A CONNECTION BETWEEN $p$-ADIC BANACH SPACES AND LOCALLY CONVEX
COMPACTOIDS

by

W.H. SCHIKHOF

Report 8736
December 1987

DEPARTMENT OF MATHEMATICS
CATHOLIC UNIVERSITY
Toernooiveld
6525 ED Nijmegen
The Netherlands
A CONNECTION BETWEEN p-ADIC BANACH SPACES AND LOCALLY
CONVEX COMPACTOIDS

by

W.H. Schikhof

ABSTRACT. For a vector space $E$ over a non-archimedean valued field $K$ a correspondence $p \mapsto p^0$ is established between seminorms $p$ on $E$ and compactoids $p^0$ in $E^*$. Examination of it yields the solution of two open problems (see §4 and §8) and a reformulation of Serre's renorming problem (see §2). As a by-product results on metrizable compactoids are obtained (see §6).

§0 THE CORRESPONDENCE $p \mapsto p^0$.

Throughout this note $K$ is a non-archimedean valued field, complete with respect to the metric induced by the nontrivial valuation $| |$.

Let $E$ be a $K$-vector space, let $E^*$ be its algebraic dual. A (non-archimedean) seminorm $p$ on $E$ is polar ([3], Definition 3.1), if

$$p = \sup \{ |f| : f \in E^*, |f| \leq p \}$$

Let $P_E$ be the set of all polar seminorms on $E$.

For each $p \in P_E$ we set

$$p^0 = \{ f \in E^* : |f| \leq p \}$$

Then $p^0$ is an absolutely convex, edged ([3],§1b) subset of $E^*$. It is easy to see that $p^0$ is a closed compactoid ([3],§1e) with respect to the topology $\sigma(E^*,E)$, hence complete.
Let $C_\mathcal{E}^*$ be the set of all closed absolutely convex, edged compactoids in $E^*$ with respect to $\sigma(E^*,E)$.

**PROPOSITION 0.** The map $p \mapsto p^0$ is a bijection of $P_\mathcal{E}$ onto $C_\mathcal{E}^*$. Its inverse assigns to every $A \in C_\mathcal{E}^*$ the seminorm $p$ given by

$$p(x) = \sup \{|f(x)| : f \in A\} \quad (x \in E)$$

**Proof.** We shall prove surjectivity of $p \mapsto p^0$ leaving the (easy) rest of the proof to the reader. So, let $A \in C_\mathcal{E}^*$; we shall prove that $A = p^0$ where $p(x) = \sup \{|f(x)| : f \in A\}$.

Obviously, $A \subset p^0$. Now let $g \in E^* \setminus A$, we prove that $g \not\in p^0$. The space $E^*$ is of countable type hence strongly polar ([3],Theorem 4.4). So by [3], Theorem 4.7, there exists a $0 \in (E^*, \sigma(E^*,E)^*)'$ such that $|0| \leq 1$ on $A$, $|0(g)| > 1$. But, by [3], lemma 7.1, $\theta$ has the form $f \mapsto f(x)$ for some $x \in E$. Thus, $|f(x)| \leq 1$ for $f \in A$, $|g(x)| > 1$ i.e., $p(x) \leq 1$ and $|g(x)| > 1$ and it follows that $g \not\in p^0$.

**Remarks.**

1. Let $K$ be spherically (= maximally) complete. Then each nonarchimedean seminorm $p$ on $E$ for which $p(x) \in |K|$ $(x \in E)$ is polar ([3], Remark following 3.1).

2. Let $\tau$ be the locally convex topology on $E$ induced by all nonarchimedean seminorms i.e., $\tau$ is the strongest among all locally convex topologies on $E$. It is not hard to see that $(E,\tau)$ is a complete polar ([3],Definition 3.5) space and that $(E,\tau)$ and $(E^*,\sigma(E^*,E))$ are each others strong dual spaces.
§1 NORMS $p$ FOR WHICH $p^0$ IS $c'$-COMPACT

Recall that an absolutely convex subset $A$ of a locally convex space $F$ over $K$ is $c'$-compact if for each neighbourhood $U$ of 0 in $F$ there exist $x_1, \ldots, x_n \in A$ (rather than $x_1, \ldots, x_n \in F$) such that $A \subseteq U + \text{co} \{x_1, \ldots, x_n\}$. (Here $\text{co}$ indicates the absolutely convex hull)

THEOREM 1.1. For a polar seminorm $p$ on a $K$-vector space $E$ the following are equivalent.

(a) $p(x) \in |K|$ for each $x \in E$. Each one-dimensional subspace of $E$ has a $p$-orthocomplement.
(b) $p^0$ is $c'$-compact.

Proof. (a) $\Rightarrow$ (b). By [7], Theorem 3.2, it suffices to prove that for each $\phi \in (E^*, \sigma(E^*, E))$

$$\max \{|\phi(f)| : f \in p^0\}$$

exists. Since $\phi$ is an evaluation map we therefore have to show that

$$\max \{|f(x)| : f \in p^0\}$$

exists for each $x \in E$. This is obviously true if $p(x) = 0$. So assume $p(x) > 0$. Since $p(x) \in |K|$ we may assume that $p(x) = 1$. For such $x$ we must prove

$$\max \{|f(x)| : f \in p^0\} = 1$$

By (a), $Kx$ has a $p$-orthocomplement $H$. The function

$$f : \lambda x + h \mapsto \lambda \quad (\lambda \in K, h \in H)$$
is in $E^*$. We have $|f(x)| = 1$. For $\lambda \in K$, $h \in H$

$$|f(\lambda x + h)| = |\lambda| = p(\lambda x) \leq \max(p(\lambda x), p(h)) = p(\lambda x + h)$$

so that $f \in p^0$.

(\(\delta\)) $\Rightarrow$ (\(\alpha\)). Let $x \in E$. The map $f \mapsto |f(x)|$ is a continuous seminorm on $(E^*, \sigma(E^*, E))$. By $c'$-compactness its restriction to $p^0$ has a maximum so there exists a $g \in p^0$ with $|g(x)| = p(x)$ (It follows that $p(x) \in |K|$). We prove that $\ker g$ is a $p$-orthocomplement of $Kx$. In fact, for $z \in \ker g$ we have

$$p(x+z) \geq |g(x+z)| = |g(x)| = p(x)$$

Then also

$$p(x+z) \geq p(z)$$

completing the proof of Theorem 1.1.

Note. It is not hard to see that (\(\alpha\)) of above is equivalent too.

(\(\gamma\)) For each $x \in E$ there exists an $f \in E^*$ with $|f(x)| = p(x)$ and $|f| \leq p$.

For spherically complete $K$ we obtain a simpler form of Theorem 1.1.

COROLLARY 1.2. Let $K$ be spherically complete, let $p$ be a seminorm on $E$ for which $p(x) \in |K|$ for all $x \in E$.

Then the following are equivalent.

(\(\alpha\)) $p(x) \in |K|$ for each $x \in E$.

(\(\beta\)) $p^0$ is $c'$-compact.

Proof. By [1], lemma 4.35, each one-dimensional subspace has a $p$-orthocomplement.
§2 APPLICATION: A NEW LIGHT ON SERRE'S RENORMING PROBLEM.

Consider the following two statements (*) and (**).

(*) Let $E$ be a $K$-vector space and let $|| \cdot ||$ be a norm on $E$. Then there exists a norm $|| \cdot ||'$ on $E$, equivalent to $|| \cdot ||$, such that $||x||' \leq |K|$ for all $x \in E$.

(**) Let $K$ be spherically complete and let $A$ be a complete absolutely convex compactoid in a Hausdorff locally convex space over $K$. Then there exist $\lambda \in K$ with $|\lambda| > 1$ and a $c'$-compact $B$ such that $A \subset B \subset \lambda A$.

The question as to whether (*) is true or not is known as Serre's renorming problem. See [2] for more details. We are able to reformulate this problem in terms of compactoids:

PROPOSITION 2.1. The above statements (*) and (**) are equivalent.

Proof. Assume (*). To prove (**) we may assume that $A$ is edged. By [8], Theorem 3, $A$, as a topological module over $B(0,1) := \{ \lambda \in K : |\lambda| \leq 1 \}$, is isomorphic to a bounded submodule of $K^I$ for some set $I$. Let $E$ be the algebraic direct sum $\bigoplus_{i \in I} K_i$ where $K_i = K$ for all $i \in I$.

Then $(E^*, \sigma(E^*, E))$ is in a natural way isomorphic to $K^I$ with the product topology. So we may assume that $A = p^0$ where $p$ is a seminorm on $E$.

By (*) there exists a seminorm $q$, equivalent to $p$, such that $q(x) \in |K|$ for all $x \in E$. By a suitable scalar multiplication we can arrange that, in addition, $p \leq q \leq |\lambda|p$ for some $\lambda \in K$, $|\lambda| > 1$. Then

$$p^0 \leq q^0 \leq \lambda p^0$$
and $q^0$ is $c'$-compact by Corollary 1.2. This proves (**1). Now assume (**2).

To prove (*) we may assume (see [2]), that $K$ is spherically complete.

Let $p$ be a norm on $E$. By (**2) there is a $c'$-compact $B$ and a $\lambda \in K$, $|\lambda| > 1$
with $p^0 \subset B \subset \lambda p^0$. Then $B = q^0$ for some seminorm $q$ on $E$. We have

$$p \leq q \leq |\lambda|p$$

and $q(x) \in |K|$ for all $x \in E$ by Corollary 1.2.

Note. Serre's renorming problem is still unsettled as far as I know.

§3 NORMS $p$ FOR WHICH $p^0$ IS A KREIN-MILMAN COMPACTOID.

Recall that an absolutely convex subset $A$ of a locally convex space
over $K$ is a KM-compactoid if it is complete and if $A = \overline{\text{co}} X$ where $X$ is compact. (Here $\overline{\text{co}} X$ is the closure of $\text{co} X$).

Before stating the theorem we first make some simple observations. Let

$p$ be a norm on $E$. We say that a collection $(e_i)$ in $E$ is a

$p$-orthonormal base of $E$ if for each $x \in E$ there exist a unique

$(\lambda_i)_{i \in I} \subset K^I$ such that $\{i \in I, |\lambda_i| \geq \epsilon\}$ is finite for each $\epsilon > 0$ and

$$x = \sum_{i \in I} \lambda_i e_i$$

$$p(x) = \max_{i} |\lambda_i|$$

If $(E,p)$ is complete this definition coincides with the usual one.

**Lemma 3.1.** Let $(E,p)$ be a normed space, let $(\hat{E},\hat{p})$ be its completion. Then $(E,p)$ has a $p$-orthonormal base if and only if $(\hat{E},\hat{p})$ has a $\hat{p}$-orthonormal base.
Proof. It is not hard to see that each \( p \)-orthonormal base of \((E,p)\) is also a \( p \)-orthonormal base of \((E',p')\). Conversely, let \( (e_i) \) be a \( p \)-orthonormal base of \((E',p')\). For each \( i \in I \), choose an \( f_i \in E \) with
\[
p (e_i - f_i) \leq \frac{1}{2}.
\]
By [1], Exercise 5.C, \( (f_i) \) is a \( p \)-orthonormal base of \((E',p')\).

Clearly \( (f_i) \) is a \( p \)-orthonormal base of \((E,p)\).

**Theorem 3.2.** For a polar norm \( p \) on a \( K \)-vector space \( E \) the following are equivalent.

(a) \((E,p)\) has a \( p \)-orthonormal base

(b) \( p^0 \) is a KM-compactoid.

Proof. (a) \( \Rightarrow \) (b). Let \( (e_i) \) be a \( p \)-orthonormal base of \((E,p)\). The formula
\[
\phi(f) = (f(e_i))_{i \in I}
\]
defines a map \( \phi : p^0 \to B(0,1)^I \). Straightforward verifications show that \( \phi \) is an isomorphism of topological \( B(0,1) \)-modules. From [8], Theorem 16 we obtain that \( B(0,1)^I \), hence \( p^0 \), is a KM-compactoid.

(b) \( \Rightarrow \) (a). Suppose \( p^0 = \text{co} X \) where \( X \) is a compact subset of \( E^* \).

Let \( C(X+K) \) be the Banach space of all continuous functions \( X + K \), with the supremum norm \( \| \cdot \|_\infty \). Then \( C(X+K) \) has an orthonormal base. ([1], Theorem 5.22).

The formula
\[
\phi(x)(f) = f(x) \quad (f \in X)
\]
defines a \( K \)-linear map \( \phi : E \to C(X+K) \). From
\[ \| \phi(x) \|_w = \max_{f \in X} \| f(x) \| = \sup_{f \in \mathcal{F}} \| f(x) \| = \sup_{f \in \mathcal{F}} \| f(x) \| = p(x) \]

we obtain that $\phi$ is an isometry $(E,p) \rightarrow (C(X^K), \| \|_w)$. By Gruson's Theorem ([1], 5.9) $\phi(E)$ has an orthonormal base. Then so has $\phi(E)$ by Lemma 3.1 and has $E$.

§4 APPLICATION: A COMPLETE $c'$-COMPACT SET WHICH IS NOT A KM-COMPACTOID.

We shall give a negative answer to the Problem following Theorem 1.7 of [6].

PROPOSITION 4.1. Let $K$ be spherically complete, let $|K| = [0,\infty)$. Then there exist a locally convex space $F$ over $K$ and a complete $c'$-compact subset $A \subset F$ which is not a KM-compactoid.

Proof. Let $E := l^\infty$ and let $F := (l^\infty)^*$ (with the topology we agreed upon in §0). Let $p$ be the standard norm on $l^\infty$, and set $A := p^0$. Since, trivially, $p(x) \in |K|$ for all $x \in l^\infty$, we have that $p^0$ is $c'$-compact (Corollary 1.2).

However, it is known ([1], Cor. 5.19) that $l^\infty$ has no orthogonal base so that (Theorem 3.2) $p^0$ is not a KM-compactoid.

§5 NORMS $p$ FOR WHICH $p^0$ IS METRIZABLE.

THEOREM 5.1. For a polar seminorm $p$ on a $K$-vector space $E$ the following are equivalent.

(a) $(E,p)$ is of countable type ([3], Definition 4.3).

(β) $p^0$ is metrizable.
Proof. (a) ⇒ (b). There exist \( e_1, e_2, \ldots \) in \( E \) with \( p(e_i) \leq 1 \) for each \( i \) such that the \( K \)-linear span of \( e_1, e_2, \ldots \) is \( p \)-dense in \( E \). The formula

\[
\phi(f) = (f(e_1), f(e_2), \ldots)
\]

defines a map \( \phi : p^0 \to B(0,1)^\mathbb{N} \). Straightforward verifications show that \( \phi \) is an isomorphism of topological \( B(0,1) \)-modules of \( p^0 \) onto a submodule of \( B(0,1)^\mathbb{N} \).

Now \( B(0,1)^\mathbb{N} \) is metrizable (the product topology is induced by the metric

\[
(a,b) \mapsto \sup_{i \in \mathbb{N}} |a_i - b_i| 2^{-i}
\]

hence so is \( p^0 \).

(b) ⇒ (a). Let \( \lambda \in K, \ |\lambda| > 1 \). Since \( p^0 \) is a metrizable compactoid there exist, by \([3]\), Proposition 8.2, \( f_1, f_2, \ldots \in \lambda p^0 \) with \( \lim_{n \to \infty} f_n = 0 \) such that

\[
p^0 \leq \sup \{ f_1, f_2, \ldots \} < \lambda p^0
\]

The map

\[
\phi : x \mapsto (f_1(x), f_2(x), \ldots) \quad (x \in E)
\]

is \( K \)-linear, \( \phi(E) \subset c_0 \). We have for \( x \in E \)

\[
||\phi(x)|| = \sup_{n \in \mathbb{N}} |f_n(x)| = \sup \{ |g(x)| : g \in \overline{\text{co}} \{ f_1, f_2, \ldots \} \}
\]

It follows that

\[
p(x) \leq ||\phi(x)|| \leq |\lambda|p(x)
\]

so that \( p \) is equivalent to \( x \mapsto ||\phi(x)|| \), a seminorm of countable type. Hence, \( p \) is of countable type.
THEOREM 6.1. Let \( A \) be an absolutely convex subset of a Hausdorff locally convex space \( F \) over \( K \). The following are equivalent.

(a) \( A \) is a metrizable compactoid.

(b) As a topological \( B(0,1) \)-module, \( A \) is isomorphic to a submodule of \( B(0,1)^\mathbb{N} \).

(c) As a topological \( B(0,1) \)-module, \( A \) is isomorphic to a compactoid in \( C_0 \).

(d) For each \( \lambda \in K, |\lambda| > 1 \) then exist \( e_1, e_2, \ldots \in \lambda A \) with \( \lim_{n \to \infty} e_n = 0 \) and \( A \subseteq \overline{\{e_1, e_2, \ldots \}} \).

(e) There exist \( e_1, e_2, \ldots \in F \) with \( \lim_{n \to \infty} e_n = 0 \) and \( A \subseteq \overline{\{e_1, e_2, \ldots \}} \).

(f) There exists an ultrametrizable compact \( X \subseteq F \) with \( A \subseteq \overline{co X} \).

Proof. (a) \( \Rightarrow \) (b). It is not hard to see, by using the absolute convexity of \( A \), that \( \overline{A} \) is also metrizable. As there is no harm in taking \( F \) complete we therefore may assume that \( A \) is complete. To prove (b) we also may assume that \( A \) is edged. By [8], Theorem 3, \( A \subseteq B(0,1)^I \subseteq K^I \) for some set \( I \). Like in the proof of Proposition 2.1 we may conclude that \( A = p^0 \) where \( p \) is a polar seminorm on \( \oplus K_i \) (\( K_i = K \) for each \( i \)). Then \( p \) is of countable type by Theorem 5.1. From the proof of (a) \( \Rightarrow \) (b) of that Theorem we obtain an isomorphism \( A = p^0 \cong B(0,1)^\mathbb{N} \).

(b) \( \Rightarrow \) (c). Choose \( \lambda_1, \lambda_2, \ldots \in K, |\lambda_1| > |\lambda_2| > \ldots, \lim_{n \to \infty} \lambda_n = 0 \). The formula

\[
\phi((a_i)_{i \in \mathbb{N}}) = (\lambda_1 a_1, \lambda_2 a_2, \ldots) \in c_0
\]

defines a \( B(0,1) \)-module isomorphism of \( B(0,1)^\mathbb{N} \) onto \( C := \overline{co \{\lambda_1 e_1, \lambda_2 e_2, \ldots\}} \) where \( e_1, e_2, \ldots \) are the standard unit vectors in \( c_0 \). \( \phi \) is a homeomorphism
$B(0,1)^\mathbb{N} \to \mathbb{C}$, and maps $A$ onto a compactoid in $c_0$.

$(\gamma) \Rightarrow (\delta)$. See [3], Proposition 8.2.

$(\delta) \Rightarrow (\epsilon)$ is trivial.

$(\epsilon) \Rightarrow (n), \{0,e_1,e_2,\ldots\}$ is compact and ultrametrizable.

$(n) \Rightarrow (\alpha)$. We may assume that $F$ is complete. It suffices to prove the

metrizability of $B := co X$.

$B$ is a complete, edged compactoid. As before we may assume that $B = p^0$

for some polar seminorm $p$ on some $K$-vector space $E$ while $B \subseteq E^*$. The

map $\phi : E \to C(X \times K)$ defined by

$$
\phi(x)(f) = f(x) \quad (f \in X)
$$

is an isometry $(E,p) \to (C(X \times K), ||\cdot||_w)$.

Now $X$ is ultrametrizable so by [1], Exercise 3.5, $C(X \times K)$ is of

countable type. Hence so is $p$. By Theorem 5.1, $B = p^0$ is metrizable.

§7 NORMS $p$ FOR WHICH $(p^0)_i$ IS OF FINITE TYPE.

Recall that an absolutely convex set $A$ in a locally convex space $F$ over

$K$ is of finite type if for each zero neighbourhood $U$ in $F$ there exists

a finite-dimensional bounded set $S \subseteq A$ such that $A \subseteq U + S$.

Let us say that a seminorm $p$ on a $K$-vector space $E$ is of finite type

if $\ker p = \{x \in E : p(x) = 0\}$ has finite codimension.

**Lemma 7.1.** Let $A$ be an absolutely convex subset of a locally convex

space $F$ whose topology is generated by a collection of seminorms of

finite type. Then the following are equivalent.

(a) $A$ is a compactoid of finite type.

(b) For each closed linear subspace $H$ of finite codimension there is a

finite dimensional bounded set $S \subseteq A$ with $A \subseteq H + S$. 
Proof. (α) ⇒ (β). (Note. This implication holds for any locally convex space $F$.) We may assume $[A] = F$.

$H$ has the form $D^\perp := \{x \in F : f(x) = 0 \text{ for all } f \in D\}$ where $D$ is a finite dimensional subspace of $F'$. Let $f_1, \ldots, f_n$ be a base of $D$. There exist $x_1, \ldots, x_n \in F$ with $f_i(x_j) = \delta_{ij}$ ($i, j \in \{1, \ldots, n\}$). Since $[A] = F$ there exists a $\lambda \in K$, $\lambda \neq 0$ such that $\lambda x_i \in A$ for each $i \in \{1, \ldots, n\}$.

Set

$$U := \bigcap_{i=1}^{n} \{x \in F : |f_i(x)| \leq |\lambda|\}$$

Then $U$ is a zero neighbourhood in $F$. $A$ is a compactoid of finite type, so there exists a finite dimensional set $S_1 \subseteq A$ with $A \subseteq U + S_1$. Let $x \in U$. Write $x = y + z$ where $y := x - \sum_{i=1}^{n} f_i(x) x_i$

$z := \sum_{i=1}^{n} f_i(x) x_i$

Now, since $x \in U$, $|f_i(x)| \leq |\lambda|$ for each $i$ so that $z = \sum_{i=1}^{n} f_i(x) x_i \in A$.

Further, for each $j \in \{1, \ldots, n\}$

$$f_j(y) = f_j(x) - \sum_{i=1}^{n} f_i(x) f_j(x_i) = f_j(x) - f_j(x) = 0$$

and it follows that $y \in D^\perp = H$. So $x = y + z \in H + [x_1, \ldots, x_n] \cap A$. We see that

$$A \subset U + S_1 \subset H + S_2 + S_1$$

where $S_2 := [x_1, \ldots, x_n] \cap A$. Then (β) is proved with $S := S_1 + S_2$.

(β) ⇒ (α). Let $U$ be a zero neighbourhood in $F$. Since continuous seminorms are of finite type, $U$ contains a closed subspace $H$ of finite codimension. By (β) there exists a finite dimensional set $S \subseteq A$ with $S$ bounded and
THEOREM 7.2. Let \( p \) be a polar norm on a \( K \)-vector space \( E \). Then the following are equivalent.

(a) For each finite dimensional subspace \( D \) of \( E \) there exists a seminorm \( q \) on \( E \), \( q \) of finite type, \( q \leq p \) and \( q = p \) on \( D \).

(b) \( (p^0)^i \) is of finite type.

Proof. (a) \( \Rightarrow \) (b). As each continuous seminorm on \( E^* \) is of finite type it suffices to prove, by Lemma 7.1, that for a closed subspace \( H \) of \( E^* \) of finite codimension there exists a finite dimensional set \( S \subseteq (p^0)^i \) such that \( (p^0)^i \subseteq H + S \).

Now, by (a), there is a seminorm \( q \) of finite type, \( q \leq p \) on \( E \) and \( q = p \) on \( D := H^1 \). Let

\[ S_1 := \{ f \in E^* : |f| \leq q \}. \]

We see that \( S_1 \) is finite dimensional and since \( q \leq p \) we have \( S_1 \subseteq p^0 \).

We now shall prove that \( (p^0)^i \subseteq H + S \) where \( S := (S_1)^i \).

In fact, let \( f \in (p^0)^i \). Then there is a \( \lambda \in K \), \( 0 < |\lambda| < 1 \) with \( |f| \leq |\lambda| p \).

Choose \( \lambda' \in K \) with \( |\lambda| < |\lambda'| < 1 \).

We have \( |f| \leq |\lambda| q \) on \( D \) (since \( p = q \) on \( D \)) so we can extend \( f \) to a \( g \in E^* \) with \( |g| \leq |\lambda'| q \) on \( E \). (This is because \( q \) is of finite type so that \( (E,q) \) is strongly polar.) Now write

\[ f = f - g + g \]

Since \( f = g \) on \( D \) we have \( f - g \in D^1 = H \).
Also, \( |(\lambda')^{-1}g| \leq q \) so that \( (\lambda')^{-1}g \in S_1 \) i.e. \( g \in (S_1)^i = S \).

(\beta) \Rightarrow (a). By lemma 7.1 there exists a finite dimensional set \( S \subset (p^0)^i \) so that \( (p^0)^i = D \cap (p^0)^i + S \).

Set \( q(x) := \sup_{h \in S} |h(x)|. \quad (x \in E) \).

Then \( q(x) = 0 \) for all \( x \) in the space \( S^\perp \) which has finite codimension.

So \( q \) is of finite type.

Further, for \( x \in E \) we have

\[
q(x) = \sup_{h \in S} |h(x)| \leq \sup_{h \in (p^0)^i} |h(x)| = \sup_{h \notin (p^0)^i} |h(x)| = p(x),
\]

so \( q \leq p \). Finally, if \( x \in D \) then

\[
p(x) = \sup_{f \in p^0} |f(x)| = \sup_{f \in (p^0)^i} |f(x)| = \sup_{h \in D \cap (p^0)^i} |h(x) + t(x)|
\]

\[
= \sup_{t \in S} |t(x)| = q(x). \quad \text{Hence, } p = q \text{ on } D.
\]

§8 APPLICATION: A COMPLETE COMPACTOID IN \( c_0 \) THAT IS NOT OF FINITE TYPE.

If \( K \) is spherically complete each complete absolutely convex compactoid
is of finite type (See [4], 2.3).

If \( K \) is not spherically complete the unit ball of \( c_0 \) is a complete
compactoid for the weak topology but not of finite type (See [5], 1.6).

This is a non-metrizable compactoid. A compactoid in \( (c_0, ||| |||) \), not
of finite type, is given in [5], 1.4. However this compactoid is not
closed. The following example provides an answer to the Problem
following 1.5 in [5].
PROPOSITION 8.1. Let K be not spherically complete. Then there exists an absolutely convex complete compactoid in $c_0$ that is not of finite type.

Proof. Let $(K^v, | |)$ be the spherical completion of $(K, | |)$ in the sense of [1], Theorem 4.49. Let $E$ be a K-subspace of $K^v$ of countably infinite dimension and let $p$ be the valuation $| |$ restricted to $E$. Then $x, y \in E$, $x \perp y$ in the sense of $p$ implies $x = 0$ or $y = 0$. Obviously, the norm $p$ is of countable type (hence polar) so, by Theorem 5.1, $p^0$ is metrizable and is by Theorem 6.1, isomorphic to a compactoid in $c_0$.

Suppose $p^0$ were of finite type. Then so would $(p^0)_{1}$ ([5], Proposition 2.4). By Theorem 7.2 we would have a seminorm $q$ on $E$, $q \leq p$, $q$ of finite type, $q(x) = p(x)$ for some $x \in E$, $x \neq 0$. But then $x \perp \ker q$ in the sense of $p$. (If $q(y) = 0$ then $p(x-y) \geq q(x-y) = q(x) = p(x)$) which is impossible. So, $p^0$ is not of finite type.
REFERENCES


