A CONNECTION BETWEEN p-ADIC BANACH SPACES AND LOCALLY

CONVEX COMPACTOIDS

by

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ABSTRACT. For a vector space E over a non-archimedean valued field K a correspondence \( p \leftrightarrow p^0 \) is established between seminorms \( p \) on \( E \) and compactoids \( p^0 \) in \( E^* \).

Examination of it yields the solution of two open problems (see §4 and §8) and a reformulation of Serre's renorming problem (see §2). As a by-product results on metrizable compactoids are obtained (see §6).

§0 THE CORRESPONDENCE \( p \leftrightarrow p^0 \).

Throughout this note \( K \) is a non-archimedean valued field, complete with respect to the metric induced by the nontrivial valuation \( | \cdot | \).

Let \( E \) be a \( K \)-vector space, let \( E^* \) be its algebraic dual. A (non-archimedean) seminorm \( p \) on \( E \) is polar ([3], Definition 3.1), if

\[
p = \sup \{ |f| : f \in E^*, |f| \leq p \}
\]

Let \( P_E \) be the set of all polar seminorms on \( E \).

For each \( p \in P_E \) we set

\[
p^0 = \{ f \in E^* : |f| \leq p \}
\]

Then \( p^0 \) is an absolutely convex, edged ([3],§1b) subset of \( E^* \). It is easy to see that \( p^0 \) is a closed compactoid ([3],§1e) with respect to the topology \( \sigma(E^*,E) \), hence complete.
Let \( C_E^* \) be the set of all closed absolutely convex, edged compactoids in \( E^* \) with respect to \( \sigma(E^*, E) \).

**PROPOSITION 0.** The map \( p \mapsto p^0 \) is a bijection of \( P_E \) onto \( C_E^* \). Its inverse assigns to every \( A \in C_E^* \) the seminorm \( p \) given by

\[
p(x) = \sup \{|f(x)| : f \in A\} \quad (x \in E)
\]

**Proof.** We shall prove surjectivity of \( p \mapsto p^0 \) leaving the (easy) rest of the proof to the reader. So, let \( A \in C_E^* \); we shall prove that \( A = p^0 \) where \( p(x) = \sup \{|f(x)| : f \in A\} \).

Obviously, \( A \subset p^0 \). Now let \( g \in E^* \backslash A \), we prove that \( g \notin p^0 \). The space \( E^* \) is of countable type hence strongly polar ([3], Theorem 4.4). So by [3], Theorem 4.7, there exists a \( 0 \in (E^*, \sigma(E^*, E))' \) such that \( |0| \leq 1 \) on \( A \), \( |0(g)| > 1 \). But, by [3], Lemma 7.1, \( \theta \) has the form \( f \mapsto f(x) \) for some \( x \in E \). Thus, \( |f(x)| \leq 1 \) for \( f \in A \), \( |g(x)| > 1 \) i.e., \( p(x) \leq 1 \) and \( |g(x)| > 1 \) and it follows that \( g \notin p^0 \).

**Remarks.**

1. Let \( K \) be spherically (= maximally) complete. Then each nonarchimedean seminorm \( p \) on \( E \) for which \( p(x) \in |K| \) \( (x \in E) \) is polar ([3], Remark following 3.1).

2. Let \( \tau \) be the locally convex topology on \( E \) induced by all nonarchimedean seminorms i.e., \( \tau \) is the strongest among all locally convex topologies on \( E \). It is not hard to see that \( (E, \tau) \) is a complete polar \( ([3], \text{Definition 3.5}) \) space and that \( (E, \tau) \) and \( (E^*, \sigma(E^*, E)) \) are each others strong dual spaces.
§1 NORMS $p$ FOR WHICH $p^0$ IS $c'$-COMPACT

Recall that an absolutely convex subset $A$ of a locally convex space $F$ over $K$ is $c'$-compact if for each neighbourhood $U$ of 0 in $F$ there exist $x_1, \ldots, x_n \in A$ (rather than $x_1, \ldots, x_n \in F$) such that $A \subset U + \text{co} \{x_1, \ldots, x_n\}$. (Here $\text{co}$ indicates the absolutely convex hull)

**Theorem 1.1.** For a polar seminorm $p$ on a $K$-vector space $E$ the following are equivalent.

(a) $p(x) \in |K|$ for each $x \in E$. Each one-dimensional subspace of $E$ has a $p$-orthocomplement.

(β) $p^0$ is $c'$-compact.

**Proof.** (a) ⇒ (β). By [7], Theorem 3.2, it suffices to prove that for each $\phi \in (E^*, \sigma(E^*,E))^*$

$$\max \{ |\phi(f)| : f \in p^0 \}$$

exists. Since $\phi$ is an evaluation map we therefore have to show that

$$\max \{ |f(x)| : f \in p^0 \}$$

exists for each $x \in E$. This is obviously true if $p(x) = 0$. So assume $p(x) > 0$. Since $p(x) \in |K|$ we may assume that $p(x) = 1$. For such $x$ we must prove

$$\max \{ |f(x)| : f \in p^0 \} = 1$$

By (a), $Kx$ has a $p$-orthocomplement $H$. The function

$$f : \lambda x + h \mapsto \lambda \quad (\lambda \in K, h \in H)$$
is in $E^*$. We have $|f(x)| = 1$. For $\lambda \in K$, $h \in H$

$$|f(\lambda x + h)| = |\lambda| = p(\lambda x) \leq \max(p(\lambda x), p(h)) = p(\lambda x + h)$$

so that $f \in p^0$.

(β) $\Rightarrow$ (α). Let $x \in E$. The map $f \mapsto |f(x)|$ ($f \in E^*$)
is a continuous seminorm on $(E^*, \sigma(E^*, E))$. By $c'$-compactness its
restriction to $p^0$ has a maximum so there exists a $g \in p^0$ with $|g(x)| = p(x)$
(It follows that $p(x) \in |K|$). We prove that Ker$\cdot g$ is a $p$-orthocomplement
of $Kx$. In fact, for $z \in \text{Ker} g$ we have

$$p(x+z) \geq |g(x+z)| = |g(x)| = p(x)$$

Then also

$$p(x+z) \geq p(z)$$

completing the proof of Theorem 1.1.

Note. It is not hard to see that (α) of above is equivalent too.

(γ) For each $x \in E$ there exists an $f \in E^*$ with $|f(x)| = p(x)$ and $|f| \leq p$.

For spherically complete $K$ we obtain a simpler form of Theorem 1.1.

COROLLARY 1.2. Let $K$ be spherically complete, let $p$ be a seminorm on
$E$ for which $p(x) \in |K|$ for all $x \in E$.

Then the following are equivalent.

(α) $p(x) \in |K|$ for each $x \in E$.

(β) $p^0$ is $c'$-compact.

Proof. By [1], lemma 4.35, each onedimensional subspace has a $p$-ortho-
complement.
§2 APPLICATION: A NEW LIGHT ON SERRE'S RENORMING PROBLEM.

Consider the following two statements (*) and (**) .

(*) Let $E$ be a $K$-vector space and let $|| \cdot ||$ be a norm on $E$. Then there exists a norm $|| \cdot ||'$ on $E$, equivalent to $|| \cdot ||$, such that $||x||' \leq |K|$ for all $x \in E$.

(**) Let $K$ be spherically complete and let $A$ be a complete absolutely convex compactoid in a Hausdorff locally convex space over $K$. Then there exist a $\lambda \in K$ with $|\lambda| > 1$ and a $c'$-compact $B$ such that $A \subseteq B \subseteq \lambda A$.

The question as to whether (*) is true or not is known as Serré's renorming problem. See [2] for more details. We are able to reformulate this problem in terms of compactoids:

**PROPOSITION 2.1.** The above statements (*) and (**) are equivalent.

**Proof.** Assume (*). To prove (**) we may assume that $A$ is edged. By [8], Theorem 3, $A$, as a topological module over $B(0,1) := \{\lambda \in K : |\lambda| \leq 1\}$, is isomorphic to a bounded submodule of $K^I$ for some set $I$. Let $E$ be the algebraic direct sum $\oplus K_i$ where $K_i = K$ for all $i \in I$.

Then $(E^*, \sigma(E^*, E))$ is in a natural way isomorphic to $K^I$ with the product topology. So we may assume that $A = p^0$ where $p$ is a seminorm on $E$.

By (*) there exists a seminorm $q$, equivalent to $p$, such that $q(x) \leq |K|$ for all $x \in E$. By a suitable scalar multiplication we can arrange that, in addition, $p \leq q \leq |\lambda|p$ for some $\lambda \in K$, $|\lambda| > 1$. Then

$$p^0 \leq q^0 \leq \lambda p^0$$
and \(q^0\) is \(c'\)-compact by Corollary 1.2. This proves (**). Now assume (**).

To prove (*) we may assume (see [2]), that \(K\) is spherically complete.

Let \(p\) be a norm on \(E\). By (**) there is a \(c'\)-compact \(B\) and a \(\lambda \in K, |\lambda| > 1\)

with \(p^0 \subset B \subset \lambda p^0\). Then \(B = q^0\) for some seminorm \(q\) on \(E\). We have

\[
p \leq q \leq |\lambda| p
\]

and \(q(x) \in |K|\) for all \(x \in E\) by Corollary 1.2.

Note. Serre's renorming problem is still unsettled as far as I know.

§3 NORMS \(p\) FOR WHICH \(p^0\) IS A KREIN-MILMAN COMPACTOID.

Recall that an absolutely convex subset \(A\) of a locally convex space

over \(K\) is a \(KM\)-compactoid if it is complete and if \(A = \overline{co X}\) where \(X\) is compact. (Here \(\overline{co X}\) is the closure of \(co X\)).

Before stating the theorem we first make some simple observations. Let \(p\) be a norm on \(E\). We say that a collection \((e_i)_{i \in I}\) in \(E\) is a

\(p\)-orthonormal base of \(E\) if for each \(x \in E\) there exist a unique

\((\lambda_i)_{i \in I} \subset K^\ast\) such that \(\{i \in I, |\lambda_i| \geq \varepsilon\}\) is finite for each \(\varepsilon > 0\) and

\[
x = \sum_{i \in I} \lambda_i e_i
\]

\[
p(x) = \max_{i} |\lambda_i|
\]

If \((E, p)\) is complete this definition coincides with the usual one.

**Lemma 3.1.** Let \((E, p)\) be a normed space, let \((\hat{E}, \hat{p})\) be its completion.

Then \((E, p)\) has a \(p\)-orthonormal base if and only if \((\hat{E}, \hat{p})\) has a

\(\hat{p}\)-orthonormal base.
Proof. It is not hard to see that each $p$-orthonormal base of $(E,p)$ is also a $\hat{p}$-orthonormal base of $(\hat{E},\hat{p})$. Conversely, let $(e_i)$ be a $\hat{p}$-orthonormal base of $(\hat{E},\hat{p})$. For each $i \in I$, choose an $f_i \in E$ with $p(e_i - f_i) \leq \frac{1}{2}$.

By [1], Exercise 5.C, $(f_i)_{i \in I}$ is a $\hat{p}$-orthonormal base of $(\hat{E},\hat{p})$.

Clearly $(f_i)_{i \in I}$ is a $p$-orthonormal base of $(E,p)$.

THEOREM 3.2. For a polar norm $p$ on a $K$-vector space $E$ the following are equivalent.

(a) $(E,p)$ has a $p$-orthonormal base

(b) $p^0$ is a KM-compactoid.

Proof. (a) $\Rightarrow$ (b). Let $(e_i)_{i \in I}$ be a $p$-orthonormal base of $(E,p)$. The formula

$$\phi(f) = (f(e_i))_{i \in I}$$

defines a map $\phi : p^0 \rightarrow B(0,1)^I$. Straightforward verifications show that $\phi$ is an isomorphism of topological $B(0,1)$-modules. From [8], Theorem 16 we obtain that $B(0,1)^I$, hence $p^0$, is a KM-compactoid.

(b) $\Rightarrow$ (a). Suppose $p^0 = \text{co} X$ where $X$ is a compact subset of $E^*$. Let $C(X+K)$ be the Banach space of all continuous functions $X + K$, with the supremum norm $\|\cdot\|_\infty$. Then $C(X+K)$ has an orthonormal base. ([1], Theorem 5.22).

The formula

$$\phi(x)(f) = f(x) \quad (f \in X)$$

defines a $K$-linear map $\phi : E \rightarrow C(X+K)$. From
\[ ||f(x)||_\infty = \max_{f \in X} |f(x)| = \sup_{f \in \mathcal{C}(X)} |f(x)| = \sup_{f \in \mathcal{C}(X)} |f(x)| = p(x) \]

we obtain that \( \phi \) is an isometry \((E, p) \rightarrow (C(X; K), || ||_\infty)\).

By Gruson's Theorem ([1], 5.9) \( \phi(E) \) has an orthonormal base. Then so has \( \phi(E) \) by Lemma 3.1 and has \( E \).

§4 APPLICATION: A COMPLETE c'-COMPACT SET WHICH IS NOT A KM-COMPACTOID.

We shall give a negative answer to the Problem following Theorem 1.7 of [6].

PROPOSITION 4.1. Let \( K \) be spherically complete, let \( |K| = [0, \infty) \).

Then there exist a locally convex space \( F \) over \( K \) and a complete

c'-compact subset \( A \subset F \) which is not a KM-compactoid.

**Proof.** Let \( E := l^\infty \) and let \( F := (l^\infty)^* \) (with the topology we agreed upon in §0). Let \( p \) be the standard norm on \( l^\infty \), and set \( A := p^0 \). Since, trivially, \( p(x) \in |K| \) for all \( x \in l^\infty \), we have that \( p^0 \) is c'-compact (Corollary 1.2).

However, it is known ([1], Cor. 5.19) that \( l^\infty \) has no orthogonal base so that (Theorem 3.2) \( p^0 \) is not a KM-compactoid.

§5 NORMS \( p \) FOR WHICH \( p^0 \) IS METRIZABLE.

THEOREM 5.1. For a polar seminorm \( p \) on a \( K \)-vector space \( E \) the following are equivalent.

(a) \((E, p)\) is of countable type ([3], Definition 4.3).

(b) \( p^0 \) is metrizable.
Proof. (a) ⇒ (b). There exist \( e_1, e_2, \ldots \) in \( E \) with \( p(e_i) \leq 1 \) for each \( i \) such that the \( K \)-linear span of \( e_1, e_2, \ldots \) is \( p \)-dense in \( E \). The formula

\[
\phi(f) = (f(e_1), f(e_2), \ldots)
\]

defines a map \( \phi : p^0 \to B(0,1)^\mathbb{N} \). Straightforward verifications show that \( \phi \) is an isomorphism of topological \( B(0,1) \)-modules of \( p^0 \) onto a submodule of \( B(0,1)^\mathbb{N} \).

Now \( B(0,1)^\mathbb{N} \) is metrizable (the product topology is induced by the metric

\[(a,b) \mapsto \sup_{i \in \mathbb{N}} |a_i - b_i| 2^{-i}\]

hence so is \( p^0 \).

(b) ⇒ (a). Let \( \lambda \in K, |\lambda| > 1 \). Since \( p^0 \) is a metrizable compactoid there exist, by [3], Proposition 8.2, \( f_1, f_2, \ldots \in \lambda p^0 \) with \( \lim_{n \to \infty} f_n = 0 \) such that

\[
p^0 \subseteq \overline{\text{co} \{f_1, f_2, \ldots \}} \subseteq \lambda p^0
\]

The map

\[
\phi : x \mapsto (f_1(x), f_2(x), \ldots) \quad (x \in E)
\]

is \( K \)-linear, \( \phi(E) \subseteq c_0 \). We have for \( x \in E \)

\[
||\phi(x)|| = \sup_{n \in \mathbb{N}} |f_n(x)| = \sup \{ |g(x)| : g \in c_0 \{f_1, f_2, \ldots \} \}
\]

It follows that

\[
p(x) \leq ||\phi(x)|| \leq \lambda p(x)
\]

so that \( p \) is equivalent to \( x \mapsto ||\phi(x)|| \), a seminorm of countable type. Hence, \( p \) is of countable type.
§6 APPLICATION: DESCRIPTION OF METRIZABLE COMPACTOIDS.

THEOREM 6.1. Let \( A \) be an absolutely convex subset of a Hausdorff locally convex space \( F \) over \( K \). The following are equivalent.

(a) \( A \) is a metrizable compactoid.

(b) As a topological \( B(0,1) \)-module, \( A \) is isomorphic to a submodule of \( B(0,1)^\mathbb{N} \).

(c) As a topological \( B(0,1) \)-module, \( A \) is isomorphic to a compactoid in \( C^0 \).

(d) For each \( \lambda \in K, |\lambda| > 1 \) there exist \( e_1, e_2, \ldots, e_\lambda \in A \) with \( \lim_{n \to \infty} e_n = 0 \) and \( A \in \overline{\{e_1, e_2, \ldots\}} \).

(e) There exist \( e_1, e_2, \ldots \in F \) with \( \lim_{n \to \infty} e_n = 0 \) and \( A \in \overline{\{e_1, e_2, \ldots\}} \).

(f) There exists an ultrametrizable compact \( X \subseteq F \) with \( A \subseteq \overline{X} \).

Proof. (a) \( \Rightarrow \) (b). It is not hard to see, by using the absolute convexity of \( A \), that \( \overline{A} \) is also metrizable. As there is no harm in taking \( F \) complete we therefore may assume that \( A \) is complete. To prove (b) we also may assume that \( A \) is edged. By [8], Theorem 3, \( A \subseteq B(0,1)^I \subseteq K^I \) for some set \( I \). Like in the proof of Proposition 2.1 we may conclude that \( A = p^0 \) where \( p \) is a polar seminorm on \( \otimes K_i (K_i = K \) for each \( i \)). Then \( p \) is of countable type by Theorem 5.1. From the proof of (a) \( \Rightarrow \) (b) of that Theorem we obtain an isomorphism \( A = p^0 \cong B(0,1)^\mathbb{N} \).

(b) \( \Rightarrow \) (c). Choose \( \lambda_1, \lambda_2, \ldots \in K, |\lambda_1| > |\lambda_2| > \ldots \), \( \lim_{n \to \infty} \lambda_n = 0 \). The formula

\[
\phi((a_i)_{i \in \mathbb{N}}) = (\lambda_1 a_1, \lambda_2 a_2, \ldots) \in c_0
\]

defines a \( B(0,1) \)-module isomorphism of \( B(0,1)^\mathbb{N} \) onto \( C := \overline{\{e_1, e_2, e_3, \ldots\}} \) where \( e_1, e_2, \ldots \) are the standard unit vectors in \( c_0 \). \( \phi \) is a homeomorphism.
$B(0,1)^\mathbb{N} \to C$, and maps $A$ onto a compactoid in $c_0$.

(γ) $\Rightarrow$ (δ). See [3], Proposition 8.2.

(δ) $\Rightarrow$ (ε) is trivial.

(ε) $\Rightarrow$ (η). $\{0,e_1,e_2,\ldots\}$ is compact and ultrametrizable.

(η) $\Rightarrow$ (α). We may assume that $F$ is complete. It suffices to prove the

metrizability of $B := \overline{co X}$.

$B$ is a complete, edged compactoid. As before we may assume that $B = p^0$

for some polar seminorm $p$ on some $K$-vector space $E$ while $B \subseteq E^*$. The

map $\phi : E \to C(X + K)$ defined by

$$
\phi(x)(f) = f(x) \quad (f \in X)
$$

is an isometry $(E,p) \to (C(X + K), ||\ ||_\omega)$.

Now $X$ is ultrametrizable so by [1], Exercise 3.5, $C(X + K)$ is of

countable type. Hence so is $p$. By Theorem 5.1, $B = p^0$ is metrizable.

§7 NORMS $p$ FOR WHICH $(p^0)_1$ IS OF FINITE TYPE.

Recall that an absolutely convex set $A$ in a locally convex space $F$ over

$K$ is of finite type if for each zero neighbourhood $U$ in $F$ there exists

a finite-dimensional bounded set $S \subset A$ such that $A \subset U + S$.

Let us say that a seminorm $p$ on a $K$-vector space $E$ is of finite type

if $\text{Ker } p = \{x \in E : p(x) = 0\}$ has finite codimension.

**LEMMA 7.1.** Let $A$ be an absolutely convex subset of a locally convex

space $F$ whose topology is generated by a collection of seminorms of

finite type. Then the following are equivalent.

(a) $A$ is a compactoid of finite type.

(b) For each closed linear subspace $H$ of finite codimension there is a

finite dimensional bounded set $S \subset A$ with $A \subset H + S$. 

Proof. \((a) \Rightarrow (b)\). (Note. This implication holds for any locally convex space \(F\).) We may assume \(\|A\| = F\).

\(H\) has the form \(D^1 := \{x \in F : f(x) = 0 \text{ for all } f \in D\}\) where \(D\) is a finite dimensional subspace of \(F'\). Let \(f_1, \ldots, f_n\) be a base of \(D\). There exist \(x_1, \ldots, x_n \in F\) with \(f_i(x_j) = \delta_{ij} (i, j \in \{1, \ldots, n\})\). Since \(\|A\| = F\) there exists a \(\lambda \in K, \lambda \neq 0\) such that \(\lambda x_i \in A\) for each \(i \in \{1, \ldots, n\}\).

Set
\[
U := \bigcap_{i=1}^{n} \{x \in F : |f_i(x)| \leq |\lambda|\}
\]

Then \(U\) is a zero neighbourhood in \(F\). \(A\) is a compactoid of finite type, so there exists a finite dimensional set \(S_1 \subset A\) with \(A \subset U + S_1\). Let \(x \in U\). Write \(x = y + z\) where
\[
y := x - \sum_{i=1}^{n} f_i(x) x_i
\]
\[
z := \sum_{i=1}^{n} f_i(x) x_i
\]
Now, since \(x \in U\), \(|f_i(x)| \leq |\lambda|\) for each \(i\) so that \(z = \sum_{i=1}^{n} f_i(x) x_i \in A\).

Further, for each \(j \in \{1, \ldots, n\}\)
\[
f_j(y) = f_j(x) - \sum_{i=1}^{n} f_i(x)f_j(x_i) = f_j(x) - f_j(x) = 0
\]
and it follows that \(y \in D^1 = H\). So \(x = y + z\) \(\in H + [x_1, \ldots, x_n] \cap A\). We see that
\[
A \subset U + S_1 \subset H + S_2 + S_1
\]
where \(S_2 := [x_1, \ldots, x_n] \cap A\). Then \((b)\) is proved with \(S := S_1 + S_2\).

\((b) \Rightarrow (a)\). Let \(U\) be a zero neighbourhood in \(F\). Since continuous seminorms are of finite type, \(U\) contains a closed subspace \(H\) of finite codimension.

By \((b)\) there exists a finite dimensional set \(S \subset A\) with \(S\) bounded and
A ⊂ H + S. Then A ⊂ U + S.

From now on we assume that the valuation on K is dense.

Recall that for an absolutely convex set B we have $B^\perp := \bigcup_{|\lambda| < 1} \lambda B$.

**THEOREM 7.2.** Let p be a polar norm on a K-vector space E. Then the following are equivalent.

(a) For each finite dimensional subspace D of E there exists a seminorm $q$ on E, q of finite type, $q \leq p$ and $q = p$ on D.

(b) $(p^0)^1$ is of finite type.

**Proof.** (a) $\Rightarrow$ (b). As each continuous seminorm on $E^*$ is of finite type it suffices to prove, by Lemma 7.1, that for a closed subspace H of $E^*$ of finite codimension there exists a finite dimensional set $S \in (p^0)^1$ such that $(p^0)^1 \subset H + S$.

Now, by (a), there is a seminorm $q$ of finite type, $q \leq p$ on E and $q = p$ on $D := H^\perp$. Let

$$S_1 := \{f \in E^*: |f| \leq q\}.$$ 

We see that $S_1$ is finite dimensional and since $q \leq p$ we have $S_1 \subset p^0$.

We now shall prove that $(p^0)^1 \subset H + S$ where $S := (S_1)^1$.

In fact, let $f \in (p^0)^1$. Then there is a $\lambda \in K$, $0 < |\lambda| < 1$ with $|f| \leq |\lambda| p$.

Choose $\lambda' \in K$ with $|\lambda| < |\lambda'| < 1$.

We have $|f| \leq |\lambda| q$ on D (since $p = q$ on D) so we can extend $f$ to a $g \in E^*$ with $|g| \leq |\lambda'| q$ on E. (This is because $q$ is of finite type so that $(E, q)$ is strongly polar.) Now write

$$f = f - g + g$$

Since $f = g$ on D we have $f - g \in D^\perp = H.$
Also, \((\lambda')^{-1} g \leq q\) so that \((\lambda')^{-1} g \in S_1\) i.e. \(g \in (S_1)^i = S\).

By lemma 7.1 there exists a finite dimensional set \(S \subset (p^0)^i\) so that \((p^0)^i = D^i \cap (p^0)^i + S\).

Set \(q(x) := \sup_{h \in S} |h(x)|.\) \((x \in E).\)

Then \(q(x) = 0\) for all \(x\) in the space \(S^\perp\) which has finite codimension. So \(q\) is of finite type.

Further, for \(x \in E\) we have
\[
q(x) = \sup_{h \in S} |h(x)| = \sup_{h \in (p^0)^i} |h(x)| = p(x),
\]
so \(q \leq p.\) Finally, if \(x \in D\) then
\[
p(x) = \sup_{f \in p} |f(x)| = \sup_{f \in (p^0)^i} |f(x)| = \sup_{h \in D^i \cap (p^0)^i} |h(x) + t(x)|
\]
\[
\sup_{t \in S} |t(x)| = q(x).\) Hence, \(p = q\) on \(D.\)

\[8\text{ APPLICATION: A COMPLETE COMPACTOID IN } c_0 \text{ THAT IS NOT OF FINITE TYPE.}\]

If \(K\) is spherically complete each complete absolutely convex compactoid is of finite type (See [4], 2.3).

If \(K\) is not spherically complete the unit ball of \(c_0\) is a complete compactoid for the weak topology but not of finite type (See [5], 1.6).

This is a non-metrizable compactoid. A compactoid in \((c_0, ||||),\) not of finite type, is given in [5], 1.4. However this compactoid is not closed. The following example provides an answer to the Problem following 1.5 in [5].
PROPOSITION 8.1. Let $K$ be not spherically complete. Then there exists an absolutely convex complete compactoid in $c_0$ that is not of finite type.

Proof. Let $(V, | |)$ be the spherical completion of $(K, | |)$ in the sense of [1], Theorem 4.49. Let $E$ be a $K$-subspace of $V$ of countably infinite dimension and let $p$ be the valuation $| |$ restricted to $E$. Then $x, y \in E$, $x \perp y$ in the sense of $p$ implies $x = 0$ or $y = 0$. Obviously, the norm $p$ is of countable type (hence polar) so, by Theorem 5.1, $p^0$ is metrizable and is by Theorem 6.1, isomorphic to a compactoid in $c_0$. Suppose $p^0$ were of finite type. Then so would $(p^0)^1$ ([5], Proposition 2.4). By Theorem 7.2 we would have a seminorm $q$ on $E$, $q \leq p$, $q$ of finite type, $q(x) = p(x)$ for some $x \in E$, $x \neq 0$. But then $x \perp \text{Ker} q$ in the sense of $p$ (If $q(y) = 0$ then $p(x-y) \geq q(x-y) = q(x) = p(x)$) which is impossible. So, $p^0$ is not of finite type.
REFERENCES


