 4.4). Now apply Propositions 6.2 and 6.3. □

Problem. Let \( K \) be not spherically complete. Do there exist nonmetrizable monocompact sets in \( C_K \)? (This is a restatement of the last problem mentioned in Remark 1.)

8. Stability of compactoids under (almost) surjections

Let us say that a compactoid \( A \in C_K \) is epicompact if for each \( B \in B_K \) and \( \varphi \in \text{Hom}(A, B) \) the module \( \varphi(A)^e \) is closed (or complete). If, in addition, we have \( \varphi(A)^e = \varphi(A) \) we call \( A \) strictly epicompact. Like monocompactness these notions can, with the help of Theorem 4.6, be expressed in terms of absolute convexity in locally convex spaces as follows. An \( A \in C_K \) is epicompact (strictly epicompact) if for every Hausdorff locally convex ‘overspace’ \( E \supseteq A \), for each locally convex space \( X \), for each continuous linear map \( S : E \to X \), the set \( S(A)^e \) is complete (the set \( S(A) \) is complete and edged).

It is known that, if \( K \) is spherically complete, every \( A \in C_K \) is strictly epicompact (if \( B \in B_K \) and \( \varphi \in \text{Hom}(A, B) \) then \( \varphi(A) \) is c-compact hence complete. A direct proof ([11]) can be given of \( \varphi(A)^e = \varphi(A) \)). If \( K \) is not spherically complete it is proved by Van Rooij in [2], 6.28 that if a metrizable \( A \in C_K \) is strictly epicompact then \( \text{dim} \ A < \infty \). (It is not hard to extend this result to all edged complete absolutely convex compactoids, see Corollary 7.4.) Fortunately, epicompactness is less stringent. In fact, it is shown in [8] that metrizable \( A \in C_K \) are epicompact. This will also follow from the next general criterion for epicompactness.

Proposition 8.1. (See also Theorem 8.2). A compactoid in \( C_K \) is epicompact if and only if it is isomorphic to the weak star unit ball of the dual of a Banach space \( E \in B_K \) satisfying one of the following equivalent conditions (see 2.3).

\[
\begin{align*}
(\alpha) & \ E \text{ is strongly normpolar.} \\
(\beta) & \ E \text{ has HB}^+. \\
(\gamma) & \text{For each closed subspace } D \text{ the adjoint } E' \to D' \text{ of the inclusion map } D \hookrightarrow E \text{ is a quotient map.} \\
(\delta) & \text{For each closed subspace } D, \text{ for each } f \in D' \text{ with } \|f\| < 1 \text{ there is an extension } \tilde{f} \in E' \text{ of } f \text{ with } \|\tilde{f}\| < 1.
\end{align*}
\]

Proof. We leave the proof of the equivalences of (\( \alpha \)) – (\( \delta \)) to the reader. Now suppose \( B_{E'} \) is epicompact; we prove (\( \delta \)). Let \( T : D \to E \) be the inclusion map. By Theorem 5.1, \( (T_d^b B_{E'})^e = B_{D'} \); epicompactness implies \( (T_d^b B_{E'})^e = B_{D'} \). But this implies \( T_d^b B_{E'} = B_{D'} \) which is (\( \delta \)). Conversely, suppose (\( \delta \)) and let \( \varphi \in \text{Hom}(B_{E'}, X) \) where \( X \in C_K \). Then \( Y := (\varphi(B_{E'}))^e \) is in \( C_K \) and has the form \( B_{D'} \) where \( D \in B_K \). Let \( T : D \to E \) be \( \varphi^d \) (where \( \varphi \) is considered as a map \( B_{E'} \to B_{D'} \)). By Theorem 5.1 \( T \)
is an isometry and so, by ($\delta$), $T^d(B_{E'}) = B_{D'}$ implying $\varphi(B_{E'}) = T^d(B_{E'}) = B_{D'}$. We see that $B_{E'}$ is epicompact. □

**Corollary 3.**

(i) A metrizable $A \in \mathcal{C}_K$ is epicompact.

(ii) An epicompact $A \in \mathcal{C}_K$ is monocompact.

**Problem.** Let $K$ be not spherically complete. Do there exist nonmetrizable epicompact $A \in \mathcal{C}_K$? (This problem is of course equivalent to: Is every strongly normpolar space of countable type?)

By making the obvious modifications in the proof of Proposition 7.1 one arrives easily at the following criterion for strict epicompactness.

**Proposition 8.2.** A compactoid in $\mathcal{C}_K$ is strictly epicompact if and only if it is isomorphic to the weak star unit ball of $E'$ where $E \in \mathcal{B}_K$ satisfies the following condition.

($\delta'$) For each closed subspace $D$, for each $f \in D'$ with $\|f\| \leq 1$ there is an extension $\tilde{f} \in E'$ of $f$ with $\|\tilde{f}\| \leq 1$.

**Corollary 4.** If $K$ is spherically complete every $A \in \mathcal{C}_K$ is strictly epicompact. If $K$ is not spherically complete any strictly epicompact $A \in \mathcal{C}_K$ is finite dimensional.

**Proof.** We only have to consider the second statement. We shall prove that a Banach space $E \in \mathcal{B}_K$ with property ($\delta'$) of Proposition 7.3 is finite dimensional. To this end first observe that ($\delta'$) is stable for the forming of closed subspaces and quotients by closed subspaces. Thus, if there exists an infinite dimensional $E \in \mathcal{B}_K$ with ($\delta'$) then we may assume that $E$ is of countable type. By [2], 3.1, every space of countable type, in particular the space $K_2^2$ of [3], p.68, is a quotient of $E$ implying that $K_2^2$ has ($\delta'$), a contradiction. □

**Problem.** Let $K$ be not spherically complete. Which finite dimensional edged compactoids are strictly epicompact? (Equivalently, which finite dimensional $K$-Banach spaces $E$ have the extension property ($\delta'$) of Proposition 7.3.?)

9. **Almost openness of almost surjections between compactoids**

Let $A, B \in \mathcal{C}_K$, let $\varphi \in \text{Hom}(A, B)$. If $\varphi(A) \in \mathcal{C}_K$ it is natural to ask whether $\varphi : A \to \varphi(A)$ is an open mapping. The example of §1 has showed us that the answer is "no" in general. A further complication is that, if $K$ is not spherically complete, $\varphi(A)$ may be neither complete nor edged. In this section we show that we can obtain satisfactory results by relaxing the condition of openness to 'for each open zero neighbourhood $U \subset A$ and each $\lambda \in B_K$ the set $\varphi(U) \cap \lambda \varphi(A)$ is open in $\lambda \varphi(A)$'. In §10 we shall consider 'ordinary' openness of $\varphi$ in case $K$ is spherically complete once more by formulating its dual property for $\varphi^d$ (9.5).

Let $A, B \in \mathcal{C}_K$, let $\varphi \in \text{Hom}(A, B)$. We shall say that $\varphi$ is almost pre-open if for each $\lambda \in B_K$ and each open neighbourhood $U$ of 0 in $A$ the set $\varphi(U) \cap \lambda \varphi(A)$
is open in $\lambda \varphi(A)$. We shall call $\varphi$ almost open if for each $\lambda \in B_K^-$ and each open neighbourhood $U$ of 0 in $A$ the set $\varphi(U) \cap \lambda B$ is open in $\lambda B$. (If in the above, we take $\lambda := 1$ we arrive at the definitions of 'pre-open' and 'open', see §9.) Obviously, almost openness implies almost pre-openness.

**Proposition 9.1.** Let $D$ be a closed subspace of a Banach space $E \in B_K$. Let $T : D \to E$ be the inclusion map with adjoint $T^d : B_{E'} \to B_{D'}$. Then

(i) If the valuation of $K$ is discrete then $T^d$ is surjective and open.

(ii) If $K$ is spherically complete then $T^d$ is surjective and almost open.

In general we have

(iii) $T^d$ is almost pre-open if and only if $E/D$ is normpolar (i.e. $E/D \in B_K$).

(iv) $T^d$ is almost open if and only if $E/D$ is normpolar and, for each $\epsilon > 0$, every $f \in D'$ can be extended to an $\tilde{f} \in E'$ such that $\|\tilde{f}\| \leq (1+\epsilon)\|f\|$.

**Proof.** We prove (iii) and (iv) providing a proof of (i) at the same time. Then we will be done as (ii) follows from (iv).

(iii) Suppose $T^d$ is almost pre-open. To prove that $E/D$ is normpolar we may assume that $K$ is not spherically complete hence that the valuation is dense. It suffices, for a given $a \in E\setminus D$ and $s \in (0,1)$, to establish an $f \in E'$, $f$ zero on $D$, such that

$$|f(a)| > s\|f\| \operatorname{dist}(a,D).$$

To this end, choose $t \in (0,1)$ such that $s < t^3 < 1$. Without harm, assume $\operatorname{dist}(a,D) \geq t\|a\|$. Now

$$U := \{ g \in B_{E'} : |g(a)| \leq t^3\|a\| \}$$

is a $w'$-neighbourhood of 0 in $B_{E'}$. Choose any $\lambda \in B_K$ with $t < |\lambda| < 1$. By assumption $T^d(U) \cap \lambda T^d(B_{E'})$ is open in $\lambda T^d(B_{E'})$ so there is a finite set $X \subset D$ such that

$$(*) \quad \{ \theta \in T^d(B_{E'}) : |\theta| \leq 1 \text{ on } X \} \cap \lambda T^d(B_{E'}) \subset T^d(U) \cap \lambda T^d(B_{E'}).$$

Now choose a $\mu \in K$ such that $t^3\|a\| < |\mu| < |\lambda|t^2\|a\|$, and consider the linear map $K\lambda + [X] \to K$ given by

$$\xi a + x \mapsto \xi \mu \quad (\xi \in K, \ x \in [X]).$$

Because $|\xi \mu| = |\mu| \|a\|^{-1} \|\xi a\| \leq t^{-1}|\mu| \|a\|^{-1} \|\xi a + x\|$ its norm is $\leq t^{-1}|\mu| \|a\|^{-1}$, so by normpolarity of $E$ it can be extended to an $h \in E'$ with $\|h\| \leq t^{-2}|\mu| \|a\|^{-1}$ which is $\leq |\lambda|$ so $T^d h \in \lambda T^d B_{E'}$. By construction, $T^d h = 0$ on $X$ so, by $(*)$ (with $\theta = T^d h$), $T^d h = T^d g$ where $g \in U$. Now set $f := h - g$. Then $f = 0$ on $D$. We have $|h(a)| = |\mu|$ while $|g(a)| \leq t^3\|a\| < |u|$. Hence, $|f(a)| = \max(|h(a)|, |g(a)|) = |\mu| > t^3\|a\| \geq t^3\|f\| \|a\| \geq s\|f\| \|a\| \geq s\|f\| \operatorname{dist}(a,D)$ and we are done.
Now suppose that $E/D$ is normpolar. First assume that the valuation of $K$ is dense. Let $\lambda \in B_K^-$, let $U$ be an open neighbourhood of 0 in $B_{E'}$; we shall prove the existence of a finite set $X \subset D$ such that

\[ T^d(U) \supset \{ f \in \lambda T^d(B_{E'}) : |f| \leq 1 \text{ on } X \}. \]

There is a finite set $Y \subset E$ such that $U \supset \{ g \in B_{E'} : |g| \leq 1 \text{ on } Y \}$. Choose a $t \in (0,1)$ with $t^{-2} |\lambda| \leq 1$ and let $P$ be a projection of $[Y] + D$ onto $D$ with norm $\leq t^{-1}$. Let $X := PY$, let also $Z := (I-P)Y$. Then certainly

\[ \{ g \in B_{E'} : |g| \leq 1 \text{ on } X, |g| \leq 1 \text{ on } Z \} \subset U. \]

We now prove (**) . Let $f \in \lambda T^d(B_{E'})$, $|f| \leq 1$ on $X$. Then $f = T^d g$ where $g \in E'$, $\|g\| \leq |\lambda|$. The map

\[ [Y] + D \xrightarrow{I-P} [Z] \xrightarrow{g} K \]

is a linear function on $[Y] + D$, zero on $D$, and has norm $\leq \|I-P\| \|g\| \leq t^{-1} |\lambda|$ so, using normpolarity of $E/D$, we can extend it to an $h \in E'$ with $\|h\| \leq t^{-2} |\lambda| \leq 1$.

Now set $\tilde{f} := g - h$. Then $\tilde{f} \in B_{E'}$. By construction, $g = h$ on $Z$ so $\tilde{f} = 0$ on $Z$. Also $X \subset D$, $h = 0$ on $D$ so $\tilde{f} = g = f$ on $X$ and hence $|\tilde{f}| \leq 1$ on $X$. We see that $\tilde{f} \in U$ and $f = T^d g = T^d \tilde{f} \in T^d(U)$ and (**) is proved. If the valuation of $K$ is discrete we can repeat the above proof for $A = 1$, $t = 1$ (the existence of the projection $P$ with norm $\leq 1$ follows from $\|E\| \leq |K|$), where $h$ is chosen such that $\|h\| = \|g(I-P)\|$ (with the Hahn-Banach Theorem). The conclusion is that $T^d$ is pre-open rather than almost pre-open!

(iv) Let $T^d$ be almost open. Normpolarity of $E/D$ follows from (iii). To prove the extension property we may assume that $K$ is not spherically complete hence that the valuation of $K$ is dense. Let $\lambda \in B_K^-$, $\lambda \neq 0$. Then $T^d B_{E'} \cap \lambda B_{D'}$ is open in $\lambda B_{D'}$ hence closed. Then $\lambda^{-1} T^d B_{E'} \cap B_{D'} = \lambda^{-1} (T^d B_{E'} \cap \lambda B_{D'})$ is closed. By taking the intersection over all nonzero $\lambda \in B_K^-$ we find $(T^d B_{E'})^e = (T^d B_{E'})^e \cap B_{D'}$ is closed. But by Theorem 5.1 (ii), $(T^d B_{E'})^e = B_{D'}$ so that $B_{D'} = (T^d B_{E'})^e$ implying that $T^d : E' \to D'$ is a quotient map.

Conversely, suppose $E/D$ is normpolar and $T^d : E' \to D'$ is a quotient map. If the valuation of $K$ is discrete we have pre-openness of $T^d$ by the proof of (iii). The Hahn-Banach Theorem yields $T^d B_{E'} = B_{D'}$ and $T^d$ is an open mapping which proves (i). If the valuation of $K$ is dense we have by assumption $T^d B_{E'} \supset B_{D'}$. Let $\lambda \in B_K^-$ and choose $\mu \in B_K^-$ with $|\lambda| < |\mu|$. Let $U$ be an open zero neighbourhood in $B_{E'}$. By (iii) the set $T^d U \cap \mu T^d B_{E'}$ is open in $\mu T^d B_{E'}$. After taking the intersections with $\lambda B_{D'}$ we find that $T^d U \cap \lambda B_{D'}$ is open in $\lambda B_{D'}$ and we are done. \( \square \)

**Theorem 3.** (See also Proposition 7.1.) For a compactoid $A \in C_K$ the following are equivalent.

(\( \alpha \)) $A$ is epicom pact.

(\( \beta \)) If $X \in C_K$, $\varphi \in Hom(A, X)$ then $\varphi$ is almost pre-open.
(γ) If $B \in \mathcal{C}_K$, $\varphi \in \text{Hom}(A, B)$, $\overline{\varphi(A)^e} = B$ then $\varphi$ is almost open.

**Proof.** $(\alpha) \implies (\gamma)$. By Proposition 7.1 $A = B_E$ when $E \in \mathcal{B}_K$ is strongly normpolar and, equivalently, has $HB^\dagger$. By Proposition 8.1 (iv) for each linear isometry $T : D \leftrightarrow E$ the adjoint $T^d$ is almost open. But the map $\varphi$ in (γ) is such an adjoint by Theorem 5.1.

$(\gamma) \implies (\beta)$ is obvious. To show $(\beta) \implies (\alpha)$ let $A = B_E$ where $E \in \mathcal{B}_K$. Proposition 8.1 (iii) yields that $E$ is strongly normpolar which implies epicompactness by Proposition 7.1. □

**Remark 8.** If $K$ is spherically complete $(\alpha)$, $(\beta)$, $(\gamma)$ are true for each $A \in \mathcal{C}_K$. If, in addition, the valuation of $K$ is discrete we may drop the word 'almost' in $(\beta)$ and in $(\gamma)$.

10. **Which surjections between compactoids are open?**

From the last remark of the previous section we infer that if the valuation of $K$ is discrete every continuous surjection in $\mathcal{C}_K$ is open. So in this section we only have to worry about dense valuations.

**Lemma 1.** Let $D$ be a closed subspace of a Banach space $E \in \mathcal{B}_K$ with inclusion map $T : D \to E$. Suppose there exists an $r \in (0, 1]$ such that, with $M$ equal to either $B_E(0, r)$ or $\overline{B_E(0, r)}$, the restriction of $T^d$ is an open mapping $M \to T^d M$. Then for each $a \in E$ there exists a finite dimensional subspace $F$ of $D$ such that $\text{dist}(a, D) = \text{dist}(a, F)$.

**Proof.** I. Suppose the conclusion were false; we shall derive a contradiction. We have an $a \in E$ for which $\text{dist}(a, D) < \text{dist}(a, F)$ for each finite dimensional space $F \subset D$. Choose $\varepsilon := r \text{ dist}(a, D)$. The valuation of $K$ is dense.

II. Consider the obvious decomposition

\[
M \xrightarrow{T^d} T^d M
\]

\[
\pi \downarrow \quad \nearrow \varphi
\]

\[
S := M/(\text{Ker } T^d) \cap M
\]

where the $B_K$-module $S$ carries the quotient topology. Then $\varphi$ is a homomorphism by assumption. The seminorm $f \mapsto |f(a)|$ is $w'$-continuous on $E'$ hence on $M$. Therefore the quotient 'seminorm' $q$ on $S$ defined by

\[
q : \pi(f) \mapsto \inf \{|f(a) - h(a)| : h \in \text{Ker } \pi\}
\]

is continuous. Then $q \circ \varphi^{-1}$ is continuous on $T^d M$ so that \{ $h \in T^d M : (q \circ \varphi^{-1})(h) < \varepsilon$ \} is an open zero neighbourhood and therefore is the intersection of $T^d M$ and a $w'$-zero neighbourhood in $D'$. In other words, there exists a finite set $X \subset D$ such that

\[(*) \quad h \in T^d M, \ |h| \leq 1 \text{ on } X \to (q \circ \varphi^{-1})(h) < \varepsilon.
\]
III. By the assumption of I there is a $c \in K$ with

$$\text{dist}(a, D) < |c| < \text{dist}(a, [X]).$$

The map $\lambda a + v \mapsto \lambda c$ ($\lambda \in K$, $v \in [X]$) is easily seen to be an element of $(Ka + [X])'$ with norm $\leq |c|/\text{dist}(a, [X]) < 1$. By normpolarity it can be extended to an $f \in E'$ whose norm is $< 1$. Choose a $\xi \in K$ with $|\xi| \geq r$, $\|\xi f\| < r$. Then $\xi f \in M$, $T^d(\xi f) = 0$ on $X$ so from (*) we obtain $(q \circ \varphi^{-1})T^d(\xi f) < \varepsilon$ i.e. $q(\pi(\xi f)) < \varepsilon$ (see diagram), so there exists an $h \in \text{Ker} \pi$ with $|\xi f(a) - h(a)| < \varepsilon$. Now we have $|\xi f(a)| = |\xi||c| > r \text{ dist}(a, D) = \varepsilon$ while $h \in \text{Ker} \pi$ implies $h \in M$, so $\|h\| \leq r$ and $h = 0$ on $D$. Then $|h(a)| = \inf\{|h(a-d)| : d \in D\} \leq r \text{ dist}(a, D) = \varepsilon$. Then $|\xi f(a) - h(a)| = \max(|\xi f(a)|, |h(a)|) = \varepsilon$, a contradiction. □

**Lemma 2.** Let $E$ be a Banach space over a densely valued field $K$ with the following property (*).

\[
(*) \quad \text{For each } a \in E \text{ and each closed subspace } D \subset E \text{ there is a finite}
\text{dimensional space } F \subset D \text{ such that } \text{dist}(a, D) = \text{dist}(a, F).
\]

Then $E$ is finite dimensional.

**Proof.** Direct verifications show that (*) is stable for the forming of closed subspaces and quotients by closed subspaces. So suppose there exists an infinite dimensional space $E$ with (*). Then there exists one of countable type $E_1$. Now $c_0$ is a quotient of $E_1$ ([2], 3.1) hence $c_0$ has (*). But every finite dimensional subspace of $c_0$ has an orthogonal base so, for every finite dimensional subspace $F$ of $c_0 \text{ dist}(a, F)$ is attained for every $a \in c_0$. Then (*) implies that $\text{dist}(a, D)$ is attained for every closed subspace $D$ of $c_0$. But this is impossible (let $D$ be the closed linear span of a maximal orthogonal set that is not an orthogonal base, see [3], p.73). □

The next corollary obtains.

**Theorem 4.** Let the valuation of $K$ be dense. For an $A \in \mathcal{C}_K$ the following are equivalent.

($\alpha$) For all $B \in \mathcal{C}_K$ and $\varphi \in \text{Hom}(A, B)$, $\varphi$ is an open mapping $A \to \varphi(A)$.

($\beta$) $A$ is finite dimensional.

**Proof.** We only prove ($\alpha$) $\implies$ ($\beta$). Let $A = B_{E'}$ where $E \in \mathcal{B}_K$, let $D \subset E$ be a closed subspace with inclusion map $T$. Then by ($\alpha$) the map $T^d : B_{E'} \to T^dB_{E'}$ is open. By Lemmas 9.1 and 9.2 $E$ is finite dimensional hence so is $E'$ and is $A$. □

For spherically complete $K$ we have $\varphi(A) \in \mathcal{C}_K$ in ($\alpha$) so we can formulate

**Corollary 5.** (Compare Example 1.1.) Let $K$ be spherically complete with a dense valuation. Then for each infinite dimensional $A \in \mathcal{C}_K$ there exists a $B \in \mathcal{C}_K$ and a surjective continuous homomorphism $A \to B$ that is not an open mapping.
Also we have the following sharper version of Lemma 9.1.

**Theorem 5.** Let $K$ be spherically complete. Let $E \in \mathcal{B}_K$ and let $D$ be a closed subspace with inclusion map $T : D \to E$. Then $T^d : B_{E'} \to B_{D'}$ is open if and only if the quotient map $\pi : E \to E/D$ is strict.

**Proof.** Suppose $T^d$ is open. Let $a \in E$. From Lemma 9.1 $\text{dist}(a,D) = \text{dist}(a,F)$ for some finite dimensional $F \subset D$. Now $F$ is spherically complete so $\text{dist}(a,F)$ is attained. Hence $\pi$ is strict.

Conversely, suppose $\pi : E \to E/D$ is strict.

I. We first prove that for each subspace $D_1 \supset D$ such that $\dim D_1/D < \infty$, $D$ has an orthocomplement in $D_1$. In fact, $\pi(D_1)$ is finite dimensional so it has an orthogonal base $z_1, \ldots, z_n$. By strictness there are $e_1, \ldots, e_n \in D_1$ with $\pi(e_i) = z_i$, $\|e_i\| = \|z_i\|$ for each $i \in \{1, \ldots, n\}$. One proves easily that $F := [e_1, \ldots, e_n]$ is an orthocomplement of $D$ in $D_1$.

II. Let $U \subset B_{E'}$ be an open zero neighbourhood. Then there is a finite set $Z \subset E$ such that $U \supset \{f \in B_{E'} : |f(Z)| \leq 1\}$. By I there is a projection $P : [Z] + D \to D$ such that $\|P\| = 1$. Set $X := PZ$, $Y := (1-P)Z$. Then $X \subset D$; we prove that $T^d U \supset \{f \in B_{D'} : |f(X)| \leq 1\}$. In fact, let $f \in B_{D'}$, $|f(X)| \leq 1$. Then $f \circ P \in ([Z] + D)'$, $\|f \circ P\| = \|f\|$. This $f \circ P$ extends to an $\tilde{f} \in E'$ with $\|\tilde{f}\| = \|f \circ P\| = \|f\| \leq 1$. Then, $T^d(\tilde{f}) = f$. Further, $|\tilde{f}(X)| = |f(X)| \leq 1$ and $\tilde{f}(Y) = 0$ hence $|\tilde{f}| \leq 1$ on $X + Y \supset Z$. Then $\tilde{f} \in U$ and we are done. □

**Remark 9.** By using the ‘full’ Lemma 9.1 one easily obtains the following more refined version of Theorem 9.5. With $K, E, D, T, \pi$ as in 9.5, the following assertions are equivalent.

1. There exists an $r \in (0,1]$ such that $T^d$ is an open map $B_{E'}(0,r) \to B_{D'}(0,r)$.
2. There exists an $r \in (0,1]$ such that $T^d$ is an open map $B_{E'}(0,r) \to B_{D'}^-(0,r)$.
3. For all $r \in (0,1]$, $T^d : B_{E'}(0,r) \to B_{D'}(0,r)$ and $T^d : B_{E'}^-(0,r) \to B_{D'}^-(0,r)$ are open.
4. $\pi : E \to E/D$ is strict.
5. $D$ has an orthocomplement in every finite linear extension $D_1 \subset E$ of $D$.

**Problem.** For non-spherically complete $K$ and $E \in \mathcal{B}_K$, characterize the closed subspaces $D$ of $E$ for which $B_{E'} \to B_{D'}$ (the adjoint of the inclusion map $D \hookrightarrow E$, restricted to the unit ball) is surjective and open.

**11. Reflexivity**

We prove that reflexivity of $E$ is equivalent to automatic continuity of homomorphisms with domain $B_{E'}$:

**Theorem 6.** For a compactoid $A \in \mathcal{C}_K$ the following are equivalent.

1. For every $B \in \mathcal{C}_K$ each module homomorphism $A \to B$ is continuous.
(β) For every absolutely convex compactoid subset B of a Hausdorff locally convex space over K each module homomorphism $A \rightarrow B$ is continuous.

(γ) There is no strictly stronger topology on A for which it is an embeddable compactoid.

(δ) A is isomorphic to the weak star unit ball of the dual of a reflexive space.

(ε) A is isomorphic to the weak unit ball of a reflexive space.

Proof. (α) $\iff$ (β) and (δ) $\iff$ (ε) are almost obvious. We prove (β) $\implies$ (γ) $\implies$ (δ) $\implies$ (α). Suppose (β) and let τ be a stronger topology on A for which $(A, \tau)$ is an embeddable compactoid. Then the identity $A \rightarrow (A, \tau)$ is continuous by (β), hence τ is also weaker than the initial topology and we have (γ). Assume (γ). Let $A = B_{E'}$ for some Banach space $E \in \mathcal{B}_K$. The identity $(B_{E'}, \sigma(E', E') \mid B_{E'}) \rightarrow (B_{E'}, \sigma(E', E) \mid B_{E'})$ is continuous hence a homeomorphism by (γ) so for each $\theta \in E''$ the set Ker $\theta \cap B_{E'}$ is $w'$-closed. Then Ker $\theta$ is $w'$-closed by [9], 3.1 and $\theta$ is $w'$-continuous. We see that $E'' = j_E(E)$; i.e. $E$ is reflexive and (δ) is proved. Finally to show (δ) $\implies$ (α), let $A = B_{E'}, B = B_{X'}$ where $E, X \in \mathcal{B}_K$, let $\varphi : B_{E'} \rightarrow B_{X'}$ be any $B_K$-module homeomorphism. Let $i \mapsto f_i$ be a net in $B_{E'}$ such that $f_i \rightarrow 0$ in $w'$. Then, by reflexivity, $f_i \rightarrow 0$ weakly. Now $\varphi$ is norm continuous hence weakly continuous so $\varphi(f_i) \rightarrow 0$ weakly. Then certainly $\varphi(f_i) \rightarrow 0$ in $w'$. We see that $\varphi$ is $w'$-continuous yielding (α). □

Corollary 6. (i) Let $K$ be spherically complete. Then the only members $A$ of $\mathcal{C}_K$ satisfying (α)-(ε) are the finite dimensional ones.

(ii) Let $K$ be not spherically complete. Then every metrizable $A \in \mathcal{C}_K$ satisfies (α)-(ε).

Proof. Infinite dimensional $K$-Banach spaces are reflexive if and only if $K$ is not spherically complete [3]. □

12. Orthogonal Bases

We characterize the spaces in $\mathcal{B}_K$ with an orthogonal base in terms of the dual category $\mathcal{C}_K$.

Theorem 7. For a normpolar $K$-Banach space the following are equivalent.

(α) $E$ has an orthogonal base

(β) $B_{E'}$ is a product of edged bounded discs in $K$.

Proof. (α) $\implies$ (β). Let $\{e_i : i \in I\}$ be an orthogonal base of $E$. The formula

$$\phi(f) = \left(f(e_i)\right)_{i \in I}$$
defines a homomorphism $B_{E'} \rightarrow \Pi_{i \in I} C_i$ where $C_i := \{ \lambda \in K : |\lambda| \leq \|e_i\| \}$. It is easily seen to be a homeomorphism into. To prove surjectivity let $\eta_i \in C_i$ for each $i \in I$. The formula

$$f(\sum \lambda_i e_i) = \sum \lambda_i \eta_i \quad (\lambda_i \in K, \|\lambda_i e_i\| \rightarrow 0)$$

defines a linear map $E \rightarrow K$. We have, with $x = \sum \lambda_i e_i$,

$$|f(x)| \leq \sup_i |\lambda_i n_i| \leq \sup_i |\lambda_i| \|e_i\| = \|x\|.$$ 

It follows that $f \in B_{E'}$ and that $\phi(f) = (\eta_i)_{i \in I}$.

$(\beta) \implies (\alpha)$. Suppose $B_{E'} = \Pi_{i \in I} C_i$ where, for each $i \in I$, $C_i = \{ \lambda \in K : |\lambda| \leq r_i \}$. We identify $E$ to $\text{Hom}(V_{E'}, K)$ with the sup norm (Remark 3 following 4.7). For each $j \in I$ let $e_j \in E$ be the $j^{th}$ coordinate function $\Pi_{i \in I} C_i \rightarrow K$ (then $\|e_i\| = r_i$ for each $i$) and let $\pi_j : \Pi_{i \in I} C_i \rightarrow \Pi_{i \in I} C_i$ be the map defined by

$$(\pi_j(c))_i = \begin{cases} e_i & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

For each $x \in E$ there exist $\xi_i \in K$ such that $x \circ \pi_i = \xi_i e_i$ for each $i \in I$. Straightforward verifications show that $(e_i)_{i \in I}$ is orthogonal and that $x = \sum_{i \in I} \xi_i e_i$ with respect to the sup norm. Hence $E$ has an orthogonal base. □

**Corollary 7.** For a normpolar $K$-Banach space the following are equivalent.

$(\alpha)$ $E$ has an orthonormal base.

$(\beta)$ $B_{E'}$ is a power of $B_K$.

**Proof.** It suffices to read the previous proof for the special case where $\|e_i\| = 1$ and $\|r_i\| = 1$ for all $i$. □

**References**


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