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EPSILON STABILITY OF $p$-ADIC CHARACTERS

by

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ABSTRACT. The following problem is solved (see Theorem 1.1 below). Let $G$ be an abelian group, written additively, and let $K$ be an algebraically closed nonarchimedean valued field which is complete for the valuation $| |$. Let $f : G \to T_K := \{ \lambda \in K : |\lambda| = 1 \}$ be an $\varepsilon$-character i.e.

$$|f(x+y)-f(x)f(y)| \leq \varepsilon \quad (x,y \in G)$$

where $0 \leq \varepsilon < 1$. Does there exist a character $\alpha : G \to T_K$ for which $|\alpha(x)-f(x)| \leq \varepsilon \quad (x \in G)$? Is it unique?

NOTES. The results of this Report will be used in a future paper on $p$-adic almost periodic functions [4]. The expression 'epsilon stability' is taken from [1], p.11, where a closely related problem is discussed.
§1 THE THEOREM

Throughout, let $G, K, T_K$ be as above. For trivially valued fields $K$ the above problem has a trivial solution. So from now on we assume that the valuation of $K$ is non-trivial. The residue class field of $K$ is $k$. The characteristic of a field $L$ is denoted $\text{char} L$.

Let $0 < \varepsilon < 1$. A function $f : G \to T_K$ is an $\varepsilon$-character if 
$$|f(x+y)-f(x)f(y)| \leq \varepsilon \text{ for all } x, y \in G.$$ As usual, we shall say 'character' instead of '0-character'.

Consider the following statements (E) and (U).

(E) For each $\varepsilon \in [0,1)$ and each $\varepsilon$-character $f : G \to T_K$ there exists a character $\alpha : G \to T_K$ for which $|f(x)-\alpha(x)| \leq \varepsilon$ ($x \in G$).

(U) For each $\varepsilon \in [0,1)$ and each $\varepsilon$-character $f : G \to T_K$ there exists at most one character $\alpha : G \to T_K$ for which $|f(x)-\alpha(x)| \leq \varepsilon$ ($x \in G$).

The purpose of this note is to prove the following Theorem.

THEOREM 1.1.

(i) Let $\text{char} k = 0$. Then (E) holds for any $G$, (U) holds if and only if $G$ is a torsion group.

(ii) Let $\text{char} K = 0$, $\text{char} k = p \neq 0$. Then (E) holds if and only if $G$ has no subgroups of order $p$, (U) holds if and only if $G$ has no subgroups of index $p$.

(iii) Let $\text{char} K = p \neq 0$. Then (E) holds if and only if $G$ has no subgroups of order $p$, (U) holds if and only if no subgroup of $G/\mathbb{Z}_p$ where \[ G_p := \{x \in G : p^n x = 0 \text{ for some } n \in \mathbb{N}\}, \] has index $p$. 


We have the following trivial corollary.

**COROLLARY 1.2.**

(i) Let char \( k = 0 \). Then both \((E)\) and \((U)\) hold if and only if \( G \) is a torsion group.

(ii) Let char \( k = p \neq 0 \). Then both \((E)\) and \((U)\) hold if and only if \( G \) has neither subgroups of order \( p \), nor subgroups of index \( p \).

**REMARK.** The statement '\( G \) has no subgroups of order \( p \)' is obviously equivalent to 'the map \( x \mapsto px \ (x \in G) \) is injective'. It is not hard to see that '\( G \) has no subgroups of index \( p \)' is equivalent to 'the map \( x \mapsto px \ (x \in G) \) is surjective'.

**EXAMPLES.** If \( G \) is \( p \)-free (i.e. if \( H_1 \subset H_2 \) are subgroups then the index \( [H_2 : H_1] \), whenever finite, is not divisible by \( p \)) then \( x \mapsto px \) is a bijection. But this conclusion holds also for the additive group of the \( p \)-adic numbers \( \mathbb{Q}_p \). On \( \mathbb{Z}_p := \{ x \in \mathbb{Q}_p : |x|_p \leq 1 \} \) the map \( x \mapsto px \) is injective but not surjective, on \( \mathbb{Q}_p/\mathbb{Z}_p \) the map \( x \mapsto px \) is surjective but not injective.

§2 PRELIMINARIES

**LEMMA 2.1.** (Elementary properties of \( \varepsilon \)-characters). Let \( 0 < \varepsilon < 1 \).

(i) Let \( f : G \rightarrow T_K \) be an \( \varepsilon \)-character. Then

(a) \( |f(0)-1| \leq \varepsilon \).

(b) If \( x_1, \ldots, x_n \in G \) then \( |f(x_1+\cdots+x_n)-f(x_1)f(x_2)\cdots f(x_n)| \leq \varepsilon \).

(c) If \( g : G \rightarrow K \), \( |g(x)| \leq \varepsilon \) for all \( x \in G \), then \( f+g \) is an \( \varepsilon \)-character.

(ii) \( B_1(\varepsilon) := \{ \lambda \in K : |1-\lambda| \leq \varepsilon \} \) is a subgroup of \( T_K \). Let \( \pi : T_K \rightarrow T_K/B_1(\varepsilon) \) be the quotient map. Then \( f : G \rightarrow T_K \) is an \( \varepsilon \)-character if and only if \( \pi \circ f : G \rightarrow T_K/B_1(\varepsilon) \) is a homomorphism.

**Proof.** Straightforward.
PROPOSITION 2.2. (Extension of ε-characters) Let \( H \) be a subgroup of \( G \) and let, for some \( \epsilon \in (0,1) \), \( f : H \rightarrow T_K \) be an ε-character. Then \( f \) can be extended to an ε-character \( \tilde{f} : G \rightarrow T_K \).

Proof. By lemma 2.1 (i) (c) it suffices to find an ε-character \( \tilde{f} : G \rightarrow T_K \) for which \( |\tilde{f}(h) - f(h)| < \epsilon \) \( (h \in H) \). With the notations as in Lemma 2.1 (ii) we have that \( \pi \circ f \) is a homomorphism \( H \rightarrow T_K / B_1(\epsilon) \). As \( K \) is algebraically closed the group \( T_K / B_1(\epsilon) \) is divisible. Therefore, \( \pi \circ f \) can be extended to a homomorphism \( g : G \rightarrow T_K / B_1(\epsilon) \). Choose any \( \rho : T_K / B_1(\epsilon) \rightarrow T_K \) for which \( \pi \circ \rho \) is the identity. Then \( \tilde{f} := \rho \circ g \) has the required properties.

We quote a lemma needed for the proof of Proposition 3.2.

LEMMA 2.3. Let \( \text{char } k = p \neq 0 \). Let \( a, b \in T_K \) such that \( 0 < |a-b| < \epsilon < 1 \). Then \( |a^p - b^p| < \tau \epsilon \) where \( \tau := \max(\epsilon, |p|) \). In particular, \( |a^p - b^p| < |a-b| \).

Proof. [2], Lemma 32.1.

§3 EXISTENCE

In this section we prove the 'existence part' of Theorem 1.1 i.e. the statements involving (E).

For the case \( \text{char } k = 0 \) we have quite standard methods:

PROPOSITION 3.1. Let \( \text{char } k = 0 \). Then, for any \( G \), for each \( \epsilon \in (0,1) \) and each ε-character \( f : G \rightarrow T_K \) there exists a character \( \alpha : G \rightarrow T_K \) for which \( |f(x) - \alpha(x)| < \epsilon \) \( (x \in G) \).

Proof. We start the construction of \( \alpha \) by setting \( \alpha(0) := 1 \). Then, by Lemma 2.1 (i)(a), \( |f-\alpha| \leq \epsilon \) on the zero group. Now suppose we have a subgroup \( H \) of \( G \) and a character \( \alpha : H \rightarrow T_K \) such that \( |f(h) - \alpha(h)| \leq \epsilon \) for all \( h \in H \). Let \( x \in G \setminus H \). We prove that \( \alpha \) can be extended to a character \( \tilde{\alpha} \) defined on the group \( H' \) generated by \( H \) and \( \{x\} \) such that \( |f(h') - \tilde{\alpha}(h')| \leq \epsilon \) for all \( h' \in H' \). (A simple application of Zorn's Lemma then may complete the proof.)
If $H'/H \cong \mathbb{Z}$ we define $\tilde{\alpha}$ by

$$\tilde{\alpha}(nx+h) := f(x)^n\alpha(h) \quad (n \in \mathbb{Z}, h \in H)$$

One proves easily, by using Lemma 2.1(i)(b), that $\tilde{\alpha}$ satisfies the requirements. If $H'/H \cong \mathbb{Z}/q\mathbb{Z}$ for some $q \in \mathbb{N}, q > 1$ we set

$$\tilde{\alpha}(nx+h) := \theta^n\alpha(h) \quad (n \in \{0,1,\ldots,q-1\}, h \in H)$$

where $\theta \in K$ is chosen such that $\theta^q = \alpha(qx)$ (then $\tilde{\alpha}$ is a character) and such that $|\theta-f(x)| \leq \varepsilon$ (then $|\tilde{\alpha}(h')-f(h')| \leq \varepsilon$ for all $h' \in H'$). To see such a $\theta$ exists consider the polynomial $p = x^q - \alpha(qx) \in K[x]$. We have $|P(f(x))| = |q||f(x)q^{-1}| = 1$ (here we use the assumption $char k = 0$). By Hensel's Lemma there exists a $\theta \in K$ with $P(\theta) = 0$ and $|\theta-f(x)| \leq \varepsilon$.

**REMARKS.**

1. The algebraic closedness of $K$ has not been used in the above proof.

2. Let $B_1(\varepsilon), \pi$ be as in Lemma 2.1. Then any $\rho : T_K/B_1(\varepsilon) \to T_K$ for which $\pi \circ \rho$ is the identity, is an $\varepsilon$-character of $T_K/B_1(\varepsilon)$. Proposition 3.1 tells that there exists a homomorphism $\tilde{\rho} : T_K/B_1(\varepsilon) \to T_K$ such that $\pi \circ \tilde{\rho}$ is the identity. As a corollary we obtain that if $char k = 0$ then $B_1(\varepsilon)$ is a factor in $T_K$.

Next we turn to the case where $char k = p \neq 0$. To cover also groups like $\mathbb{Q}_p$ we shall use a technique different from the above one.

**PROPOSITION 3.2.** Let $char k = p \neq 0$. Suppose $x \mapsto px$ is a bijection $G \to G$.

Then for each $\varepsilon \in [0,1)$ and each $\varepsilon$-character $f : G \to T_K$ there exists a character $\alpha : G \to T_K$ such that $|f(x) - \alpha(x)| \leq \varepsilon$ ($x \in G$).
Proof. For each \( n \in \mathbb{N} \) let \( x \mapsto p^{-n}x \) \((x \in G)\) be the inverse of \( x \mapsto p^n x \).

For any \( n \in \mathbb{N}, x \in G \) we have

\[
|f(p^{-n-1}x)^p - f(p^{-n}x)| < \varepsilon
\]

so that, by Lemma 2.3,

\[
|f(p^{-n-1}x)^{p^{n+1}} - f(p^{-n}x)^p| < \varepsilon^n.
\]

By completeness of \( K \)

\[
a(x) := \lim_{n \to \infty} f(p^{-n}x)^p
\]

exists (uniformly in \( x \in G \)). For each \( n \in \mathbb{N}, x \in G \) we have

\[
|f(x) - f(p^{-n}x)^p| < \varepsilon
\]

hence also

\[
|f(x) - a(x)| < \varepsilon
\]

To see that \( a \) is a character, let \( x, y \in G, n \in \mathbb{N} \). We have

\[
|f(p^{-n}(x+y)) - f(p^{-n}x)f(p^{-n}y)| < \varepsilon
\]

Again by Lemma 2.3,

\[
|f(p^{-n}(x+y))^p - f(p^{-n}x)^p f(p^{-n}y)^p| < \varepsilon^n
\]

hence also

\[
|a(x+y) - a(x)a(y)| < 0.
\]

Proposition 3.2 is a stepping stone for
PROPOSITION 3.3. Let $\text{char } k = p \neq 0$. Suppose $x \mapsto px$ is an injection $G \to G$. Then the conclusion of Proposition 3.2 holds.

Proof. $G$ can be embedded into a divisible group $D$. Set $G_1 := D/D_p$ (for $D_p$ see Theorem 1.1(iii)). Then $G + D + G_1$ is injective and $x \mapsto px$ is a bijection $G_1 \to G$. We may assume $G \subseteq G_1$. By Proposition 2.2 $f$ can be extended to an $\varepsilon$-character $\hat{f}$ on $G_1$. By Proposition 3.2 there is a character $\tilde{a} : G_1 \to \mathbb{T}_k$ for which $|\hat{f}(x) - \tilde{a}(x)| \leq \varepsilon$ ($x \in G_1$). Set $a := \tilde{a}|_G$.

We now prove the converse to Proposition 3.3.

PROPOSITION 3.4. Let $\text{char } k = p \neq 0$. Suppose $x \mapsto px$ ($x \in G$) is not injective. Then there exists an $\varepsilon \in (0,1)$ and an $\varepsilon$-character $f : G + T_K$ such that for every character $a : G + T_K$ there exists an $x \in G$ with $|f(x) - a(x)| > \varepsilon$.

Proof. Choose an element $x \in G$ of order $p$ and a $b \in K$ with $0 < |1-b| < 1$.

If $\text{char } K = 0$ we assume in addition that $|1-b| < |1-\theta|$ where $\theta$ is a primitive $p$th root of unity. Consider the map $g : nx \mapsto b^n$ ($n \in \{0,1,\ldots,p-1\}$)

defined on the group generated by $x$. For $n,m \in \{0,1,\ldots,p-1\}$ we have

$$|g(nx+mx) - g(nx)g(mx)| = \begin{cases} 0 & \text{if } n+m \leq p-1 \\ |b^{n+m-p} - b^{n-m}| & \text{if } n+m > p \end{cases}$$

so we see that $g$ is an $\varepsilon$-character where $\varepsilon = |b^{p-1}| = |b^{p-1}|$.

By Proposition 2.2 $g$ extends to an $\varepsilon$-character $f : G + T_K$. Now let $a : G + T_K$ be a character. If $a = 1$ on $H$ we have $|f(x) - a(x)| = |b-1| > |b^{p-1}| = \varepsilon$ (Lemma 2.3). Otherwise we have $a(x) = \theta$ where $\theta$ is a primitive $p$th root of unity (and $\text{char } K = 0$). Then we have, since $|1-b| < |1-\theta|$, $|f(x) - a(x)| = |b-\theta| = \max(|b-1|,|1-\theta|) = |1-\theta| > |1-b| > \varepsilon$. 
§4 UNIQUENESS

In this section we prove the 'uniqueness part' of Theorem 1.1 i.e. the statements involving (U).

PROPOSITION 4.1.

(i) Let char $k = 0$. Then (U) holds if and only if $G$ is a torsion group.

(ii) Let char $K = 0$, char $k = p 
eq 0$. Then (U) holds if and only if $G$ has no subgroup of index $p$.

(iii) Let char $K = p 
eq 0$. Then (U) holds if and only if each homomorphism $G \to \mathbb{Z}_p$ is zero.

Proof. Statement (U) is equivalent to "if $\alpha, \beta$ are distinct characters then $\sup\{|\alpha(x) - \beta(x)| : x \in G\} = 1$", which is by [3], Proposition 1.1, the same as "the characters form an orthogonal set with respect to the sup norm".

Now apply [3], Theorem 3.1, Theorem 2.2, Theorem 4.3 (where $G$ has the discrete topology). Proposition 4.1 follows.

To complete the proof of Theorem 1.1 the following lemma remains to be shown.

LEMMA 4.2. Let $p$ be a prime number. The following are equivalent.

(a) Any homomorphism $G \to \mathbb{Z}_p$ is zero.

(b) If $H$ is a subgroup of $G/G_p$ then $H$ does not have index $p$.

The heart of the proof is contained in

LEMMA 4.3. Let $x \mapsto px$ ($x \in G$) be injective but not surjective. Then there exists a nonzero homomorphism $G \to \mathbb{Z}_p$.

Proof. Let $\pi : G \to G/pG$ be the quotient map. As $pG \neq G$ the group $G/pG$ is in a natural way a nonzero vector space over the field of $p$ elements.

So there exists an indexed set $(e_i)_{i \in I}$ in $G$ where $I \neq \emptyset$ such that

$\{\pi(e_i) : i \in I\}$ is a base of the vector space $G/pG$. It follows that for
each \( x \in G \) there exist unique \((\lambda^{(1)}_i)_{i \in I}\) where \( \lambda^{(1)}_i \in \{0, 1, \ldots, p-1\} \subset \mathbb{Z} \) and \( \{i \in I : \lambda^{(1)}_i \neq 0\} \) is finite such that \( x = \sum \lambda^{(1)}_i e_i = px_1 \) where \( x_1 \in G \). By injectivity of \( x \mapsto px \) also \( x_1 \) is unique. By treating \( x_1 \) in the same way as we did for \( x \) we find unique \((\lambda^{(2)}_i)_{i \in I} \in \{0, 1, \ldots, p^2-1\} \subset \mathbb{Z} \) with \( \{i \in I : \lambda^{(2)}_i \neq 0\} \) is finite such that \( x = \sum \lambda^{(2)}_i e_i = p^2 x_2 \) where \( x_2 \in G \) etc.

Thus, for each \( n \in \mathbb{N} \) there exist unique maps \( \phi^{(n)}_i : G \to \{0, 1, \ldots, p^n-1\} \subset \mathbb{Z} \) with \( \{i \in I : \phi^{(n)}_i (x) \neq 0\} \) finite for each \( x \in G \) such that

\[
x - \sum \phi^{(n)}_i (x) e_i \in p^n G \quad (x \in G)
\]

By uniqueness, for each \( i \in I, n \in \mathbb{N} \)

\[
\phi^{(n+1)}_i (x) \equiv \phi^{(n)}_i (x) \mod p^n \quad (x \in G)
\]

We see that, for any \( j \in I \), the \( p \)-adic limit

\[
\phi_j (x) = \lim_{n \to \infty} \phi^{(n)}_j (x) \quad (x \in G)
\]

exists and defines a map \( \phi_j : G \to \mathbb{Z}_p \). As \( \phi_j (e_j) = 1 \) this map is not zero.

To see that \( \phi_j \) is a homomorphism observe that for each \( n \in \mathbb{N}, x, y \in G \)

\[
\sum_{i} \left( \phi^{(n)}_i (x+y) - \phi^{(n)}_i (x) - \phi^{(n)}_i (y) \right) e_i \in p^n G
\]

By what we have proved above

\[
\phi^{(n)}_j (x+y) - \phi^{(n)}_j (x) - \phi^{(n)}_j (y) \equiv 0 \mod p^n
\]

i.e.

\[
\left| \phi^{(n)}_j (x+y) - \phi^{(n)}_j (x) - \phi^{(n)}_j (y) \right|_p \leq p^{-n}
\]

which means for \( \phi_j \) that

\[
\left| \phi_j (x+y) - \phi_j (x) - \phi_j (y) \right| \leq 0
\]
Proof of Lemma 4.2. (a) ⇒ (b). Suppose we had a subgroup \( H \) of \( G/G_p \) of index \( p \). Then, since \( G/G_p \) has no elements of order \( p \), the map \( x \mapsto px \) is injective but not surjective on \( G/G_p \), so Lemma 4.3 gives us a nontrivial homomorphism \( \phi : G/G_p \to \mathbb{Z}_p \). But then \( G + G/G_p \phi \mathbb{Z}_p \) is a nontrivial homomorphism \( G \to \mathbb{Z}_p \), which conflicts (a). To prove (b) ⇒ (a), suppose we had a nontrivial homomorphism \( \phi : G \to \mathbb{Z}_p \). Then we may assume \( 1 \in \text{Im} \phi \). It is easy to see that \( H := \phi^{-1}(p\mathbb{Z}_p) \) has index \( p \) and contains \( G_p \). This violates (a).

PROBLEM. Generalize Theorem 1.1 to topological abelian groups \( G \) and continuous \((\epsilon-)\)characters.

REFERENCES


