$p$-adic Trigonometric Polynomials

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Report 8727
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\textbf{Introduction.} Let $G$ be an abelian group, let $f$ be a bounded complex valued function on $G$ whose translates generate a finite dimensional space. It is well known ([2], 27.7) that $f$ is a linear combination of characters. This conclusion is not valid if the range of $f$ lies in a non-archimedean valued field $K$ rather than $\mathbb{C}$. For example, if $K$ contains the field $\mathbb{Q}_p$ of the $p$-adic numbers and if $G = \mathbb{Z}_p$, the additive group of the $p$-adic integers, it is easily seen that the translates of the function $f : x \mapsto x$ generate a twodimensional space over $K$ whereas $f$ is not a $K$-linear combination of $K$-valued characters (follow the proof of the implication $(\gamma) \Rightarrow (\alpha)$ of Theorem 1.4).

\textbf{Abstract.} For an abelian topological group $G$ and an algebraically closed, nontrivially valued, complete field $K$ necessary and sufficient conditions are derived for a representative function $f : G \rightarrow K$ to be a finite $K$-linear combination of $K$-valued characters (Theorems 1.4, 2.1, 2.2). Also, a complete description of the set of all representative functions $\mathbb{Z}_p \rightarrow K$ is given (Theorem 3.1).

\textbf{Terminology & Standard Facts.} Throughout this paper $G$ is an additively written abelian topological group, $K$ is an algebraically closed nontrivially valued complete field with valuation $|\cdot|$. The set $BC(G \rightarrow K)$ consisting of all bounded continuous functions $G \rightarrow K$ is a $K$-Banach algebra with respect to pointwise operations and the norm $f \mapsto \|f\|_\infty := \sup \{|f(x)| : x \in G\}$.

A \textit{character} is a nonzero element $\alpha$ of $BC(G \rightarrow K)$ for which $\alpha(x+y) = \alpha(x)\alpha(y)$ for all $x,y \in G$. Then $|\alpha(x)| = 1$ for all $x \in G$. Under pointwise multiplication the characters form a group $G^\wedge_K$. A function $f \in BC(G \rightarrow K)$ is a \textit{representative function} (or a \textit{trigonometric polynomial}) if the $K$-linear span $[f_s : s \in G]$ of $\{f_s : s \in G\}$ is finite dimensional. Here, as usual, $f_s(x) := f(s + x)$ for $x \in G$. It is not hard to prove that the collection $\mathcal{R}(G \rightarrow K)$ of all representative functions $G \rightarrow K$ is a $K$-subalgebra of $BC(G \rightarrow K)$ containing $G^\wedge_K$. 

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A $G$-module is a Banach space $E$ over $K$ together with a separately continuous structure map $G \times E \to E$

$$(s, x) \mapsto U_s(x) = sx \quad (s \in G, x \in E)$$

such that $s \mapsto U_s$ is a homomorphism of $G$ into the group of invertible (continuous) $K$-linear operators $E \to E$ and such that, for each $x \in G$, $\sup\{\|U_s x\| : s \in G\}$ is finite. In this paper we shall deal only with finite dimensional $G$-modules.

§1. THE MAIN THEOREM

**Proposition 1.1.** Let $f \in \mathcal{B}C(G \to K)$, $f \neq 0$. Then $[f_s : s \in G]$ is onedimensional if and only if $f$ is a multiple of a character.

**Proof.** If $\alpha$ is a character then for each $s \in G$ we have $\alpha_s = \alpha(s)\alpha$ and $[\alpha_s : s \in G]$ is onedimensional. Conversely, suppose $\dim[f_s : s \in G] = 1$. For each $s \in G$ there is a unique $\alpha(s) \in K$ for which $f_s = \alpha(s)f$. The equality $f_{s+t} = (f_s)_t$ yields $\alpha(s+t) = \alpha(s)\alpha(t)$ for all $s, t \in G$. From $\|f_s\|_{\infty} = |\alpha(s)||f\|_{\infty}$ we infer that $|\alpha(s)| = 1$ for all $s \in G$. So, $\alpha$ is a character and $f = f(0)\alpha$.

**Proposition 1.2.** A representative function is uniformly continuous.

**Proof.** Let $f \in \mathcal{B}R(G \to K)$, $f \neq 0$ and let $e_1, \ldots, e_n$ be a base of $E := [f_s : s \in G]$. By equivalence of norms (see [3], Theorem 3.15 for the non-archimedean case) there exists a $C > 0$ such that

$$\max_{1 \leq i \leq n} |\lambda_i| \leq C\left\|\sum_{i=1}^{n} \lambda_i e_i\right\|_{\infty}$$

for all $\lambda_1, \ldots, \lambda_n \in K$. Let $\varepsilon > 0$. There is a neighbourhood $U$ of 0 in $G$ such that for all $i \in \{1, 2, \ldots, n\}$

$$x \in U \Rightarrow |e_i(x) - e_i(0)| \leq (C n\|f\|_{\infty})^{-1}\varepsilon.$$ 

Now let $s \in G$, $t \in U$; we shall prove that $|f(s+t) - f(s)| \leq \varepsilon$. There exist $\lambda_1, \ldots, \lambda_n \in K$ (depending on $s$) such that

$$f_s = \sum_{i=1}^{n} \lambda_i e_i$$

Then

$$f_{s+t} = \sum_{i=1}^{n} \lambda_i (e_i)_t$$

We see that

$$|f(s+t) - f(s)| = |f_{s+t}(0) - f_s(0)| =$$
Proposition 1.3. Let $E$ be a $G$-module of dimension $n \in \mathbb{N}$. For each $m \in \mathbb{N}$, $1 \leq m \leq n$, $E$ has a $G$-submodule of dimension $m$.

Proof. By induction on $m$. To find a one-dimensional submodule choose, among all nonzero $G$-submodules of $E$, a $G$-submodule $E_1$ with minimal dimension. Then $E_1$ is simple (i.e., the corresponding representation $s \mapsto U_s$ is irreducible). As $K$ is algebraically closed, a standard application of Schur's lemma ([2], 27.9) yields $\dim E_1 = 1$. Now let $m < n$ and let $E_m$ be an $m$-dimensional $G$-submodule of $E$. The quotient $E/E_m$ is, in an obvious way, a $G$-module of dimension $n - m \geq 1$. By the first part of the proof it has a one-dimensional $G$-submodule $D_1$. One verifies immediately that $E_{m+1} := \pi^{-1}(D_1)$, where $\pi : E \to E/E_m$ is the quotient map, is a $G$-submodule of $E$ whose dimension is $m + 1$.

We now prove the main theorem. A function $\mu : G \to K$ is additive if $\mu(st) = \mu(s) + \mu(t)$ for all $s, t \in G$.

Theorem 1.4. The following statements on $G, K$ are equivalent.

(α) Any bounded continuous additive function $G \to K$ is 0.

(β) Each nonzero finite dimensional $G$-module over $K$ is a (direct) sum of one-dimensional $G$-modules.

(γ) Each representative function $G \to K$ is a finite $K$-linear combination of $K$-valued characters.

Proof. To obtain the implication $(\alpha) \Rightarrow (\beta)$ we shall prove that

\[
\text{each } n\text{-dimensional } G\text{-module has a base } e_1, \ldots, e_n
\]

\[
\text{for which } se_i \in [e_i] \quad (s \in G) \text{ for each } i \in \{1, \ldots, n\}
\]

by induction on $n$. The case $n = 1$ is trivial, so suppose (*) is true for some $n$ and let $E$ be an $(n+1)$-dimensional $G$-module. According to Proposition 1.3 $E$ has an $n$-dimensional $G$-submodule $D$ which, by the induction hypothesis, has a base $e_1, \ldots, e_n$ such that $se_i \in [e_i]$ for all $s \in G$, all $i \in \{1, \ldots, n\}$. Choose an $x \in E \setminus D$; then $e_1, \ldots, e_n, x$ is base for $E$. With respect to this base the maps $U_s (s \in G)$ have the following matrices

\[
\sum_{i=1}^{n} \lambda_i (e_i(t) - e_i(0)) \leq n \max_{1 \leq i \leq n} |\lambda_i| |e_i(t) - e_i(0)| \leq n C \sum_{i=1}^{n} \lambda_i \|e_i\|_{\infty} (C_n \|f\|_{\infty})^{-1} = \|f_1\|_{\infty} \|f\|_{\infty}^{-1} = \varepsilon.
\]
Observe that its entries are continuous functions of \( s \) (since each of them has the form \( s \mapsto \phi(sy) \) for some \( y \in E, \phi \in E^* \), the dual space of \( E \)) and are also bounded by our definition of a \( G \)-module. Since \( U_s \) is invertible we have \( \lambda_i(s) \neq 0 \) for all \( i \in \{1,...,n+1\} \). The equality \( U_{s+u} = U_s U_t \) expressed in matrix form yields

\[
\begin{bmatrix}
\lambda_1(s) & \xi_1(s) \\
0 & \xi_1(s) \\
\lambda_2(s) & \xi_2(s) \\
\vdots & \vdots \\
\lambda_n(s) & \xi_n(s) \\
0 & \lambda_{n+1}(s)
\end{bmatrix}
\]

That is, we have to choose in such a way that

\[
(\ast\ast) \quad (\xi_i(s) + q_i(\lambda_i(s) - \lambda_{n+1}(s))) = 0 \quad (1 \leq i \leq n, s \in G).
\]

For any \( i \in \{1,...,n\} \) we distinguish two cases.

(i) \( \lambda_i(t) \neq \lambda_{n+1}(t) \) for some \( t \in G \). Then we are forced to choose

\[
q_i := (\lambda_{n+1}(t) - \lambda_i(t))^{-1}\xi_i(t)
\]

Now (\ast\ast) guarantees that for any \( s \in G \)

\[
(\lambda_{n+1}(t) - \lambda_i(t))(\xi_i(s) + q_i(\lambda_i(s) - \lambda_{n+1}(s))) = 0
\]

and (\ast\ast) follows for this \( i \).

(ii) \( \lambda_i = \lambda_{n+1} \). We shall prove that \( \xi_i(s) = 0 \) for all \( s \in G \) (so that we may choose for \( q_i \) and arbitrary element of \( K \)).

In fact, by (\ast\ast) we have
\[ \xi_t(s+t) = \lambda_t(s)\xi_t(t) + \xi_t(s)\lambda_t(t) \quad (s, t \in G) \]

After dividing by \( \lambda_t(s+t) = \lambda_t(s)\lambda_t(t) \) we obtain

\[ \mu(s+t) = \mu(s) + \mu(t) \]

where \( \mu := \lambda_t^{-1}\xi_t \) is continuous and bounded. By (a) we have \( \mu = 0 \). It follows that \( \xi_t = 0 \).

(\( \beta \)) \( \Rightarrow \) (\( \gamma \)). Let \( f \in \mathfrak{R}(G \to K) \), \( f \neq 0 \) and let \( E = \{ f_s : s \in G \} \). The structure map

\[ (s, g) \mapsto g_s \quad (s \in G, g \in E) \]

makes \( E \) into a finite dimensional \( G \)-module, taking into account that Proposition 1.2 guarantees the continuity of \( s \mapsto g_s \). By (\( \beta \)), \( E \) is the sum of one-dimensional \( G \)-modules \( \{ \alpha_1, \ldots, \alpha_n \} \), where Proposition 1.1 tells us that we may assume that \( \alpha_1, \ldots, \alpha_n \) are characters and (\( \gamma \)) follows.

(\( \gamma \)) \( \Rightarrow \) (\( \alpha \)). Let \( \mu \in \mathfrak{BC}(G \to K) \) be additive. For each \( s \in G \) we have \( \mu_s = \mu(s) \cdot 1 + \mu' \) where 1 is the function with constant value one. So, \( [\mu_s : s \in G] = [1, \mu] \) implying that \( \mu \) is a representable function. By (\( \gamma \)) there exist distinct characters \( \alpha_0, \alpha_1, \ldots, \alpha_n \), where \( \alpha_0 \) is the unit character, and \( \lambda_0, \lambda_1, \ldots, \lambda_n \in K \) such that

\[ \mu = \sum_{i=0}^{n} \lambda_i \alpha_i \]

The relation \( \mu_s = \mu(s) \alpha_0 + \mu \) yields

\[ \sum_{i=0}^{n} \lambda_i \alpha_i(s) \alpha_i = \mu(s) \alpha_0 + \sum_{i=0}^{n} \lambda_i \alpha_i \quad (s \in G) \]

By linear independence of characters we have equality of the coefficients of \( \alpha_0 \) i.e.

\[ \lambda_0 = \lambda_0 \alpha_0(s) = \mu(s) + \lambda_0 \quad (s \in G) \]

implying \( \mu(s) = 0 \) for all \( s \in G \).

§2. THE MAIN THEOREM FOR VARIOUS GROUND FIELDS

**Theorem 2.1.** If the valuation of \( K \) is archimedean then (\( \alpha \)), (\( \beta \)), (\( \gamma \)) of Theorem 1.4 hold for every topological abelian group \( G \).

**Proof.** Property (\( \alpha \)) of Theorem 1.4 follows from the fact that \( K \) has no bounded additive subgroups other than \( (0) \).

Next we turn to the case where the valuation of \( K \) is non-archimedean. First some notations. The residue class field of \( K \) is \( k \). The characteristic of a field \( L \) is \( \text{char}L \). For topological groups \( G_1, G_2 \) the set of all continuous homomorphisms \( G_1 \to G_2 \) is \( \text{Hom}(G_1, G_2) \).

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Theorem 2.2. Let the valuation of \( K \) be non-archimedean. Then \((\alpha),(\beta),(\gamma)\) of Theorem 1.4 are equivalent to

\[(\delta)' \quad \text{Hom}(G, \mathbb{Q}) = (0) \quad (\text{where } \mathbb{Q} \text{ carries the discrete topology})
\]
if \( \text{char} \mathbb{Q} = \text{char} k = 0 \),

\[(\delta)'' \quad \text{Hom}(G, \mathbb{Z}/p\mathbb{Z}) = (0)
\]
if \( \text{char} K = \text{char} k = p \neq 0 \),

\[(\delta)''' \quad \text{Hom}(G, \mathbb{Z}_p) = (0)
\]
if \( \text{char} K = 0, \text{char} k = p \neq 0 \).

Proof. (a) Assume \( \text{char} K = \text{char} k = 0 \). We have a natural embedding \( \mathbb{Q} \rightarrow K \) whose image is bounded so \((\alpha)\) of Theorem 1.4 implies \((\delta)'\). To obtain \((\delta)' \implies (\alpha)\), let \( \mu : G \rightarrow K \) be a bounded nonzero additive homomorphism; we shall prove that \( \text{Hom}(G, \mathbb{Q}) \neq (0) \). Let \( s \in G, \mu(s) \neq 0 \) and let

\[\pi : K \rightarrow K/(x \in K : |x| < |\mu(s)|) \rightarrow H\]

be the canonical quotient map. The discrete group \( H \) is torsion free so the formula

\[n \pi(\mu(s)) \mapsto n \quad (n \in \mathbb{Z})\]

defines a homomorphism of the group generated by \( \pi(\mu(s)) \) into \( \mathbb{Q} \). By divisibility of \( \mathbb{Q} \) it can be extended to a homomorphism \( \phi : H \rightarrow \mathbb{Q} \). The map \( \phi \circ \pi \circ \mu : G \rightarrow \mathbb{Q} \) is a continuous homomorphism sending \( s \) into 1. Hence \( \text{Hom}(G, \mathbb{Q}) \neq (0) \).

(b) Assume \( \text{char} K = \text{char} k = p \neq 0 \). We have a natural embedding \( \mathbb{Z}/p\mathbb{Z} \rightarrow K \) so \((\alpha)\) of Theorem 1.4 implies \((\delta)''\). To obtain \((\delta)'' \implies (\alpha)\), let \( \mu : G \rightarrow K \) be a bounded nonzero additive homomorphism; we shall prove that \( \text{Hom}(G, \mathbb{Z}/p\mathbb{Z}) \neq (0) \). Define \( s, \pi, H \) as in part (a). This time every nonzero element of \( H \) has order \( p \) so the homomorphism

\[n \pi(\mu(s)) \mapsto n \mod p \mathbb{Z} \quad (n \in \mathbb{Z})\]

can be extended to homomorphism \( \phi : H \rightarrow \mathbb{Z}/p\mathbb{Z} \). The map \( \phi \circ \pi \circ \mu : G \rightarrow \mathbb{Z}/p\mathbb{Z} \) is a continuous homomorphism sending \( s \) into \( 1 \mod p \mathbb{Z} \). Hence \( \text{Hom}(G, \mathbb{Z}/p\mathbb{Z}) \neq (0) \).

(c) Assume \( \text{char} K = 0, \text{char} k = p \neq 0 \). Then we may assume \( K \supseteq \mathbb{Q}_p \). We have a natural embedding \( \mathbb{Z}_p \rightarrow K \) so \((\alpha)\) of Theorem 1.4 implies \((\delta)'''\). To obtain \((\delta)''' \implies (\alpha)\), let \( \mu : G \rightarrow K \) be a bounded nonzero additive homomorphism; we shall prove that \( \text{Hom}(G, \mathbb{Z}_p) \neq (0) \). Now \( K \) is, in a natural way, a Banach space over \( \mathbb{Q}_p \). Since \( \mathbb{Q}_p \) is spherically complete there exists, by the non-archimedean Hahn-Banach Theorem ([3], Theorem 4.15, \((\gamma) \implies (\alpha)\)), a continuous \( \mathbb{Q}_p \)-linear map \( \phi : K \rightarrow \mathbb{Q}_p \) that does not vanish on \( \mu(G) \). Then \( \phi \circ \mu \) is a nonzero bounded continuous homomorphism \( G \rightarrow \mathbb{Q}_p \). After multiplying it by a suitable element of \( \mathbb{Q}_p \) we obtain a nonzero element of \( \text{Hom}(G, \mathbb{Z}_p) \).
Remarks.

1. It is easily seen that \( \text{Hom}(G, \mathbb{Q}) = (0)' \) is equivalent to 'for each open subgroup \( H \) of \( G \) the quotient \( G/H \) is a torsion group'. Similarly, \( \text{Hom}(G, \mathbb{Z}/p\mathbb{Z}) = (0)' \) is equivalent to '\( G \) has no open subgroups of index \( p \)'. Further observe that \( \text{Hom}(G, \mathbb{Z}/p\mathbb{Z}) = (0) \) implies \( \text{Hom}(G, \mathbb{Z}/p\mathbb{Z}) = (0) \).

2. The groups \( \mathbb{Q}_p, \mathbb{Q}_p/\mathbb{Z}_p \) have the property that each nontrivial discrete quotient is an infinite torsion group. Thus, for such groups, each representative function is a linear combination of characters, for any choice of \( K \).

3. In [5] necessary and sufficient conditions are derived on \( G, K \) in order that \( G^*_K \) be an orthonormal set. It is a striking fact that these are the same as in Theorem 2.2 but where \((\delta)'\) and \((\delta)'\) are interchanged!

§3. REPRESENTATIVE FUNCTIONS ON \( \mathbb{Z}_p \)

From the previous theory it follows that a representative function \( \mathbb{Z}_p \to K \) is a linear combination of characters if \( K \) is archimedean and also if \( K \) is non-archimedean and \( \text{char} k = p \). So one may be interested in a description of \( \mathcal{R}(\mathbb{Z}_p \to K) \) for the remaining case \( \text{char} k = p \). We shall prove the following theorem.

Theorem 3.1. Let \( f : \mathbb{Z}_p \to K \).

(i) Let \( K \supseteq \mathbb{Q}_p \). Then \( f \in \mathcal{R}(\mathbb{Z}_p \to K) \) if and only if \( f \) has the form

\[ (*) \quad f = \sum_{i=1}^{n} P_i \alpha_i \]

where \( n \in \mathbb{N} \), \( P_1, \ldots, P_n \) are polynomial functions, and \( \alpha_1, \ldots, \alpha_n \) are characters.

(ii) Let \( \text{char} K = \text{char} k = p \). Then \( f \in \mathcal{R}(\mathbb{Z}_p \to K) \) if and only if \( f \) has the form

\[ (**) \quad f = \sum_{i=1}^{n} L_i \alpha_i \]

where \( n \in \mathbb{N} \), \( L_1, \ldots, L_n \) are locally constant functions and \( \alpha_1, \ldots, \alpha_n \) are characters.

To prove this theorem we need a few lemmas. For technical reasons we shall say that an \( f : \mathbb{Z}_p \to K \) is a polycharacter if it has the form \((*)\) if \( K \supseteq \mathbb{Q}_p \) or the form \((***)\) if \( \text{char} K = \text{char} k = p \). Then Theorem 3.1 reads in short: \( f \in \mathcal{R}(\mathbb{Z}_p \to K) \Leftrightarrow f \) is a polycharacter.

One half is easy:

Lemma 3.2. Let \( \text{char} k = p \). Each polycharacter \( \mathbb{Z}_p \to K \) is a representative function.

Proof. If \( K \supseteq \mathbb{Q}_p \) the function \( x \to x \) is an additive homomorphism and therefore is a representative function. For any \( K \), a locally constant function on \( \mathbb{Z}_p \) is constant on cosets of \( p^m \mathbb{Z}_p \) for some \( m \) so its translates generate a space whose dimension is \( \leq p^m \). Now the lemma follows after observing that
\[ R(\mathbb{Z}_p \to K) \] is a \( K \)-algebra.

For the second half of Theorem 3.1 we introduce the following. A function \( f : \mathbb{N} \to K \) can be interpolated if there exists a (unique) continuous function \( \tilde{f} : \mathbb{Z}_p \to K \) whose restriction to \( \mathbb{N} \) is \( f \). We need the following result. (As usual, the symbol \( \lfloor \rfloor \) indicates the entire part.)

Lemma 3.3. Let \( \text{char } k = p \neq 0 \).

(i) For \( a \in K \), \( a \neq 0 \), the sequence \( n \mapsto a^n \) can be interpolated if and only if \( |a - 1| < 1 \).

(ii) For a continuous function \( f : \mathbb{Z}_p \to K \) the sequence 
\[ n \mapsto f(0) + f(1) + \ldots + f(n-1) \]
can be interpolated.

(iii) For each \( m \in \mathbb{N} \) the sequence \( n \mapsto \lfloor \frac{n}{p^m} \rfloor \), considered as a map \( \mathbb{N} \to \mathbb{Q}_p \) can be interpolated to a function
\[ x \mapsto \left\lfloor \frac{x}{p^m} \right\rfloor \] on \( \mathbb{Z}_p \). The function \( x \mapsto x - \left\lfloor \frac{x}{p^m} \right\rfloor p^m \) (\( x \in \mathbb{Z}_p \)) is locally constant.

Proof.

(i) See [4], Theorem 32.4.

(ii) See [4], Theorem 34.1 (the assumption \( K \supset \mathbb{Q}_p \) is not used in that proof).

(iii) Without trouble one verifies that
\[ x \mapsto \left\lfloor \frac{x}{p^m} \right\rfloor := a_m + a_{m+1}p + a_{m+2}p^2 + \ldots \]
where \( x = \sum_{i=0}^{\infty} a_ip^i \) is the standard \( p \)-adic expansion of \( x \), is the required extension.

For the continuous extension \( x \mapsto a^x \) (\( x \in \mathbb{Z}_p \)) of \( n \mapsto a^n \) in Lemma 3.3(i) we shall also write \( a^* \). The continuous extension of \( n \mapsto f(0) + f(1) + \ldots + f(n-1) \) is called the indefinite sum of \( f \), denoted by \( Sf \).

Observe that
\[ S(f_i) - Sf = f - f(0) \]

Lemma 3.4. Let \( \text{char } k = p \). The indefinite sum of a polycharacter \( \mathbb{Z}_p \to K \) is again a polycharacter.

Proof. We consider two cases.

(i) \( K \supset \mathbb{Q}_p \). It is not hard to see that the indefinite sum of a polynomial function is again a polynomial function. By linearity it therefore suffices to prove that for each \( j \in \{0,1,2,\ldots\} \) and each \( a \in K \) with \( 0 < |1 - a| < 1 \) the function

\[ S(f_j) - Sf = f - f(0) \]
where $\omega^j$ is the polynomial $x \mapsto x^j$, is a polycharacter. We shall do this by proving the following statement (*) by induction on $j$.

There is a polynomial function $P_j$ of degree $\leq j$, whose coefficients are rational functions of $a$ and there is a rational function $Q_j$ of $a$ such that for all $n \in \mathbb{N}$ and all $a \in K$ with $0 < |1-a| < 1$

\[
S(\omega^j a^n)(n) = P_j(n) a^n + Q_j(a)
\]

For the case $j = 0$ observe that

\[
S(a^n)(n) = a^0 + a^1 + \ldots + a^{n-1} = \frac{1}{a-1} a^n + \frac{1}{1-a}
\]

So, (*) holds with $P_0(n) = \frac{1}{a-1}$, $Q_0(a) = \frac{1}{1-a}$.

Now suppose we have (*) for some $j$:

\[
S(\omega^j a^n)(n) = \sum_{i=0}^{n-1} i^j a^i = P_j(n) a^n + Q_j(a) \quad (n \in \mathbb{N})
\]

Then

\[
S(\omega^{j+1} a^n)(n) = \sum_{i=0}^{n-1} i^{j+1} a^i = a \frac{d}{da} \sum_{i=0}^{n-1} i^j a^i
\]

\[
= (a \frac{d}{da} P_j(n) + n P_j(n)) a^n + a \frac{d}{da} Q_j(a)
\]

So, if we take

\[
P_{j+1}(n) := a \frac{d}{da} P_j(n) + n P_j(n)
\]

\[
Q_{j+1}(a) = a \frac{d}{da} Q_j(a)
\]

then (*) holds for $j + 1$ in place of $j$.

(ii) $\text{char} K = p$. First we prove that $Sf$ is a polycharacter for

\[
f = \xi_p^m \mathbb{Z}_p \alpha
\]

where $m \in \mathbb{N}$, where $\xi_p^m \mathbb{Z}_p$ is the $K$-valued characteristic function of $p^m \mathbb{Z}_p$ and where $\alpha$ is a character.

We have for $n \in \mathbb{N}$

\[
(Sf)(n) = \sum_{i=0}^{n-1} \xi_p^m \mathbb{Z}_p(i) \alpha(i) = \sum_{j=0}^{\lfloor p^{-m}(n-1) \rfloor} \alpha(p^m j)
\]

If $\alpha(p^m) = 1_K$, the unit element of $K$, we obtain

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and we see that $Sf$ is a locally constant function.

If $\alpha(p^n) \neq 1_K$ then $\alpha(x) = a^x$ $(x \in \mathbb{Z}_p)$ where $a \in K$, $0 < |1_K - a| < 1$. We have, for $n \in \mathbb{N}$

$$(Sf)(n) = \frac{\alpha(p^n)^{-1}}{\alpha(p^n)^{-1}_K} = \frac{a^{n-1}_K}{a^{-1}}$$

It follows that $Sf$ is a $K$-linear combination of a constant function and the function

$$x \mapsto a^{x-1}_K \cdot 1_K$$

which is the product of the character $a^{-x}$ and a locally constant function (Lemma 3.3(iii)). Thus, we may conclude that $S(Z_p^\times \alpha)$ is a polycharacter.

By linearity of $S$ and by the remark preceding this lemma the set of all polycharacters $f$ for which $Sf$ is also a polycharacter is a linear space, invariant under translations. A standard reasoning shows that the smallest translation invariant linear space containing all $Z_p^\times \alpha$ $(m \in \mathbb{N}, \alpha$ character) is the set of all polycharacters which finishes the proof.

**Lemma 3.5.** Let $\text{char } k = p$, let $a \in K$, $|1-a| < 1$. If $f: \mathbb{Z}_p^\times \rightarrow K$ is a polycharacter and if $g$ is a continuous solution of

$$g(x+1) - ag(x) = f(x) \quad (x \in \mathbb{Z}_p)$$

then $g$ is a polycharacter.

**Proof.** Inductively we arrive easily at

$$g(n) = a^ng(0) + a^{n-1}S(a^{-1}f)(n) \quad (n \in \mathbb{N})$$

By continuity,

$$g = a^ng(0) + a^{n-1}S(a^{-1}f)$$

which is a polycharacter by Lemma 3.4.

Let $L$ denote the operator $BC(\mathbb{Z}_p \rightarrow K) \rightarrow BC(\mathbb{Z}_p \rightarrow K)$ sending $f$ into $f_1$ (recall that $f_1(x) = f(x+1)$).

**Lemma 3.6.** If, for some $a \in K$, the operator $L - af$ is not injective then $|a-1| < 1$.

**Proof.** Let $f \in BC(\mathbb{Z}_p \rightarrow K)$, $f \neq 0$ such that $Lf - af = 0$. Then $f(x+1) = af(x)$ for all $x \in \mathbb{Z}_p$ so that $f(n) = a^n f(0)$ for all $n \in \mathbb{N}$. We have $f(0) \neq 0$ and, by continuity of $f$, the sequence $x \mapsto a^n$ can be interpolated. By Lemma 3.3(i), $|1-a| < 1$. 

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Proof of Theorem 3.1. Let \( f \) be a representative function, \( f \neq 0 \); we shall prove that \( f \) is a polycharacter. The sequence \( f, Lf, L^2f, \ldots \) lies in a finite dimensional space so there is an \( n \in \mathbb{N} \) such that \( L^n f \) is a \( K \)-linear combination of \( f, Lf, \ldots, L^{n-1}f \). We may choose \( n \) minimal. In other words, we have a monic polynomial \( P \in K[X] \) with \( P(L)(f) = 0 \) with minimal degree \( n \). As \( K \) is algebraically closed \( P \) decomposes into linear factors \( X - a_1, \ldots, X - a_n \) so we have

\[
(L-a_1I)(L-a_2I) \cdots (L-a_nI)(f) = 0
\]

The operators \( L-a_i \) commute and \( n \) is minimal so no \( L-a_iI \) is injective. By Lemma 3.6, \( |a_i - 1| < 1 \) for \( i \in \{1, \ldots, n\} \).

Lemma 3.5, applied for \( a = a_1, g = (L-a_2I), \ldots, (L-a_nI)f \) and \( f = 0 \) yields

\[
(L-a_2I)(L-a_2I) \cdots (L-a_nI)(f) = g
\]

where \( g \) is a polycharacter. By repeated application of Lemma 3.5 we can remove all \( L-a_iI \) obtaining that \( f \) is a polycharacter.

Note. For results on closely related matters see [1].

References


