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$p$-ADIC TRIGONOMETRIC POLYNOMIALS

by

W.H. SCHIKHOF

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\textbf{Introduction.} Let \( G \) be an abelian group, let \( f \) be a bounded complex valued function on \( G \) whose translates generate a finite dimensional space. It is well known ([2], 27.7) that \( f \) is a linear combination of characters. This conclusion is not valid if the range of \( f \) lies in a non-archimedean valued field \( K \) rather than \( \mathbb{C} \). For example, if \( K \) contains the field \( \mathbb{Q}_p \) of the \( p \)-adic numbers and if \( G = \mathbb{Z}_p \), the additive group of the \( p \)-adic integers, it is easily seen that the translates of the function \( f : x \mapsto x \) generate a two-dimensional space over \( K \) whereas \( f \) is not a \( K \)-linear combination of \( K \)-valued characters (follow the proof of the implication (\( \gamma \)) \( \Rightarrow (\alpha) \) of Theorem 1.4).

\textbf{Abstract.} For an abelian topological group \( G \) and an algebraically closed, nontrivially valued, complete field \( K \) necessary and sufficient conditions are derived for a representative function \( f : G \to K \) to be a finite \( K \)-linear combination of \( K \)-valued characters (Theorems 1.4, 2.1, 2.2). Also, a complete description of the set of all representative functions \( \mathbb{Z}_p \to K \) is given (Theorem 3.1).

\textbf{Terminology & Standard Facts.} Throughout this paper \( G \) is an additively written abelian topological group, \( K \) is an algebraically closed nontrivially valued complete field with valuation \( | \cdot | \). The set \( BC(G \to K) \) consisting of all bounded continuous functions \( G \to K \) is a \( K \)-Banach algebra with respect to pointwise operations and the norm \( f \mapsto \|f\|_\infty := \sup \{|f(x)| : x \in G\} \).

A \textit{character} is a nonzero element \( \alpha \) of \( BC(G \to K) \) for which \( \alpha(x+y) = \alpha(x)\alpha(y) \) for all \( x, y \in G \). Then \( |\alpha(x)| = 1 \) for all \( x \in G \). Under pointwise multiplication the characters form a group \( G^\wedge \). A function \( f \in BC(G \to K) \) is a \textit{representative function} (or a \textit{trigonometric polynomial}) if the \( K \)-linear span \( \{f_s : s \in G\} \) of \( \{f_s : s \in G\} \) is finite dimensional. Here, as usual, \( f_s(x) := f(s+x) \) for \( x \in G \). It is not hard to prove that the collection \( \mathcal{R}(G \to K) \) of all representative functions \( G \to K \) is a \( K \)-subalgebra of \( BC(G \to K) \) containing \( G^\wedge \).
A $G$-module is a Banach space $E$ over $K$ together with a separately continuous structure map $G \times E \rightarrow E$

$$(s, x) \mapsto U_s(x) = sx \quad (s \in G, x \in E)$$

such that $s \mapsto U_s$ is a homomorphism of $G$ into the group of invertible (continuous) $K$-linear operators $E \rightarrow E$ and such that, for each $x \in G$, sup$\{\|U_s x\| : s \in G\}$ is finite. In this paper we shall deal only with finite dimensional $G$-modules.

§1. THE MAIN THEOREM

Proposition 1.1. Let $f \in BC(G \rightarrow K)$, $f \neq 0$. Then $[f_s : s \in G]$ is onedimensional if and only if $f$ is a multiple of a character.

Proof. If $\alpha$ is a character then for each $s \in G$ we have $\alpha_s = \alpha(s)\alpha$ and $[\alpha_s : s \in G]$ is onedimensional. Conversely, suppose dim$[f_s : s \in G] = 1$. For each $s \in G$ there is a unique $\alpha(s) \in K$ for which $f_s = \alpha(s)f$. The equality $f_{s+t} = (f_s)_t$ yields $\alpha(s+t) = \alpha(s)\alpha(t)$ for all $s, t \in G$. From $\|f_s\|_\infty = |\alpha(s)||f|_\infty$ we infer that $|\alpha(s)| = 1$ for all $s \in G$. So, $\alpha$ is a character and $f = f(0)\alpha$.

Proposition 1.2. A representative function is uniformly continuous.

Proof. Let $f \in \mathcal{B}(G \rightarrow K)$, $f \neq 0$ and let $e_1, \ldots, e_n$ be a base of $E := [f_s : s \in G]$. By equivalence of norms (see [3], Theorem 3.15 for the non-archimedean case) there exists a $C > 0$ such that

$$\max_{1 \leq i \leq n} |\lambda_i| \leq C\|\sum_{i=1}^n \lambda_i e_i\|_\infty$$

for all $\lambda_1, \ldots, \lambda_n \in K$. Let $\varepsilon > 0$. There is a neighbourhood $U$ of $0$ in $G$ such that for all $i \in \{1, 2, \ldots, n\}$

$$x \in U \Rightarrow |e_i(x) - e_i(0)| \leq (C n \|f\|_\infty)^{-1} \varepsilon.$$

Now let $s \in G$, $t \in U$; we shall prove that $|f(s+t) - f(s)| \leq \varepsilon$. There exist $\lambda_1, \ldots, \lambda_n \in K$ (depending on $s$) such that

$$f_s = \sum_{i=1}^n \lambda_i e_i$$

Then

$$f_{s+t} = \sum_{i=1}^n \lambda_i (e_i)_t$$

We see that

$$|f(s+t) - f(s)| = |f_{s+t}(0) - f_s(0)| =$$
\[
\left| \sum_{i=1}^{n} \lambda_i (e_i(t) - e_i(0)) \right| \leq n \max_{1 \leq i \leq n} \left| \lambda_i \right| \left| e_i(t) - e_i(0) \right| \leq \\
n C \left\| \sum_{i=1}^{n} \lambda_i e_i \right\| \infty \left( Cn \| f \| \infty \right)^{-1} \varepsilon = \| f \| \| f \|^{-1} \varepsilon = \varepsilon.
\]

**Proposition 1.3.** Let \( E \) be a \( G \)-module of dimension \( n \in \mathbb{N} \). For each \( m \in \mathbb{N} \), \( 1 \leq m \leq n \), \( E \) has a \( G \)-submodule of dimension \( m \).

**Proof.** By induction on \( m \). To find a onedimensional submodule choose, among all nonzero \( G \)-submodules of \( E \), a \( G \)-submodule \( E_1 \) with minimal dimension. Then \( E_1 \) is simple (i.e., the corresponding representation \( s \mapsto U_s \) is irreducible). As \( K \) is algebraically closed, a standard application of Schur's lemma ([2], 27.9) yields \( \dim E_1 = 1 \). Now let \( m < n \) and let \( E_m \) be an \( m \)-dimensional \( G \)-submodule of \( E \). The quotient \( E/E_m \) is, in an obvious way, a \( G \)-module of dimension \( n - m \geq 1 \). By the first part of the proof it has a onedimensional \( G \)-submodule \( D_1 \). One verifies immediately that \( E_{m+1} := \pi^{-1}(D_1) \), where \( \pi : E \rightarrow E/E_m \) is the quotient map, is a \( G \)-submodule of \( E \) whose dimension is \( m + 1 \).

We now prove the main theorem. A function \( \mu : G \rightarrow K \) is *additive* if \( \mu(s + t) = \mu(s) + \mu(t) \) for all \( s, t \in G \).

**Theorem 1.4.** The following statements on \( G, K \) are equivalent.

1. Any bounded continuous additive function \( G \rightarrow K \) is 0.
2. Each nonzero finite dimensional \( G \)-module over \( K \) is a (direct) sum of onedimensional \( G \)-modules.
3. Each representative function \( G \rightarrow K \) is a finite \( K \)-linear combination of \( K \)-valued characters.

**Proof.** To obtain the implication \((\alpha) \Rightarrow (\beta)\) we shall prove that

\[
\left\{ \begin{array}{l}
\text{each } n \text{-dimensional } G \text{-module has a base } e_1, \ldots, e_n \\
\text{for which } se \in [e_i] \text{ (} s \in G \text{) for each } i \in \{1, \ldots, n\}
\end{array} \right.
\]

by induction on \( n \). The case \( n = 1 \) is trivial, so suppose \((\ast)\) is true for some \( n \) and let \( E \) be an \((n+1)\)-dimensional \( G \)-module. According to Proposition 1.3 \( E \) has an \( n \)-dimensional \( G \)-submodule \( D \) which, by the induction hypothesis, has a base \( e_1, \ldots, e_n \) such that \( se \in [e_i] \) for all \( s \in G \), all \( i \in \{1, \ldots, n\} \). Choose an \( x \in E \setminus D \); then \( e_1, \ldots, e_n, x \) is base for \( E \). With respect to this base the maps \( U_s (s \in G) \) have the following matrices
Observe that its entries are continuous functions of $s$ (since each of them has the form $s \mapsto \phi(sy)$ for some $y \in E$, $\phi \in E^*$, the dual space of $E$) and are also bounded by our definition of a $G$-module. Since $U_s$ is invertible we have $\lambda_i(s) \neq 0$ for all $i \in \{1, \ldots, n+1\}$. The equality $U_{s+t}=U_s U_t$ expressed in matrix form yields

$$h(s+t) = h(s)h(0 (s, t \in G)$$

(so, each $\lambda_i$ is a character) and

\[ \xi_i(s+t) = \lambda_i(s)\xi_i(t) + \xi_i(s)\lambda_n+1(t) = \lambda_i(t)\xi_i(s) + \xi_i(t)\lambda_n+1(s) \quad (s, t \in G) \]

for $i \in \{1, \ldots, n\}$. We now complete the proof of $(\alpha) \Rightarrow (\beta)$ by defining $q_1, \ldots, q_n \in K$ such that for

\[ e_{n+1} := x + \sum_{i=1}^{n} q_i e_i \]

we have $se_{n+1} = \lambda_{n+1}(s)e_{n+1}$ ($s \in G$). That is, we have to choose $q_1, \ldots, q_n$ in such a way that

\[ \xi_i(s) + q_i(\lambda_i(s) - \lambda_{n+1}(s)) = 0 \quad (1 \leq i \leq n, s \in G). \]

For any $i \in \{1, \ldots, n\}$ we distinguish two cases.

(i) $\lambda_i(t) \neq \lambda_{n+1}(t)$ for some $t \in G$. Then we are forced to choose

\[ q_i := (\lambda_{n+1}(t) - \lambda_i(t))^{-1} \xi_i(t) \]

Now $(\ast\ast)$ guarantees that for any $s \in G$

\[ (\lambda_{n+1}(t) - \lambda_i(t)) (\xi_i(s) + q_i(\lambda_i(s) - \lambda_{n+1}(s))) = 0 \]

and $(\ast\ast\ast)$ follows for this $i$.

(ii) $\lambda_i = \lambda_{n+1}$. We shall prove that $\xi_i(s)=0$ for all $s \in G$ (so that we may choose for $q_i$ and arbitrary element of $K$).

In fact, by $(\ast\ast)$ we have

\[
\begin{bmatrix}
\lambda_1(s) & \xi_1(s) \\
0 & \\
\lambda_2(s) & \xi_2(s) \\
& \ddots \\
\lambda_n(s) & \xi_n(s) \\
0 & \\
\end{bmatrix}
\]

-- 4 --
\[\xi_{t}(s+t) = \lambda_{t}(s)\xi_{t}(t) + \xi_{t}(s)\lambda_{t}(t) \quad (s, t \in G)\]

After dividing by \(\lambda_{t}(s+t) = \lambda_{t}(s)\lambda_{t}(t)\) we obtain

\[\mu(s+t) = \mu(s) + \mu(t)\]

where \(\mu := \lambda_{t}^{-1}\xi_{t}\) is continuous and bounded. By (\(\alpha\)) we have \(\mu = 0\). It follows that \(\xi_{t} = 0\).

(\(\beta\)) \(\Rightarrow\) (\(\gamma\)). Let \(f \in \mathcal{M}(G \to K), f \neq 0\) and let \(E = \{f_{s} : s \in G\}\). The structure map

\[(s, g) \mapsto g_{s} \quad (s \in G, g \in E)\]

makes \(E\) into a finite dimensional \(G\)-module, taking into account that Proposition 1.2 guarantees the continuity of \(s \mapsto g_{s}\). By (\(\beta\)), \(E\) is the sum of one-dimensional \(G\)-modules \([\alpha_{1}], \ldots, [\alpha_{n}]\), where Proposition 1.1 tells us that we may assume that \(\alpha_{1}, \ldots, \alpha_{n}\) are characters and (\(\gamma\)) follows.

(\(\gamma\)) \(\Rightarrow\) (\(\alpha\)). Let \(\mu \in \mathcal{BC}(G \to K)\) be additive. For each \(s \in G\) we have \(\mu_{s} = \mu(s) \cdot 1 + \mu\) where 1 is the function with constant value one. So, \([\mu_{s} : s \in G] = [1, \mu]\) implying that \(\mu\) is a representative function. By (\(\gamma\)) there exist distinct characters \(\alpha_{0}, \alpha_{1}, \ldots, \alpha_{n}\), where \(\alpha_{0}\) is the unit character, and \(\lambda_{0}, \lambda_{1}, \ldots, \lambda_{n} \in K\) such that

\[\mu = \sum_{i=0}^{n} \lambda_{i} \alpha_{i}\]

The relation \(\mu_{s} = \mu(s) \alpha_{0} + \mu\) yields

\[\sum_{i=0}^{n} \lambda_{i} \alpha_{i}(s) \alpha_{i} = \mu(s) \alpha_{0} + \sum_{i=0}^{n} \lambda_{i} \alpha_{i} \quad (s \in G)\]

By linear independence of characters we have equality of the coefficients of \(\alpha_{0}\) i.e.

\[\lambda_{0} = \lambda_{0} \alpha_{0}(s) = \mu(s) + \lambda_{0} \quad (s \in G)\]

implying \(\mu(s) = 0\) for all \(s \in G\).

§2. THE MAIN THEOREM FOR VARIOUS GROUND FIELDS

Theorem 2.1. If the valuation of \(K\) is archimedean then (\(\alpha\)),(\(\beta\)),(\(\gamma\)) of Theorem 1.4 hold for every topological abelian group \(G\).

Proof. Property (\(\alpha\)) of Theorem 1.4 follows from the fact that \(K\) has no bounded additive subgroups other than (0).

Next we turn to the case where the valuation of \(K\) is non-archimedean. First some notations. The residue class field of \(K\) is \(k\). The characteristic of a field \(L\) is \(\text{char} L\). For topological groups \(G_{1}, G_{2}\) the set of all continuous homomorphisms \(G_{1} \to G_{2}\) is \(\text{Hom}(G_{1}, G_{2})\).
Theorem 2.2. Let the valuation of \( K \) be non-archimedean. Then (\( \alpha \)), (\( \beta \)), (\( \gamma \)) of Theorem 1.4 are equivalent to

\( \delta' \) \quad \text{Hom}(G, \mathbb{Q}) = (0) \quad \text{(where \( \mathbb{Q} \) carries the discrete topology)}

if \( \text{char} K = \text{char} k = 0 \),

\( \delta'' \) \quad \text{Hom}(G, \mathbb{Z}/p\mathbb{Z}) = (0)

if \( \text{char} K = \text{char} k = p \neq 0 \),

\( \delta''' \) \quad \text{Hom}(G, \mathbb{Z}_p) = (0)

if \( \text{char} K = 0 \), \( \text{char} k = p = 0 \).

Proof. (a) Assume \( \text{char} K = \text{char} k = 0 \). We have a natural embedding \( \mathbb{Q} \rightarrow K \) whose image is bounded so (\( \alpha \)) of Theorem 1.4 implies (\( \delta' \)). To obtain (\( \delta' \))\( \Rightarrow \)\( (\alpha) \), let \( \mu : G \rightarrow K \) be a bounded nonzero additive homomorphism; we shall prove that \( \text{Hom}(G, \mathbb{Q}) \neq (0) \). Let \( s \in G \), \( \mu(s) \neq 0 \) and let

\[ \pi : K \rightarrow K/\{ x \in K : |x| < |\mu(s)| \} \]

be the canonical quotient map. The discrete group \( H \) is torsion free so the formula

\[ n \pi(\mu(s)) \mapsto n \quad (n \in \mathbb{Z}) \]

defines a homomorphism of the group generated by \( \pi(\mu(s)) \) into \( \mathbb{Q} \). By divisibility of \( \mathbb{Q} \) it can be extended to a homomorphism \( \phi : H \rightarrow \mathbb{Q} \). The map \( \phi \circ \pi \circ \mu : G \rightarrow \mathbb{Q} \) is a continuous homomorphism sending \( s \) into 1. Hence \( \text{Hom}(G, \mathbb{Q}) \neq (0) \).

(b) Assume \( \text{char} K = \text{char} k = p \neq 0 \). We have a natural embedding \( \mathbb{Z}/p\mathbb{Z} \rightarrow K \) so (\( \alpha \)) of Theorem 1.4 implies (\( \delta'' \)). To obtain (\( \delta'' \))\( \Rightarrow \)\( (\alpha) \), let \( \mu : G \rightarrow K \) be a bounded nonzero additive homomorphism; we shall prove that \( \text{Hom}(G, \mathbb{Z}/p\mathbb{Z}) \neq (0) \). Define \( s, \pi, H \) as in part (a). This time every nonzero element of \( H \) has order \( p \) so the homomorphism

\[ n \pi(\mu(s)) \mapsto n \mod p \mathbb{Z} \quad (n \in \mathbb{Z}) \]

can be extended to homomorphism \( \phi : H \rightarrow \mathbb{Z}/p\mathbb{Z} \). The map \( \phi \circ \pi \circ \mu : G \rightarrow \mathbb{Z}/p\mathbb{Z} \) is a continuous homomorphism sending \( s \) into 1 mod \( p \mathbb{Z} \). Hence \( \text{Hom}(G, \mathbb{Z}/p\mathbb{Z}) \neq (0) \).

(c) Assume \( \text{char} K = 0 \), \( \text{char} k = p \neq 0 \). Then we may assume \( K \supseteq \mathbb{Q}_p \). We have a natural embedding \( \mathbb{Z}_p \rightarrow K \) so (\( \alpha \)) of Theorem 1.4 implies (\( \delta''' \)). To obtain (\( \delta''' \))\( \Rightarrow \)\( (\alpha) \), let \( \mu : G \rightarrow K \) be a bounded nonzero additive homomorphism; we shall prove that \( \text{Hom}(G, \mathbb{Z}_p) \neq (0) \). Now \( K \) is, in a natural way, a Banach space over \( \mathbb{Q}_p \). Since \( \mathbb{Q}_p \) is spherically complete there exists, by the non-archimedean Hahn-Banach Theorem ([3], Theorem 4.15, (\( \gamma \))\( \Rightarrow \)\( (\alpha) \)), a continuous \( \mathbb{Q}_p \)-linear map \( \phi : K \rightarrow \mathbb{Q}_p \) that does not vanish on \( \mu(G) \). Then \( \phi \circ \mu \) is a nonzero bounded continuous homomorphism \( G \rightarrow \mathbb{Q}_p \). After multiplying it by a suitable element of \( \mathbb{Q}_p \) we obtain a nonzero element of \( \text{Hom}(G, \mathbb{Z}_p) \).
Remarks.

1. It is easily seen that \( \text{Hom}(G,\mathbb{Q})=(0) \)' is equivalent to 'for each open subgroup \( H \) of \( G \) the quotient \( G/H \) is a torsion group'. Similarly, \( \text{Hom}(G,\mathbb{Z}/p\mathbb{Z})=(0) \)' is equivalent to '\( G \) has no open subgroups of index \( p \)'. Further observe that \( \text{Hom}(G,\mathbb{Z}/p\mathbb{Z})=(0) \) implies \( \text{Hom}(G,\mathbb{Z}_p)=(0) \).

2. The groups \( \mathbb{Q}_p, \mathbb{Q}_p/\mathbb{Z}_p \) have the property that each nontrivial discrete quotient is an infinite torsion group. Thus, for such groups, each representative function is a linear combination of characters, for any choice of \( K \).

3. In [5] necessary and sufficient conditions are derived on \( G, K \) in order that \( G* \) be an orthonormal set. It is a striking fact that these are the same as in Theorem 2.2 but where (\( \delta \))'' and (\( \delta \))''' are interchanged!

§3. REPRESENTATIVE FUNCTIONS ON \( \mathbb{Z}_p \)

From the previous theory it follows that a representative function \( \mathbb{Z}_p \rightarrow K \) is a linear combination of characters if \( K \) is archimedean and also if \( K \) is non-archimedean and char \( k \neq p \). So one may be interested in a description of \( \mathcal{R}(\mathbb{Z}_p \rightarrow K) \) for the remaining case \( \text{char} k = p \). We shall prove the following theorem.

**Theorem 3.1.** Let \( f : \mathbb{Z}_p \rightarrow K \).

(i) Let \( K \supseteq \mathbb{Q}_p \). Then \( f \in \mathcal{R}(\mathbb{Z}_p \rightarrow K) \) if and only if \( f \) has the form

\[
(*) \quad f = \sum_{i=1}^{n} P_i \alpha_i
\]

where \( n \in \mathbb{N} \), \( P_1, \ldots, P_n \) are polynomial functions, and \( \alpha_1, \ldots, \alpha_n \) are characters.

(ii) Let \( \text{char} K = \text{char} k = p \). Then \( f \in \mathcal{R}(\mathbb{Z}_p \rightarrow K) \) if and only if \( f \) has the form

\[
(**) \quad f = \sum_{i=1}^{n} L_i \alpha_i
\]

where \( n \in \mathbb{N}, L_1, \ldots, L_n \) are locally constant functions and \( \alpha_1, \ldots, \alpha_n \) are characters.

To prove this theorem we need a few lemmas. For technical reasons we shall say that an \( f : \mathbb{Z}_p \rightarrow K \) is a polycharacter if it has the form \( (*) \) if \( K \supseteq \mathbb{Q}_p \) or the form \( (**) \) if \( \text{char} K = \text{char} k = p \). Then Theorem 3.1 reads in short: \( f \in \mathcal{R}(\mathbb{Z}_p \rightarrow K) \Leftrightarrow f \) is a polycharacter.

One half is easy:

**Lemma 3.2.** Let \( \text{char} k = p \). Each polycharacter \( \mathbb{Z}_p \rightarrow K \) is a representative function.

**Proof.** If \( K \supseteq \mathbb{Q}_p \) the function \( x \rightarrow x \) is an additive homomorphism and therefore is a representative function. For any \( K \), a locally constant function on \( \mathbb{Z}_p \) is constant on cosets of \( p^m \mathbb{Z}_p \) for some \( m \) so its translates generate a space whose dimension is \( \leq p^m \). Now the lemma follows after observing that
\( \mathcal{R}(\mathbb{Z}_p \to K) \) is a \( K \)-algebra.

For the second half of Theorem 3.1 we introduce the following. A function \( f : \mathbb{N} \to K \) can be interpolated if there exists a (unique) continuous function \( \bar{f} : \mathbb{Z}_p \to K \) whose restriction to \( \mathbb{N} \) is \( f \). We need the following result. (As usual, the symbol \( [ \ ] \) indicates the entire part.)

**Lemma 3.3.** Let \( \text{char } k = p \neq 0 \).

(i) For \( a \in K \), \( a \neq 0 \), the sequence \( n \mapsto a^n \) can be interpolated if and only if \( |a - 1| < 1 \).

(ii) For a continuous function \( f : \mathbb{Z}_p \to K \) the sequence

\[
n \mapsto f(0) + f(1) + \ldots + f(n-1)
\]

can be interpolated.

(iii) For each \( m \in \mathbb{N} \) the sequence \( n \mapsto [\frac{n}{p^m}] \), considered as a map \( \mathbb{N} \to \mathbb{Q}_p \) can be interpolated to a function \( x \mapsto x \cdot \frac{x}{p^m} \) on \( \mathbb{Z}_p \). The function \( x \mapsto x \cdot \frac{x}{p^m} \) \( (x \in \mathbb{Z}_p) \) is locally constant.

**Proof.**

(i) See [4], Theorem 32.4.

(ii) See [4], Theorem 34.1 (the assumption \( K \supseteq \mathbb{Q}_p \) is not used in that proof).

(iii) Without trouble one verifies that

\[
x \mapsto [\frac{x}{p^m}] := a_m + a_{m+1}p + a_{m+2}p^2 + \ldots
\]

where \( x = \sum_{i=0}^{\infty} a_i p^i \) is the standard \( p \)-adic expansion of \( x \), is the required extension.

For the continuous extension \( x \mapsto a^x \) \( (x \in \mathbb{Z}_p) \) of \( n \mapsto a^n \) in Lemma 3.3(i) we shall also write \( a^x \). The continuous extension of \( n \mapsto f(0) + f(1) + \ldots + f(n-1) \) is called the indefinite sum of \( f \), denoted by \( Sf \). Observe that

\[
S(f_1) - Sf = f - f(0)
\]

**Lemma 3.4.** Let \( \text{char } k = p \). The indefinite sum of a polycharacter \( \mathbb{Z}_p \to K \) is again a polycharacter.

**Proof.** We consider two cases.

(i) \( K \supseteq \mathbb{Q}_p \). It is not hard to see that the indefinite sum of a polynomial function is again a polynomial function. By linearity it therefore suffices to prove that for each \( j \in \{0,1,2,\ldots\} \) and each \( a \in K \) with \( 0 < |1 - a| < 1 \) the function

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where \( \omega^j \) is the polynomial \( x \mapsto x^j \), is a polycharacter. We shall do this by proving the following statement (*) by induction on \( j \).

There is a polynomial function \( P_j \) of degree \( \leq j \), whose coefficients are rational functions of \( a \) and there is a rational function \( Q_j \) of \( a \) such that for all \( n \in \mathbb{N} \) and all \( a \in K \) with \( 0 < |1-a| < 1 \)

\[
S(\omega^j a^*) = P_j(n) a^n + Q_j(a)
\]

For the case \( j=0 \) observe that

\[
S(a^*)(n) = a^0 + a^1 + \ldots + a^{n-1} = \frac{1}{a-1} a^n + \frac{1}{1-a}
\]

So, (*) holds with \( P_0(n) = \frac{1}{a-1} \), \( Q_0(a) = \frac{1}{1-a} \).

Now suppose we have (*) for some \( j \):

\[
S(\omega^{j+1} a^*)(n) = \sum_{i=0}^{n-1} i^j a^i = P_j(n) a^n + Q_j(a) \quad (n \in \mathbb{N})
\]

Then

\[
S(\omega^{j+1} a^*)(n) = \sum_{i=0}^{n-1} i^{j+1} a^i = a \frac{d}{da} \sum_{i=0}^{n-1} i^j a^i
\]

\[
= (a \frac{d}{da} P_j(n) + n P_j(n)) a^n + a \frac{d}{da} Q_j(a)
\]

So, if we take

\[
P_{j+1}(n) := a \frac{d}{da} P_j(n) + n P_j(n)
\]

\[
Q_{j+1}(a) = a \frac{d}{da} Q_j(a)
\]

then (*) holds for \( j+1 \) in place of \( j \).

(ii) \( \text{char} K = p \). First we prove that \( Sf \) is a polycharacter for

\[
f = \xi_{p^m} \alpha
\]

where \( m \in \mathbb{N} \), where \( \xi_{p^m} \) is the \( K \)-valued characteristic function of \( p^m \mathbb{Z}_p \) and where \( \alpha \) is a character.

We have for \( n \in \mathbb{N} \)

\[
(Sf)(n) = \sum_{i=0}^{n-1} \xi_{p^m} \alpha(i) = \frac{|p^m(n-1)|}{p^m(n)} \alpha(p^m j)
\]

If \( \alpha(p^m) = 1_K \), the unit element of \( K \), we obtain
\[ (Sf)(n) = \left( \frac{n-1}{p^n} \right) + 1 \cdot 1_K \]

and we see that \( Sf \) is a locally constant function.

If \( a(p^n) \neq 1_K \) then \( \alpha(x) = a^x \ (x \in \mathbb{Z}_p) \) where \( a \in K, \ 0 < |1_K - a| < 1. \) We have, for \( n \in \mathbb{N} \)

\[ (Sf)(n) = \frac{\alpha(p^n) - 1}{\alpha(p^n) - 1_K} = \frac{a^{n-1}p^n - 1}{ap^n - 1_K} \]

It follows that \( Sf \) is a \( K \)-linear combination of a constant function and the function

\[ x \mapsto a^{\frac{x}{p^n} - 1} = a^{-x} \cdot a^{\frac{x-1}{p^n}} \]

which is the product of the character \( a^{-x} \) and a locally constant function (Lemma 3.3(iii)). Thus, we may conclude that \( S(\Xi_{p^n \mathbb{Z}_p} \alpha) \) is a polycharacter.

By linearity of \( S \) and by the remark preceding this lemma the set of all polycharacters \( f \) for which \( Sf \) is also a polycharacter is a linear space, invariant under translations. A standard reasoning shows that the smallest translation invariant linear space containing all \( \Xi_{p^n \mathbb{Z}_p} \alpha \) \((m \in \mathbb{N}, \alpha \text{ character})\) is the set of all polycharacters which finishes the proof.

**Lemma 3.5.** Let \( \text{char} K = p, \) let \( a \in K, \ |1 - a| < 1. \) If \( f : \mathbb{Z}_p \to K \) is a polycharacter and if \( g \) is a continuous solution of

\[ g(x+1) - ag(x) = f(x) \ (x \in \mathbb{Z}_p) \]

then \( g \) is a polycharacter.

**Proof.** Inductively we arrive easily at

\[ g(n) = a^n g(0) + a^{n-1} S(a^{-x} f)(n) \ (n \in \mathbb{N}) \]

By continuity,

\[ g = a^n g(0) + a^{n-1} S(a^{-x} f) \]

which is a polycharacter by Lemma 3.4.

Let \( L \) denote the operator \( BC(\mathbb{Z}_p \to K) \to BC(\mathbb{Z}_p \to K) \) sending \( f \) into \( f_1 \) (recall that \( f_1(x) = f(x+1) \)).

**Lemma 3.6.** If, for some \( a \in K, \) the operator \( L - af \) is not injective then \( |a - 1| < 1. \)

**Proof.** Let \( f \in BC(\mathbb{Z}_p \to K), \ f \neq 0 \) be such that \( Lf - af = 0. \) Then \( f(x+1) = af(x) \) for all \( x \in \mathbb{Z}_p \) so that \( f(n) = a^n f(0) \) for all \( n \in \mathbb{N}. \) We have \( f(0) \neq 0 \) and, by continuity of \( f, \) the sequence \( x \mapsto a^n \) can be interpolated. By Lemma 3.3(i), \( |1 - a| < 1. \)
Proof of Theorem 3.1. Let $f$ be a representative function, $f \neq 0$; we shall prove that $f$ is a polycharacter. The sequence $f, Lf, L^2f, \ldots$ lies in a finite dimensional space so there is an $n \in \mathbb{N}$ such that $L^n f$ is a $K$-linear combination of $f, Lf, \ldots, L^{n-1}f$. We may choose $n$ minimal. In other words, we have a monic polynomial $P \in K[X]$ with $P(L)(f) = 0$ with minimal degree $n$. As $K$ is algebraically closed $P$ decomposes into linear factors $X-a_1, \ldots, X-a_n$ so we have

$$(L-a_1I)(L-a_2I) \ldots (L-a_nI)(f) = 0$$

The operators $L-a_i$ commute and $n$ is minimal so no $L-a_iI$ is injective. By Lemma 3.6, $|a_i-1|<1$ for $i \in \{1, \ldots, n\}$.

Lemma 3.5, applied for $a=a_1$, $g=(L-a_2I)\ldots(L-a_nI)f$ and $f=0$ yields

$$(L-a_2I)(L-a_2I) \ldots (L-a_nI)(f) = g$$

where $g$ is a polycharacter. By repeated application of Lemma 3.5 we can remove all $L-a_iI$ obtaining that $f$ is a polycharacter.

Note. For results on closely related matters see [1].

References


