EPSILON STABILITY OF $p$-ADIC CHARACTERS

by

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ABSTRACT. The following problem is solved (see Theorem 1.1 below). Let $G$ be an abelian group, written additively, and let $K$ be an algebraically closed nonarchimedean valued field which is complete for the valuation $| |$. Let $f : G \to T_K := \{\lambda \in K : |\lambda| = 1\}$ be an $\epsilon$-character i.e.

$$|f(x+y)-f(x)f(y)| \leq \epsilon \quad (x,y \in G)$$

where $0 \leq \epsilon < 1$. Does there exist a character $\alpha : G \to T_K$ for which $|\alpha(x)-f(x)| \leq \epsilon \quad (x \in G)$? Is it unique?

NOTES. The results of this Report will be used in a future paper on $p$-adic almost periodic functions [4]. The expression 'epsilon stability' is taken from [1], p.11, where a closely related problem is discussed.
§1 THE THEOREM

Throughout, let $G, K, T$ be as above. For trivially valued fields $K$ the above problem has a trivial solution. So from now on we assume that the valuation of $K$ is non-trivial. The residue class field of $K$ is $k$. The characteristic of a field $L$ is denoted $\text{char} L$.

Let $0 < \varepsilon < 1$. A function $f : G \to T_k$ is an $\varepsilon$-character if

$$|f(x+y) - f(x)f(y)| \leq \varepsilon \quad \text{for all } x, y \in G.$$  

As usual, we shall say 'character' instead of '0-character'.

Consider the following statements (E) and (U).

(E)

For each $\varepsilon \in [0,1)$ and each $\varepsilon$-character $f : G \to T_k$ there exists a character $\alpha : G \to T_k$ for which $|f(x) - \alpha(x)| \leq \varepsilon \quad (x \in G)$.

(U)

For each $\varepsilon \in [0,1)$ and each $\varepsilon$-character $f : G \to T_k$ there exists at most one character $\alpha : G \to T_k$ for which $|f(x) - \alpha(x)| \leq \varepsilon \quad (x \in G)$.

The purpose of this note is to prove the following Theorem.

THEOREM 1.1.

(i) Let $\text{char } k = 0$. Then (E) holds for any $G$, (U) holds if and only if $G$ is a torsion group.

(ii) Let $\text{char } K = 0$, $\text{char } k = p \neq 0$. Then (E) holds if and only if $G$ has no subgroups of order $p$, (U) holds if and only if $G$ has no subgroups of index $p$.

(iii) Let $\text{char } K = p \neq 0$. Then (E) holds if and only if $G$ has no subgroups of order $p$, (U) holds if and only if $G/G_p$, where $G_p := \{x \in G : p^n x = 0 \text{ for some } n \in \mathbb{N}\}$, has index $p$. 
We have the following trivial corollary.

**COROLLARY 1.2.**

(i) Let \( \text{char } k = 0 \). Then both (E) and (U) hold if and only if \( G \) is a torsion group.

(ii) Let \( \text{char } k = p \neq 0 \). Then both (E) and (U) hold if and only if \( G \) has neither subgroups of order \( p \), nor subgroups of index \( p \).

**REMARK.** The statement '\( G \) has no subgroups of order \( p \)' is obviously equivalent to 'the map \( x \mapsto px \ (x \in G) \) is injective'. It is not hard to see that '\( G \) has no subgroups of index \( p \)' is equivalent to 'the map \( x \mapsto px \ (x \in G) \) is surjective'.

**EXAMPLES.** If \( G \) is \( p \)-free (i.e. if \( H_1 \subseteq H_2 \) are subgroups then the index \([H_2 : H_1]\), whenever finite, is not divisible by \( p \)) then \( x \mapsto px \) is a bijection. But this conclusion holds also for the additive group of the \( p \)-adic numbers \( \mathbb{Q}_p \). On \( \mathbb{Z}_p := \{x \in \mathbb{Q}_p : |x|_p \leq 1\} \) the map \( x \mapsto px \) is injective but not surjective, on \( \mathbb{Q}_p/\mathbb{Z}_p \) the map \( x \mapsto px \) is surjective but not injective.

§2 PRELIMINARIES

**LEMMA 2.1.** (Elementary properties of \( \varepsilon \)-characters). Let \( 0 \leq \varepsilon < 1 \).

(i) Let \( f : G \to T_K \) be an \( \varepsilon \)-character. Then

(a) \( |f(0)-1| \leq \varepsilon \).

(b) If \( x_1, \ldots, x_n \in G \) then \( |f(x_1+\ldots+x_n)-f(x_1)f(x_2)\ldots f(x_n)| \leq \varepsilon \).

(c) If \( g : G \to K \), \( |g(x)| \leq \varepsilon \) for all \( x \in G \), then \( f+g \) is an \( \varepsilon \)-character.

(ii) \( B_1(\varepsilon) := \{\lambda \in K : |1-\lambda| \leq \varepsilon\} \) is a subgroup of \( T_K \). Let \( \pi : T_K \to T_K/B_1(\varepsilon) \) be the quotient map. Then \( f : G \to T_K \) is an \( \varepsilon \)-character if and only if \( \pi \circ f : G \to T_K/B_1(\varepsilon) \) is a homomorphism.

**Proof.** Straightforward.
PROPOSITION 2.2. (Extension of ε-characters) Let H be a subgroup of G and let, for some ε ∈ (0,1), ϕ : H → Tk be an ε-character. Then ϕ can be extended to an ε-character \( \tilde{\phi} : G \rightarrow T_K \).

Proof. By lemma 2.1 (i) (c) it suffices to find an ε-character \( \tilde{\phi} : G \rightarrow T_K \) for which \( |\tilde{\phi}(h) - \phi(h)| < \varepsilon \) (h ∈ H). With the notations as in Lemma 2.1 (ii) we have that \( \pi \circ \phi \) is a homomorphism \( H \rightarrow T_K/B_1(\varepsilon) \). As K is algebraically closed the group \( T_K' \), hence \( T_K/B_1(\varepsilon) \), is divisible. Therefore, \( \pi \circ \phi \) can be extended to a homomorphism \( g : G \rightarrow T_K/B_1(\varepsilon) \). Choose any \( \rho : T_K/B_1(\varepsilon) \rightarrow T_K \) for which \( \pi \circ \rho \) is the identity. Then \( \tilde{\phi} := \rho \circ g \) has the required properties.

We quote a lemma needed for the proof of Proposition 3.2.

LEMMA 2.3. Let char k = p \neq 0. Let a, b ∈ Tk such that 0 < |a - b| < \varepsilon < 1. Then \( |a^p - b^p| < \tau \varepsilon \) where \( \tau := \max(\varepsilon, |p|) \). In particular, \( |a^p - b^p| < |a - b| \).

Proof. [2], Lemma 32.1.

§3 existence

In this section we prove the 'existence part' of Theorem 1.1 i.e. the statements involving (E).

For the case char k = 0 we have quite standard methods:

PROPOSITION 3.1. Let char k = 0. Then, for any G, for each ε ∈ (0,1) and each ε-character \( \phi : G \rightarrow T_K \) there exists a character \( \alpha : G \rightarrow T_K \) for which \( |\phi(x) - \alpha(x)| < \varepsilon \) (x ∈ G).

Proof. We start the construction of \( \alpha \) by setting \( \alpha(0) := 1 \). Then, by Lemma 2.1 (i)(a), \( |\phi - \alpha| < \varepsilon \) on the zero group. Now suppose we have a subgroup H of G and a character \( \alpha : H \rightarrow T_K \) such that \( |\phi(h) - \alpha(h)| < \varepsilon \) for all h ∈ H. Let x ∈ G\H. We prove that \( \alpha \) can be extended to a character \( \tilde{\alpha} \) defined on the group H' generated by H and \{x\} such that \( |\phi(h') - \tilde{\alpha}(h')| < \varepsilon \) for all h' ∈ H. (A simple application of Zorn's Lemma then may complete the proof.)
If $H'/H \cong \mathbb{Z}$ we define $\tilde{a}$ by

$$
\tilde{a}(nx+h) := f(x)^n a(h) \quad (n \in \mathbb{Z}, h \in H)
$$

One proves easily, by using Lemma 2.1(i)(b), that $\tilde{a}$ satisfies the requirements. If $H'/H \cong \mathbb{Z}/q\mathbb{Z}$ for some $q \in \mathbb{N}, q > 1$ we set

$$
\tilde{a}(nx+h) := \theta^n a(h) \quad (n \in \{0,1,\ldots,q-1\}, h \in H)
$$

where $\theta \in K$ is chosen such that $\theta^q = a(qx)$ (then $\tilde{a}$ is a character) and such that $|\theta-f(x)| \leq \varepsilon$ (then $|\tilde{a}(h')-f(h')| \leq \varepsilon$ for all $h' \in H'$). To see such a $\theta$ exists consider the polynomial $p = x^q - a(qx) \in K[x]$. We have $|p(f(x))| \leq \varepsilon$, $|p'(f(x))| = |q||f(x)q^q - 1| = 1$ (here we use the assumption $\text{char } k = 0$). By Hensel's Lemma there exists a $\theta \in K$ with $p(\theta) = 0$ and $|\theta-f(x)| \leq \varepsilon$.

**REMARKS.**

1. The algebraic closedness of $K$ has not been used in the above proof.

2. Let $B_1(\varepsilon)$, $\pi$ be as in Lemma 2.1. Then any $\rho : T_K/B_1(\varepsilon) \to T_K$ for which $\pi \circ \rho$ is the identity, is an $\varepsilon$-character of $T_K/B_1(\varepsilon)$. Proposition 3.1 tells that there exists a homomorphism $\tilde{\rho} : T_K/B_1(\varepsilon) \to T_K$ such that $\pi \circ \tilde{\rho}$ is the identity. As a corollary we obtain that if $\text{char } k = 0$ then $B_1(\varepsilon)$ is a factor in $T_K$.

Next we turn to the case where $\text{char } k = p \neq 0$. To cover also groups like $\mathbb{Q}_p$ we shall use a technique different from the above one.

**PROPOSITION 3.2.** Let $\text{char } k = p \neq 0$. Suppose $x \mapsto px$ is a bijection $G \to G$.

Then for each $\varepsilon \in [0,1)$ and each $\varepsilon$-character $f : G \to T_K$ there exists a character $a : G + T_K$ such that $|f(x)-a(x)| \leq \varepsilon$ $(x \in G)$.
Proof. For each $n \in \mathbb{N}$ let $x \mapsto p^{-n}x$ ($x \in G$) be the inverse of $x \mapsto p^nx$.

For any $n \in \mathbb{N}, x \in G$ we have

$$|f(p^{-n-1}x)p - f(p^{-n}x)| \leq \epsilon$$

so that, by Lemma 2.3,

$$|f(p^{-n-1}x)p^{n+1} - f(p^{-n}x)p^n| \leq \epsilon.$$ 

By completeness of $K$

$$\alpha(x) := \lim_{n \to \infty} f(p^{-n}x)p^n$$

exists (uniformly in $x \in G$). For each $n \in \mathbb{N}, x \in G$ we have

$$|f(x) - f(p^{-n}x)p^n| \leq \epsilon$$

hence also

$$|f(x) - \alpha(x)| \leq \epsilon$$

To see that $\alpha$ is a character, let $x, y \in G, n \in \mathbb{N}$. We have

$$|f(p^{-n}(x+y)) - f(p^{-n}x)f(p^{-n}y)| \leq \epsilon$$

Again by Lemma 2.3,

$$|f(p^{-n}(x+y))p^n - f(p^{-n}x)p^n f(p^{-n}y)p^n| \leq \epsilon$$

hence also

$$|\alpha(x+y) - \alpha(x)\alpha(y)| \leq 0.$$ 

Proposition 3.2 is a stepping stone for
PROPOSITION 3.3. Let char $k = p \neq 0$. Suppose $x \mapsto px$ is an injection $G \rightarrow G$. Then the conclusion of Proposition 3.2 holds.

Proof. $G$ can be embedded into a divisible group $D$. Set $G_1 := D/D_p$ (for $D_p$ see Theorem 1.1(iii)). Then $G + D + G_1$ is injective and $x \mapsto px$ is a bijection $G_1 \rightarrow G_1$. We may assume $G \subseteq G_1$. By Proposition 2.2 $f$ can be extended to an $\epsilon$-character $\tilde{f}$ on $G_1$. By Proposition 3.2 there is a character $\tilde{\alpha} : G_1 \rightarrow \mathbb{T}_k$ for which $|\tilde{f}(x) - \tilde{\alpha}(x)| \leq \epsilon$ ($x \in G_1$). Set $\alpha := \tilde{\alpha}|G$.

We now prove the converse to Proposition 3.3.

PROPOSITION 3.4. Let char $k = p \neq 0$. Suppose $x \mapsto px$ ($x \in G$) is not injective. Then there exists an $\epsilon \in (0,1)$ and an $\epsilon$-character $f : G + T_K$ such that for every character $\alpha : G + T_K$ there exists an $x \in G$ with $|f(x) - \alpha(x)| > \epsilon$.

Proof. Choose an element $x \in G$ of order $p$ and a $b \in K$ with $0 < |1-b| < 1$.

If char $K = 0$ we assume in addition that $|1-b| < |1-\theta|$ where $\theta$ is a primitive $p^{th}$ root of unity. Consider the map

$$g : nx \mapsto b^n \quad (n \in \{0,1,\ldots,p-1\})$$

defined on the group generated by $x$. For $n,m \in \{0,1,\ldots,p-1\}$ we have

$$|g(nx+mx) - g(nx)g(mx)| = \begin{cases} 0 & \text{if } n+m \leq p-1 \\ |b^{n+m}-b^nb^m| & \text{if } n+m > p \end{cases}$$

so we see that $g$ is an $\epsilon$-character where $\epsilon = |b^{P-1}| = |b^P-1|.$

By Proposition 2.2 $g$ extends to an $\epsilon$-character $f : G + T_K$. Now let $\alpha : G + T_K$ be a character. If $\alpha = 1$ on $H$ we have $|f(x) - \alpha(x)| = |b-1| > |b^P-1| = \epsilon$ (Lemma 2.3). Otherwise we have $\alpha(x) = \theta$ where $\theta$ is a primitive $p^{th}$ root of unity (and char $K = 0$). Then we have, since $|1-b| < |1-\theta|$, $|f(x) - \alpha(x)| = |b-\theta| = \max(|b-1|,|1-\theta|) = |1-\theta| > |1-b| > \epsilon$. 
§4 UNIQUENESS

In this section we prove the 'uniqueness part' of Theorem 1.1 i.e. the statements involving (U).

PROPOSITION 4.1.

(i) Let char \( k = 0 \). Then (U) holds if and only if \( G \) is a torsion group.

(ii) Let char \( k = 0 \), char \( k = p \neq 0 \). Then (U) holds if and only if \( G \) has no subgroup of index \( p \).

(iii) Let char \( k = p \neq 0 \). Then (U) holds if and only if each homomorphism \( G \rightarrow \mathbb{Z}_p \) is zero.

Proof. Statement (U) is equivalent to "if \( \alpha, \beta \) are distinct characters then \( \sup\{|\alpha(x) - \beta(x)| : x \in G\} = 1 \)" which is by [3], Proposition 1.1, the same as "the characters form an orthogonal set with respect to the sup norm".

Now apply [3], Theorem 3.1, Theorem 2.2, Theorem 4.3 (where \( G \) has the discrete topology). Proposition 4.1 follows.

To complete the proof of Theorem 1.1 the following lemma remains to be shown.

LEMMA 4.2. Let \( p \) be a prime number. The following are equivalent.

(a) Any homomorphism \( G \rightarrow \mathbb{Z}_p \) is zero.

(b) If \( H \) is a subgroup of \( G/G_p \), then \( H \) does not have index \( p \).

The heart of the proof is contained in

LEMMA 4.3. Let \( x \mapsto px \ (x \in G) \) be injective but not surjective. Then there exists a nonzero homomorphism \( G \rightarrow \mathbb{Z}_p \).

Proof. Let \( \pi : G \rightarrow G/pG \) be the quotient map. As \( pG \neq G \) the group \( G/pG \) is in a natural way a nonzero vector space over the field of \( p \) elements.

So there exists an indexed set \( \{e_i\}_{i \in I} \) in \( G \) where \( I \neq \emptyset \) such that \( \{\pi(e_i) : i \in I\} \) is a base of the vector space \( G/pG \). It follows that for
each \( x \in G \) there exist unique \( (\lambda^{(1)}_i)_{i \in I} \) where \( \lambda^{(1)}_i \in \{0, 1, \ldots, p-1\} \subset \mathbb{Z} \) and \( \{i \in I : \lambda^{(1)}_i \neq 0\} \) is finite such that \( x = \sum \lambda^{(1)}_i x_i = px_i \) where \( x_i \in \mathbb{Z} \). By injectivity of \( x \mapsto px \) also \( x_i \) is unique. By treating \( x_i \) in the same way as we did for \( x \) we find unique \( (\lambda^{(2)}_i)_{i \in I} \in \{0, 1, \ldots, p^2-1\} \subset \mathbb{Z} \), \( \{i \in I : \lambda^{(2)}_i \neq 0\} \) is finite such that \( x = \sum \lambda^{(2)}_i x_i = p^2x_2 \) where \( x_2 \in \mathbb{Z} \) etc.

Thus, for each \( n \in \mathbb{N} \) there exist unique maps \( \phi^{(n)}_i : G \rightarrow \{0, 1, \ldots, p^n-1\} \subset \mathbb{Z} \) with \( \{i \in I : \phi^{(n)}_i(x) \neq 0\} \) finite for each \( x \in G \) such that

\[
x - \sum \phi^{(n)}_i(x) e_i \in p^nG \quad (x \in G)
\]

By uniqueness, for each \( i \in I, n \in \mathbb{N} \)

\[
\phi^{(n+1)}_i(x) \equiv \phi^{(n)}_i(x) \mod p^n \quad (x \in G)
\]

We see that, for any \( j \in I \), the \( p \)-adic limit

\[
\phi^{(n)}_j(x) = \lim_{n \to \infty} \phi^{(n)}_j(x) \quad (x \in G)
\]

exists and defines a map \( \phi_j : G \rightarrow \mathbb{Z}_p \). As \( \phi_j(e_j) = 1 \) this map is not zero.

To see that \( \phi_j \) is a homomorphism observe that for each \( n \in \mathbb{N}, x, y \in G \)

\[
\sum \phi^{(n)}_i(x+y) - \phi^{(n)}_i(x) - \phi^{(n)}_i(y)) e_i \in p^nG
\]

By what we have proved above

\[
\phi^{(n)}_j(x+y) - \phi^{(n)}_j(x) - \phi^{(n)}_j(y) \equiv 0 \mod p^n
\]

i.e.

\[
|\phi^{(n)}_j(x+y) - \phi^{(n)}_j(x) - \phi^{(n)}_j(y)|_p \leq p^{-n}
\]

which means for \( \phi_j \) that

\[
|\phi_j(x+y) - \phi_j(x) - \phi_j(y)| \leq 0
\]
Proof of Lemma 4.2. \((a) \Rightarrow (b)\). Suppose we had a subgroup \(H\) of \(G/G_p\) of index \(p\). Then, since \(G/G_p\) has no elements of order \(p\), the map \(x \mapsto px\) is injective but not surjective on \(G/G_p\), so Lemma 4.3 gives us a nontrivial homomorphism \(\phi : G/G_p \rightarrow \mathbb{Z}_p\). But then \(G \rightarrow G/G_p \phi \mathbb{Z}_p\) is a nontrivial homomorphism \(G \rightarrow \mathbb{Z}_p\) which conflicts \((a)\). To prove \((b) \Rightarrow (a)\), suppose we had a nontrivial homomorphism \(\phi : G \rightarrow \mathbb{Z}_p\). Then we may assume \(1 \in \text{Im}\phi\). It is easy to see that \(H := \phi^{-1}(p\mathbb{Z}_p)\) has index \(p\) and contains \(G_p\). This violates \((a)\).

**PROBLEM.** Generalize Theorem 1.1 to topological abelian groups \(G\) and continuous \((\varepsilon-)\)characters.

**REFERENCES**


