A CONNECTION BETWEEN $p$-ADIC BANACH SPACES AND LOCALLY CONVEX COMPACTOIDS

by

W.H. SCHIKHOF

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DEPARTMENT OF MATHEMATICS
CATHOLIC UNIVERSITY
Toernooiveld
6525 ED Nijmegen
The Netherlands
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ABSTRACT. For a vector space $E$ over a non-archimedean valued field $K$ a correspondence $p \leftrightarrow p^0$ is established between seminorms $p$ on $E$ and compactoids $p^0$ in $E^\ast$.

Examination of it yields the solution of two open problems (see §4 and §8) and a reformulation of Serre's renorming problem (see §2). As a by-product results on metrizable compactoids are obtained (see §6).

§0 THE CORRESPONDENCE $p \leftrightarrow p^0$.

Throughout this note $K$ is a non-archimedean valued field, complete with respect to the metric induced by the nontrivial valuation $| |$.

Let $E$ be a $K$-vector space, let $E^\ast$ be its algebraic dual. A (non-archimedean) seminorm $p$ on $E$ is polar ([3], Definition 3.1), if

$$p = \sup \{ |f| : f \in E^\ast, |f| \leq p \}$$

Let $P_E$ be the set of all polar seminorms on $E$.

For each $p \in P_E$ we set

$$p^0 = \{ f \in E^\ast : |f| \leq p \}$$

Then $p^0$ is an absolutely convex, edged ([3],§1b) subset of $E^\ast$. It is easy to see that $p^0$ is a closed compactoid ([3],§1e) with respect to the topology $\sigma(E^\ast,E)$, hence complete.
Let $C_E^*$ be the set of all closed absolutely convex, edged compactoids in $E^*$ with respect to $\sigma(E^*,E)$.

**Proposition 0.** The map $p \mapsto p^0$ is a bijection of $P_E$ onto $C_E^*$. Its inverse assigns to every $A \in C_E^*$ the seminorm $p$ given by

$$p(x) = \sup \{|f(x)| : f \in A\} \quad (x \in E)$$

**Proof.** We shall prove surjectivity of $p \mapsto p^0$ leaving the (easy) rest of the proof to the reader. So, let $A \in C_E^*$; we shall prove that $A = p^0$ where $p(x) = \sup \{|f(x)| : f \in A\}$.

Obviously, $A \subset p^0$. Now let $g \in E^* \setminus A$, we prove that $g \not\in p^0$. The space $E^*$ is of countable type hence strongly polar ([3], Theorem 4.4). So by [3], Theorem 4.7, there exists a $\theta \in (E^*, \sigma(E^*,E))^\prime$ such that $|\theta| \leq 1$ on $A$, $|\theta(g)| > 1$. But, by [3], lemma 7.1, $\theta$ has the form $f \mapsto f(x)$ for some $x \in E$. Thus, $|f(x)| \leq 1$ for $f \in A$, $|g(x)| > 1$ i.e., $p(x) \leq 1$ and $|g(x)| > 1$ and it follows that $g \not\in p^0$.

**Remarks.**

1. Let $K$ be spherically (= maximally) complete. Then each nonarchimedean seminorm $p$ on $E$ for which $p(x) \in |K|$ (i.e., $x \in E$) is polar ([3], Remark following 3.1).

2. Let $\tau$ be the locally convex topology on $E$ induced by all nonarchimedean seminorms i.e., $\tau$ is the strongest among all locally convex topologies on $E$. It is not hard to see that $(E,\tau)$ is a complete polar space ([3], Definition 3.5) space and that $(E,\tau)$ and $(E^*, \sigma(E^*,E))$ are each others strong dual spaces.
§ 1 NORMS p FOR WHICH p^0 IS c'-COMPACT

Recall that an absolutely convex subset A of a locally convex space F over K is c'-compact if for each neighbourhood U of 0 in F there exist x_1, ..., x_n ∈ A (rather than x_1, ..., x_n ∈ F) such that A ⊂ U + co{x_1, ..., x_n}. (Here co indicates the absolutely convex hull)

THEOREM 1.1. For a polar seminorm p on a K-vector space E the following are equivalent.

(a) p(x) ∈ |K| for each x ∈ E. Each one-dimensional subspace of E has a p-orthocomplement.

(β) p^0 is c'-compact.

Proof. (α) ⇒ (β). By [7], Theorem 3.2, it suffices to prove that for each < f > ∈ (E*, σ(E*,E))'

\[ \max \{ |\phi(f)| : f \in p^0 \} \]

exists. Since ϕ is an evaluation map we therefore have to show that

\[ \max \{ |f(x)| : f \in p^0 \} \]

exists for each x ∈ E. This is obviously true if p(x) = 0. So assume p(x) > 0. Since p(x) ∈ |K| we may assume that p(x) = 1. For such x we must prove

\[ \max \{ |f(x)| : f \in p^0 \} = 1 \]

By (α), Kx has a p-orthocomplement H. The function

f : λx + h ↦ λ \quad (λ ∈ K, h ∈ H)
is in $E^\ast$. We have $|f(x)| = 1$. For $\lambda \in K$, $h \in H$

$$|f(\lambda x + h)| = |\lambda| = p(\lambda x) \leq \max(p(\lambda x), p(h)) = p(\lambda x + h)$$

so that $f \in p^0$.

($\beta$) $\Rightarrow$ ($\alpha$). Let $x \in E$. The map $f \mapsto |f(x)|$ ($f \in E^\ast$)
is a continuous seminorm on $(E^\ast, c(E^\ast, E))$. By $c'$-compactness its
restriction to $p^0$ has a maximum so there exists a $g \in p^0$ with $|g(x)| = p(x)$
(It follows that $p(x) \in |K|$). We prove that $\text{Ker} \cdot g$ is a $p$-orthocomplement
of $Kx$. In fact, for $z \in \text{Ker} \cdot g$ we have

$$p(x+z) \geq |g(x+z)| = |g(x)| = p(x)$$

Then also

$$p(x+z) \geq p(z)$$

completing the proof of Theorem 1.1.

Note. It is not hard to see that ($\alpha$) of above is equivalent too.

($\gamma$) For each $x \in E$ there exists an $f \in E^\ast$ with $|f(x)| = p(x)$ and $|f| \leq p$.

For spherically complete $K$ we obtain a simpler form of Theorem 1.1.

**COROLLARY 1.2.** Let $K$ be spherically complete, let $p$ be a seminorm on
$E$ for which $p(x) \in |K|$ for all $x \in E$.

Then the following are equivalent.

($\alpha$) $p(x) \in |K|$ for each $x \in E$.

($\beta$) $p^0$ is $c'$-compact.

**Proof.** By [1], lemma 4.35, each onedimensional subspace has a $p$-orthocomplement.
§2 APPLICATION: A NEW LIGHT ON SERRE'S RENORMING PROBLEM.

Consider the following two statements (*) and (**).

(*) Let $E$ be a $K$-vector space and let $|| \cdot ||$ be a norm on $E$. Then there exists a norm $|| \cdot ||'$ on $E$, equivalent to $|| \cdot ||$, such that $||x||' \leq |K|$ for all $x \in E$.

(**) Let $K$ be spherically complete and let $A$ be a complete absolutely convex compactoid in a Hausdorff locally convex space over $K$. Then there exist a $\lambda \in K$ with $|\lambda| > 1$ and a $C'$-compact $B$ such that $A \subset B \subset \lambda A$.

The question as to whether (*) is true or not is known as Serré's renorming problem. See [2] for more details. We are able to reformulate this problem in terms of compactoids:

**Proposition 2.1.** The above statements (*) and (**) are equivalent.

**Proof.** Assume (*). To prove (**) we may assume that $A$ is edged. By [8], Theorem 3, $A$, as a topological module over $B(0,1) := \{ \lambda \in K : |\lambda| \leq 1 \}$, is isomorphic to a bounded submodule of $K^I$ for some set $I$. Let $E$ be the algebraic direct sum $\oplus K_i$ where $K_i = K$ for all $i \in I$.

Then $(E^*, \sigma(E^*, E))$ is in a natural way isomorphic to $K^I$ with the product topology. So we may assume that $A = p^0$ where $p$ is a seminorm on $E$.

By (*) there exists a seminorm $q$, equivalent to $p$, such that $q(x) \in |K|$ for all $x \in E$. By a suitable scalar multiplication we can arrange that, in addition, $p \leq q \leq |\lambda|p$ for some $\lambda \in K$, $|\lambda| > 1$. Then

$$p^0 \leq q^0 \leq \lambda p^0$$
and $q^0$ is $c'$-compact by Corollary 1.2. This proves (**). Now assume (**). To prove (*) we may assume (see [2]), that $K$ is spherically complete.

Let $p$ be a norm on $E$. By (**) there is a $c'$-compact $B$ and a $\lambda \in K$, $|\lambda| > 1$ with $p^0 \subseteq B \subseteq \lambda p^0$. Then $B = q^0$ for some seminorm $q$ on $E$. We have

$$p \leq q \leq |\lambda| p$$

and $q(x) \in |K|$ for all $x \in E$ by Corollary 1.2.

Note. Serre's renorming problem is still unsettled as far as I know.

§3 NORMS $p$ FOR WHICH $p^0$ IS A KREIN-MILMAN COMPACTOID.

Recall that an absolutely convex subset $A$ of a locally convex space over $K$ is a KM-compactoid if it is complete and if $A = \overline{\text{co} X}$ where $X$ is compact. (Here $\text{co} X$ is the closure of $\text{co} X$).

Before stating the theorem we first make some simple observations. Let $p$ be a norm on $E$. We say that a collection $(e_i)_{i \in I}$ in $E$ is a $p$-orthonormal base of $E$ if for each $x \in E$ there exist a unique $(\lambda_i)_{i \in I} \subseteq K^+$ such that $\{i \in I, |\lambda_i| \geq \epsilon\}$ is finite for each $\epsilon > 0$ and

$$x = \sum_{i \in I} \lambda_i e_i$$

$$p(x) = \max_{i} |\lambda_i|$$

If $(E, p)$ is complete this definition coincides with the usual one.

**Lemma 3.1.** Let $(E, p)$ be a normed space, let $(E^\sim, p^\sim)$ be its completion. Then $(E, p)$ has a $p$-orthonormal base if and only if $(E^\sim, p^\sim)$ has a $p^\sim$-orthonormal base.
Proof. It is not hard to see that each p-orthonormal base of \((E,p)\) is also a p'-orthonormal base of \((E',p')\). Conversely, let \((e_i)_{i \in I}\) be a p'-orthonormal base of \((E',p')\). For each \(i \in I\), choose an \(f_i \in E\) with \(p'(e_i - f_i) \leq \frac{1}{2}\).

By [1], Exercise 5.C, \((f_i)_{i \in I}\) is a p'-orthonormal base of \((E',p')\).

Clearly \((f_i)_{i \in I}\) is a p-orthonormal base of \((E,p)\).

**THEOREM 3.2.** For a polar norm \(p\) on a K-vector space \(E\) the following are equivalent.

(a) \((E,p)\) has a p-orthonormal base

(b) \(p^0\) is a KM-compactoid.

Proof. (a) \(\Rightarrow\) (b). Let \((e_i)_{i \in I}\) be a p-orthonormal base of \((E,p)\). The formula

\[ \phi(f) = (f(e_i))_{i \in I} \]

defines a map \(\phi : p^0 \to B(0,1)^I\). Straightforward verifications show that \(\phi\) is an isomorphism of topological \(B(0,1)^I\)-modules. From [8], Theorem 16 we obtain that \(B(0,1)^I\), hence \(p^0\), is a KM-compactoid.

(b) \(\Rightarrow\) (a). Suppose \(p^0 = \text{co} X\) where \(X\) is a compact subset of \(E^*\).

Let \(C(X^*K)\) be the Banach space of all continuous functions \(X^*K\), with the supremum norm \(||\cdot||_\omega\). Then \(C(X^*K)\) has an orthonormal base. ([1], Theorem 5.22).

The formula

\[ \phi(x)(f) = f(x) \quad (f \in X) \]

defines a K-linear map \(\phi : E \to C(X^*K)\). From
\[ ||f(x)||_\infty = \max_{f \in X} |f(x)| = \sup_{f \in \mathcal{C}(X^K)} |f(x)| = \sup_{f \in \mathcal{P}} |f(x)| = p(x) \]

we obtain that \( f \) is an isometry \( (E,p) \to (\mathcal{C}(X^K), ||\cdot||_\infty) \).

By Gruson's Theorem ([1], 5.9) \( \hat{f}(E) \) has an orthonormal base. Then so has \( \hat{f}(E) \) by Lemma 3.1 and has \( E \).

§4 APPLICATION: A COMPLETE c'-COMPACT SET WHICH IS NOT A KM-COMPACTOID.

We shall give a negative answer to the Problem following Theorem 1.7 of [6].

PROPOSITION 4.1. Let \( K \) be spherically complete, let \( |K| = (0,\infty) \).

Then there exist a locally convex space \( F \) over \( K \) and a complete c'-compact subset \( A \subset F \) which is not a KM-compactoid.

**Proof.** Let \( E := c_0 \) and let \( F := (c_0)^* \) (with the topology we agreed upon in §0). Let \( p \) be the standard norm on \( c_0 \), and set \( A := p^0 \). Since, trivially, \( p(x) \in |K| \) for all \( x \in c_0 \), we have that \( p^0 \) is c'-compact (Corollary 1.2).

However, it is known ([1], Cor. 5.19) that \( c_0 \) has no orthogonal base so that (Theorem 3.2) \( p^0 \) is not a KM-compactoid.

§5 NORMS \( p \) FOR WHICH \( p^0 \) IS METRIZABLE.

THEOREM 5.1. For a polar seminorm \( p \) on a \( K \)-vector space \( E \) the following are equivalent.

(a) \( (E,p) \) is of countable type ([3], Definition 4.3).

(b) \( p^0 \) is metrizable.
Proof. \((\alpha) \Rightarrow (\beta)\). There exist \(e_1, e_2, \ldots\) in \(E\) with \(p(e_i) \leq 1\) for each \(i\) such that the \(K\)-linear span of \(e_1, e_2, \ldots\) is \(p\)-dense in \(E\). The formula

\[\phi(f) = (f(e_1), f(e_2), \ldots)\]

defines a map \(\phi : p^0 \rightarrow B(0,1)^N\). Straightforward verifications show that \(\phi\) is an isomorphism of topological \(B(0,1)\)-modules of \(p^0\) onto a submodule of \(B(0,1)^N\).

Now \(B(0,1)^N\) is metrizable (the product topology is induced by the metric \((a,b) \mapsto \sup_{i \in \mathbb{N}} |a_i - b_i| 2^{-i}\) hence so is \(p^0\).

\((\beta) \Rightarrow (\alpha)\). Let \(\lambda \in K\), \(|\lambda| > 1\). Since \(p^0\) is a metrizable compactoid there exist, by [3], Proposition 8.2, \(f_1, f_2, \ldots \in \lambda p^0\) with \(\lim_{n \to \infty} f_n = 0\) such that

\[p^0 \subset \overline{\text{co}} \{f_1, f_2, \ldots\} \subset \lambda p^0\]

The map

\[\phi : x \mapsto (f_1(x), f_2(x), \ldots)\quad (x \in E)\]

is \(K\)-linear, \(\phi(E) \subset c_0\). We have for \(x \in E\)

\[||\phi(x)|| = \sup_{n \in \mathbb{N}} |f_n(x)| = \sup \{|g(x)| : g \in \overline{\text{co}} \{f_1, f_2, \ldots\}\}\]

It follows that

\[p(x) \leq ||\phi(x)|| \leq |\lambda| p(x)\]

so that \(p\) is equivalent to \(x \mapsto ||\phi(x)||\), a seminorm of countable type. Hence, \(p\) is of countable type.
§6 APPLICATION: DESCRIPTION OF METRIZABLE COMPACTOIDS.

THEOREM 6.1. Let $A$ be an absolutely convex subset of a Hausdorff locally convex space $F$ over $K$. The following are equivalent.

(a) $A$ is a metrizable compactoid.

(b) As a topological $B(0,1)$-module, $A$ is isomorphic to a submodule of $B(0,1)^\mathbb{N}$.

(c) As a topological $B(0,1)$-module, $A$ is isomorphic to a compactoid in $C_0$.

(d) For each $\lambda \in K$, $|\lambda| > 1$ then exist $e_1, e_2, \ldots \in \lambda A$ with $\lim_{n \to \infty} e_n = 0$ and $A \subseteq \overline{\{e_1, e_2, \ldots\}}$.

(e) There exist $e_1, e_2, \ldots \in F$ with $\lim_{n \to \infty} e_n = 0$ and $A \subseteq \overline{\{e_1, e_2, \ldots\}}$.

(f) There exists an ultrametrizable compact $X \subseteq F$ with $A \subseteq \overline{X}$.

Proof. (a) $\Rightarrow$ (b). It is not hard to see, by using the absolute convexity of $A$, that $\overline{A}$ is also metrizable. As there is no harm in taking $F$ complete we therefore may assume that $A$ is complete. To prove (b) we also may assume that $A$ is edged. By [8], Theorem 3, $A \subseteq B(0,1)^I \subseteq K^I$ for some set $I$. Like in the proof of Proposition 2.1 we may conclude that $A = p^{0I}$ where $p$ is a polar seminorm on $\oplus K_i (K_i = K$ for each $i$). Then $p$ is of countable type by Theorem 5.1. From the proof of (a) $\Rightarrow$ (b) of that Theorem we obtain an isomorphism $A = p^{0I} \to B(0,1)^\mathbb{N}$.

(b) $\Rightarrow$ (c). Choose $\lambda_1, \lambda_2, \ldots \in K$, $|\lambda_1| > |\lambda_2| > \ldots$, $\lim_{n \to \infty} \lambda_n = 0$. The formula

$$\phi((a_i)_{i \in \mathbb{N}}) = (\lambda_1 a_1, \lambda_2 a_2, \ldots) \in c_0$$

defines a $B(0,1)$-module isomorphism of $B(0,1)^\mathbb{N}$ onto $C := \overline{\{\lambda_1 e_1, \lambda_2 e_2, \ldots\}}$ where $e_1, e_2, \ldots$ are the standard unit vectors in $c_0$. $\phi$ is a homeomorphism.
B(0,1)^\mathbb{N} \to C$, and maps $A$ onto a compactoid in $c_0$.

(γ) ⇒ (δ). See [3], Proposition 8.2.

(δ) ⇒ (ε) is trivial.

(ε) ⇒ (η), \{0,e_1,e_2,...\} is compact and ultrametrizable.

(η) ⇒ (α). We may assume that $F$ is complete. It suffices to prove the metrizability of $B := \co X$.

$B$ is a complete, edged compactoid. As before we may assume that $B = p^0$ for some polar seminorm $p$ on some K-vector space $E$ while $B \subset E^*$. The map $\phi : E \to C(X \to K)$ defined by

$$\phi(x)(f) = f(x) \quad (f \in X)$$

is an isometry $(E, p) \cong (C(X \to K), \|\cdot\|_\infty)$.

Now $X$ is ultrametrizable so by [1], Exercise 3.5, $C(X \to K)$ is of countable type. Hence so is $p$. By Theorem 5.1, $B = p^0$ is metrizable.

§7 Norms $p$ for which $(p^0)^i$ is of finite type.

Recall that an absolutely convex set $A$ in a locally convex space $F$ over $K$ is of finite type if for each zero neighbourhood $U$ in $F$ there exists a finite-dimensional bounded set $S \subset A$ such that $A \subset U + S$.

Let us say that a seminorm $p$ on a K-vector space $E$ is of finite type if $\text{Ker } p = \{x \in E : p(x) = 0\}$ has finite codimension.

**Lemma 7.1.** Let $A$ be an absolutely convex subset of a locally convex space $F$ whose topology is generated by a collection of seminorms of finite type. Then the following are equivalent.

(a) $A$ is a compactoid of finite type.

(b) For each closed linear subspace $H$ of finite codimension there is a finite dimensional bounded set $S \subset A$ with $A \subset H + S$. 
Proof. $(a) \Rightarrow (\beta).$ (Note. This implication holds for any locally convex space $F$.) We may assume $\{A\} = F.$

$H$ has the form $D^1 := \{x \in F : f(x) = 0 \text{ for all } f \in D\}$ where $D$ is a finite dimensional subspace of $F'.$ Let $f_1, \ldots, f_n$ be a base of $D.$ There exist $x_1, \ldots, x_n \in F$ with $f_i(x_j) = \delta_{ij}$ ($i, j \in \{1, \ldots, n\}$). Since $\{A\} = F$ there exists a $\lambda \in K, \lambda \neq 0$ such that $\lambda x_i \in A$ for each $i \in \{1, \ldots, n\}.$

Set

$$U := \bigcap_{i=1}^{n} \{x \in F : |f_i(x)| \leq |\lambda|\}$$

Then $U$ is a zero neighbourhood in $F.$ $A$ is a compactoid of finite type, so there exists a finite dimensional set $S_1 \subseteq A$ with $A \subseteq U + S_1.$ Let $x \in U.$ Write $x = y + z$ where

$$y := x - \sum_{i=1}^{n} f_i(x) x_i$$
$$z := \sum_{i=1}^{n} f_i(x) x_i$$

Now, since $x \in U,$ $|f_i(x)| \leq |\lambda|$ for each $i$ so that $z = \sum_{i=1}^{n} f_i(x) x_i \in A.$

Further, for each $j \in \{1, \ldots, n\}$

$$f_j(y) = f_j(x) - \sum_{i=1}^{n} f_i(x) f_j(x_i) = f_j(x) - f_j(x) = 0$$

and it follows that $y \in D^1 = H.$ So $x = y + z \in H + [x_1, \ldots, x_n] \cap A.$ We see that

$$A \subseteq U + S_1 \subseteq H + S_2 + S_1$$

where $S_2 := \{x_1, \ldots, x_n\} \cap A.$ Then $(\beta)$ is proved with $S := S_1 + S_2.$

$(\beta) \Rightarrow (a).$ Let $U$ be a zero neighbourhood in $F.$ Since continuous seminorms are of finite type, $U$ contains a closed subspace $H$ of finite codimension. By $(\beta)$ there exists a finite dimensional set $S \subseteq A$ with $S$ bounded and
A \subset H + S. Then A \subset U + S.

From now on we assume that the valuation on K is dense.

Recall that for an absolutely convex set B we have $B^\perp := \bigcup_{|\lambda| < 1} \lambda B$.

**THEOREM 7.2.** Let $p$ be a polar norm on a K-vector space $E$. Then the following are equivalent.

(a) For each finite dimensional subspace $D$ of $E$ there exists a seminorm $q$ on $E$, $q$ of finite type, $q \leq p$ and $q = p$ on $D$.

(b) $(p^0)^\perp$ is of finite type.

**Proof.** (a) $\Rightarrow$ (b). As each continuous seminorm on $E^*$ is of finite type it suffices to prove, by Lemma 7.1, that for a closed subspace $H$ of $E^*$ of finite codimension there exists a finite dimensional set $S \in (p^0)^\perp$ such that $(p^0)^\perp \subset H + S$.

Now, by (a), there is a seminorm $q$ of finite type, $q \leq p$ on $E$ and $q = p$ on $D := H^\perp$. Let

$$S_1 := \{f \in E^* : |f| \leq q\}.$$ 

We see that $S_1$ is finite dimensional and since $q \leq p$ we have $S_1 \subset p^0$.

We now shall prove that $(p^0)^\perp \subset H + S$ where $S := (S_1)^\perp$.

In fact, let $f \in (p^0)^\perp$. Then there is a $\lambda \in K$, $0 < |\lambda| < 1$ with $|f| \leq |\lambda|p$.

Choose $\lambda' \in K$ with $|\lambda| < |\lambda'| < 1$.

We have $|f| \leq |\lambda|$ on $D$ (since $p = q$ on $D$) so we can extend $f$ to a $g \in E^*$ with $|g| \leq |\lambda'|$ on $E$. (This is because $q$ is of finite type so that $(E,q)$ is strongly polar.) Now write

$$f = f - g + g$$

Since $f = g$ on $D$ we have $f - g \in D^\perp = H$. 


Also, \((\lambda')^{-1}g \leq q\) so that \((\lambda')^{-1}g \in S\), i.e. \(g \in (S')\) = \(S\).

(\(\beta\)) \(\Rightarrow\) (a). By lemma 7.1 there exists a finite dimensional set \(S \subset (p')\) so that \((p')\) \(\cap (p') S\).

Set \(q(x) := \sup_{h \in S} |h(x)|\) \((x \in E)\).

Then \(q(x) = 0\) for all \(x\) in the space \(S\) which has finite codimension.

So \(q\) is of finite type.

Further, for \(x \in E\) we have

\[ q(x) = \sup_{h \in S} |h(x)| = \sup_{h \in S} |h(x)| = \sup_{h \in (p')} |h(x)| = p(x), \]

so \(q \leq p\). Finally, if \(x \in D\) then

\[ p(x) = \sup_{f \in p} |f(x)| = \sup_{h \in S} |h(x) + t(x)| = \sup_{h \in (p') \cap (p')} |h(x) + t(x)| = \sup_{t \in S} |t(x)| = q(x). \]

Hence, \(p = q\) on \(D\).

§8 APPLICATION: A COMPLETE COMPACTOID IN \(c_0\) THAT IS NOT OF FINITE TYPE.

If \(K\) is spherically complete each complete absolutely convex compactoid is of finite type (See [4], 2.3).

If \(K\) is not spherically complete the unit ball of \(c_0\) is a complete compactoid for the weak topology but not of finite type (See [5], 1.6).

This is a non-metrizable compactoid. A compactoid in \((c_0, \|\|\|)\), not of finite type, is given in [5], 1.4. However this compactoid is not closed. The following example provides an answer to the Problem following 1.5 in [5].
PROPOSITION 8.1. Let $K$ be not spherically complete. Then there exists an absolutely convex complete compactoid in $c_0$ that is not of finite type.

**Proof.** Let $(K', | |)$ be the spherical completion of $(K, | |)$ in the sense of [1], Theorem 4.49. Let $E$ be a $K$-subspace of $K'$ of countably infinite dimension and let $p$ be the valuation $| |$ restricted to $E$. Then $x, y \in E$, $x \perp y$ in the sense of $p$ implies $x = 0$ or $y = 0$. Obviously, the norm $p$ is of countable type (hence polar) so, by Theorem 5.1, $p^0$ is metrizable and is by Theorem 6.1, isomorphic to a compactoid in $c_0$.

Suppose $p^0$ were of finite type. Then so would $(p^0)^i$ ([5], Proposition 2.4). By Theorem 7.2 we would have a seminorm $q$ on $E$, $q \leq p$, $q$ of finite type, $q(x) = p(x)$ for some $x \in E$, $x \neq 0$. But then $x \perp \text{Ker } q$ in the sense of $p$ (If $q(y) = 0$ then $p(x-y) \geq q(x-y) = q(x) = p(x)$) which is impossible.

So, $p^0$ is not of finite type.
REFERENCES


