$p$-ADIC TRIGONOMETRIC POLYNOMIALS

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INTRODUCTION. Let $G$ be an abelian group, let $f$ be a bounded complex valued function on $G$ whose translates generate a finite dimensional space. It is well known ([2], 27.7) that $f$ is a linear combination of characters. This conclusion is not valid if the range of $f$ lies in a non-archimedean valued field $K$ rather than $\mathbb{C}$. For example, if $K$ contains the field $\mathbb{Q}_p$ of the $p$-adic numbers and if $G=\mathbb{Z}_p$, the additive group of the $p$-adic integers, it is easily seen that the translates of the function $f : x\mapsto x$ generate a twodimensional space over $K$ whereas $f$ is not a $K$-linear combination of $K$-valued characters (follow the proof of the implication $(\gamma) \Rightarrow (\alpha)$ of Theorem 1.4).

ABSTRACT. For an abelian topological group $G$ and an algebraically closed, nontrivially valued, complete field $K$ necessary and sufficient conditions are derived for a representative function $f : G\to K$ to be a finite $K$-linear combination of $K$-valued characters (Theorems 1.4, 2.1, 2.2). Also, a complete description of the set of all representative functions $\mathbb{Z}_p\to K$ is given (Theorem 3.1).

TERMINOLOGY & STANDARD FACTS. Throughout this paper $G$ is an additively written abelian topological group, $K$ is an algebraically closed nontrivially valued complete field with valuation $|\cdot|$. The set $BC(G\to K)$ consisting of all bounded continuous functions $G\to K$ is a $K$-Banach algebra with respect to pointwise operations and the norm $f\mapsto \|f\|_K := \sup \{|f(x)| : x \in G\}$.

A character is a nonzero element $\alpha$ of $BC(G\to K)$ for which $\alpha(x+y) = \alpha(x)\alpha(y)$ for all $x,y \in G$. Then $|\alpha(x)|=1$ for all $x \in G$. Under pointwise multiplication the characters form a group $G^\wedge_K$. A function $f \in BC(G\to K)$ is a representative function (or a trigonometric polynomial) if the $K$-linear span $[f_s : s \in G]$ of $\{f_s : s \in G\}$ is finite dimensional. Here, as usual, $f_s(x) := f(s+x)$ for $x \in G$. It is not hard to prove that the collection $\mathcal{R}(G\to K)$ of all representative functions $G\to K$ is a $K$-subalgebra of $BC(G\to K)$ containing $G^\wedge_K$. 

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A $G$-module is a Banach space $E$ over $K$ together with a separately continuous structure map $G \times E \to E$

$$(s, x) \mapsto U_s(x) = sx \quad (s \in G, x \in E)$$

such that $s \mapsto U_s$ is a homomorphism of $G$ into the group of invertible (continuous) $K$-linear operators $E \to E$ and such that, for each $x \in G$, $\sup(\|U_sx\| : s \in G)$ is finite. In this paper we shall deal only with finite dimensional $G$-modules.

§1. THE MAIN THEOREM

Proposition 1.1. Let $f \in BC(G \to K)$, $f \neq 0$. Then $[f_s : s \in G]$ is onedimensional if and only if $f$ is a multiple of a character.

Proof. If $\alpha$ is a character then for each $s \in G$ we have $\alpha_s = \alpha(s)\alpha$ and $[\alpha_s : s \in G]$ is onedimensional. Conversely, suppose $\dim[f_s : s \in G] = 1$. For each $s \in G$ there is a unique $\alpha(s) \in K$ for which $f_s = \alpha(s)f$. The equality $f_{s+t} = (f_s)(t)$ yields $\alpha(s+t) = \alpha(s)\alpha(t)$ for all $s, t \in G$. From $\|f_s\|_\infty = |\alpha(s)||f|_\infty$ we infer that $|\alpha(s)| = 1$ for all $s \in G$. So, $\alpha$ is a character and $f = f(0)\alpha$.

Proposition 1.2. A representative function is uniformly continuous.

Proof. Let $f \in \mathcal{R}(G \to K)$, $f \neq 0$ and let $e_1, \ldots, e_n$ be a base of $E := [f_s : s \in G]$. By equivalence of norms (see [3], Theorem 3.15 for the non-archimedean case) there exists a $C > 0$ such that

$$\max_{1 \leq i \leq n} |\lambda_i| \leq C \left\| \sum_{i=1}^n \lambda_i e_i \right\|_\infty$$

for all $\lambda_1, \ldots, \lambda_n \in K$. Let $\varepsilon > 0$. There is a neighbourhood $U$ of 0 in $G$ such that for all $t \in \{1, 2, \ldots, n\}$

$$x \in U \Rightarrow |e_t(x) - e_t(0)| \leq (Cn\|f\|_\infty)^{-1} \varepsilon.$$

Now let $s \in G$, $t \in U$; we shall prove that $|f(s+t) - f(s)| \leq \varepsilon$. There exist $\lambda_1, \ldots, \lambda_n \in K$ (depending on $s$) such that

$$f_s = \sum_{i=1}^n \lambda_i e_i$$

Then

$$f_{s+t} = \sum_{i=1}^n \lambda_i (e_i)_t$$

We see that

$$|f(s+t) - f(s)| = |f_{s+t}(0) - f_s(0)| =$$
Proposition 1.3. Let $E$ be a $G$-module of dimension $n \in \mathbb{N}$. For each $m \in \mathbb{N}$, $1 \leq m \leq n$, $E$ has a $G$-submodule of dimension $m$.

Proof. By induction on $m$. To find a one-dimensional submodule choose, among all nonzero $G$-submodules of $E$, a $G$-submodule $E_1$ with minimal dimension. Then $E_1$ is simple (i.e., the corresponding representation $s \mapsto U_s$ is irreducible). As $K$ is algebraically closed, a standard application of Schur's lemma ([2], 27.9) yields $\dim E_1 = 1$. Now let $m < n$ and let $E_m$ be an $m$-dimensional $G$-submodule of $E$. The quotient $E/E_m$ is, in an obvious way, a $G$-module of dimension $n - m \geq 1$. By the first part of the proof it has a one-dimensional $G$-submodule $E_1$. One verifies immediately that $E_{m+1} := \pi^{-1}(D_1)$, where $\pi : E \to E/E_m$ is the quotient map, is a $G$-submodule of $E$ whose dimension is $m + 1$.

We now prove the main theorem. A function $\mu : G \to K$ is additive if $\mu(s + t) = \mu(s) + \mu(t)$ for all $s, t \in G$.

**Theorem 1.4.** The following statements on $G, K$ are equivalent.

- (a) Any bounded continuous additive function $G \to K$ is $0$.
- (b) Each nonzero finite dimensional $G$-module over $K$ is a (direct) sum of one-dimensional $G$-modules.
- (c) Each representative function $G \to K$ is a finite $K$-linear combination of $K$-valued characters.

Proof. To obtain the implication (a) $\Rightarrow$ (b) we shall prove that

\[
\left\{ \begin{array}{l}
\text{each } n\text{-dimensional } G\text{-module has a base } e_1, \ldots, e_n \\
\text{for which } se_i \in [e_i] \ (s \in G) \text{ for each } i \in \{1, \ldots, n\}
\end{array} \right.
\]

by induction on $n$. The case $n = 1$ is trivial, so suppose (*) is true for some $n$ and let $E$ be an $(n+1)$-dimensional $G$-module. According to Proposition 1.3 $E$ has an $n$-dimensional $G$-submodule $D$ which, by the induction hypothesis, has a base $e_1, \ldots, e_n$ such that $se_i \in [e_i]$ for all $s \in G$, all $i \in \{1, \ldots, n\}$. Choose an $x \in E \setminus D$; then $e_1, \ldots, e_n, x$ is base for $E$. With respect to this base the maps $U_s (s \in G)$ have the following matrices
Observe that its entries are continuous functions of \( s \) (since each of them has the form \( s \mapsto \phi(sy) \) for some \( y \in E, \phi \in E^* \), the dual space of \( E \)) and are also bounded by our definition of a \( G \)-module. Since \( U_s \) is invertible we have \( \lambda_i(s) \neq 0 \) for all \( i \in \{1, \ldots, n+1\} \). The equality \( U_{s+t} = U_s U_t \) expressed in matrix form yields

\[
\begin{pmatrix}
\lambda_1(s) & \xi_1(s) \\
0 & \\
\lambda_2(s) & \xi_2(s) \\
\vdots & \ddots \\
\lambda_{n}(s) & \xi_{n}(s) \\
0 & \\
\lambda_{n+1}(s)
\end{pmatrix}
\]

(\( s, t \in G \))

(so, each \( \lambda_i \) is a character) and

\[
(\star) \quad \xi_i(s+t) = \lambda_i(s)\xi_i(t) + \xi_i(s)\lambda_{n+1}(t) = \lambda_i(t)\xi_i(s) + \xi_i(t)\lambda_{n+1}(s) \quad (s, t \in G)
\]

for \( i \in \{1, \ldots, n\} \). We now complete the proof of \( (\alpha) \Rightarrow (\beta) \) by defining \( q_1, \ldots, q_n \in K \) such that for

\[
e_{n+1} := x + \sum_{i=1}^{n} q_i e_i
\]

we have \( s e_{n+1} = \lambda_{n+1}(s) e_{n+1} \quad (s \in G) \). That is, we have to choose \( q_1, \ldots, q_n \) in such a way that

\[
(\star\star) \quad \xi_i(s) + q_i(\lambda_i(s) - \lambda_{n+1}(s)) = 0 \quad (1 \leq i \leq n, s \in G).
\]

For any \( i \in \{1, \ldots, n\} \) we distinguish two cases.

(i) \( \lambda_i(t) \neq \lambda_{n+1}(t) \) for some \( t \in G \). Then we are forced to choose

\[
q_i := \left(\lambda_{n+1}(t) - \lambda_i(t)\right)^{-1}\xi_i(t)
\]

Now \( (\star) \) guarantees that for any \( s \in G \)

\[
(\lambda_{n+1}(t)-\lambda_i(t))(\xi_i(s)+q_i(\lambda_i(s)\lambda_{n+1}(s))) = 0
\]

and \( (\star\star) \) follows for this \( i \).

(ii) \( \lambda_i = \lambda_{n+1} \). We shall prove that \( \xi_i(s) = 0 \) for all \( s \in G \) (so that we may choose for \( q_i \) and arbitrary element of \( K \)).

In fact, by \( (\star) \) we have

\[
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\]
\[ \xi_t(s+t) = \lambda_t(s)\xi_t(t) + \xi_t(s)\lambda_t(t) \quad (s, t \in G) \]

After dividing by \( \lambda_t(s+t) = \lambda_t(s)\lambda_t(t) \) we obtain

\[ \mu(s+t) = \mu(s) + \mu(t) \]

where \( \mu := \lambda_t^{-1}\xi_t \) is continuous and bounded. By (α) we have \( \mu = 0 \). It follows that \( \xi_t = 0 \).

(β)⇒(γ). Let \( f \in \mathcal{R}(G \to K) \), \( f \neq 0 \) and let \( E = [f_s : s \in G] \). The structure map

\[ (s, g) \mapsto g_s \quad (s \in G, g \in E) \]

makes \( E \) into a finite dimensional \( G \)-module, taking into account that Proposition 1.2 guarantees the continuity of \( s \mapsto g_s \). By (β), \( E \) is the sum of onedimensional \( G \)-modules \([\alpha_1], \ldots, [\alpha_n]\), where Proposition 1.1 tells us that we may assume that \( \alpha_1, \ldots, \alpha_n \) are characters and (γ) follows.

(γ)⇒(α). Let \( \mu \in BC(G \to K) \) be additive. For each \( s \in G \) we have \( \mu_s = \mu(s) \cdot 1 + \mu \) where \( 1 \) is the function with constant value one. So, \([\mu_s : s \in G] = [1, \mu]\) implying that \( \mu \) is a representative function. By (γ) there exist distinct characters \( \alpha_0, \alpha_1, \ldots, \alpha_n \), where \( \alpha_0 \) is the unit character, and \( \lambda_0, \lambda_1, \ldots, \lambda_n \in K \) such that

\[ \mu = \sum_{i=0}^{n} \lambda_i \alpha_i \]

The relation \( \mu_s = \mu(s)\alpha_0 + \mu \) yields

\[ \sum_{i=0}^{n} \lambda_i \alpha_i(s)\alpha_i = \mu(s)\alpha_0 + \sum_{i=0}^{n} \lambda_i \alpha_i \quad (s \in G) \]

By linear independence of characters we have equality of the coefficients of \( \alpha_0 \) i.e.

\[ \lambda_0 = \lambda_0\alpha_0(s) = \mu(s) + \lambda_0 \quad (s \in G) \]

implying \( \mu(s) = 0 \) for all \( s \in G \).

§2. THE MAIN THEOREM FOR VARIOUS GROUND FIELDS

Theorem 2.1. If the valuation of \( K \) is archimedean then (α),(β),(γ) of Theorem 1.4 hold for every topological abelian group \( G \).

Proof. Property (α) of Theorem 1.4 follows from the fact that \( K \) has no bounded additive subgroups other than (0).

Next we turn to the case where the valuation of \( K \) is non-archimedean. First some notations. The residue class field of \( K \) is \( k \). The characteristic of a field \( L \) is \( \text{char}L \). For topological groups \( G_1, G_2 \) the set of all continuous homomorphisms \( G_1 \to G_2 \) is \( \text{Hom}(G_1, G_2) \).
Theorem 2.2. Let the valuation of $K$ be non-archimedean. Then $(\alpha),(\beta),(\gamma)$ of Theorem 1.4 are equivalent to

\begin{align*}
(\delta)' & \quad \text{Hom}(G, \mathbb{Q}) = (0) \quad \text{(where $\mathbb{Q}$ carries the discrete topology)}
\text{if char} \ K = \text{char} \ k = 0, \\
(\delta)'' & \quad \text{Hom}(G, \mathbb{Z}/p\mathbb{Z}) = (0)
\text{if char} \ K = \text{char} \ k = p \neq 0, \\
(\delta)''' & \quad \text{Hom}(G, \mathbb{Z}_p) = (0)
\text{if char} \ K = 0, \text{char} \ k = p \neq 0.
\end{align*}

Proof. (a) Assume $\text{char} \ K = \text{char} \ k = 0$. We have a natural embedding $\mathbb{Q} \to K$ whose image is bounded so $(\alpha)$ of Theorem 1.4 implies $(\delta)'$. To obtain $(\delta)' \Rightarrow (\alpha)$, let $\mu : G \to K$ be a bounded nonzero additive homomorphism; we shall prove that $\text{Hom}(G, \mathbb{Q}) \neq (0)$. Let $s \in G$, $\mu(s) \neq 0$ and let

$$
\pi : K \to K/\{x \in K : |x| < |\mu(s)|\}
$$

be the canonical quotient map. The discrete group $H$ is torsion free so the formula

$$
n \pi(n(s)) \mapsto n \quad (n \in \mathbb{Z})
$$

defines a homomorphism of the group generated by $\pi(n(s))$ into $\mathbb{Q}$. By divisibility of $\mathbb{Q}$ it can be extended to a homomorphism $\phi : H \to \mathbb{Q}$. The map $\phi \circ \pi \circ \mu : G \to \mathbb{Q}$ is a continuous homomorphism sending $s$ into 1. Hence $\text{Hom}(G, \mathbb{Q}) \neq (0)$.

(b) Assume $\text{char} \ K = \text{char} \ k = p \neq 0$. We have a natural embedding $\mathbb{Z}/p\mathbb{Z} \to K$ so $(\alpha)$ of Theorem 1.4 implies $(\delta)''$. To obtain $(\delta)'' \Rightarrow (\alpha)$, let $\mu : G \to K$ be a bounded nonzero additive homomorphism; we shall prove that $\text{Hom}(G, \mathbb{Z}/p\mathbb{Z}) \neq (0)$. Define $s, \pi, H$ as in part (a). This time every nonzero element of $H$ has order $p$ so the homomorphism

$$
n \pi(n(s)) \mapsto n \mod p \mathbb{Z} \quad (n \in \mathbb{Z})
$$

can be extended to homomorphism $\phi : H \to \mathbb{Z}/p\mathbb{Z}$. The map $\phi \circ \pi \circ \mu : G \to \mathbb{Z}/p\mathbb{Z}$ is a continuous homomorphism sending $s$ into 1 mod $p \mathbb{Z}$. Hence $\text{Hom}(G, \mathbb{Z}/p\mathbb{Z}) \neq (0)$.

(c) Assume $\text{char} \ K = 0$, $\text{char} \ k = p \neq 0$. Then we may assume $K \cong \mathbb{Q}_p$. We have a natural embedding $\mathbb{Z}_p \to K$ so $(\alpha)$ of Theorem 1.4 implies $(\delta)'''$. To obtain $(\delta)''' \Rightarrow (\alpha)$, let $\mu : G \to K$ be a bounded nonzero additive homomorphism; we shall prove that $\text{Hom}(G, \mathbb{Z}_p) \neq (0)$. Now $K$ is, in a natural way, a Banach space over $\mathbb{Q}_p$. Since $\mathbb{Q}_p$ is spherically complete there exists, by the non-archimedean Hahn-Banach Theorem ([3], Theorem 4.15, $(\gamma) \Rightarrow (\alpha)$), a continuous $\mathbb{Q}_p$-linear map $\phi : K \to \mathbb{Q}_p$ that does not vanish on $\mu(G)$. Then $\phi \circ \mu$ is a nonzero bounded continuous homomorphism $G \to \mathbb{Q}_p$. After multiplying it by a suitable element of $\mathbb{Q}_p$ we obtain a nonzero element of $\text{Hom}(G, \mathbb{Z}_p)$. 
Remarks.

1. It is easily seen that \( \text{Hom}(G, \mathbb{Q}) = (0)' \) is equivalent to 'for each open subgroup \( H \) of \( G \) the quotient \( G/H \) is a torsion group'. Similarly, \( \text{Hom}(G, \mathbb{Z}/p\mathbb{Z}) = (0)' \) is equivalent to '\( G \) has no open subgroups of index \( p \)'. Further observe that \( \text{Hom}(G, \mathbb{Z}/p\mathbb{Z}) = (0) \) implies \( \text{Hom}(G, \mathbb{Z}/p\mathbb{Z}) = (0) \).

2. The groups \( \mathbb{Q}_p \), \( \mathbb{Q}_p/\mathbb{Z}_p \) have the property that each nontrivial discrete quotient is an infinite torsion group. Thus, for such groups, each representative function is a linear combination of characters, for any choice of \( K \).

3. In [5] necessary and sufficient conditions are derived on \( G, K \) in order that \( G^*_K \) be an orthonormal set. It is a striking fact that these are the same as in Theorem 2.2 but where \((5)''\) and \((5)'''\) are interchanged!

§3. REPRESENTATIVE FUNCTIONS ON \( \mathbb{Z}_p \)

From the previous theory it follows that a representative function \( \mathbb{Z}_p \rightarrow K \) is a linear combination of characters if \( K \) is archimedean and also if \( K \) is non-archimedean and \( \text{char } k = p \). So one may be interested in a description of \( \mathfrak{r}(\mathbb{Z}_p \rightarrow K) \) for the remaining case \( \text{char } k = p \). We shall prove the following theorem.

**Theorem 3.1.** Let \( f : \mathbb{Z}_p \rightarrow K \).

(i) Let \( K \subseteq \mathbb{Q}_p \). Then \( f \in \mathfrak{r}(\mathbb{Z}_p \rightarrow K) \) if and only if \( f \) has the form

\[
(*) \quad f = \sum_{i=1}^{n} P_i \alpha_i
\]

where \( n \in \mathbb{N} \), \( P_1, \ldots, P_n \) are polynomial functions, and \( \alpha_1, \ldots, \alpha_n \) are characters.

(ii) Let \( \text{char } K = \text{char } k = p \). Then \( f \in \mathfrak{r}(\mathbb{Z}_p \rightarrow K) \) if and only if \( f \) has the form

\[
(**) \quad f = \sum_{i=1}^{n} L_i \alpha_i
\]

where \( n \in \mathbb{N} \), \( L_1, \ldots, L_n \) are locally constant functions and \( \alpha_1, \ldots, \alpha_n \) are characters.

To prove this theorem we need a few lemmas. For technical reasons we shall say that an \( f : \mathbb{Z}_p \rightarrow K \) is a polycharacter if it has the form \( (*) \) if \( K \subseteq \mathbb{Q}_p \), or the form \( (**) \) if \( \text{char } K = \text{char } k = p \). Then Theorem 3.1 reads in short: \( f \in \mathfrak{r}(\mathbb{Z}_p \rightarrow K) \iff f \) is a polycharacter.

One half is easy:

**Lemma 3.2.** Let \( \text{char } k = p \). Each polycharacter \( \mathbb{Z}_p \rightarrow K \) is a representative function.

**Proof.** If \( K \subseteq \mathbb{Q}_p \) the function \( x \mapsto x \) is an additive homomorphism and therefore is a representative function. For any \( K \), a locally constant function on \( \mathbb{Z}_p \) is constant on cosets of \( p^m \mathbb{Z}_p \) for some \( m \) so its translates generate a space whose dimension is \( \leq p^m \). Now the lemma follows after observing that

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$\mathcal{R}(\mathbb{Z}_p \rightarrow K)$ is a $K$-algebra.

For the second half of Theorem 3.1 we introduce the following. A function $f : \mathbb{N} \rightarrow K$ can be interpolated if there exists a (unique) continuous function $\tilde{f} : \mathbb{Z}_p \rightarrow K$ whose restriction to $\mathbb{N}$ is $f$. We need the following result. (As usual, the symbol $[\cdot]$ indicates the entire part.)

**Lemma 3.3.** Let char $k = p \neq 0$.

(i) For $a \in K$, $a \neq 0$, the sequence $n \mapsto a^n$ can be interpolated if and only if $|a - 1| < 1$.

(ii) For a continuous function $f : \mathbb{Z}_p \rightarrow K$ the sequence

$$n \mapsto f(0) + f(1) + \ldots + f(n-1)$$

can be interpolated.

(iii) For each $m \in \mathbb{N}$ the sequence $n \mapsto \frac{n}{p^m}$, considered as a map $\mathbb{N} \rightarrow \mathbb{Q}_p$ can be interpolated to a function $x \mapsto x \cdot \frac{x}{p^m}$ on $\mathbb{Z}_p$. The function $x \mapsto x - \frac{x}{p^m} (x \in \mathbb{Z}_p)$ is locally constant.

**Proof.**

(i) See [4], Theorem 32.4.

(ii) See [4], Theorem 34.1 (the assumption $K \supset \mathbb{Q}_p$ is not used in that proof).

(iii) Without trouble one verifies that

$$x \mapsto [\frac{x}{p^m}] := a_m + a_{m+1}p + a_{m+2}p^2 + \ldots$$

where $x = \sum_{i=0}^{\infty} a_i p^i$ is the standard $p$-adic expansion of $x$, is the required extension.

For the continuous extension $x \mapsto a^x$ ($x \in \mathbb{Z}_p$) of $n \mapsto a^n$ in Lemma 3.3(i) we shall also write $a^*$. The continuous extension of $n \mapsto f(0) + f(1) + \ldots + f(n-1)$ is called the indefinite sum of $f$, denoted by $Sf$.

Observe that

$$S(f_i) - Sf = f - f(0)$$

**Lemma 3.4.** Let char $k = p$. The indefinite sum of a polycharacter $\mathbb{Z}_p \rightarrow K$ is again a polycharacter.

**Proof.** We consider two cases.

(i) $K \supset \mathbb{Q}_p$. It is not hard to see that the indefinite sum of a polynomial function is again a polynomial function. By linearity it therefore suffices to prove that for each $j \in \{0, 1, 2, \ldots\}$ and each $a \in K$ with $0 < |1 - a| < 1$ the function

$$a^x$$

is a $K$-algebra.
\[ S(\omega^j a^*) \]

where \( \omega^j \) is the polynomial \( x \mapsto x^j \), is a polycharacter. We shall do this by proving the following statement (*) by induction on \( j \).

There is a polynomial function \( P_j \) of degree \( \leq j \), whose coefficients are rational functions of \( a \) and there is a rational function \( Q_j \) of \( a \) such that for all \( n \in \mathbb{N} \) and all \( a \in K \) with \( 0 < |1-a| < 1 \)

\[ S(\omega^j a^*)(n) = P_j(n)a^n + Q_j(a) \]

For the case \( j = 0 \) observe that

\[ S(a^*)(n) = a^0 + a^1 + \ldots + a^{n-1} = \frac{1}{a-1}a^n + \frac{1}{1-a} \]

So, (*) holds with \( P_0(n) = \frac{1}{a-1}, Q_0(a) = \frac{1}{1-a} \).

Now suppose we have (*) for some \( j \):

\[ S(\omega^j a^*)(n) = \sum_{i=0}^{n-1} i^j a^i = P_j(n)a^n + Q_j(a) \quad (n \in \mathbb{N}) \]

Then

\[ S(\omega^{j+1} a^*)(n) = \sum_{i=0}^{n-1} (i+1)^j a^i = a \frac{d}{da} \sum_{i=0}^{n-1} i^j a^i \]

\[ = (a \frac{d}{da} P_j(n) + n P_j(n))a^n + a \frac{d}{da} Q_j(a) \]

So, if we take

\[ P_{j+1}(n) := a \frac{d}{da} P_j(n) + n P_j(n) \]

\[ Q_{j+1}(a) = a \frac{d}{da} Q_j(a) \]

then (*) holds for \( j + 1 \) in place of \( j \).

(ii) \( \text{char} K = p \). First we prove that \( Sf \) is a polycharacter for

\[ f = \xi_p^m Z_p \alpha \]

where \( m \in \mathbb{N} \), where \( \xi_p^m Z_p \) is the \( K \)-valued characteristic function of \( p^m \mathbb{Z}_p \) and where \( \alpha \) is a character.

We have for \( n \in \mathbb{N} \)

\[ (Sf)(n) = \sum_{i=0}^{n-1} \xi_p^m Z_p(i) \alpha(i) = \sum_{j=0}^{[p^{-m}(n-1)]} \alpha(p^m j) \]

If \( \alpha(p^m) = 1_K \), the unit element of \( K \), we obtain
and we see that $Sf$ is a locally constant function.

If $\alpha(p^m) \neq 1_K$ then $\alpha(x) = a^x$ ($x \in \mathbb{Z}_p$) where $a \in K$, $0 < |1_K - a| < 1$. We have, for $n \in \mathbb{N}$

$$(Sf)(n) = \left( \frac{a^{n-1}}{p^n} \right) 1_K$$

It follows that $Sf$ is a $K$-linear combination of a constant function and the function

$$x \mapsto a^{x-1} \cdot p^n$$

which is the product of the character $a^{-x}$ and a locally constant function (Lemma 3.3(iii)). Thus, we may conclude that $S(\Xi_{p^n})_\alpha$ is a polycharacter.

By linearity of $S$ and by the remark preceding this lemma the set of all polycharacters $f$ for which $Sf$ is also a polycharacter is a linear space, invariant under translations. A standard reasoning shows that the smallest translation invariant linear space containing all $\Xi_{p^n} \alpha$ ($m \in \mathbb{N}$, $\alpha$ character) is the set of all polycharacters which finishes the proof.

Lemma 3.5. Let $\text{char } k = p$, let $a \in K$, $|1-a| < 1$. If $f : \mathbb{Z}_p \to K$ is a polycharacter and if $g$ is a continuous solution of

$$g(x+1) - ag(x) = f(x) \quad (x \in \mathbb{Z}_p)$$

then $g$ is a polycharacter.

Proof. Inductively we arrive easily at

$$g(n) = a^n g(0) + a^{n-1} S(a^{-x}f)(n) \quad (n \in \mathbb{N})$$

By continuity,

$$g = a^n g(0) + a^{n-1} S(a^{-x}f)$$

which is a polycharacter by Lemma 3.4.

Let $L$ denote the operator $BC(\mathbb{Z}_p \to K) \to BC(\mathbb{Z}_p \to K)$ sending $f$ into $f_1$ (recall that $f_1(x) = f(x+1)$).

Lemma 3.6. If, for some $a \in K$, the operator $L - aI$ is not injective then $|a-1| < 1$.

Proof. Let $f \in BC(\mathbb{Z}_p \to K)$, $f \neq 0$ be such that $Lf - af = 0$. Then $f(x+1) = af(x)$ for all $x \in \mathbb{Z}_p$ so that $f(n) = a^n f(0)$ for all $n \in \mathbb{N}$. We have $f(0) \neq 0$ and, by continuity of $f$, the sequence $x \mapsto a^x$ can be interpolated. By Lemma 3.3(i), $|1-a| < 1$. 

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Proof of Theorem 3.1. Let $f$ be a representative function, $f \neq 0$; we shall prove that $f$ is a polycharacter. The sequence $f, Lf, L^2f, \ldots$ lies in a finite dimensional space so there is an $n \in \mathbb{N}$ such that $L^n f$ is a $K$-linear combination of $f, Lf, \ldots, L^{n-1}f$. We may choose $n$ minimal. In other words, we have a monic polynomial $P \in K[X]$ with $P(L)(f) = 0$ with minimal degree $n$. As $K$ is algebraically closed $P$ decomposes into linear factors $X - a_1, \ldots, X - a_n$ so we have

$$(L-a_1I)(L-a_2I) \ldots (L-a_nI)(f) = 0$$

The operators $L - a_i$ commute and $n$ is minimal so no $L - a_iI$ is injective. By Lemma 3.6, $|a_i - 1| < 1$ for $i \in \{1, \ldots, n\}$.

Lemma 3.5, applied for $a = a_1, g = (L-a_2I) \ldots (L-a_nI)f$ and $f = 0$ yields

$$(L-a_2I)(L-a_2I) \ldots (L-a_nI)(f) = g$$

where $g$ is a polycharacter. By repeated application of Lemma 3.5 we can remove all $L - a_iI$ obtaining that $f$ is a polycharacter.

Note. For results on closely related matters see [1].

References


