A CONNECTION BETWEEN $p$-ADIC BANACH SPACES AND LOCALLY CONVEX COMPACTOIDS

by

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Report 8736
December 1987

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ABSTRACT. For a vector space \( E \) over a non-archimedean valued field \( K \) a correspondence \( p \mapsto p^0 \) is established between seminorms \( p \) on \( E \) and compactoids \( p^0 \) in \( E^* \).

Examination of it yields the solution of two open problems (see §4 and §8) and a reformulation of Serre's renorming problem (see §2). As a by-product results on metrizable compactoids are obtained (see §6).

§0 THE CORRESPONDENCE \( p \mapsto p^0 \).

Throughout this note \( K \) is a non-archimedean valued field, complete with respect to the metric induced by the nontrivial valuation \( | \cdot | \).

Let \( E \) be a \( K \)-vector space, let \( E^* \) be its algebraic dual. A (non-archimedean) seminorm \( p \) on \( E \) is polar ([3], Definition 3.1), if

\[
p = \sup \{ |f| : f \in E^*, |f| \leq p \}
\]

Let \( P_E \) be the set of all polar seminorms on \( E \).

For each \( p \in P_E \) we set

\[
p^0 = \{ f \in E^* : |f| \leq p \}
\]

Then \( p^0 \) is an absolutely convex, edged ([3],§1b) subset of \( E^* \). It is easy to see that \( p^0 \) is a closed compactoid ([3],§1e) with respect to the topology \( \sigma(E^*,E) \), hence complete.
Let $C_E^*$ be the set of all closed absolutely convex, edged compactoids in $E^*$ with respect to $\sigma(E^*,E)$.

**PROPOSITION 0.** The map $p \mapsto p^0$ is a bijection of $P_E$ onto $C_E^*$. Its inverse assigns to every $A \in C_E^*$ the seminorm $p$ given by

$$p(x) = \sup \{|f(x)| : f \in A\} \quad (x \in E)$$

**Proof.** We shall prove surjectivity of $p \mapsto p^0$ leaving the (easy) rest of the proof to the reader. So, let $A \in C_E^*$; we shall prove that $A = p^0$ where $p(x) = \sup \{|f(x)| : f \in A\}$.

Obviously, $A \subset p^0$. Now let $g \in E^* \setminus A$, we prove that $g \not\in p^0$. The space $E^*$ is of countable type hence strongly polar ([3], Theorem 4.4). So by [3], Theorem 4.7, there exists a $0 \in (E^*, \sigma(E^*,E))'$ such that $|0| \leq 1$ on $A$, $|0(g)| > 1$. But, by [3], lemma 7.1, $0$ has the form $f \mapsto f(x)$ for some $x \in E$. Thus, $|f(x)| \leq 1$ for $f \in A$, $|g(x)| > 1$ i.e., $p(x) \leq 1$ and $|g(x)| > 1$ and it follows that $g \not\in p^0$.

**Remarks.**

1. Let $K$ be spherically (= maximally) complete. Then each nonarchimedean seminorm $p$ on $E$ for which $p(x) \in \overline{|K|}$ ($x \in E$) is polar ([3], Remark following 3.1).

2. Let $\tau$ be the locally convex topology on $E$ induced by all nonarchimedean seminorms i.e., $\tau$ is the strongest among all locally convex topologies on $E$. It is not hard to see that $(E,\tau)$ is a complete polar space and that $(E,\tau)$ and $(E^*,\sigma(E^*,E))$ are each others strong dual spaces.
§1 NORMS p FOR WHICH p^0 IS c'-COMPACT

Recall that an absolutely convex subset A of a locally convex space F over K is c'-compact if for each neighbourhood U of 0 in F there exist \( x_1, \ldots, x_n \in A \) (rather than \( x_1, \ldots, x_n \in F \)) such that \( A \subset U + \text{co} \{ x_1, \ldots, x_n \} \).

(Here \( \text{co} \) indicates the absolutely convex hull)

THEOREM 1.1. For a polar seminorm p on a K-vector space E the following are equivalent.

(a) \( p(x) \leq |K| \) for each \( x \in E \). Each one-dimensional subspace of E has a p-orthocomplement.

(β) \( p^0 \) is c'-compact.

Proof. (a) ⇒ (β). By [7], Theorem 3.2, it suffices to prove that for each \( \Phi \in (E^*, \sigma(E^*,E))^\prime \)

\[
\max \{|\Phi(f)| : f \in p^0\}
\]

exists. Since \( \Phi \) is an evaluation map we therefore have to show that

\[
\max \{|f(x)| : f \in p^0\}
\]

exists for each \( x \in E \). This is obviously true if \( p(x) = 0 \). So assume \( p(x) > 0 \). Since \( p(x) \leq |K| \) we may assume that \( p(x) = 1 \). For such \( x \) we must prove

\[
\max \{|f(x)| : f \in p^0\} = 1
\]

By (a), Kx has a p-orthocomplement H. The function

\[
f : \lambda x + h \mapsto \lambda \quad (\lambda \in K, h \in H)
\]
is in $E^*$. We have $|f(x)| = 1$. For $\lambda \in K$, $h \in H$

$$|f(\lambda x + h)| = |\lambda| = p(\lambda x) \leq \max(p(\lambda x), p(h)) = p(\lambda x + h)$$

so that $f \in p^0$.

$(f) \Rightarrow (a)$. Let $x \in E$. The map $f \mapsto |f(x)|$ ($f \in E^*$) is a continuous seminorm on $(E^*, c(E^*, E))$. By $c'$-compactness its restriction to $p^0$ has a maximum so there exists a $g \in p^0$ with $|g(x)| = p(x)$ (It follows that $p(x) \in |K|$). We prove that $\ker g$ is a $p$-orthocomplement of $Kx$. In fact, for $z \in \ker g$ we have

$$p(x+z) \leq |g(x+z)| = |g(x)| = p(x)$$

Then also

$$p(x+z) \leq p(z)$$

completing the proof of Theorem 1.1.

Note. It is not hard to see that $(a)$ of above is equivalent too.

$(\gamma)$ For each $x \in E$ there exists an $f \in E^*$ with $|f(x)| = p(x)$ and $|f| \leq p$.

For spherically complete $K$ we obtain a simpler form of Theorem 1.1.

COROLLARY 1.2. Let $K$ be spherically complete, let $p$ be a seminorm on $E$ for which $p(x) \in |K|$ for all $x \in E$.

Then the following are equivalent.

(a) $p(x) \in |K|$ for each $x \in E$.

$(\beta) p^0$ is $c'$-compact.

Proof. By [1], lemma 4.35, each onedimensional subspace has a $p$-orthocomplement.
§2 APPLICATION: A NEW LIGHT ON SERRE'S RENORMING PROBLEM.

Consider the following two statements (*) and (**).

(*) Let $E$ be a $K$-vector space and let $||| \cdot |||$ be a norm on $E$. Then there exists a norm $||| \cdot |||$' on $E$, equivalent to $||| \cdot |||$, such that $|||x|||' \leq |x|$ for all $x \in E$.

(**) Let $K$ be spherically complete and let $A$ be a complete absolutely convex compactoid in a Hausdorff locally convex space over $K$. Then there exist a $\lambda \in K$ with $|\lambda| > 1$ and a $c'$-compact $B$ such that $A \subset B \subset \lambda A$.

The question as to whether (*) is true or not is known as Serré's renorming problem. See [2] for more details. We are able to reformulate this problem in terms of compactoids:

**PROPOSITION 2.1.** The above statements (*) and (**) are equivalent.

**Proof.** Assume (*). To prove (**) we may assume that $A$ is edged. By [8], Theorem 3, $A$, as a topological module over $B(0,1) := \{ \lambda \in K : |\lambda| \leq 1 \}$, is isomorphic to a bounded submodule of $K^I$ for some set $I$. Let $E$ be the algebraic direct sum $\bigoplus_{i \in I} K_i$ where $K_i = K$ for all $i \in I$.

Then $(E^*, \sigma(E^*, E))$ is in a natural way isomorphic to $K^I$ with the product topology. So we may assume that $A = p^0$ where $p$ is a seminorm on $E$.

By (*) there exists a seminorm $q$, equivalent to $p$, such that $q(x) \in |K|$ for all $x \in E$. By a suitable scalar multiplication we can arrange that, in addition, $p \leq q \leq |\lambda|p$ for some $\lambda \in K$, $|\lambda| > 1$. Then

$$p^0 \leq q^0 \leq \lambda p^0$$
and $q^0$ is $c'$-compact by Corollary 1.2. This proves (**). Now assume (**).

To prove (*) we may assume (see [2]), that $K$ is spherically complete.

Let $p$ be a norm on $E$. By (**) there is a $c'$-compact $B$ and a $\lambda \in K$, $|\lambda| > 1$
with $p^0 < B < \lambda p^0$. Then $B = q^0$ for some seminorm $q$ on $E$. We have

$$p \leq q \leq |\lambda|p$$

and $q(x) < |K|$ for all $x \in E$ by Corollary 1.2.

Note. Serre's renorming problem is still unsettled as far as I know.

§3 NORMS $p$ FOR WHICH $p^0$ IS A KREIN-MILMAN COMPACTOID.

Recall that an absolutely convex subset $A$ of a locally convex space over $K$ is a $KM$-compactoid if it is complete and if $A = \overline{co\, X}$ where $X$ is compact. (Here $\overline{co\, X}$ is the closure of $co\, X$).

Before stating the theorem we first make some simple observations. Let $p$ be a norm on $E$. We say that a collection $(e_i)_{i \in I}$ in $E$ is a $p$-orthonormal base of $E$ if for each $x \in E$ there exist a unique $(\lambda_i)_{i \in I} \subset K^I$ such that $\{i \in I, |\lambda_i| > \epsilon\}$ is finite for each $\epsilon > 0$ and

$$x = \sum_{i \in I} \lambda_i e_i$$

$$|p(x)| = \max_{i} |\lambda_i|$$

If $(E, p)$ is complete this definition coincides with the usual one.

**Lemma 3.1.** Let $(E, p)$ be a normed space, let $(E', p')$ be its completion.

Then $(E, p)$ has a $p$-orthonormal base if and only if $(E', p')$ has a $p'$-orthonormal base.
Proof. It is not hard to see that each p-orthonormal base of \((E,p)\) is also a \(p\)-orthonormal base of \((E,p')\). Conversely, let \((e_i)_{i \in I}\) be a \(p\)-orthonormal base of \((E,p')\). For each \(i \in I\), choose an \(f_i \in E\) with \(p(e_i - f_i) \leq \frac{1}{2}\).

By [1], Exercise 5.C, \((f_i)_{i \in I}\) is a \(p\)-ortonormal base of \((E,p')\).

Clearly \((f_i)_{i \in I}\) is a \(p\)-orthonormal base of \((E,p)\).

**Theorem 3.2.** For a polar norm \(p\) on a \(K\)-vector space \(E\) the following are equivalent.

(a) \((E,p)\) has a \(p\)-orthonormal base

(b) \(p^0\) is a \(KM\)-compactoid.

**Proof.** (a) \(\Rightarrow\) (b). Let \((e_i)_{i \in I}\) be a \(p\)-orthonormal base of \((E,p)\). The formula

\[
\phi(f) = (f(e_i))_{i \in I}
\]

defines a map \(\phi : p^0 \rightarrow B(0,1)^I\). Straightforward verifications show that \(\phi\) is an isomorphism of topological \(B(0,1)^I\)-modules. From [8], Theorem 16 we obtain that \(B(0,1)^I\), hence \(p^0\), is a \(KM\)-compactoid.

(b) \(\Rightarrow\) (a). Suppose \(p^0 = \text{co} X\) where \(X\) is a compact subset of \(E^*\).

Let \(C(X^*K)\) be the Banach space of all continuous functions \(X + K\), with the supremum norm \(|| \cdot ||_\infty\). Then \(C(X^*K)\) has an orthonormal base. ([1], Theorem 5.22).

The formula

\[
\phi(x)(f) = f(x) \quad (f \in X)
\]

defines a \(K\)-linear map \(\phi : E \rightarrow C(X^*K)\). From
we obtain that $\phi$ is an isometry $(E,p) \rightarrow (C(X,K), \|\|_\infty)$. By Gruson's Theorem ([1], 5.9) $\phi(E)$ has an orthonormal base. Then so has $\phi(E)$ by Lemma 3.1 and has $E$.

§4 APPLICATION: A COMPLETE $c'$-COMPACT SET WHICH IS NOT A KM-COMPACTOID.

We shall give a negative answer to the Problem following Theorem 1.7 of [6].

PROPOSITION 4.1. Let $K$ be spherically complete, let $|K| = [0,\infty)$. Then there exist a locally convex space $F$ over $K$ and a complete $c'$-compact subset $A \subseteq F$ which is not a KM-compactoid.

Proof. Let $E := l^\infty$ and let $F := (l^\infty)^*$ (with the topology we agreed upon in §0). Let $p$ be the standard norm on $l^\infty$, and set $A := p^0$. Since, trivially, $p(x) \in |K|$ for all $x \in l^\infty$, we have that $p^0$ is $c'$-compact (Corollary 1.2).

However, it is known ([1], Cor. 5.19) that $l^\infty$ has no orthogonal base so that (Theorem 3.2) $p^0$ is not a KM-compactoid.

§5 NORMS $p$ FOR WHICH $p^0$ IS MÉTRIZABLE.

THEOREM 5.1. For a polar seminorm $p$ on a $K$-vector space $E$ the following are equivalent.

(a) $(E,p)$ is of countable type ([3], Definition 4.3).

(β) $p^0$ is metrizable.
Proof. (a) ⇒ (β). There exist e₁, e₂, ... in E with p(eᵢ) ≤ 1 for each i such that the K-linear span of e₁, e₂, ... is p-dense in E. The formula

φ(f) = (f(e₁), f(e₂), ...)

defines a map φ : p₀ → B(0,1)ᴺ. Straightforward verifications show that φ is an isomorphism of topological B(0,1)-modules of p₀ onto a submodule of B(0,1)ᴺ.

Now B(0,1)ᴺ is metrizable (the product topology is induced by the metric

(a,b) → sup |aᵢ - bᵢ|2⁻ⁱ

i∈ℕ

hence so is p₀. (B) ⇒ (a). Let λ ∈ K, |λ| > 1. Since p₀ is a metrizable compactoid there exist, by [3], Proposition 8.2, f₁, f₂, ... ∈ λp₀ with limₙ→∞ fₙ = 0 such that

p₀ ⊆ sup |{f₁, f₂, ...} | ≤ λp₀

The map

φ : x ↦ (f₁(x), f₂(x), ...)

is K-linear, φ(E) ⊆ c₀. We have for x ∈ E

||φ(x)|| = sup |fₙ(x)| = sup {||g(x)|| : g ∈ co {f₁ f₂, ...}}

It follows that

p(x) ≤ ||φ(x)|| ≤ |λ|p(x)

so that p is equivalent to x ↦ ||φ(x)||, a seminorm of countable type. Hence, p is of countable type.
THEOREM 6.1. Let $A$ be an absolutely convex subset of a Hausdorff locally convex space $F$ over $K$. The following are equivalent.

(a) $A$ is a metrizable compactoid.

(b) As a topological $B(0,1)$-module, $A$ is isomorphic to a submodule of $B(0,1)^N$.

(c) As a topological $B(0,1)$-module, $A$ is isomorphic to a compactoid in $C_0$.

(d) For each $\lambda \in K$, $|\lambda| > 1$ then exist $e_1, e_2, \ldots \in A$ with $\lim_{n \to \infty} e_n = 0$ and $A \subset \overline{\operatorname{co}}\{e_1, e_2, \ldots\}$.

(e) There exist $e_1, e_2, \ldots \in F$ with $\lim_{n \to \infty} e_n = 0$ and $A \subset \overline{\operatorname{co}}\{e_1, e_2, \ldots\}$.

(f) There exists an ultrametrizable compact $X \subset F$ with $A \subset \overline{\operatorname{co}} X$.

Proof. (a) $\Rightarrow$ (b). It is not hard to see, by using the absolute convexity of $A$, that $\overline{A}$ is also metrizable. As there is no harm in taking $F$ complete we therefore may assume that $A$ is complete. To prove (b) we also may assume that $A$ is edged. By [8], Theorem 3, $A \subset B(0,1)^I \subset K^I$ for some set $I$. Like in the proof of Proposition 2.1 we may conclude that $A = p^0$ where $p$ is a polar seminorm on $\oplus_{i \in I} (K_i = K$ for each $i)$. Then $p$ is of countable type by Theorem 5.1. From the proof of (a) $\Rightarrow$ (b) of that Theorem we obtain an isomorphism $A = p^0 \cong B(0,1)^N$.

(b) $\Rightarrow$ (c). Choose $\lambda_1, \lambda_2, \ldots \in K$, $|\lambda_1| > |\lambda_2| > \ldots$, $\lim_{n \to \infty} \lambda_n = 0$. The formula

$$\phi((a_i)_{i \in \mathbb{N}}) = (\lambda_1 a_1, \lambda_2 a_2, \ldots) \in C_0$$

defines a $B(0,1)$-module isomorphism of $B(0,1)^\mathbb{N}$ onto $C := \overline{\operatorname{co}}\{\lambda_1 e_1, \lambda_2 e_2, \ldots\}$ where $e_1, e_2, \ldots$ are the standard unit vectors in $C_0$. $\phi$ is a homeomorphism.
\( B(0,1)^{\mathbb{N}} \to C \), and maps \( A \) onto a compactoid in \( c_0 \).

(\( \gamma \)) \( \Rightarrow \) (\( \delta \)). See [3], Proposition 8.2.

(\( \delta \)) \( \Rightarrow \) (\( \varepsilon \)) is trivial.

(\( \varepsilon \)) \( \Rightarrow \) (\( \eta \)). \( \{0, e_1, e_2, \ldots \} \) is compact and ultrametrizable.

(\( \eta \)) \( \Rightarrow \) (a). We may assume that \( F \) is complete. It suffices to prove the metrizability of \( B := \text{co} X \).

\( B \) is a complete, edged compactoid. As before we may assume that \( B = p^0 \) for some polar seminorm \( p \) on some \( K \)-vector space \( E \) while \( B \subset E^* \). The map \( \phi : E \to C(X \times K) \) defined by

\[
\phi(x) (f) = f(x) \quad (f \in X)
\]

is an isometry \((E, p) \to (C(X \times K), || ||_\omega)\).

Now \( X \) is ultrametrizable so by [1], Exercise 3.5, \( C(X \times K) \) is of countable type. Hence so is \( p \). By Theorem 5.1f, \( B = p^0 \) is metrizable.

\section{Norms \( p \) For Which \((p^0)_i\) Is of Finite Type.}

Recall that an absolutely convex set \( A \) in a locally convex space \( F \) over \( K \) is of finite type if for each zero neighbourhood \( U \) in \( F \) there exists a finite-dimensional bounded set \( S \subset A \) such that \( A \subset U + S \).

Let us say that a seminorm \( p \) on a \( K \)-vector space \( E \) is of finite type if \( \ker p = \{x \in E : p(x) = 0\} \) has finite codimension.

\textbf{Lemma 7.1.} Let \( A \) be an absolutely convex subset of a locally convex space \( F \) whose topology is generated by a collection of seminorms of finite type. Then the following are equivalent.

(a) \( A \) is a compactoid of finite type.

(b) For each closed linear subspace \( H \) of finite codimension there is a finite dimensional bounded set \( S \subset A \) with \( A \subset H + S \).
Proof. \((a) \Rightarrow (\beta)\). (Note. This implication holds for any locally convex space \(F\).) We may assume \(\{A\} = F\).

\(H\) has the form \(D^1 := \{x \in F : f(x) = 0 \text{ for all } f \in D\}\) where \(D\) is a finite dimensional subspace of \(F\). Let \(f_1, \ldots, f_n\) be a base of \(D\). There exist \(x_1, \ldots, x_n \in F\) with \(f_i(x_j) = \delta_{ij} (i, j \in \{1, \ldots, n\})\). Since \(\{A\} = F\) there exists a \(\lambda \in K, \lambda \neq 0\) such that \(\lambda x_i \in A\) for each \(i \in \{1, \ldots, n\}\).

Set

\[
U := \bigcap_{i=1}^{n} \{x \in F : |f_i(x)| \leq |\lambda| \}
\]

Then \(U\) is a zero neighbourhood in \(F\). \(A\) is a compactoid of finite type, so there exists a finite dimensional set \(S_1 \subset A\) with \(A \subset U + S_1\). Let \(x \in U\). Write \(x = y + z\) where

\[
y := x - \sum_{i=1}^{n} f_i(x) x_i
\]

\[
z := \sum_{i=1}^{n} f_i(x) x_i
\]

Now, since \(x \in U\), \(|f_i(x)| \leq |\lambda|\) for each \(i\) so that \(z = \sum_{i=1}^{n} f_i(x) x_i \in A\).

Further, for each \(j \in \{1, \ldots, n\}\)

\[
f_j(y) = f_j(x) - \sum_{i=1}^{n} f_i(x) f_j(x_i) = f_j(x) - f_j(x) = 0
\]

and it follows that \(y \in D^1 = H\). So \(x = y + z \in H + [x_1, \ldots, x_n] \cap A\). We see that

\(A \subset U + S_1 \subset H + S_2 + S_1\)

where \(S_2 := [x_1, \ldots, x_n] \cap A\). Then \((\beta)\) is proved with \(S := S_1 + S_2\).

\((\beta) \Rightarrow (a)\). Let \(U\) be a zero neighbourhood in \(F\). Since continuous seminorms are of finite type, \(U\) contains a closed subspace \(H\) of finite codimension.

By \((\beta)\) there exists a finite dimensional set \(S \subset A\) with \(S\) bounded and
A ⊂ H + S. Then A ⊂ U + S.

From now on we assume that the valuation on K is dense.
Recall that for an absolutely convex set B we have $B^\dagger := \bigcup_{|\lambda|<1} \lambda B$.

**Theorem 7.2.** Let $p$ be a polar norm on a K-vector space $E$. Then the following are equivalent.

(a) For each finite dimensional subspace $D$ of $E$ there exists a seminorm $q$ on $E$, $q$ of finite type, $q \leq p$ and $q = p$ on $D$.

(b) $(p^0)^\dagger$ is of finite type.

**Proof.** (a) ⇒ (b). As each continuous seminorm on $E^*$ is of finite type it suffices to prove, by Lemma 7.1, that for a closed subspace $H$ of $E^*$ of finite codimension there exists a finite dimensional set $S \in (p^0)^\dagger$ such that $(p^0)^\dagger \subset H + S$.

Now, by (a), there is a seminorm $q$ of finite type, $q \leq p$ on $E$ and $q = p$ on $D := H^\perp$. Let

$$S_1 := \{ f \in E^* : |f| \leq q \}.$$  

We see that $S_1$ is finite dimensional and since $q \leq p$ we have $S_1 \subset p^0$.

We now shall prove that $(p^0)^\dagger \subset H + S$ where $S := (S_1)^\dagger$.

In fact, let $f \in (p^0)^\dagger$. Then there is a $\lambda \in K$, $0 < |\lambda| < 1$ with $|f| \leq |\lambda| p$.

Choose $\lambda' \in K$ with $|\lambda| < |\lambda'| < 1$.

We have $|f| \leq |\lambda'| q$ on $D$ (since $p = q$ on $D$) so we can extend $f$ to a $g \in E^*$ with $|g| \leq |\lambda'| q$ on $E$. (This is because $q$ is of finite type so that $(E,q)$ is strongly polar.) Now write

$$f = f - g + g$$

Since $f = g$ on $D$ we have $f - g \in D^\perp = H$. 


Also, \( |(\lambda')^{-1} g| \leq q \) so that \((\lambda')^{-1} g \in S_1\) i.e. \(g \in (S_1)^i = S\).

(\(\beta\) \Rightarrow (\alpha)). By lemma 7.1 there exists a finite dimensional set \(S \subset (p^0)^i\) so that \((p^0)^i = D^\perp \cap (p^0)^i + S\).

Set \(q(x) := \sup_{\mathbf{h} \in S} |h(x)|,\quad (x \in E)\).

Then \(q(x) = 0\) for all \(x\) in the space \(S^\perp\) which has finite codimension. So \(q\) is of finite type.

Further, for \(x \in E\) we have

\[
q(x) = \sup_{\mathbf{h} \in S} |h(x)| \leq \sup_{\mathbf{h} \in (p^0)^i} |h(x)| = \sup_{\mathbf{h} \in (p^0)^i} |h(x)| = p(x),
\]

so \(q \leq p\). Finally, if \(x \in D\) then

\[
p(x) = \sup_{f \in p^0} |f(x)| = \sup_{f \in (p^0)^i} |f(x)| = \sup_{h \in D^\perp \cap (p^0)^i} |h(x) + t(x)| = \sup_{t \in S} |t(x)| = q(x).
\]

Hence, \(p = q\) on \(D\).

§8 APPLICATION: A COMPLETE COMPACTOID IN \(c_0\) THAT IS NOT OF FINITE TYPE.

If \(K\) is spherically complete each complete absolutely convex compactoid is of finite type (See [4], 2.3).

If \(K\) is not spherically complete the unit ball of \(c_0\) is a complete compactoid for the weak topology but not of finite type (See [5], 1.6).

This is a non-metrizable compactoid. A compactoid in \((c_0, ||||)\), not of finite type, is given in [5], 1.4. However this compactoid is not closed. The following example provides an answer to the Problem following 1.5 in [5].
PROPOSITION 8.1. Let $K$ be not spherically complete. Then there exists an absolutely convex complete compactoid in $c_0$ that is not of finite type.

Proof. Let $(K^v, | |)$ be the spherical completion of $(K, | |)$ in the sense of [1], Theorem 4.49. Let $E$ be a $K$-subspace of $K^v$ of countably infinite dimension and let $p$ be the valuation $| |$ restricted to $E$. Then $x,y \in E$, $x \perp y$ in the sense of $p$ implies $x = 0$ or $y = 0$. Obviously, the norm $p$ is of countable type (hence polar) so, by Theorem 5.1, $p^0$ is metrizable and is by Theorem 6.1, isomorphic to a compactoid in $c_0$.

Suppose $p^0$ were of finite type. Then so would $(p^0)^i$ ([5], Proposition 2.4). By Theorem 7.2 we would have a seminorm $q$ on $E$, $q \leq p$, $q$ of finite type, $q(x) = p(x)$ for some $x \in E$, $x \neq 0$. But then $x \perp \ker q$ in the sense of $p$ (If $q(y) = 0$ then $p(x-y) \geq q(x-y) = q(x) = p(x)$) which is impossible.

So, $p^0$ is not of finite type.
REFERENCES


