A CONNECTION BETWEEN p-ADIC BANACH SPACES AND LOCALLY CONVEX COMPACTOIDS

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ABSTRACT. For a vector space $E$ over a non-archimedean valued field $K$ a correspondence $p \mapsto p^0$ is established between seminorms $p$ on $E$ and compactoids $p^0$ in $E^*$. Examination of it yields the solution of two open problems (see §4 and §8) and a reformulation of Serre's renorming problem (see §2). As a by-product results on metrizable compactoids are obtained (see §6).

§0 THE CORRESPONDENCE $p \mapsto p^0$.

Throughout this note $K$ is a non-archimedean valued field, complete with respect to the metric induced by the nontrivial valuation $|\cdot|$. Let $E$ be a $K$-vector space, let $E^*$ be its algebraic dual. A (non-archimedean) seminorm $p$ on $E$ is polar ([3], Definition 3.1), if

$$p = \sup \{|f| : f \in E^*, |f| \leq p\}$$

Let $P_E$ be the set of all polar seminorms on $E$.

For each $p \in P_E$ we set

$$p^0 = \{f \in E^* : |f| \leq p\}$$

Then $p^0$ is an absolutely convex, edged ([3], §1b) subset of $E^*$. It is easy to see that $p^0$ is a closed compactoid ([3], §1e) with respect to the topology $\sigma(E^*, E)$, hence complete.
Let $C_{E^*}$ be the set of all closed absolutely convex, edged compactoids in $E^*$ with respect to $\sigma(E^*,E)$.

**Proposition 0.** The map $p \mapsto p^0$ is a bijection of $P_E$ onto $C_{E^*}$. Its inverse assigns to every $A \in C_{E^*}$ the seminorm $p$ given by

$$p(x) = \sup \{|f(x)| : f \in A\} \quad (x \in E)$$

**Proof.** We shall prove surjectivity of $p \mapsto p^0$ leaving the (easy) rest of the proof to the reader. So, let $A \in C_{E^*}$; we shall prove that $A = p^0$ where $p(x) = \sup \{|f(x)| : f \in A\}$.

Obviously, $A \subset p^0$. Now let $g \in E^* \setminus A$, we prove that $g \notin p^0$. The space $E^*$ is of countable type hence strongly polar ([3], Theorem 4.4). So by [3], Theorem 4.7, there exists a $0 \in (E^*, \sigma(E^*,E))'$ such that $|0| \leq 1$ on $A$, $|0(g)| > 1$. But, by [3], lemma 7.1, $\theta$ has the form $f \mapsto f(x)$ for some $x \in E$. Thus, $|f(x)| \leq 1$ for $f \in A$, $|g(x)| > 1$ i.e., $p(x) \leq 1$ and $|g(x)| > 1$ and it follows that $g \notin p^0$.

Remarks.

1. Let $K$ be spherically (= maximally) complete. Then each nonarchimedean seminorm $p$ on $E$ for which $p(x) \in |X|$ ($x \in E$) is polar ([3], Remark following 3.1).

2. Let $\tau$ be the locally convex topology on $E$ induced by all nonarchimedean seminorms i.e., $\tau$ is the strongest among all locally convex topologies on $E$. It is not hard to see that $(E, \tau)$ is a complete polar ([3], Definition 3.5) space and that $(E, \tau)$ and $(E^*, \sigma(E^*, E))$ are each others strong dual spaces.
§1 NORMS \( p \) FOR WHICH \( p^0 \) IS \( c' \)-COMPACT

Recall that an absolutely convex subset \( A \) of a locally convex space \( F \) over \( K \) is \( c' \)-compact if for each neighbourhood \( U \) of \( 0 \) in \( F \) there exist \( x_1, \ldots, x_n \in A \) (rather than \( x_1, \ldots, x_n \in F \)) such that \( A \subseteq U + \text{co} \{x_1, \ldots, x_n\} \).
(Here \( \text{co} \) indicates the absolutely convex hull)

THEOREM 1.1. For a polar seminorm \( p \) on a \( K \)-vector space \( E \) the following are equivalent.

(a) \( p(x) \in |K| \) for each \( x \in E \). Each onedimensional subspace of \( E \) has a \( p \)-orthocomplement.

(b) \( p^0 \) is \( c' \)-compact.

Proof. (a) \( \Rightarrow \) (b). By [7], Theorem 3.2, it suffices to prove that for each \( \phi \in (E^*, \sigma(E^*,E)) \)

\[
\max \{|\phi(f)| : f \in p^0\}
\]

exists. Since \( \phi \) is an evaluation map we therefore have to show that

\[
\max \{|f(x)| : f \in p^0\}
\]

exists for each \( x \in E \). This is obviously true if \( p(x) = 0 \). So assume \( p(x) > 0 \). Since \( p(x) \in |K| \) we may assume that \( p(x) = 1 \). For such \( x \) we must prove

\[
\max \{|f(x)| : f \in p^0\} = 1
\]

By (a), \( Kx \) has a \( p \)-orthocomplement \( H \). The function

\[
f : \lambda x + h \mapsto \lambda \quad (\lambda \in K, h \in H)
\]
is in $E^*$. We have $|f(x)| = 1$. For $\lambda \in K$, $h \in H$

$$|f(\lambda x + h)| = |\lambda| = p(\lambda x) \leq \max(p(\lambda x), p(h)) = p(\lambda x + h)$$

so that $f \in p^0$.

(f) $\Rightarrow$ (a). Let $x \in E$. The map $f \mapsto |f(x)|$ ($f \in E^*$) is a continuous seminorm on $(E^*, o(E^*,E))$. By $c'$-compactness its restriction to $p^0$ has a maximum so there exists a $g \in p^0$ with $|g(x)| = p(x)$ (It follows that $p(x) \in |K|$). We prove that $\ker g$ is a $p$-orthocomplement of $Kx$. In fact, for $z \in \ker g$ we have

$$p(x+z) \geq |g(x+z)| = |g(x)| = p(x)$$

Then also

$$p(x+z) \geq p(z)$$

completing the proof of Theorem 1.1.

Note. It is not hard to see that (a) of above is equivalent too.

(y) For each $x \in E$ there exists an $f \in E^*$ with $|f(x)| = p(x)$ and $|f| \leq p$.

For spherically complete $K$ we obtain a simpler form of Theorem 1.1.

COROLLARY 1.2. Let $K$ be spherically complete, let $p$ be a seminorm on $E$ for which $p(x) \in |K|$ for all $x \in E$.

Then the following are equivalent.

(a) $p(x) \in |K|$ for each $x \in E$.

(b) $p^0$ is $c'$-compact.

Proof. By [1], lemma 4.35, each onedimensional subspace has a $p$-orthocomplement.
§2 APPLICATION: A NEW LIGHT ON SERRE'S RENORMING PROBLEM.

Consider the following two statements (*) and (**).

(*) Let $E$ be a $K$-vector space and let $||\cdot||$ be a norm on $E$. Then there exists a norm $||\cdot||'$ on $E$, equivalent to $||\cdot||$, such that $||x||' \leq |K|$ for all $x \in E$.

(**) Let $K$ be spherically complete and let $A$ be a complete absolutely convex compactoid in a Hausdorff locally convex space over $K$. Then there exist a $\lambda \in K$ with $|\lambda| > 1$ and a $c'$-compact $B$ such that $A \subset B \subset \lambda A$.

The question as to whether (*) is true or not is known as Serre's renorming problem. See [2] for more details. We are able to reformulate this problem in terms of compactoids:

PROPOSITION 2.1. The above statements (*) and (**) are equivalent.

Proof. Assume (*). To prove (**) we may assume that $A$ is edged. By [8], Theorem 3, $A$, as a topological module over $B(0,1) := \{ \lambda \in K : |\lambda| \leq 1 \}$, is isomorphic to a bounded submodule of $K^I$ for some set $I$. Let $E$ be the algebraic direct sum $\oplus K_i$ where $K_i = K$ for all $i \in I$.

Then $(E^*, \sigma(E^*, E))$ is in a natural way isomorphic to $K^I$ with the product topology. So we may assume that $A = p^0$ where $p$ is a seminorm on $E$.

By (*) there exists a seminorm $q$, equivalent to $p$, such that $q(x) \in |K|$ for all $x \in E$. By a suitable scalar multiplication we can arrange that, in addition, $p \leq q \leq |\lambda|p$ for some $\lambda \in K$, $|\lambda| > 1$. Then

$$p^0 \leq q^0 \leq \lambda p^0$$
and $q^0$ is $c'$-compact by Corollary 1.2. This proves (**'). Now assume (**).

To prove (*) we may assume (see [2]), that $K$ is spherically complete.

Let $p$ be a norm on $E$. By (**') there is a $c'$-compact $B$ and a $\lambda \in K$, $|\lambda| > 1$ with $p^0 \subseteq B \subseteq \lambda p^0$. Then $B = q^0$ for some seminorm $q$ on $E$. We have

$$p \leq q \leq |\lambda|p$$

and $q(x) \in \{K\}$ for all $x \in E$ by Corollary 1.2.

Note. Serre's renorming problem is still unsettled as far as I know.

§3 NORMS $p$ FOR WHICH $p^0$ IS A KREIN-MILMAN COMPACTOID.

Recall that an absolutely convex subset $A$ of a locally convex space over $K$ is a KM-compactoid if it is complete and if $A = \overline{\text{co}} X$ where $X$ is compact. (Here $\overline{\text{co}} X$ is the closure of $\text{co} X$).

Before stating the theorem we first make some simple observations. Let $p$ be a norm on $E$. We say that a collection $(e_i)_{i \in I}$ in $E$ is a $p$-orthonormal base of $E$ if for each $x \in E$ there exist a unique $(\lambda_i)_{i \in I} \subseteq K^+$ such that $\{i \in I, |\lambda_i| \geq \varepsilon\}$ is finite for each $\varepsilon > 0$ and

$$x = \sum_{i \in I} \lambda_i e_i$$

$$p(x) = \max_{i} |\lambda_i|$$

If $(E, p)$ is complete this definition coincides with the usual one.

**Lemma 3.1.** Let $(E, p)$ be a normed space, let $(\hat{E}, \hat{p})$ he its completion. Then $(E, p)$ has a $p$-orthonormal base if and only if $(\hat{E}, \hat{p})$ has a $\hat{p}$-orthonormal base.
Proof. It is not hard to see that each \( p \)-orthonormal base of \((E, p)\) is also a \( \hat{p} \)-orthonormal base of \((E, \hat{p})\). Conversely, let \((e_i)_{i \in I}\) be a \( \hat{p} \)-orthonormal base of \((E, \hat{p})\). For each \( i \in I \), choose an \( f_i \in E \) with 
\[
\hat{p}(e_i - f_i) \leq \frac{1}{2}.
\]
By [1], Exercise 5.C, \((f_i)_{i \in I}\) is a \( \hat{p} \)-orthonormal base of \((E, \hat{p})\).

Clearly \((f_i)_{i \in I}\) is a \( p \)-orthonormal base of \((E, p)\).

**Theorem 3.2.** For a polar norm \( p \) on a \( K \)-vector space \( E \) the following are equivalent.

1. \((E, p)\) has a \( p \)-orthonormal base
2. \( p^0 \) is a KM-compactoid.

**Proof.** (a) \( \Rightarrow \) (b). Let \((e_i)_{i \in I}\) be a \( p \)-orthonormal base of \((E, p)\). The formula
\[
\phi(f) = (f(e_i))_{i \in I}
\]
defines a map \( \phi : p^0 \to B(0,1)^I \). Straightforward verifications show that \( \phi \) is an isomorphism of topological \( B(0,1) \)-modules. From [8], Theorem 16 we obtain that \( B(0,1)^I \), hence \( p^0 \), is a KM-compactoid.

(b) \( \Rightarrow \) (a). Suppose \( p^0 = \text{co} X \) where \( X \) is a compact subset of \( E^* \).

Let \( C(X \to K) \) be the Banach space of all continuous functions \( X \to K \), with the supremum norm \( || \cdot ||_\infty \). Then \( C(X \to K) \) has an orthonormal base. ([1], Theorem 5.22).

The formula
\[
\phi(x)(f) = f(x) \quad (f \in X)
\]
defines a \( K \)-linear map \( \phi : E \to C(X \to K) \). From
\[
\|\phi(x)\|_\infty = \max_{f \in X} |f(x)| = \sup_{f \in \mathcal{C}(X,K)} |f(x)| = p(x)
\]
we obtain that \( \phi \) is an isometry \((E, p) \to (\mathcal{C}(X,K), \| \cdot \|_\infty)\).

By Gruson's Theorem ([1], 5.9) \( \overline{\phi(E)} \) has an orthonormal base. Then so has \( \phi(E) \) by Lemma 3.1 and has \( E \).

§4 APPLICATION: A COMPLETE c'-COMPACT SET WHICH IS NOT A KM-COMPACTOID.

We shall give a negative answer to the Problem following Theorem 1.7 of [6].

**PROPOSITION 4.1.** Let \( K \) be spherically complete, let \( |K| = [0, \infty) \).

Then there exist a locally convex space \( F \) over \( K \) and a complete c'-compact subset \( A \subset F \) which is not a KM-compactoid.

**Proof.** Let \( E := l^\infty \) and let \( F := (l^\infty)^* \) (with the topology we agreed upon in §0). Let \( p \) be the standard norm on \( l^\infty \), and set \( A := p^0 \). Since, trivially, \( p(x) \leq |K| \) for all \( x \in l^\infty \), we have that \( p^0 \) is c'-compact (Corollary 1.2).

However, it is known ([1], Cor. 5.19) that \( l^\infty \) has no orthogonal base so that (Theorem 3.2) \( p^0 \) is not a KM-compactoid.

§5 NORMS \( p \) FOR WHICH \( p^0 \) IS METRIZABLE.

**THEOREM 5.1.** For a polar seminorm \( p \) on a \( K \)-vector space \( E \) the following are equivalent.

(a) \((E, p)\) is of countable type ([3], Definition 4.3).

(b) \( p^0 \) is metrizable.
Proof. (a) ⇒ (b). There exist \( e_1, e_2, \ldots \) in \( E \) with \( p(e_i) \leq 1 \) for each \( i \) such that the \( K \)-linear span of \( e_1, e_2, \ldots \) is \( p \)-dense in \( E \). The formula

\[
\phi(f) = (f(e_1), f(e_2), \ldots)
\]

defines a map \( \phi : p^0 \to B(0,1)^{\mathbb{N}} \). Straightforward verifications show that \( \phi \) is an isomorphism of topological \( B(0,1) \)-modules of \( p^0 \) onto a submodule of \( B(0,1)^{\mathbb{N}} \).

Now \( B(0,1)^{\mathbb{N}} \) is metrizable (the product topology is induced by the metric

\[
(a,b) \mapsto \sup_{i \in \mathbb{N}} |a_i - b_i| 2^{-i}
\]

hence so is \( p^0 \).

\( (B) \Rightarrow (a) \). Let \( \lambda \in K, |\lambda| > 1 \). Since \( p^0 \) is a metrizable compactoid there exist, by [3], Proposition 8.2, \( f_1, f_2, \ldots \in \lambda p^0 \) with \( \lim_{n \to \infty} f_n = 0 \) such that

\[
p^0 = \overline{\co \{f_1, f_2, \ldots\}} \subset \lambda p^0
\]

The map

\[
\phi : x \mapsto (f_1(x), f_2(x), \ldots) \quad (x \in E)
\]

is \( K \)-linear, \( \phi(E) \subset c_0 \). We have for \( x \in E \)

\[
|\phi(x)| = \sup_{n \in \mathbb{N}} |f_n(x)| = \sup_{n \in \mathbb{N}} \{ |g(x)| : g \in \overline{\co \{f_1, f_2, \ldots\}} \}
\]

It follows that

\[
p(x) \leq ||\phi(x)|| \leq |\lambda| p(x)
\]

so that \( p \) is equivalent to \( x \mapsto ||\phi(x)|| \), a seminorm of countable type.

Hence, \( p \) is of countable type.
§6 APPLICATION: DESCRIPTION OF METRIZABLE COMPACTOIDS.

THEOREM 6.1. Let A be an absolutely convex subset of a Hausdorff locally convex space F over K. The following are equivalent.

(a) A is a metrizable compactoid.

(b) As a topological B(0,1)-module, A is isomorphic to a submodule of \(B(0,1)^\infty\).

(c) As a topological B(0,1)-module, A is isomorphic to a compactoid in \(c_0\).

(d) For each \(\lambda \in K, |\lambda| > 1\) then exist \(e_1, e_2, \ldots \in A\) with \(\lim_{n \to \infty} e_n = 0\) and \(A \subseteq \overline{\operatorname{co}} \{e_1, e_2, \ldots\}\).

(e) There exist \(e_1, e_2, \ldots \in F\) with \(\lim_{n \to \infty} e_n = 0\) and \(A \subseteq \overline{\operatorname{co}} \{e_1, e_2, \ldots\}\).

(n) There exists an ultrametrizable compact \(X \subset F\) with \(A \subseteq \overline{\operatorname{co}} X\).

Proof. (a) \(\Rightarrow\) (b). It is not hard to see, by using the absolute convexity of \(A\), that \(\overline{A}\) is also metrizable. As there is no harm in taking \(F\) complete we therefore may assume that \(A\) is complete. To prove (b) we also may assume that \(A\) is edged. By [8], Theorem 3, \(A = B(0,1)^I \subset K^I\) for some set \(I\). Like in the proof of Proposition 2.1 we may conclude that \(A = p^0\), where \(p\) is a polar seminorm on \(\oplus K_i (K_i = K\) for each \(i\)). Then \(p\) is of countable type by Theorem 5.1. From the proof of (a) \(\Rightarrow\) (b) of that Theorem we obtain an isomorphism \(A = p^0 \cong B(0,1)^\infty\).

(b) \(\Rightarrow\) (c). Choose \(\lambda_1, \lambda_2, \ldots \in K, |\lambda_1| > |\lambda_2| > \ldots, \lim_{n \to \infty} \lambda_n = 0\). The formula

\[
\phi((a_i)_{i \in \mathbb{N}}) = (\lambda_1 a_1, \lambda_2 a_2, \ldots) \in c_0
\]

defines a B(0,1)-module isomorphism of \(B(0,1)^\infty\) onto \(C := \overline{\operatorname{co}} \{\lambda_1 e_1, \lambda_2 e_2, \ldots\}\) where \(e_1, e_2, \ldots\) are the standard unit vectors in \(c_0\). \(\phi\) is a homeomorphism.
B(0,1) \mathbb{N} \to C, and maps A onto a compactoid in c_0.

(\gamma) \Rightarrow (\delta). See [3], Proposition 8.2.

(\delta) \Rightarrow (\varepsilon) is trivial.

(\varepsilon) \Rightarrow (\eta). \{0, e_1, e_2, \ldots\} is compact and ultrametrizable.

(\eta) \Rightarrow (\alpha). We may assume that F is complete. It suffices to prove the

metrizability of B := \text{co} X.

B is a complete, edged compactoid. As before we may assume that B = p^0 for some polar seminorm p on some K-vector space E while B \subseteq E^*. The

c = 0, C(X; \alpha), \|\| \alpha). The

map \phi : E \to C(X; K) defined by

\phi(x)(f) = f(x) \quad (f \in X)

is an isometry (E, p) \alpha (C(X; K), \|\| \alpha).

Now X is ultrametrizable so by [1], Exercise 3.5, C(X; K) is of

countable type. Hence so is p. By Theorem 5.1, B = p^0 is metrizable.

§7 Norms p for which (p^0)^*_1 is of finite type.

Recall that an absolutely convex set A in a locally convex space F over

K is of finite type if for each zero neighbourhood U in F there exists

a finite-dimensional bounded set S \subseteq A such that A \subseteq U + S.

Let us say that a seminorm p on a K-vector space E is of finite type

if Ker p = \{x \in E : p(x) = 0\} has finite codimension.

Lemma 7.1. Let A be an absolutely convex subset of a locally convex

space F whose topology is generated by a collection of seminorms of

finite type. Then the following are equivalent.

(a) A is a compactoid of finite type.

(b) For each closed linear subspace H of finite codimension there is a

finite dimensional bounded set S \subseteq A with A \subseteq H + S.
Proof. (a) ⇒ (β). (Note. This implication holds for any locally convex space F.) We may assume \( [A] = F \).

Set \( H \) has the form \( D^1 := \{ x \in F : f(x) = 0 \text{ for all } f \in D \} \) where \( D \) is a finite dimensional subspace of \( F' \). Let \( f_1, \ldots, f_n \) be a base of \( D \). There exist \( x_1, \ldots, x_n \in F \) with \( f_i(x_j) = \delta_{ij} (i, j \in \{1, \ldots, n\}) \). Since \( [A] = F \) there exists a \( \lambda \in K, \lambda \neq 0 \) such that \( \lambda x_i \in A \) for each \( i \in \{1, \ldots, n\} \).

Then \( U \) is a zero neighbourhood in \( F \). \( A \) is a compactoid of finite type, so there exists a finite dimensional set \( S_1 \subseteq A \) with \( A \subseteq U + S_1 \). Let \( x \in U \). Write \( x = y + z \) where:

\[
\begin{align*}
y &= x - \sum_{i=1}^{n} f_i(x) x_i \\
z &= \sum_{i=1}^{n} f_i(x) x_i
\end{align*}
\]

Now, since \( x \in U \), \( |f_i(x)| \leq |\lambda| \) for each \( i \) so that \( z = \sum_{i=1}^{n} f_i(x) x_i \in A \).

Further, for each \( j \in \{1, \ldots, n\} \)

\[
f_j(y) = f_j(x) - \sum_{i=1}^{n} f_i(x) f_j(x_i) = f_j(x) - f_j(x) = 0
\]

and it follows that \( y \in D^1 = H \). So \( x = y + z \in H + [x_1, \ldots, x_n] \cap A \). We see that

\[
A \subseteq U + S_1 \subseteq H + S_2 + S_1
\]

where \( S_2 := [x_1, \ldots, x_n] \cap A \). Then (β) is proved with \( S := S_1 + S_2 \).

(β) ⇒ (a). Let \( U \) be a zero neighbourhood in \( F \). Since continuous seminorms are of finite type, \( U \) contains a closed subspace \( H \) of finite codimension. By (β) there exists a finite dimensional set \( S \subseteq A \) with \( S \) bounded and
A \subset H + S. Then A \subset U + S.

From now on we assume that the valuation on K is dense.

Recall that for an absolutely convex set B we have \( B^i := \bigcup \lambda B \). \(|\lambda| < 1\)

**THEOREM 7.2.** Let p be a polar norm on a K-vector space E. Then the following are equivalent.

(a) For each finite dimensional subspace D of E there exists a seminorm q on E, q of finite type, q \leq p and q = p on D.

(β) \((p^0)^i\) is of finite type.

**Proof.** (a) \implies (β). As each continuous seminorm on \( E^* \) is of finite type it suffices to prove, by Lemma 7.1, that for a closed subspace H of \( E^* \) of finite codimension there exists a finite dimensional set \( S \in (p^0)^i \) such that \((p^0)^i \subset H + S\).

Now, by (a), there is a seminorm q of finite type, q \leq p on E and q = p on D := H^i. Let

\[ S_1 := \{ f \in E^* : |f| \leq q \}. \]

We see that \( S_1 \) is finite dimensional and since q \leq p we have \( S_1 \subset p^0 \).

We now shall prove that \((p^0)^i \subset H + S\) where \( S := (S_1)^i \).

In fact, let \( f \in (p^0)^i \). Then there is a \( \lambda \in K \), 0 < |\lambda| < 1 with |\( f^\lambda | \leq |\lambda| \cdot p^\lambda \).

Choose \( \lambda' \in K \) with |\( \lambda | < |\lambda' | < 1 \).

We have \(|f| \leq |\lambda| \cdot q \) on D (since p = q on D) so we can extend f to a g \in E^* with |g| \leq |\lambda'| \cdot q on E. (This is because q is of finite type so that (E,q) is strongly polar.) Now write

\[ f = f - g + g \]

Since f = g on D we have f - g \in D^i = H.
Also, \( |(\lambda')^{-1}g| \leq q \) so that \( (\lambda')^{-1}g \in S_1 \) i.e. \( g \in (S_1)^\dagger = S \).

(\beta) \Rightarrow (\alpha). By lemma 7.1 there exists a finite dimensional set \( S \subset (p_0)^\dagger \) so that \( (p_0)^\dagger = D^\perp \cap (p_0)^\dagger + S \).

Set \( q(x) := \sup_{h \in S} |h(x)| \) \( (x \in E) \).

Then \( q(x) = 0 \) for all \( x \) in the space \( S^\perp \) which has finite codimension.

So \( q \) is of finite type.

Further, for \( x \in E \) we have

\[
q(x) = \sup_{h \in S} |h(x)| \leq \sup_{h \in (p_0)^\dagger} |h(x)| = \sup_{h \in p_0} |h(x)| = p(x),
\]

so \( q \leq p \). Finally, if \( x \in D \) then

\[
p(x) = \sup_{f \in p_0} |f(x)| = \sup_{f \in (p_0)^\dagger} |f(x)| = \sup_{h \in D^\perp \cap (p_0)^\dagger} |h(x) + t(x)| \leq q(x).
\]

Hence, \( p = q \) on \( D \).

§8 APPLICATION: A COMPLETE COMPACTOID IN \( c_0 \) THAT IS NOT OF FINITE TYPE.

If \( K \) is spherically complete each complete absolutely convex compactoid is of finite type (See [4], 2.3).

If \( K \) is not spherically complete the unit ball of \( c_0 \) is a complete compactoid for the weak topology but not of finite type (See [5], 1.6).

This is a non-metrizable compactoid. A compactoid in \( (c_0, ||||) \), not of finite type, is given in [5], 1.4. However this compactoid is not closed. The following example provides an answer to the Problem following 1.5 in [5].
PROPOSITION 8.1. Let K be not spherically complete. Then there exists an absolutely convex complete compactoid in $c_0$ that is not of finite type.

Proof. Let $(K^v, | |)$ be the spherical completion of $(K, | |)$ in the sense of [1], Theorem 4.49. Let $E$ be a K-subspace of $K^v$ of countably infinite dimension and let $p$ be the valuation $| |$ restricted to $E$. Then $x, y \in E$, $x \perp y$ in the sense of $p$ implies $x = 0$ or $y = 0$. Obviously, the norm $p$ is of countable type (hence polar) so, by Theorem 5.1, $p^0$ is metrizable and is by Theorem 6.1, isomorphic to a compactoid in $c_0$. Suppose $p^0$ were of finite type. Then so would $(p^0)^1$ ([5], Proposition 2.4). By Theorem 7.2 we would have a seminorm $q$ on $E$, $q \leq p$, $q$ of finite type, $q(x) = p(x)$ for some $x \in E$, $x \neq 0$. But then $x \perp \text{Ker } q$ in the sense of $p$ (If $q(y) = 0$ then $p(x-y) \geq q(x-y) = q(x) = p(x)$) which is impossible. So, $p^0$ is not of finite type.
REFERENCES


