A CONNECTION BETWEEN $p$-ADIC BANACH SPACES AND LOCALLY CONVEX
COMPACTOIDS

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ABSTRACT. For a vector space $E$ over a non-archimedean valued field $K$ a correspondence $p \mapsto p^0$ is established between seminorms $p$ on $E$ and compactoids $p^0$ in $E^*$. Examination of it yields the solution of two open problems (see §4 and §8) and a reformulation of Serre’s renorming problem (see §2). As a by-product results on metrizable compactoids are obtained (see §6).

$\S0$ THE CORRESPONDENCE $p \mapsto p^0$.

Throughout this note $K$ is a non-archimedean valued field, complete with respect to the metric induced by the nontrivial valuation $|\cdot|$. Let $E$ be a $K$-vector space, let $E^*$ be its algebraic dual. A (non-archimedean) seminorm $p$ on $E$ is polar ([3], Definition 3.1), if

$$p = \sup \{|f| : f \in E^*, |f| \leq p\}$$

Let $P_E$ be the set of all polar seminorms on $E$.

For each $p \in P_E$ we set

$$p^0 = \{f \in E^* : |f| \leq p\}$$

Then $p^0$ is an absolutely convex, edged ([3],§1b) subset of $E^*$. It is easy to see that $p^0$ is a closed compactoid ([3],§1e) with respect to the topology $\sigma(E^*,E)$, hence complete.
Let $C_*$ be the set of all closed absolutely convex, edged compactoids in $E^*$ with respect to $\sigma(E^*, E)$.

**PROPOSITION 0.** The map $p \mapsto p^0$ is a bijection of $P_E$ onto $C_*$*. Its inverse assigns to every $A \in C_*$ the seminorm $p$ given by

$$
p(x) = \sup \{|f(x)| : f \in A\} \quad (x \in E)
$$

**Proof.** We shall prove surjectivity of $p \mapsto p^0$ leaving the (easy) rest of the proof to the reader. So, let $A \in C_*$; we shall prove that $A = p^0$ where $p(x) = \sup \{|f(x)| : f \in A\}$.

Obviously, $A \subset p^0$. Now let $g \in E^* \setminus A$, we prove that $g \notin p^0$. The space $E^*$ is of countable type hence strongly polar ([3], Theorem 4.4). So by [3], Theorem 4.7, there exists a $0 \in (E^*, \sigma(E^*, E)^*)'$ such that $|0| < 1$ on $A$, $|0(g)| > 1$. But, by [3], lemma 7.1, $0$ has the form $f \mapsto f(x)$ for some $x \in E$. Thus, $|f(x)| \leq 1$ for $f \in A$, $|g(x)| > 1$ i.e., $p(x) \leq 1$ and $|g(x)| > 1$ and it follows that $g \notin p^0$.

**Remarks.**

1. Let $K$ be spherically (= maximally) complete. Then each nonarchimedean seminorm $p$ on $E$ for which $p(x) \in |x|$ ($x \in E$) is polar ([3], Remark following 3.1).

2. Let $\tau$ be the locally convex topology on $E$ induced by all nonarchimedean seminorms i.e., $\tau$ is the strongest among all locally convex topologies on $E$. It is not hard to see that $(E, \tau)$ is a complete polar ([3], Definition 3.5) space and that $(E, \tau)$ and $(E^*, \sigma(E^*, E))$ are each others strong dual spaces.
§1 NORMS p FOR WHICH $p^0$ IS $c'$-COMPACT

Recall that an absolutely convex subset $A$ of a locally convex space $F$ over $K$ is $c'$-compact if for each neighbourhood $U$ of 0 in $F$ there exist $x_1, \ldots, x_n \in A$ (rather than $x_1, \ldots, x_n \in F$) such that $A \subseteq U + \text{co} \{x_1, \ldots, x_n\}$.

(Here co indicates the absolutely convex hull)

THEOREM 1.1. For a polar seminorm $p$ on a $K$-vector space $E$ the following are equivalent.

(a) $p(x) \in |K|$ for each $x \in E$. Each one-dimensional subspace of $E$ has a $p$-orthocomplement.

(β) $p^0$ is $c'$-compact.

Proof. (a) $\Rightarrow$ (β). By [7], Theorem 3.2, it suffices to prove that for each $\phi \in (E^*, \sigma(E^*, E))'$

$$\max \{ |\phi(f)| : f \in p^0 \}$$

exists. Since $\phi$ is an evaluation map we therefore have to show that

$$\max \{ |f(x)| : f \in p^0 \}$$

exists for each $x \in E$. This is obviously true if $p(x) = 0$. So assume $p(x) > 0$. Since $p(x) \in |K|$ we may assume that $p(x) = 1$. For such $x$ we must prove

$$\max \{ |f(x)| : f \in p^0 \} = 1$$

By (α), $Kx$ has a $p$-orthocomplement $H$. The function

$$f : \lambda x + h \mapsto \lambda \quad (\lambda \in K, h \in H)$$
is in $E^*$. We have $|f(x)| = 1$. For $\lambda \in K$, $h \in H$

$$|f(\lambda x + h)| = |\lambda| = p(\lambda x) \leq \max(p(\lambda x), p(h)) = p(\lambda x + h)$$

so that $f \in \mathcal{P}$. 

(\$) $\Rightarrow$ (a). Let $x \in E$. The map $f \mapsto |f(x)|$ is a continuous seminorm on $(E^*, c(E^*, E))$. By $c'$-compactness its restriction to $\mathcal{P}$ has a maximum so there exists a $g \in \mathcal{P}$ with $|g(x)| = p(x)$ (It follows that $p(x) \in |K|$). We prove that $\ker g$ is a $p$-orthocomplement of $Kx$. In fact, for $z \in \ker g$ we have

$$p(x+z) \leq |g(x+z)| = |g(x)| = p(x)$$

Then also

$$p(x+z) \geq p(z)$$

completing the proof of Theorem 1.1.

**Note.** It is not hard to see that (a) of above is equivalent too.

(\$) For each $x \in E$ there exists an $f \in E^*$ with $|f(x)| = p(x)$ and $|f| \leq p$.

For spherically complete $K$ we obtain a simpler form of Theorem 1.1.

**Corollary 1.2.** Let $K$ be spherically complete, let $p$ be a seminorm on $E$ for which $p(x) \in |K|$ for all $x \in E$.

Then the following are equivalent.

(a) $p(x) \in |K|$ for each $x \in E$.

(b) $\mathcal{P}$ is $c'$-compact.

**Proof.** By [1], lemma 4.35, each onedimensional subspace has a $p$-orthocomplement.
§2 APPLICATION: A NEW LIGHT ON SERRE'S RENORMING PROBLEM.

Consider the following two statements (*) and (**).

(*) Let $E$ be a $K$-vector space and let $\| \cdot \|$ be a norm on $E$. Then there exists a norm $\| \cdot \|'$ on $E$, equivalent to $\| \cdot \|$, such that $\|x\|' \leq |\lambda| \|x\|$ for all $x \in E$.

(**) Let $K$ be spherically complete and let $A$ be a complete absolutely convex compactoid in a Hausdorff locally convex space over $K$. Then there exist a $\lambda \in K$ with $|\lambda| > 1$ and a c'-compact $B$ such that $A \subset B \subset \lambda A$.

The question as to whether (*) is true or not is known as Serre's renorming problem. See [2] for more details. We are able to reformulate this problem in terms of compactoids:

PROPOSITION 2.1. The above statements (*) and (**) are equivalent.

Proof. Assume (*). To prove (**) we may assume that $A$ is edged. By [8], Theorem 3, $A$, as a topological module over $B(0,1) := \{\lambda \in K : |\lambda| \leq 1\}$, is isomorphic to a bounded submodule of $K^I$ for some set $I$. Let $E$ be the algebraic direct sum $\bigoplus_{i \in I} K_i$ where $K_i = K$ for all $i \in I$.

Then $(E^*, c(E^*, E))$ is in a natural way isomorphic to $K^I$ with the product topology. So we may assume that $A = p^0$ where $p$ is a seminorm on $E$.

By (*) there exists a seminorm $q$, equivalent to $p$, such that $q(x) \leq |\lambda|x$ for all $x \in E$. By a suitable scalar multiplication we can arrange that, in addition, $p \leq q \leq |\lambda|p$ for some $\lambda \in K$, $|\lambda| > 1$. Then

$$p^0 \leq q^0 \leq \lambda p^0$$
and \( q^0 \) is \( c' \)-compact by Corollary 1.2. This proves (**). Now assume (**).

To prove (*) we may assume (see [2]), that \( K \) is spherically complete.

Let \( p \) be a norm on \( E \). By (**) there is a \( c' \)-compact \( B \) and a \( \lambda \in K, \ |\lambda| > 1 \)
with \( p^0 \subset B \subset \lambda p^0 \). Then \( B = q^0 \) for some seminorm \( q \) on \( E \). We have

\[
p \leq q \leq |\lambda| p
\]

and \( q(x) \in |K| \) for all \( x \in E \) by Corollary 1.2.

Note. Serre's renorming problem is still unsettled as far as I know.

§3 NORMS \( p \) FOR WHICH \( p^0 \) IS A KREIN-MILMAN COMPACTOID.

Recall that an absolutely convex subset \( A \) of a locally convex space
over \( K \) is a \( KM \)-compactoid if it is complete and if \( A = \overline{co} X \) where \( X \) is
compact. (Here \( co X \) is the closure of \( co X \)).

Before stating the theorem we first make some simple observations. Let
\( p \) be a norm on \( E \). We say that a collection \( (e_i) \) in \( E \) is a
\( p \)-orthonormal base of \( E \) if for each \( x \in E \) there exist a unique
\( (\lambda_i)_{i \in I} \subset K^i \) such that \( \{ i \in I, \ |\lambda_i| > \varepsilon \} \) is finite for each \( \varepsilon > 0 \) and

\[
x = \sum_{i \in I} \lambda_i e_i
\]

\[
p(x) = \max_{i} |\lambda_i|
\]

If \( (E,p) \) is complete this definition coincides with the usual one.

**Lemma 3.1.** Let \( (E,p) \) be a normed space, let \( \hat{(E,p)} \) be its completion.

Then \( (E,p) \) has a \( p \)-orthonormal base if and only if \( \hat{(E,p)} \) has a
\( \hat{p} \)-orthonormal base.
Proof. It is not hard to see that each p-orthonormal base of \((E,p)\) is also a \(\hat{p}\)-orthonormal base of \((\hat{E},\hat{p})\). Conversely, let \((e_i)\) be a \(\hat{p}\)-orthonormal base of \((\hat{E},\hat{p})\). For each \(i \in I\), choose an \(f_i \in E\) with 
\[ p(e_i - f_i) \leq \frac{1}{2}. \]
By [1], Exercise 5.C, \((f_i)_{i \in I}\) is a \(\hat{p}\)-orthonormal base of \((\hat{E},\hat{p})\).
Clearly \((f_i)_{i \in I}\) is a p-orthonormal base of \((E,p)\).

THEOREM 3.2. For a polar norm \(p\) on a \(K\)-vector space \(E\) the following are equivalent.

(a) \((E,p)\) has a \(p\)-orthonormal base

(b) \(p^0\) is a KM-compactoid.

Proof. (a) \(\Rightarrow\) (b). Let \((e_i)_{i \in I}\) be a \(p\)-orthonormal base of \((E,p)\). The formula
\[
\phi(f) = (f(e_i))_{i \in I}
\]
defines a map \(\phi : p^0 \rightarrow B(0,1)^I\). Straightforward verifications show that \(\phi\) is an isomorphism of topological \(B(0,1)\)-modules. From [8], Theorem 16 we obtain that \(B(0,1)^I\), hence \(p^0\), is a KM-compactoid.

(b) \(\Rightarrow\) (a). Suppose \(p^0 = \text{co} X\) where \(X\) is a compact subset of \(E^*\). Let \(C(X^*K)\) be the Banach space of all continuous functions \(X \rightarrow K\), with the supremum norm \(\|\cdot\|_\infty\). Then \(C(X^*K)\) has an orthonormal base. ([1], Theorem 5.22).

The formula
\[
\phi(x)(f) = f(x) \quad (f \in X)
\]
defines a \(K\)-linear map \(\phi : E \rightarrow C(X^*K)\). From
\[ ||\phi(x)||_\infty = \max \{ |f(x)| : f \in X \} = \sup_{f \in X} |f(x)| = \sup_{f \in E} |f(x)| = p(x) \]

we obtain that $\phi$ is an isometry $(E, p) \rightarrow (C(X^+K), ||\cdot||_\infty)$.

By Gruson's Theorem ([1], 5.9) $\phi(E)$ has an orthonormal base. Then so has $\phi(E)$ by Lemma 3.1 and has $E$.

\section{APPLICATION: A COMPLETE $c'$-COMPACT SET WHICH IS NOT A KM-COMPACTOID.}

We shall give a negative answer to the Problem following Theorem 1.7 of [6].

\textbf{PROPOSITION 4.1.} Let $K$ be spherically complete, let $|K| = (0, \infty)$. Then there exist a locally convex space $F$ over $K$ and a complete $c'$-compact subset $A \subset F$ which is not a KM-compactoid.

\textbf{Proof.} Let $E := l^\infty$ and let $F := (l^\infty)^*$ (with the topology we agreed upon in §0). Let $p$ be the standard norm on $l^\infty$, and set $A := p^0$. Since, trivially, $p(x) \leq |K|$ for all $x \in l^\infty$, we have that $p^0$ is $c'$-compact (Corollary 1.2).

However, it is known ([1], Cor. 5.19) that $l^\infty$ has no orthogonal base so that (Theorem 3.2) $p^0$ is not a KM-compactoid.

\section{NORMS $p$ FOR WHICH $p^0$ IS METRIZABLE.}

\textbf{THEOREM 5.1.} For a polar seminorm $p$ on a $K$-vector space $E$ the following are equivalent.

(a) $(E, p)$ is of countable type ([3], Definition 4.3).

(\beta) $p^0$ is metrizable.
Proof. (a) ⇒ (b). There exist $e_1, e_2, \ldots$ in $E$ with $p(e_i) \leq 1$ for each $i$ such that the $K$-linear span of $e_1, e_2, \ldots$ is $p$-dense in $E$. The formula

$$\phi(f) = (f(e_1), f(e_2), \ldots)$$

defines a map $\phi : p^0 \to B(0,1)^{\mathbb{N}}$. Straightforward verifications show that $\phi$ is an isomorphism of topological $B(0,1)$-modules of $p^0$ onto a submodule of $B(0,1)^{\mathbb{N}}$.

Now $B(0,1)^{\mathbb{N}}$ is metrizable (the product topology is induced by the metric

$$(a,b) \mapsto \sup_{i \in \mathbb{N}} |a_i - b_i| 2^{-i}$$

hence so is $p^0$.

(b) ⇒ (a). Let $\lambda \in K$, $|\lambda| > 1$. Since $p^0$ is a metrizable compactoid there exist, by [3], Proposition 8.2, $f_1, f_2, \ldots \in \lambda p^0$ with $\lim_{n \to \infty} f_n = 0$ such that

$$p^0 \subseteq \overline{\{f_1, f_2, \ldots\}} \subseteq \lambda p^0$$

The map

$$\phi : x \mapsto (f_1(x), f_2(x), \ldots) \quad (x \in E)$$

is $K$-linear, $\phi(E) \subseteq c_0$. We have for $x \in E$

$$||\phi(x)|| = \sup_{n \in \mathbb{N}} |f_n(x)| = \sup \{ |g(x)| : g \in \overline{\{f_1, f_2, \ldots\}} \}$$

It follows that

$$p(x) \leq ||\phi(x)|| \leq |\lambda| p(x)$$

so that $p$ is equivalent to $x \mapsto ||\phi(x)||$, a seminorm of countable type. Hence, $p$ is of countable type.
§6 APPLICATION: DESCRIPTION OF METRIZABLE COMPACTOIDS.

THEOREM 6.1. Let A be an absolutely convex subset of a Hausdorff locally convex space F over K. The following are equivalent.

(a) A is a metrizable compactoid.

(β) As a topological B(0,1)-module, A is isomorphic to a submodule of B(0,1)^\mathbb{N}.

(γ) As a topological B(0,1)-module, A is isomorphic to a compactoid in C^0.

(δ) For each λ ∈ K, |λ| > 1 then exist e_1, e_2, ..., ∈ λ A with lim_{n→∞} e_n = 0 and A ⊂ co \{e_1, e_2, ....\}.

(ε) There exist e_1, e_2, ..., ∈ F with lim_{n→∞} e_n = 0 and A ⊂ co \{e_1, e_2, ....\}.

(ν) There exists an ultrametrizable compact X ⊂ F with A ⊂ co X.

Proof. (a) ⇒ (β). It is not hard to see, by using the absolute convexity of A, that A is also metrizable. As there is no harm in taking F complete we therefore may assume that A is complete. To prove (β) we also may assume that A is edged. By [8], Theorem 3, A ⊂ B(0,1)^I ⊂ K^I for some set I. Like in the proof of Proposition 2.1 we may conclude that A = p^0 where p is a polar seminorm on Φ \bigoplus_{i ∈ I} K_i (K_i = K for each i). Then p is of countable type by Theorem 5.1. From the proof of (a) ⇒ (β) of that Theorem we obtain an isomorphism A = p^0 \cong B(0,1)^\mathbb{N}.

(β) ⇒ (γ). Choose λ_1, λ_2, ..., ∈ K, |λ_1| > |λ_2| > ..., lim_{n→∞} λ_n = 0. The formula

φ((a_i)_{i ∈ \mathbb{N}}) = (λ_1a_1, λ_2a_2, ...) ∈ C^0

defines a B(0,1)-module isomorphism of B(0,1)^\mathbb{N} onto C := co \{λ_1e_1, λ_2e_2, ....\}

where e_1, e_2, ... are the standard unit vectors in C^0. φ is a homeomorphism.
$B(0,1)^\mathbb{N} \to \mathbb{C}$, and maps $A$ onto a compactoid in $c_0$.

$(\gamma) \Rightarrow (\delta)$. See [3], Proposition 8.2.

$(\delta) \Rightarrow (\varepsilon)$ is trivial.

$(\varepsilon) \Rightarrow (\eta), \{0, e_1, e_2, \ldots\}$ is compact and ultrametrizable.

$(\eta) \Rightarrow (\alpha).$ We may assume that $F$ is complete. It suffices to prove the metrizability of $B := \text{co} X$.

$B$ is a complete, edged compactoid. As before we may assume that $B = p^0$ for some polar seminorm $p$ on some $K$-vector space $E$ while $B \subset E^*$. The map $\phi : E \to C(X \to K)$ defined by

$$\phi(x) (f) = f(x) \quad (f \in X)$$

is an isometry $(E, p) \to (C(X \to K), ||| |||_w)$.

Now $X$ is ultrametrizable so by [1], Exercise 3.5, $C(X \to K)$ is of countable type. Hence so is $p$. By Theorem 5.1, $B = p^0$ is metrizable.

§7 NORMS $p$ FOR WHICH $(p^0)^1$ IS OF FINITE TYPE.

Recall that an absolutely convex set $A$ in a locally convex space $F$ over $K$ is of finite type if for each zero neighbourhood $U$ in $F$ there exists a finite-dimensional bounded set $S \subset A$ such that $A \subset U + S$.

Let us say that a seminorm $p$ on a $K$-vector space $E$ is of finite type if $\text{Ker} p = \{x \in E : p(x) = 0\}$ has finite codimension.

**Lemma 7.1.** Let $A$ be an absolutely convex subset of a locally convex space $F$ whose topology is generated by a collection of seminorms of finite type. Then the following are equivalent.

$(a)$ $A$ is a compactoid of finite type.

$(\beta)$ For each closed linear subspace $H$ of finite codimension there is a finite dimensional bounded set $S \subset A$ with $A \subset H + S$. 

Proof. (a) ⇒ (β). (Note. This implication holds for any locally convex space F.) We may assume \([A] = F\).

H has the form \(D^1 := \{x \in F : f(x) = 0 \text{ for all } f \in D\}\) where D is a finite dimensional subspace of \(F'\). Let \(f_1, \ldots, f_n\) be a base of D. There exist \(x_1, \ldots, x_n \in F\) with \(f_i(x_j) = \delta_{ij} (i, j \in \{1, \ldots, n\})\). Since \([A] = F\) there exists a \(\lambda \in K\), \(\lambda \neq 0\) such that \(\lambda x_i \in A\) for each \(i \in \{1, \ldots, n\}\).

Set

\[
U := \bigcap_{i=1}^{n} \{x \in F : |f_i(x)| \leq |\lambda|\}
\]

Then \(U\) is a zero neighbourhood in \(F\). A is a compactoid of finite type, so there exists a finite dimensional set \(S_1 \subseteq A\) with \(A \subseteq U + S_1\). Let \(x \in U\). Write \(x = y + z\) where

\[
y := x - \sum_{i=1}^{n} f_i(x) x_i
\]

\[
z := \sum_{i=1}^{n} f_i(x) x_i
\]

Now, since \(x \in U\), \(|f_i(x)| \leq |\lambda|\) for each \(i\) so that \(z = \sum_{i=1}^{n} f_i(x) x_i \in A\).

Further, for each \(j \in \{1, \ldots, n\}\)

\[
f_j(y) = f_j(x) - \sum_{i=1}^{n} f_i(x) f_j(x_i) = f_j(x) - f_j(x) = 0
\]

and it follows that \(y \in D^1 = H\). So \(x = y + z \in H + [x_1, \ldots, x_n] \cap A\). We see that

\[
A \subseteq U + S_1 \subseteq H + S_2 + S_1
\]

where \(S_2 := [x_1, \ldots, x_n] \cap A\). Then (β) is proved with \(S := S_1 + S_2\).

(β) ⇒ (α). Let \(U\) be a zero neighbourhood in \(F\). Since continuous seminorms are of finite type, \(U\) contains a closed subspace \(H\) of finite codimension. By (β) there exists a finite dimensional set \(S \subseteq A\) with \(S\) bounded and
A \subset H + S. Then A \subset U + S.

From now on we assume that the valuation on K is dense.

Recall that for an absolutely convex set B we have $B^\perp := \bigcup_{|\lambda| < 1} \lambda B$.

THEOREM 7.2. Let p be a polar norm on a K-vector space E. Then the following are equivalent.

(a) For each finite dimensional subspace D of E there exists a seminorm q on E, q of finite type, q \leq p and q = p on D.

(b) $(p^0)^\perp$ is of finite type.

Proof. (a) \Rightarrow (b). As each continuous seminorm on $E^*$ is of finite type it suffices to prove, by Lemma 7.1, that for a closed subspace H of $E^*$ of finite codimension there exists a finite dimensional set $S \subset (p^0)^\perp$ such that $(p^0)^\perp \subset H + S$.

Now, by (a), there is a seminorm q of finite type, q \leq p on E and q = p on D := H$. Let

$$S_1 := \{f \in E^* : |f| \leq q\}.$$

We see that $S_1$ is finite dimensional and since q \leq p we have $S_1 \subset p^0$.

We now shall prove that $(p^0)^\perp \subset H + S$ where $S := (S_1)^\perp$.

In fact, let $f \in (p^0)^\perp$. Then there is a $\lambda \in K$, $0 < |\lambda| < 1$ with $|f| \leq |\lambda| p$.

Choose $\lambda' \in K$ with $|\lambda| < |\lambda'| < 1$.

We have $|f| \leq |\lambda| q$ on D (since p = q on D) so we can extend f to a $g \in E^*$ with $|g| \leq |\lambda'| q$ on E. (This is because q is of finite type so that $(E, q)$ is strongly polar.) Now write

$$f = f - g + g$$

Since $f = g$ on D we have $f - g \in D^\perp = H$. 

Also, \(|(\lambda')^{-1} g| \leq q\) so that \((\lambda')^{-1} g \in S_1\), i.e. \(g \in (S_1)^\perp = S\).

(B) \(\Rightarrow\) (a). By Lemma 7.1 there exists a finite dimensional set \(S \subset (p_0)^\perp\) so that \((p_0)^\perp = D^\perp \cap (p_0)^\perp + S\).

Set \(q(x) := \sup_{h \in S} |h(x)|\) \((x \in E)\).

Then \(q(x) = 0\) for all \(x\) in the space \(S^\perp\) which has finite codimension. So \(q\) is of finite type.

Further, for \(x \in E\) we have

\[ q(x) = \sup_{h \in S} |h(x)| \leq \sup_{h \in (p_0)^\perp} |h(x)| = \sup_{h \in (p_0)^\perp} |h(x)| = p(x), \]

so \(q \leq p\). Finally, if \(x \in D\) then

\[ p(x) = \sup_{f \in p} |f(x)| = \sup_{f \in (p_0)^\perp} |f(x)| = \sup_{h \in D^\perp \cap (p_0)^\perp} |h(x) + t(x)| = \sup_{t \in S} |t(x)| = q(x). \]

Hence, \(p = q\) on \(D\).

§8 APPLICATION: A COMPLETE COMPACTOID IN \(c_0\) THAT IS NOT OF FINITE TYPE.

If \(K\) is spherically complete each complete absolutely convex compactoid is of finite type (See [4], 2.3).

If \(K\) is not spherically complete the unit ball of \(c_0\) is a complete compactoid for the weak topology but not of finite type (See [5], 1.6).

This is a non-metrizable compactoid. A compactoid in \((c_0, ||| |||),\) not of finite type, is given in [5], 1.4. However this compactoid is not closed. The following example provides an answer to the Problem following 1.5 in [5].
PROPOSITION 8.1. Let $K$ be not spherically complete. Then there exists an absolutely convex complete compactoid in $c_0$ that is not of finite type.

Proof. Let $(K^v, | |)$ be the spherical completion of $(K, | |)$ in the sense of [1], Theorem 4.49. Let $E$ be a $K$-subspace of $K^v$ of countably infinite dimension and let $p$ be the valuation $| |$ restricted to $E$. Then $x, y \in E$, $x \perp y$ in the sense of $p$ implies $x = 0$ or $y = 0$. Obviously, the norm $p$ is of countable type (hence polar) so, by Theorem 5.1, $p^0$ is metrizable and is by Theorem 6.1, isomorphic to a compactoid in $c_0$.

Suppose $p^0$ were of finite type. Then so would $(p^0)^1$ ([5], Proposition 2.4). By Theorem 7.2 we would have a seminorm $q$ on $E$, $q \leq p$, $q$ of finite type, $q(x) = p(x)$ for some $x \in E$, $x \neq 0$. But then $x \not\in \text{Ker } q$ in the sense of $p$ (If $q(y) = 0$ then $p(x-y) \geq q(x-y) = q(x) = p(x)$) which is impossible.

So, $p^0$ is not of finite type.
REFERENCES


