EPSILON STABILITY OF p-ADIC CHARACTERS

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ABSTRACT. The following problem is solved (see Theorem 1 below). Let $G$ be an abelian group, written additively, and let $K$ be an algebraically closed nonarchimedean valued field which is complete for the valuation $| |$. Let $f : G \to T_K := \{ \lambda \in K : |\lambda| = 1 \}$ be an $\varepsilon$-character i.e.

$$|f(x+y) - f(x)f(y)| \leq \varepsilon \quad (x, y \in G)$$

where $0 \leq \varepsilon < 1$. Does there exist a character $\alpha : G \to T_K$ for which

$$|\alpha(x) - f(x)| \leq \varepsilon \quad (x \in G)? \text{ Is it unique?}$$

NOTES. The results of this Report will be used in a future paper on p-adic almost periodic functions [4]. The expression 'epsilon stability' is taken from [1], p.11, where a closely related problem is discussed.
§1 THE THEOREM

Throughout, let \( G, K, T_K \) be as above. For trivially valued fields \( K \) the above problem has a trivial solution. So from now on we assume that the valuation of \( K \) is non-trivial. The residue class field of \( K \) is \( k \). The characteristic of a field \( L \) is denoted char \( L \).

Let \( 0 < \varepsilon < 1 \). A function \( f : G \rightarrow T_K \) is an \( \varepsilon \)-character if
\[
|f(x+y)-f(x)f(y)| \leq \varepsilon \quad \text{for all } x, y \in G.
\]
As usual, we shall say 'character' instead of '0-character'.

Consider the following statements (E) and (U).

\[
\text{(E)} \quad \text{For each } \varepsilon \in [0,1) \text{ and each } \varepsilon \text{-character } f : G \rightarrow T_K \text{ there exists a character } \alpha : G \rightarrow T_K \text{ for which } |f(x)-\alpha(x)| \leq \varepsilon \quad (x \in G).
\]

\[
\text{(U)} \quad \text{For each } \varepsilon \in [0,1) \text{ and each } \varepsilon \text{-character } f : G \rightarrow T_K \text{ there exists at most one character } \alpha : G \rightarrow T_K \text{ for which } |f(x)-\alpha(x)| \leq \varepsilon \quad (x \in G).
\]

The purpose of this note is to prove the following Theorem.

THEOREM 1.1.

(i) Let char \( k = 0 \). Then (E) holds for any \( G \), (U) holds if and only if \( G \) is a torsion group.

(ii) Let char \( K = 0 \), char \( k \neq 0 \). Then (E) holds if and only if \( G \) has no subgroups of order \( p \), (U) holds if and only if \( G \) has no subgroups of index \( p \).

(iii) Let char \( K = p \neq 0 \). Then (E) holds if and only if \( G \) has no subgroups of order \( p \), (U) holds if and only if no subgroup of \( G/G_p \) where \( G_p := \{ x \in G : p^n x = 0 \text{ for some } n \in \mathbb{N} \} \), has index \( p \).
COROLLARY 1.2.

(i) Let char $k = 0$. Then both $(E)$ and $(U)$ hold if and only if $G$ is a torsion group.

(ii) Let char $k = p \neq 0$. Then both $(E)$ and $(U)$ hold if and only if $G$ has neither subgroups of order $p$, nor subgroups of index $p$.

REMARK. The statement 'G has no subgroups of order $p$' is obviously equivalent to 'the map $x \mapsto px$ ($x \in G$) is injective'. It is not hard to see that 'G has no subgroups of index $p$' is equivalent to 'the map $x \mapsto px$ ($x \in G$) is surjective'.

EXAMPLES. If $G$ is $p$-free (i.e. if $H_1 \subset H_2$ are subgroups then the index $[H_2 : H_1]$, whenever finite, is not divisible by $p$) then $x \mapsto px$ is a bijection. But this conclusion holds also for the additive group of the $p$-adic numbers $\mathbb{Q}_p$. On $\mathbb{Z}_p := \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$ the map $x \mapsto px$ is injective but not surjective, on $\mathbb{Q}_p/\mathbb{Z}_p$ the map $x \mapsto px$ is surjective but not injective.

§2 PRELIMINARIES

LEMMA 2.1. (Elementary properties of $\varepsilon$-characters). Let $0 < \varepsilon < 1$.

(i) Let $f : G \to T_K$ be an $\varepsilon$-character. Then

(a) $|f(0) - 1| \leq \varepsilon$.

(b) If $x_1, \ldots, x_n \in G$ then $|f(x_1 + \ldots + x_n) - f(x_1)f(x_2)\ldots f(x_n)| \leq \varepsilon$.

(c) If $g : G \to K$, $|g(x)| \leq \varepsilon$ for all $x \in G$, then $f + g$ is an $\varepsilon$-character.

(ii) $B_1(\varepsilon) := \{\lambda \in K : |1 - \lambda| < \varepsilon\}$ is a subgroup of $T_K$. Let $\pi : T_K \to T_K/B_1(\varepsilon)$ be the quotient map. Then $f : G \to T_K$ is an $\varepsilon$-character if and only if $\pi \circ f : G \to T_K/B_1(\varepsilon)$ is a homomorphism.

Proof. Straightforward.
PROPOSITION 2.2. (Extension of $\varepsilon$-characters) Let $H$ be a subgroup of $G$ and let, for some $\varepsilon \in (0,1)$, $f : H \to T_K$ be an $\varepsilon$-character. Then $f$ can be extended to an $\varepsilon$-character $\tilde{f} : G \to T_K$.

Proof. By lemma 2.1 (i) (c) it suffices to find an $\varepsilon$-character $\tilde{f} : G \to T_K$ for which $|\tilde{f}(h) - f(h)| < \varepsilon$ $(h \in H)$. With the notations as in Lemma 2.1 (ii) we have that $\pi \circ f$ is a homomorphism $H \to T_K / B_1(\varepsilon)$. As $K$ is algebraically closed the group $T_K$, hence $T_K / B_1(\varepsilon)$, is divisible. Therefore, $\pi \circ f$ can be extended to a homomorphism $g : G \to T_K / B_1(\varepsilon)$. Choose any $\rho : T_K / B_1(\varepsilon) \to T_K$ for which $\pi \circ \rho$ is the identity. Then $\tilde{f} := \rho \circ g$ has the required properties.

We quote a lemma needed for the proof of Proposition 3.2.

LEMMA 2.3. Let $\text{char } k = p \neq 0$. Let $a, b \in T_K$ such that $0 < |a - b| < \varepsilon < 1$. Then $|a^p - b^p| < \tau \varepsilon$ where $\tau := \max(\varepsilon, |p|)$. In particular, $|a^p - b^p| < |a - b|$.

Proof. [2], Lemma 32.1.

§3 EXISTENCE

In this section we prove the 'existence part' of Theorem 1.1 i.e. the statements involving (E).

For the case char $k = 0$ we have quite standard methods:

PROPOSITION 3.1. Let char $k = 0$. Then, for any $G$, for each $\varepsilon \in (0,1)$ and each $\varepsilon$-character $f : G \to T_K$ there exists a character $\alpha : G \to T_K$ for which $|f(x) - \alpha(x)| < \varepsilon$ $(x \in G)$.

Proof. We start the construction of $\alpha$ by setting $\alpha(0) := 1$. Then, by Lemma 2.1 (i) (a), $|f - \alpha| < \varepsilon$ on the zero group. Now suppose we have a subgroup $H$ of $G$ and a character $\alpha : H \to T_K$ such that $|f(h) - \alpha(h)| < \varepsilon$ for all $h \in H$. Let $x \in G \setminus H$. We prove that $\alpha$ can be extended to a character $\tilde{\alpha}$ defined on the group $H'$ generated by $H$ and $\{x\}$ such that $|f(h') - \tilde{\alpha}(h')| < \varepsilon$ for all $h' \in H$. (A simple application of Zorn's Lemma then may complete the proof.)
If \( H'/H = \mathbb{Z} \) we define \( \tilde{\alpha} \) by

\[
\tilde{\alpha}(nx+h) := f(x)^n\alpha(h) \quad (n \in \mathbb{Z}, h \in H)
\]

One proves easily, by using Lemma 2.1(i)(b), that \( \tilde{\alpha} \) satisfies the requirements. If \( H'/H = \mathbb{Z}/q\mathbb{Z} \) for some \( q \in \mathbb{N}, q > 1 \) we set

\[
\tilde{\alpha}(nx+h) := \theta^n\alpha(h) \quad (n \in \{0,1,\ldots,q-1\}, h \in H)
\]

where \( \theta \in K \) is chosen such that \( \theta^q = \alpha(qx) \) (then \( \tilde{\alpha} \) is a character) and such that \( |\theta-f(x)| \leq \varepsilon \) (then \( |\tilde{\alpha}(h')-f(h')| \leq \varepsilon \) for all \( h' \in H' \)). To see such a \( \theta \) exists consider the polynomial \( p = x^q-\alpha(qx) \in K[X] \). We have

\[
|p(f(x))| \leq \varepsilon, \quad |p'(f(x))| = |q||f(x)q^{-1}| = 1 \quad \text{(here we use the assumption char } k = 0). \quad \text{By Hensel's Lemma there exists a } \theta \in K \text{ with } p(\theta) = 0 \text{ and } |\theta-f(x)| \leq \varepsilon.
\]

REMARKS.

1. The algebraic closedness of \( K \) has not been used in the above proof.

2. Let \( B_1(\varepsilon), \pi \) be as in Lemma 2.1. Then any \( \rho : T_K/B_1(\varepsilon) \to T_K \) for which \( \pi \circ \rho \) is the identity, is an \( \varepsilon \)-character of \( T_K/B_1(\varepsilon) \). Proposition 3.1 tells that there exists a homomorphism \( \tilde{\rho} : T_K/B_1(\varepsilon) \to T_K \) such that \( \pi \circ \tilde{\rho} \) is the identity. As a corollary we obtain that if char \( k = 0 \) then \( B_1(\varepsilon) \) is a factor in \( T_K \).

Next we turn to the case where char \( k = p \neq 0 \). To cover also groups like \( \mathbb{Z}_p \) we shall use a technique different from the above one.

**Proposition 3.2.** Let \( \text{char } k = p \neq 0 \). Suppose \( x \mapsto px \) is a bijection \( G \to G \).

Then for each \( \varepsilon \in [0,1) \) and each \( \varepsilon \)-character \( f : G \to T_K \) there exists a character \( \alpha : G \to T_K \) such that \( |f(x)-\alpha(x)| \leq \varepsilon \quad (x \in G) \).
Proof. For each \( n \in \mathbb{N} \) let \( x \mapsto p^{-n}x \ (x \in G) \) be the inverse of \( x \mapsto p^n x \).

For any \( n \in \mathbb{N}, x \in G \) we have

\[
|f(p^{-n-1}x)^p - f(p^{-n}x)| \leq \varepsilon
\]

so that, by Lemma 2.3,

\[
|f(p^{-n-1}x)^{p^{n+1}} - f(p^{-n}x)^p| \leq \varepsilon^n.
\]

By completeness of \( K \)

\[
a(x) := \lim_{n \to \infty} f(p^{-n}x)^p
\]

exists (uniformly in \( x \in G \)). For each \( n \in \mathbb{N}, x \in G \) we have

\[
|f(x) - f(p^{-n}x)^p| \leq \varepsilon
\]

hence also

\[
|f(x) - a(x)| \leq \varepsilon
\]

To see that \( a \) is a character, let \( x, y \in G, n \in \mathbb{N} \). We have

\[
|f(p^{-n}(x+y)) - f(p^{-n}x)f(p^{-n}y)| \leq \varepsilon
\]

Again by Lemma 2.3,

\[
|f(p^{-n}(x+y))^p - f(p^{-n}x)^p f(p^{-n}y)^p| \leq \varepsilon^n
\]

hence also

\[
|a(x+y) - a(x)a(y)| \leq 0.
\]

Proposition 3.2 is a stepping stone for
PROPOSITION 3.3. Let char $k = p \neq 0$. Suppose $x \mapsto px$ is an injection $G \to G$. Then the conclusion of Proposition 3.2 holds.

Proof. $G$ can be embedded into a divisible group $D$. Set $G_1 := D/D_p$ (for $D_p$ see Theorem 1.1(iii)). Then $G + D + G_1$ is injective and $x \mapsto px$ is a bijection $G_1 \to G_1$. We may assume $G \subset G_1$. By Proposition 2.2 $f$ can be extended to an $\varepsilon$-character $\tilde{f}$ on $G_1$. By Proposition 3.2 there is a character $\tilde{\alpha} : G_1 \to \mathbb{T}_k$ for which $|\tilde{f}(x) - \tilde{\alpha}(x)| \leq \varepsilon$ $(x \in G_1)$. Set $\alpha := \tilde{\alpha}|_G$.

We now prove the converse to Proposition 3.3.

PROPOSITION 3.4. Let char $k = p \neq 0$. Suppose $x \mapsto px$ $(x \in G)$ is not injective. Then there exists an $\varepsilon \in (0,1)$ and an $\varepsilon$-character $f : G + T_K$ such that for every character $\alpha : G + T_K$ there exists an $x \in G$ with $|f(x) - \alpha(x)| > \varepsilon$.

Proof. Choose an element $x \in G$ of order $p$ and a $b \in K$ with $0 < |1-b| < 1$. If char $K = 0$ we assume in addition that $|1-b| < |1-\theta|$ where $\theta$ is a primitive $p^{th}$ root of unity. Consider the map

$$g : nx \mapsto b^n \quad (n \in \{0,1,\ldots,p-1\})$$

developed on the group generated by $x$. For $n,m \in \{0,1,\ldots,p-1\}$ we have

$$|g(nx+mx) - g(nx)g(mx)| = \begin{cases} 0 & \text{if } n+m \leq p-1 \\ |b^{n+m-p}b^m| & \text{if } n+m > p \end{cases}$$

so we see that $g$ is an $\varepsilon$-character where $\varepsilon = |b^{-p-1}| = |b^{p-1}|$.

By Proposition 2.2 $g$ extends to an $\varepsilon$-character $f : G + T_K$. Now let $\alpha : G + T_K$ be a character. If $\alpha = 1$ on $H$ we have $|f(x) - \alpha(x)| = |b-1| > |b^{p-1}| = \varepsilon$ (Lemma 2.3). Otherwise we have $\alpha(x) = \theta$ where $\theta$ is a primitive $p^{th}$ root of unity (and char $K = 0$). Then we have, since $|1-b| < |1-\theta|$, 

$$|f(x) - \alpha(x)| = |b-\theta| = \max(|b-1|,|1-\theta|) = |1-\theta| > |1-b| > \varepsilon.$$
§4 UNIQUENESS

In this section we prove the 'uniqueness part' of Theorem 1.1 i.e. the statements involving (U).

PROPOSITION 4.1.

(i) Let char \( k = 0 \). Then (U) holds if and only if \( G \) is a torsion group.

(ii) Let char \( K = 0 \), char \( k \neq p \). Then (U) holds if and only if \( G \) has no subgroup of index \( p \).

(iii) Let char \( K = p \neq 0 \). Then (U) holds if and only if each homomorphism \( G \rightarrow \mathbb{Z}_p \) is zero.

Proof. Statement (U) is equivalent to "if \( \alpha, \beta \) are distinct characters then \( \sup \{|\alpha(x) - \beta(x)| : x \in G\} = 1 \)", which is by [3], Proposition 1.1, the same as "the characters form an orthogonal set with respect to the sup norm".

Now apply [3], Theorem 3.1, Theorem 2.2, Theorem 4.3 (where \( G \) has the discrete topology). Proposition 4.1 follows.

To complete the proof of Theorem 1.1 the following lemma remains to be shown.

LEMMA 4.2. Let \( p \) be a prime number. The following are equivalent.

(a) Any homomorphism \( G \rightarrow \mathbb{Z}_p \) is zero.

(b) If \( H \) is a subgroup of \( G \) then \( H \) does not have index \( p \).

The heart of the proof is contained in

LEMMA 4.3. Let \( x \mapsto px \ (x \in G) \) be injective but not surjective. Then there exists a nonzero homomorphism \( G \rightarrow \mathbb{Z}_p \).

Proof. Let \( \tau : G \rightarrow G/pG \) be the quotient map. As \( pG \neq G \) the group \( G/pG \) is in a natural way a nonzero vector space over the field of \( p \) elements.

So there exists an indexed set \( \{e_i\}_{i \in I} \) in \( G \) where \( I \neq \emptyset \) such that \( \{\tau(e_i) : i \in I\} \) is a base of the vector space \( G/pG \). It follows that for
each \( x \in G \) there exist unique \( (\lambda^{(1)}_i)_{i \in I} \) where \( \lambda^{(1)}_i \in \{0, 1, \ldots, p - 1\} \subset \mathbb{N} \) and \( \{i \in I : \lambda^{(1)}_i \neq 0\} \) is finite such that \( x = \sum \lambda^{(1)}_i e_i = px \) where \( x \in G \).

By injectivity of \( x \mapsto px \) also \( x \) is unique. By treating \( x \) in the same way as we did for \( x \) we find unique \( (\lambda^{(2)}_i)_{i \in I} \in \{0, 1, \ldots, p^2 - 1\} \subset \mathbb{N} \), \( \{i \in I : \lambda^{(2)}_i \neq 0\} \) is finite such that \( x = \sum \lambda^{(2)}_i e_i = p^2 x_2 \) where \( x_2 \in G \) etc.

Thus, for each \( n \in \mathbb{N} \) there exist unique maps \( \phi^{(n)}_i : G \to \{0, 1, \ldots, p^n - 1\} \subset \mathbb{Z} \) with \( \{i \in I : \phi^{(n)}_i(x) \neq 0\} \) finite for each \( x \in G \) such that

\[
\phi^{(n)}_i(x) = \sum (e_i | \phi^{(n)}_i(x)) e_i \in p^n G \quad (x \in G)
\]

By uniqueness, for each \( i \in I \), \( n \in \mathbb{N} \)

\[
\phi^{(n+1)}_i(x) \equiv \phi^{(n)}_i(x) \mod p^n \quad (x \in G)
\]

We see that, for any \( j \in I \), the \( p \)-adic limit

\[
\phi_j(x) = \lim_{n \to \infty} \phi^{(n)}_j(x) \quad (x \in G)
\]

exists and defines a map \( \phi_j : G \to \mathbb{Z}_p \). As \( \phi_j(e_j) = 1 \) this map is not zero.

To see that \( \phi_j \) is a homomorphism observe that for each \( n \in \mathbb{N} \), \( x, y \in G \)

\[
\sum (\phi^{(n)}_i(x+y) - \phi^{(n)}_i(x) - \phi^{(n)}_i(y)) e_i \in p^n G
\]

By what we have proved above

\[
\phi^{(n)}_j(x+y) - \phi^{(n)}_j(x) - \phi^{(n)}_j(y) \equiv 0 \mod p^n
\]

i.e.

\[
|\phi^{(n)}_j(x+y) - \phi^{(n)}_j(x) - \phi^{(n)}_j(y)|_p \leq p^{-n}
\]

which means for \( \phi_j \) that

\[
|\phi_j(x+y) - \phi_j(x) - \phi_j(y)| \leq 0
\]
Proof of Lemma 4.2. (a) $\Rightarrow$ (b). Suppose we had a subgroup $H$ of $G/G_p$ of index $p$. Then, since $G/G_p$ has no elements of order $p$, the map $x \mapsto px$ is injective but not surjective on $G/G_p$, so Lemma 4.3 gives us a nontrivial homomorphism $\phi : G/G_p \to \mathbb{Z}_p$. But then $G + G/G_p \not\cong \mathbb{Z}_p$ is a nontrivial homomorphism $G \to \mathbb{Z}_p$ which conflicts (a). To prove (b) $\Rightarrow$ (a), suppose we had a nontrivial homomorphism $\phi : G \to \mathbb{Z}_p$. Then we may assume $1 \in \text{Im}\phi$. It is easy to see that $H := \phi^{-1}(p\mathbb{Z}_p)$ has index $p$ and contains $G_p$. This violates (a).

PROBLEM. Generalize Theorem 1.1 to topological abelian groups $G$ and continuous ($\varepsilon$-)characters.

REFERENCES


