EPSILON STABILITY OF $p$-ADIC CHARACTERS

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ABSTRACT. The following problem is solved (see Theorem 1.1 below). Let $G$ be an abelian group, written additively, and let $K$ be an algebraically closed nonarchimedean valued field which is complete for the valuation $|\cdot|$. Let $f : G \to T_K := \{\lambda \in K : |\lambda| = 1\}$ be an $\varepsilon$-character i.e.

$$|f(x+y)-f(x)f(y)| \leq \varepsilon \quad (x, y \in G)$$

where $0 \leq \varepsilon < 1$. Does there exist a character $\alpha : G \to T_K$ for which $|\alpha(x)-f(x)| \leq \varepsilon$ $(x \in G)$? Is it unique?

NOTES. The results of this Report will be used in a future paper on p-adic almost periodic functions [4]. The expression 'epsilon stability' is taken from [1], p.11, where a closely related problem is discussed.
§1 THE THEOREM

Throughout, let $G, K, T_K$ be as above. For trivially valued fields $K$ the above problem has a trivial solution. So from now on we assume that the valuation of $K$ is non-trivial. The residue class field of $K$ is $k$. The characteristic of a field $L$ is denoted $\text{char } L$.

Let $0 < \varepsilon < 1$. A function $f : G \rightarrow T_K$ is an $\varepsilon$-character if $|f(x+y)-f(x)f(y)| \leq \varepsilon$ for all $x,y \in G$. As usual, we shall say 'character' instead of '0-character'.

Consider the following statements (E) and (U).

$(E)$ For each $\varepsilon \in [0,1)$ and each $\varepsilon$-character $f : G \rightarrow T_K$ there exists a character $\alpha : G \rightarrow T_K$ for which $|f(x)-\alpha(x)| \leq \varepsilon$ (for $x \in G$).

$(U)$ For each $\varepsilon \in [0,1)$ and each $\varepsilon$-character $f : G \rightarrow T_K$ there exists at most one character $\alpha : G \rightarrow T_K$ for which $|f(x)-\alpha(x)| \leq \varepsilon$ (for $x \in G$).

The purpose of this note is to prove the following Theorem.

**THEOREM 1.1.**

(i) Let $\text{char } k = 0$. Then $(E)$ holds for any $G$, $(U)$ holds if and only if $G$ is a torsion group.

(ii) Let $\text{char } K = 0$, $\text{char } k \neq 0$. Then $(E)$ holds if and only if $G$ has no subgroups of order $p$, $(U)$ holds if and only if $G$ has no subgroups of index $p$.

(iii) Let $\text{char } K = p \neq 0$. Then $(E)$ holds if and only if $G$ has no subgroups of order $p$, $(U)$ holds if and only if no subgroup of $G/G_p$, where $G_p := \{x \in G : p^n x = 0 \text{ for some } n \in \mathbb{N}\}$, has index $p$. 
We have the following trivial corollary.

COROLLARY 1.2.

(i) Let $\text{char } k = 0$. Then both (E) and (U) hold if and only if $G$ is a torsion group.

(ii) Let $\text{char } k = p \neq 0$. Then both (E) and (U) hold if and only if $G$ has neither subgroups of order $p$, nor subgroups of index $p$.

REMARK. The statement 'G has no subgroups of order $p$' is obviously equivalent to 'the map $x \mapsto px$ ($x \in G$) is injective'. It is not hard to see that 'G has no subgroups of index $p$' is equivalent to 'the map $x \mapsto px$ ($x \in G$) is surjective'.

EXAMPLES. If $G$ is $p$-free (i.e. if $H_1 \subset H_2$ are subgroups then the index $[H_2 : H_1]$, whenever finite, is not divisible by $p$) then $x \mapsto px$ is a bijection. But this conclusion holds also for the additive group of the $p$-adic numbers $\mathbb{Q}_p$. On $\mathbb{Z}_p := \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$ the map $x \mapsto px$ is injective but not surjective, on $\mathbb{Q}_p / \mathbb{Z}_p$ the map $x \mapsto px$ is surjective but not injective.

§2 PRELIMINARIES

LEMMA 2.1. (Elementary properties of $\varepsilon$-characters). Let $0 \leq \varepsilon < 1$.

(i) Let $f : G \to T_K$ be an $\varepsilon$-character. Then

(a) $|f(0)-1| \leq \varepsilon$.

(b) If $x_1,\ldots,x_n \in G$ then $|f(x_1+\ldots+x_n)-f(x_1)f(x_2)\ldots f(x_n)| \leq \varepsilon$.

(c) If $g : G \to K$, $|g(x)| \leq \varepsilon$ for all $x \in G$, then $f+g$ is an $\varepsilon$-character.

(ii) $B_1(\varepsilon) := \{\lambda \in K : |1-\lambda| \leq \varepsilon\}$ is a subgroup of $T_K$. Let $\pi : T_K \to T_K / B_1(\varepsilon)$ be the quotient map. Then $f : G \to T_K$ is an $\varepsilon$-character if and only if $\pi \circ f : G \to T_K / B_1(\varepsilon)$ is a homomorphism.

Proof. Straightforward.
PROPOSITION 2.2. (Extension of \(\varepsilon\)-characters) Let \(H\) be a subgroup of \(G\) and let, for some \(\varepsilon \in (0,1)\), \(f : H \to T_K\) be an \(\varepsilon\)-character. Then \(f\) can be extended to an \(\varepsilon\)-character \(\tilde{f} : G \to T_K\).

Proof. By lemma 2.1 (i) (c) it suffices to find an \(\varepsilon\)-character \(\tilde{f} : G \to T_K\) for which 
\[
|\tilde{f}(h) - f(h)| < \varepsilon \quad (h \in H).
\]
With the notations as in Lemma 2.1 (ii) we have that \(\pi \tilde{f}\) is a homomorphism \(H \to T_K/B_1(\varepsilon)\). As \(K\) is algebraically closed the group \(T_K\), hence \(T_K/B_1(\varepsilon)\), is divisible. Therefore, \(\pi \tilde{f}\) can be extended to a homomorphism \(g : G \to T_K/B_1(\varepsilon)\). Choose any \(\rho : T_K/B_1(\varepsilon) \to T_K\) for which \(\pi \rho\) is the identity. Then \(\tilde{f} := \rho \circ g\) has the required properties.

We quote a lemma needed for the proof of Proposition 3.2.

LEMMA 2.3. Let \(\text{char } k = p \neq 0\). Let \(a, b \in T_K\) such that \(0 < |a - b| < \varepsilon < 1\). Then 
\[
|a^p - b^p| < \tau \varepsilon \quad \text{where } \tau := \max(\varepsilon, |p|). \quad \text{In particular, } |a^p - b^p| < |a - b|.
\]

Proof. [2], Lemma 32.1.

§3 EXISTENCE

In this section we prove the 'existence part' of Theorem 1.1 i.e. the statements involving \((E)\).

For the case \(\text{char } k = 0\) we have quite standard methods:

PROPOSITION 3.1. Let \(\text{char } k = 0\). Then, for any \(G\), for each \(\varepsilon \in (0,1)\) and each \(\varepsilon\)-character \(f : G \to T_K\) there exists a character \(\alpha : G \to T_K\) for which
\[
|f(x) - \alpha(x)| < \varepsilon \quad (x \in G).
\]

Proof. We start the construction of \(\alpha\) by setting \(\alpha(0) := 1\). Then, by Lemma 2.1 (i) (a), \(|f - \alpha| < \varepsilon\) on the zero group. Now suppose we have a subgroup \(H\) of \(G\) and a character \(\alpha : H \to T_K\) such that \(|f(h) - \alpha(h)| < \varepsilon\) for all \(h \in H\). Let \(x \in G \setminus H\). We prove that \(\alpha\) can be extended to a character \(\tilde{\alpha}\) defined on the group \(H'\) generated by \(H\) and \(\{x\}\) such that \(|f(h') - \tilde{\alpha}(h')| < \varepsilon\) for all \(h' \in H\). (A simple application of Zorn's Lemma then may complete the proof.)
If $H'/H \cong \mathbb{Z}$ we define $\tilde{\alpha}$ by

$$\tilde{\alpha}(nx+h) := f(x)^n \alpha(h) \quad (n \in \mathbb{Z}, h \in H)$$

One proves easily, by using Lemma 2.1(i)(b), that $\tilde{\alpha}$ satisfies the requirements. If $H'/H \cong \mathbb{Z}/q\mathbb{Z}$ for some $q \in \mathbb{N}, q > 1$ we set

$$\tilde{\alpha}(nx+h) := \theta^n \alpha(h) \quad (n \in \{0,1,\ldots,q-1\}, h \in H)$$

where $\theta \in K$ is chosen such that $\theta^q = \alpha(qx)$ (then $\tilde{\alpha}$ is a character) and such that $|\theta-f(x)| \leq \varepsilon$ (then $|\tilde{\alpha}(h')-f(h')| \leq \varepsilon$ for all $h' \in H'$). To see such a $\theta$ exists consider the polynomial $P(x) = x^q-\alpha(qx) \in K[x]$. We have $|P(f(x))| \leq \varepsilon$, $|P'(f(x))| = |q||f(x)^q-1| = 1$ (here we use the assumption $\text{char } k = 0$). By Hensel's Lemma there exists a $\theta \in K$ with $P(\theta) = 0$ and $|\theta-f(x)| \leq \varepsilon$.

**REMARKS.**

1. The algebraic closedness of $K$ has not been used in the above proof.
2. Let $B_1(\varepsilon), \pi$ be as in Lemma 2.1. Then any $\rho : T_K/B_1(\varepsilon) \to T_K$ for which $\pi \circ \rho$ is the identity, is an $\varepsilon$-character of $T_K/B_1(\varepsilon)$. Proposition 3.1 tells that there exists a homomorphism $\tilde{\rho} : T_K/B_1(\varepsilon) \to T_K$ such that $\pi \circ \tilde{\rho}$ is the identity. As a corollary we obtain that if $\text{char } k = 0$ then $B_1(\varepsilon)$ is a factor in $T_K$.

Next we turn to the case where $\text{char } k = p \neq 0$. To cover also groups like $\mathbb{Q}_p$ we shall use a technique different from the above one.

**PROPOSITION 3.2.** Let $\text{char } k = p \neq 0$. Suppose $x \mapsto px$ is a bijection $G \to G$.

Then for each $\varepsilon \in [0,1)$ and each $\varepsilon$-character $f : G \to T_K$ there exists a character $\alpha : G \to T_K$ such that $|f(x)-\alpha(x)| \leq \varepsilon$ $(x \in G)$.
Proof. For each \( n \in \mathbb{N} \) let \( x \mapsto p^{-n}x \) \((x \in G)\) be the inverse of \( x \mapsto p^n x \).

For any \( n \in \mathbb{N}, x \in G \) we have
\[
|f(p^{-n-1}x)^p - f(p^{-n}x)| \leq \varepsilon
\]
so that, by Lemma 2.3,
\[
|f(p^{-n-1}x)^{p^{n+1}} - f(p^{-n}x)^p| \leq \tau^n \varepsilon.
\]

By completeness of \( K \)
\[
a(x) := \lim_{n \to \infty} f(p^{-n}x)^{p^n}
\]
exists (uniformly in \( x \in G \)). For each \( n \in \mathbb{N}, x \in G \) we have
\[
|f(x) - f(p^{-n}x)^{p^n}| \leq \varepsilon
\]
hence also
\[
|f(x) - a(x)| \leq \varepsilon
\]

To see that \( a \) is a character, let \( x, y \in G, n \in \mathbb{N} \). We have
\[
|f(p^{-n}(x+y)) - f(p^{-n}x)f(p^{-n}y)| \leq \varepsilon
\]
Again by Lemma 2.3,
\[
|f(p^{-n}(x+y))^{p^n} - f(p^{-n}x)^{p^n}f(p^{-n}y)^{p^n}| \leq \tau^n \varepsilon
\]
hence also
\[
|a(x+y) - a(x)a(y)| \leq 0.
\]

Proposition 3.2 is a stepping stone for
PROPOSITION 3.3. Let char $k = p \neq 0$. Suppose $x \mapsto px$ is an injection $G \to G$. Then the conclusion of Proposition 3.2 holds.

Proof. $G$ can be embedded into a divisible group $D$. Set $G_1 := D/D_p$ (for $D_p$ see Theorem 1.1(iii)). Then $G + D + G_1$ is injective and $x \mapsto px$ is a bijection $G_1 \to G_1$. We may assume $G \subseteq G_1$. By Proposition 2.2 $f$ can be extended to an $e$-character $\tilde{f}$ on $G_1$. By Proposition 3.2 there is a character $\tilde{\alpha} : G_1 \to T_k$ for which $|f(x) - \tilde{\alpha}(x)| \leq e$ ($x \in G_1$). Set $\alpha := \tilde{\alpha}|G$.

We now prove the converse to Proposition 3.3.

PROPOSITION 3.4. Let char $k = p \neq 0$. Suppose $x \mapsto px$ ($x \in G$) is not injective. Then there exists an $e \in (0,1)$ and an $e$-character $f : G \to T_k$ such that for every character $\alpha : G \to T_k$ there exists an $x \in G$ with $|f(x) - \alpha(x)| > e$.

Proof. Choose an element $x \in G$ of order $p$ and a $b \in k$ with $0 < |1-b| < 1$.

If char $K = 0$ we assume in addition that $|1-b| < |1-\theta|$, where $\theta$ is a primitive $p$th root of unity. Consider the map

$$g : nx \mapsto b^n \quad (n \in \{0,1,\ldots,p-1\})$$

defined on the group generated by $x$. For $n,m \in \{0,1,\ldots,p-1\}$ we have

$$|g(nx+mx) - g(nx)g(mx)| = \begin{cases} 0 & \text{if } n+m \leq p-1 \\ |b^{n+m-p}b^nm| & \text{if } n+m > p \end{cases}$$

so we see that $g$ is an $e$-character where $e = |b^{-p+1}| = |b^{p-1}|$.

By Proposition 2.2 $g$ extends to an $e$-character $f : G \to T_k$. Now let $\alpha : G \to T_k$ be a character. If $\alpha = 1$ on $H$ we have $|f(x) - \alpha(x)| = |b-1| > |b^{p-1}| = e$ (Lemma 2.3). Otherwise we have $\alpha(x) = \theta$ where $\theta$ is a primitive $p$th root of unity (and char $K = 0$). Then we have, since $|1-b| < |1-\theta|$, $|f(x) - \alpha(x)| = |b-\theta| = \max(|b-1|,|1-\theta|) = |1-\theta| > |1-b| > e$. 
§4 UNIQUENESS

In this section we prove the 'uniqueness part' of Theorem 1.1 i.e. the statements involving (U).

PROPOSITION 4.1.

(i) Let char \( k = 0 \). Then (U) holds if and only if \( G \) is a torsion group.

(ii) Let char \( K = 0, \) char \( k \neq p \neq 0 \). Then (U) holds if and only if \( G \) has no subgroup of index \( p \).

(iii) Let char \( K = p \neq 0 \). Then (U) holds if and only if each homomorphism \( G \to \mathbb{Z}_p \) is zero.

Proof. Statement (U) is equivalent to "if \( \alpha, \beta \) are distinct characters then \( \sup\{|\alpha(x) - \beta(x)| : x \in G\} = 1 \)"; which is by [3], Proposition 1.1, the same as "the characters form an orthogonal set with respect to the sup norm".

Now apply [3], Theorem 3.1, Theorem 2.2, Theorem 4.3 (where \( G \) has the discrete topology). Proposition 4.1 follows.

To complete the proof of Theorem 1.1 the following lemma remains to be shown.

LEMMA 4.2. Let \( p \) be a prime number. The following are equivalent.

(a) Any homomorphism \( G \to \mathbb{Z}_p \) is zero.

(b) If \( H \) is a subgroup of \( G/G_p \) then \( H \) does not have index \( p \).

The heart of the proof is contained in

LEMMA 4.3. Let \( x \mapsto px \ (x \in G) \) be injective but not surjective. Then there exists a nonzero homomorphism \( G \to \mathbb{Z}_p \).

Proof. Let \( \tau : G \to G/pG \) be the quotient map. As \( pG \neq G \) the group \( G/pG \) is in a natural way a nonzero vector space over the field of \( p \) elements.

So there exists an indexed set \( \{e_i\}_{i \in I} \) in \( G \) where \( I \neq \emptyset \) such that \( \{\tau(e_i) : i \in I\} \) is a base of the vector space \( G/pG \). It follows that for
each \( x \in G \) there exist unique \( (\lambda^{(1)}_i)_{i \in I} \) \( \lambda^{(1)}_i \in \{0,1,\ldots,p-1\} \subset \mathbb{Z} \) and \( \{i \in I : \lambda^{(1)}_i \neq 0\} \) is finite such that \( x = \sum \lambda^{(1)}_i e_i = px^1 \) where \( x_1 \in G \).

By injectivity of \( x \mapsto px \) also \( x_1 \) is unique. By treating \( x_1 \) in the same way as we did for \( x \) we find unique \( (\lambda^{(2)}_i)_{i \in I} \in \{0,1,\ldots,p^2-1\} \subset \mathbb{Z} \), \( \{i \in I : \lambda^{(2)}_i \neq 0\} \) is finite such that \( x = \sum \lambda^{(2)}_i e_i = p^2x^2 \) where \( x_2 \in G \) etc.

Thus, for each \( n \in \mathbb{N} \) there exist unique maps \( \phi_i^{(n)} : G \to \{0,1,\ldots,p^n-1\} \subset \mathbb{Z} \) with \( \{i \in I : \phi_i^{(n)}(x) \neq 0\} \) finite for each \( x \in G \) such that

\[
x - \sum \phi_i^{(n)}(x)e_i \in p^nG \quad (x \in G)
\]

By uniqueness, for each \( i \in I, n \in \mathbb{N} \)

\[
\phi_i^{(n+1)}(x) \equiv \phi_i^{(n)}(x) \mod p^n \quad (x \in G)
\]

We see that, for any \( j \in I \), the \( p \)-adic limit

\[
\phi_j(x) = \lim_{n \to \infty} \phi_j^{(n)}(x) \quad (x \in G)
\]

exists and defines a map \( \phi_j : G \to \mathbb{Z}_p \). As \( \phi_j(e_j) = 1 \) this map is not zero.

To see that \( \phi_j \) is a homomorphism observe that for each \( n \in \mathbb{N}, x, y \in G \)

\[
\sum_i (\phi_i^{(n)}(x+y) - \phi_i^{(n)}(x) - \phi_i^{(n)}(y))e_i \in p^nG
\]

By what we have proved above

\[
\phi_j^{(n)}(x+y) - \phi_j^{(n)}(x) - \phi_j^{(n)}(y) \equiv 0 \mod p^n
\]

i.e.

\[
|\phi_j^{(n)}(x+y) - \phi_j^{(n)}(x) - \phi_j^{(n)}(y)|_p \leq p^{-n}
\]

which means for \( \phi_j \) that

\[
|\phi_j(x+y) - \phi_j(x) - \phi_j(y)| \leq 0
\]
Proof of Lemma 4.2. (a) ⇒ (b). Suppose we had a subgroup \( H \) of \( G/G_p \) of index \( p \). Then, since \( G/G_p \) has no elements of order \( p \), the map \( x \mapsto px \) is injective but not surjective on \( G/G_p \), so Lemma 4.3 gives us a nontrivial homomorphism \( \phi : G/G_p \to \mathbb{Z}_p \). But then \( G + G/G_p \not\cong \mathbb{Z}_p \) is a nontrivial homomorphism \( G \to \mathbb{Z}_p \), which conflicts (a). To prove (b) ⇒ (a), suppose we had a nontrivial homomorphism \( \phi : G \to \mathbb{Z}_p \). Then we may assume 1 ∈ Im\( \phi \).

It is easy to see that \( H := \phi^{-1}(p\mathbb{Z}_p) \) has index \( p \) and contains \( G_p \). This violates (a).

**PROBLEM.** Generalize Theorem 1.1 to topological abelian groups \( G \) and continuous (\( \epsilon \)-)characters.

**REFERENCES**


