$p$-ADIC TRIGONOMETRIC POLYNOMIALS

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**INTRODUCTION.** Let $G$ be an abelian group, let $f$ be a bounded complex valued function on $G$ whose translates generate a finite dimensional space. It is well known ([2], 27.7) that $f$ is a linear combination of characters. This conclusion is not valid if the range of $f$ lies in a non-archimedean valued field $K$ rather than $\mathbb{C}$. For example, if $K$ contains the field $\mathbb{Q}_p$ of the $p$-adic numbers and if $G=\mathbb{Z}_p$, the additive group of the $p$-adic integers, it is easily seen that the translates of the function $f : x \mapsto x$ generate a twodimensional space over $K$ whereas $f$ is not a $K$-linear combination of $K$-valued characters (follow the proof of the implication $(\gamma) \Rightarrow (\alpha)$ of Theorem 1.4).

**ABSTRACT.** For an abelian topological group $G$ and an algebraically closed, nontrivially valued, complete field $K$ necessary and sufficient conditions are derived for a representative function $f : G \rightarrow K$ to be a finite $K$-linear combination of $K$-valued characters (Theorems 1.4, 2.1, 2.2). Also, a complete description of the set of all representative functions $\mathbb{Z}_p \rightarrow K$ is given (Theorem 3.1).

**TERMINOLOGY & STANDARD FACTS.** Throughout this paper $G$ is an additively written abelian topological group, $K$ is an algebraically closed nontrivially valued complete field with valuation $| |$. The set $BC(G \rightarrow K)$ consisting of all bounded continuous functions $G \rightarrow K$ is a $K$-Banach algebra with respect to pointwise operations and the norm $f \mapsto \|f\|_\infty := \sup \{|f(x)| : x \in G\}$.

A character is a nonzero element $\alpha$ of $BC(G \rightarrow K)$ for which $\alpha(x+y) = \alpha(x) \alpha(y)$ for all $x, y \in G$. Then $|\alpha(x)| = 1$ for all $x \in G$. Under pointwise multiplication the characters form a group $G^\ast$. A function $f \in BC(G \rightarrow K)$ is a representative function (or a trigonometric polynomial) if the $K$-linear span $\{f_s : s \in G\}$ of $\{f_s : s \in G\}$ is finite dimensional. Here, as usual, $f_s(x) := f(s+x)$ for $x \in G$. It is not hard to prove that the collection $\mathcal{R}(G \rightarrow K)$ of all representative functions $G \rightarrow K$ is a $K$-subalgebra of $BC(G \rightarrow K)$ containing $G^\ast$. 

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A \textit{G-module} is a Banach space \( E \) over \( K \) together with a separately continuous structure map 
\[ G \times E \to E \]
\[ (s, x) \mapsto U_s(x) = sx \quad (s \in G, x \in E) \]
such that \( s \mapsto U_s \) is a homomorphism of \( G \) into the group of invertible (continuous) \( K \)-linear operators \( E \to E \) and such that, for each \( x \in G \), \( \sup \{ \| U_s x \| : s \in G \} \) is finite. In this paper we shall deal only with finite dimensional \( G \)-modules.

\section{The Main Theorem}

\textbf{Proposition 1.1.} Let \( f \in BC(G \to K) \), \( f \neq 0 \). Then \( \{ f_s : s \in G \} \) is onedimensional if and only if \( f \) is a multiple of a character.

\textbf{Proof.} If \( \alpha \) is a character then for each \( s \in G \) we have \( \alpha_s = \alpha(s) \alpha \) and \( \{ \alpha_s : s \in G \} \) is onedimensional. Conversely, suppose \( \dim \{ f_s : s \in G \} = 1 \). For each \( s \in G \) there is a unique \( \alpha(s) \in K \) for which \( f_s = \alpha(s)f \). The equality \( f_{s+t} = (f_s)_t \) yields \( \alpha(s+t) = \alpha(s)\alpha(t) \) for all \( s, t \in G \). From \( \| f_s \|_\infty = |\alpha(s)|\| f \|_\infty \) we infer that \( |\alpha(s)| = 1 \) for all \( s \in G \). So, \( \alpha \) is a character and \( f = f(0)\alpha \).

\textbf{Proposition 1.2.} A representative function is uniformly continuous.

\textbf{Proof.} Let \( f \in R(G \to K) \), \( f \neq 0 \) and let \( e_1, \ldots, e_n \) be a base of \( E := \{ f_s : s \in G \} \). By equivalence of norms (see [3], Theorem 3.15 for the non-archimedean case) there exists a \( C > 0 \) such that

\[ \max_{1 \leq i \leq n} |\lambda_i| \leq C \| \sum_{i=1}^{n} \lambda_i e_i \|_\infty \]

for all \( \lambda_1, \ldots, \lambda_n \in K \). Let \( \varepsilon > 0 \). There is a neighbourhood \( U \) of \( 0 \) in \( G \) such that for all \( i \in \{1, 2, \ldots, n\} \)

\[ x \in U \Rightarrow |e_i(x) - e_i(0)| \leq (C n \| f \|_\infty)^{-1} \varepsilon. \]

Now let \( s \in G \), \( t \in U \); we shall prove that \( |f(s+t) - f(s)| \leq \varepsilon \). There exist \( \lambda_1, \ldots, \lambda_n \in K \) (depending on \( s \)) such that

\[ f_s = \sum_{i=1}^{n} \lambda_i e_i \]

Then

\[ f_{s+t} = \sum_{i=1}^{n} \lambda_i (e_i)_t \]

We see that

\[ |f(s+t) - f(s)| = |f_{s+t}(0) - f_s(0)| = \]

--- 2 ---
\[ |\sum_{i=1}^{n} \lambda_i(e_i(t) - e_i(0))| \leq n \max_{1 \leq i \leq n} |\lambda_i| \|e_i(t) - e_i(0)\| \leq n C\|\sum_{i=1}^{n} \lambda_i e_i\|_{\infty} (C_n\|f\|_{\infty})^{-1}\|f\|_{\infty}^{-1} = \varepsilon. \]

**Proposition 1.3.** Let \( E \) be a \( G \)-module of dimension \( n \in \mathbb{N} \). For each \( m \in \mathbb{N}, 1 \leq m \leq n \), \( E \) has a \( G \)-submodule of dimension \( m \).

**Proof.** By induction on \( m \). To find a one-dimensional submodule choose, among all nonzero \( G \)-submodules of \( E \), a \( G \)-submodule \( E_1 \) with minimal dimension. Then \( E_1 \) is simple (i.e., the corresponding representation \( s \mapsto U_s \) is irreducible). As \( K \) is algebraically closed, a standard application of Schur's lemma ([2], 27.9) yields \( \dim E_1 = 1 \). Now let \( m < n \) and let \( E_m \) be an \( m \)-dimensional \( G \)-submodule of \( E \). The quotient \( E/E_m \) is, in an obvious way, a \( G \)-module of dimension \( n - m \geq 1 \). By the first part of the proof it has a one-dimensional \( G \)-submodule \( D_1 \). One verifies immediately that \( E_{m+1} := \pi^{-1}(D_1) \), where \( \pi : E \to E/E_m \) is the quotient map, is a \( G \)-submodule of \( E \) whose dimension is \( m + 1 \).

We now prove the main theorem. A function \( \mu : G \to K \) is **additive** if \( \mu(s + t) = \mu(s) + \mu(t) \) for all \( s, t \in G \).

**Theorem 1.4.** The following statements on \( G, K \) are equivalent.

1. (\( \alpha \)) Any bounded continuous additive function \( G \to K \) is 0.
2. (\( \beta \)) Each nonzero finite dimensional \( G \)-module over \( K \) is a (direct) sum of one-dimensional \( G \)-modules.
3. (\( \gamma \)) Each representative function \( G \to K \) is a finite \( K \)-linear combination of \( K \)-valued characters.

**Proof.** To obtain the implication (\( \alpha \)) \( \Rightarrow \) (\( \beta \)) we shall prove that

\[
\left\{ \begin{array}{l}
\text{each } n \text{-dimensional } G \text{-module has a base } e_1, \ldots, e_n \\
\text{for which } se_i \in [e_i] \text{ (} s \in G \text{) for each } i \in \{1, \ldots, n\}
\end{array} \right.
\]

by induction on \( n \). The case \( n = 1 \) is trivial, so suppose (\( \ast \)) is true for some \( n \) and let \( E \) be an \((n+1)\)-dimensional \( G \)-module. According to Proposition 1.3 \( E \) has an \( n \)-dimensional \( G \)-submodule \( D \) which, by the induction hypothesis, has a base \( e_1, \ldots, e_n \) such that \( se_i \in [e_i] \) for all \( s \in G \), all \( i \in \{1, \ldots, n\} \). Choose an \( x \in E \setminus D \); then \( e_1, \ldots, e_n, x \) is base for \( E \). With respect to this base the maps \( U_s (s \in G) \) have the following matrices.
Observe that its entries are continuous functions of \( s \) (since each of them has the form \( s \mapsto \phi(sy) \) for some \( y \in E, \phi \in E^* \), the dual space of \( E \)) and are also bounded by our definition of a \( G \)-module. Since \( U_s \) is invertible we have \( \lambda_i(s) \neq 0 \) for all \( i \in \{1, \ldots, n+1\} \). The equality \( U_{s+t} = U_s U_t \) expressed in matrix form yields

\[
\begin{bmatrix}
\lambda_1(s) & \xi_1(s) \\
0 & \lambda_2(s) & \xi_2(s) \\
& \ddots & \ddots \\
& & \lambda_n(s) & \xi_n(s) \\
0 & \cdots & \cdots & \lambda_{n+1}(s)
\end{bmatrix}
\]

so, each \( \lambda_i \) is a character and

\[
\xi_i(s+t) = \lambda_i(s)\xi_i(t) + \xi_i(s)\lambda_{n+1}(t) = \lambda_i(t)\xi_i(s) + \xi_i(t)\lambda_{n+1}(s) \quad (s, t \in G)
\]

for \( i \in \{1, \ldots, n\} \). We now complete the proof of \((\alpha) \Rightarrow (\beta)\) by defining \( q_1, \ldots, q_n \in K \) such that for

\[
e_{n+1} := x + \sum_{i=1}^{n} q_i e_i
\]

we have \( s e_{n+1} = \lambda_{n+1}(s) e_{n+1} \quad (s \in G) \). That is, we have to choose \( q_1, \ldots, q_n \) in such a way that

\[
\xi_i(s) + q_i(\lambda_i(s) - \lambda_{n+1}(s)) = 0 \quad (1 \leq i \leq n, s \in G).
\]

For any \( i \in \{1, \ldots, n\} \) we distinguish two cases.

(i) \( \lambda_i(t) \neq \lambda_{n+1}(t) \) for some \( t \in G \). Then we are forced to choose

\[
qu_i := (\lambda_{n+1}(t) - \lambda_i(t))^{-1}\xi_i(t)
\]

Now \((\ast)\) guarantees that for any \( s \in G \)

\[
(\lambda_{n+1}(t) - \lambda_i(t))(\xi_i(s) + q_i(\lambda_i(s) - \lambda_{n+1}(s))) = 0
\]

and \((\ast\ast)\) follows for this \( i \).

(ii) \( \lambda_i = \lambda_{n+1} \). We shall prove that \( \xi_i(s) = 0 \) for all \( s \in G \) (so that we may choose for \( q_i \) and arbitrary element of \( K \)).

In fact, by \((\ast)\) we have
After dividing by \( \lambda_t(s+\tau) = \lambda_t(s)\lambda_t(\tau) \) we obtain

\[
\mu(s+\tau) = \mu(s) + \mu(\tau)
\]

where \( \mu := \lambda_t^{-1}\xi_t \) is continuous and bounded. By (a) we have \( \mu = 0 \). It follows that \( \xi_t = 0 \).

\((\beta)\Rightarrow(\gamma)\). Let \( f \in \mathfrak{g}(G\to K) \), \( f \neq 0 \) and let \( E = \{ f_s : s \in G \} \). The structure map

\[
(s,g) \mapsto g_s \quad (s \in G, \ g \in E)
\]

makes \( E \) into a finite dimensional \( G \)-module, taking into account that Proposition 1.2 guarantees the continuity of \( s \mapsto g_s \). By (b), \( E \) is the sum of one-dimensional \( G \)-modules \([\alpha_1], \ldots, [\alpha_n]\), where Proposition 1.1 tells us that we may assume that \( \alpha_1, \ldots, \alpha_n \) are characters and (\gamma) follows.

\((\gamma)\Rightarrow(\alpha)\). Let \( \mu \in \mathfrak{g}(G\to K) \) be additive. For each \( s \in G \) we have \( \mu_s = \mu(s) \cdot 1 + \mu \) where 1 is the function with constant value one. So, \( [\mu_s : s \in G] = [1, \mu] \) implying that \( \mu \) is a representative function. By (\gamma) there exist distinct characters \( \alpha_0, \alpha_1, \ldots, \alpha_n \), where \( \alpha_0 \) is the unit character, and \( \lambda_0, \lambda_1, \ldots, \lambda_n \in K \) such that

\[
\mu = \sum_{i=0}^{n} \lambda_i \alpha_i
\]

The relation \( \mu_s = \mu(s) \alpha_0 + \mu \) yields

\[
\sum_{i=0}^{n} \lambda_i \alpha_i(s) \alpha_i = \mu(s) \alpha_0 + \sum_{i=0}^{n} \lambda_i \alpha_i \quad (s \in G)
\]

By linear independence of characters we have equality of the coefficients of \( \alpha_0 \) i.e.

\[
\lambda_0 = \lambda_0 \alpha_0(s) = \mu(s) + \lambda_0 \quad (s \in G)
\]

implying \( \mu(s) = 0 \) for all \( s \in G \).

§2. THE MAIN THEOREM FOR VARIOUS GROUND FIELDS

**Theorem 2.1.** If the valuation of \( K \) is archimedean then \((\alpha),(\beta),(\gamma)\) of Theorem 1.4 hold for every topological abelian group \( G \).

**Proof.** Property (\alpha) of Theorem 1.4 follows from the fact that \( K \) has no bounded additive subgroups other than (0).

Next we turn to the case where the valuation of \( K \) is non-archimedean. First some notations. The residue class field of \( K \) is \( k \). The characteristic of a field \( L \) is \( \text{char} L \). For topological groups \( G_1, G_2 \) the set of all continuous homomorphisms \( G_1 \to G_2 \) is \( \text{Hom}(G_1, G_2) \).
**Theorem 2.2.** Let the valuation of $K$ be non-archimedean. Then $(\alpha), (\beta), (\gamma)$ of Theorem 1.4 are equivalent to

$(\delta)'$  
$\text{Hom}(G, \mathbb{Q}) = (0)$ (where $\mathbb{Q}$ carries the discrete topology)  
if $\text{char} K = \text{char} k = 0$,

$(\delta)''$  
$\text{Hom}(G, \mathbb{Z}/p\mathbb{Z}) = (0)$  
if $\text{char} K = \text{char} k = p \neq 0$,

$(\delta)'''$  
$\text{Hom}(G, \mathbb{Z}_p) = (0)$  
if $\text{char} K = 0$, $\text{char} k = p \neq 0$.

**Proof.** (a) Assume $\text{char} K = \text{char} k = 0$. We have a natural embedding $\mathbb{Q} \to K$ whose image is bounded so $(\alpha)$ of Theorem 1.4 implies $(\delta)'$. To obtain $(\delta)' \Rightarrow (\alpha)$, let $\mu : G \to K$ be a bounded nonzero additive homomorphism; we shall prove that $\text{Hom}(G, \mathbb{Q}) \neq (0)$. Let $s \in G$, $\mu(s) \neq 0$ and let

$$
\pi : K \to K/(x \in K : |x| < |\mu(s)|) =: H
$$

be the canonical quotient map. The discrete group $H$ is torsion free so the formula

$$
n \pi(\mu(s)) \mapsto n \quad (n \in \mathbb{Z})
$$

defines a homomorphism of the group generated by $\pi(\mu(s))$ into $\mathbb{Q}$. By divisibility of $\mathbb{Q}$ it can be extended to a homomorphism $\phi : H \to \mathbb{Q}$. The map $\phi \circ \pi \circ \mu : G \to \mathbb{Q}$ is a continuous homomorphism sending $s$ into 1. Hence $\text{Hom}(G, \mathbb{Q}) \neq (0)$.

(b) Assume $\text{char} K = \text{char} k = p \neq 0$. We have a natural embedding $\mathbb{Z}/p\mathbb{Z} \to K$ so $(\alpha)$ of Theorem 1.4 implies $(\delta)''$. To obtain $(\delta)'' \Rightarrow (\alpha)$, let $\mu : G \to K$ be a bounded nonzero additive homomorphism; we shall prove that $\text{Hom}(G, \mathbb{Z}/p\mathbb{Z}) \neq (0)$. Define $s, \pi, H$ as in part (a). This time every nonzero element of $H$ has order $p$ so the homomorphism

$$
n \pi(\mu(s)) \mapsto n \mod p \mathbb{Z} \quad (n \in \mathbb{Z})
$$

can be extended to homomorphism $\phi : H \to \mathbb{Z}/p\mathbb{Z}$. The map $\phi \circ \pi \circ \mu : G \to \mathbb{Z}/p\mathbb{Z}$ is a continuous homomorphism sending $s$ into $1 \mod p \mathbb{Z}$. Hence $\text{Hom}(G, \mathbb{Z}/p\mathbb{Z}) \neq (0)$.

(c) Assume $\text{char} K = 0$, $\text{char} k = p \neq 0$. Then we may assume $K \supseteq \mathbb{Q}_p$. We have a natural embedding $\mathbb{Z}_p \to K$ so $(\alpha)$ of Theorem 1.4 implies $(\delta)'''$. To obtain $(\delta)''' \Rightarrow (\alpha)$, let $\mu : G \to K$ be a bounded nonzero additive homomorphism; we shall prove that $\text{Hom}(G, \mathbb{Z}_p) \neq (0)$. Now $K$ is, in a natural way, a Banach space over $\mathbb{Q}_p$. Since $\mathbb{Q}_p$ is spherically complete there exists, by the non-archimedean Hahn-Banach Theorem ([3], Theorem 4.15, $(\gamma) \Rightarrow (\alpha)$), a continuous $\mathbb{Q}_p$-linear map $\phi : K \to \mathbb{Q}_p$ that does not vanish on $\mu(G)$. Then $\phi \circ \mu$ is a nonzero bounded continuous homomorphism $G \to \mathbb{Q}_p$. After multiplying it by a suitable element of $\mathbb{Q}_p$ we obtain a nonzero element of $\text{Hom}(G, \mathbb{Z}_p)$. 

--- 6 ---
Remarks.

1. It is easily seen that \( \text{Hom}(G,\mathbb{Q})=(0) \)' is equivalent to 'for each open subgroup \( H \) of \( G \) the quotient \( G/H \) is a torsion group'. Similarly, \( \text{Hom}(G,\mathbb{Z}/p\mathbb{Z})=(0) \)' is equivalent to '\( G \) has no open subgroups of index \( p \)'. Further observe that \( \text{Hom}(G,\mathbb{Z}/p\mathbb{Z})=(0) \) implies \( \text{Hom}(G,\mathbb{Z}_p)=(0) \).

2. The groups \( \mathbb{Q}_p, \mathbb{Q}_p/\mathbb{Z}_p \) have the property that each nontrivial discrete quotient is an infinite torsion group. Thus, for such groups, each representative function is a linear combination of characters, for any choice of \( K \).

3. In [5] necessary and sufficient conditions are derived on \( G, K \) in order that \( G_\mathbb{K} \) be an orthonormal set. It is a striking fact that these are the same as in Theorem 2.2 but where \( (\delta)'' \) and \( (\delta)''' \) are interchanged!

§3. REPRESENTATIVE FUNCTIONS ON \( \mathbb{Z}_p \)

From the previous theory it follows that a representative function \( \mathbb{Z}_p \to K \) is a linear combination of characters if \( K \) is archimedean and also if \( K \) is non-archimedean and char \( k \neq p \). So one may be interested in a description of \( \mathfrak{R}(\mathbb{Z}_p \to K) \) for the remaining case char \( k=p \). We shall prove the following theorem.

Theorem 3.1. Let \( f : \mathbb{Z}_p \to K \).

(i) Let \( K \supseteq \mathbb{Q}_p \). Then \( f \in \mathfrak{R}(\mathbb{Z}_p \to K) \) if and only if \( f \) has the form

\[
(*) \quad f = \sum_{i=1}^{n} P_i \alpha_i
\]

where \( n \in \mathbb{N} \), \( P_1,\ldots,P_n \) are polynomial functions, and \( \alpha_1,\ldots,\alpha_n \) are characters.

(ii) Let char \( K=\text{char} \ k=p \). Then \( f \in \mathfrak{R}(\mathbb{Z}_p \to K) \) if and only if \( f \) has the form

\[
(**) \quad f = \sum_{i=1}^{n} L_i \alpha_i
\]

where \( n \in \mathbb{N} \), \( L_1,\ldots,L_n \) are locally constant functions and \( \alpha_1,\ldots,\alpha_n \) are characters.

To prove this theorem we need a few lemmas. For technical reasons we shall say that an \( f : \mathbb{Z}_p \to K \) is a polycharacter if it has the form \( (*) \) if \( K \supseteq \mathbb{Q}_p \) or the form \( (**) \) if char \( K=\text{char} \ k=p \). Then Theorem 3.1 reads in short: \( f \in \mathfrak{R}(\mathbb{Z}_p \to K) \Leftrightarrow f \) is a polycharacter.

One half is easy:

Lemma 3.2. Let char \( k=p \). Each polycharacter \( \mathbb{Z}_p \to K \) is a representative function.

Proof. If \( K \supseteq \mathbb{Q}_p \) the function \( x \to x \) is an additive homomorphism and therefore is a representative function. For any \( K \), a locally constant function on \( \mathbb{Z}_p \) is constant on cosets of \( p^m \mathbb{Z}_p \) for some \( m \) so its translates generate a space whose dimension is \( \leq p^m \). Now the lemma follows after observing that
\( R(\mathbb{Z}_p \to K) \) is a \( K \)-algebra.

For the second half of Theorem 3.1 we introduce the following. A function \( f : \mathbb{N} \to K \) can be interpolated if there exists a (unique) continuous function \( \tilde{f} : \mathbb{Z}_p \to K \) whose restriction to \( \mathbb{N} \) is \( f \). We need the following result. (As usual, the symbol \([\ ]\) indicates the entire part.)

**Lemma 3.3.** Let \( \text{char } k = p \neq 0 \).

(i) For \( a \in K, a \neq 0 \), the sequence \( n \mapsto a^n \) can be interpolated if and only if \( |a - 1| < 1 \).

(ii) For a continuous function \( f : \mathbb{Z}_p \to K \) the sequence

\[
    n \mapsto f(0) + f(1) + \ldots + f(n-1)
\]

can be interpolated.

(iii) For each \( m \in \mathbb{N} \) the sequence \( n \mapsto \frac{x}{p^m} \), considered as a map \( \mathbb{N} \to \mathbb{Q}_p \) can be interpolated to a function

\[
    x \mapsto \left[ \frac{x}{p^m} \right] \text{ on } \mathbb{Z}_p. 
\]

The function \( x \mapsto x - \left[ \frac{x}{p^m} \right] p^m \) \((x \in \mathbb{Z}_p)\) is locally constant.

**Proof.**

(i) See [4], Theorem 32.4.

(ii) See [4], Theorem 34.1 (the assumption \( K \supseteq \mathbb{Q}_p \) is not used in that proof).

(iii) Without trouble one verifies that

\[
    x \mapsto \left[ \frac{x}{p^m} \right] := a_m + a_m + 1p + a_m + 2p^2 + \ldots
\]

where \( x = \sum_{i=0}^{\infty} a_ip^i \) is the standard \( p \)-adic expansion of \( x \), is the required extension.

For the continuous extension \( x \mapsto a^x \) \((x \in \mathbb{Z}_p)\) of \( n \mapsto a^n \) in Lemma 3.3(i) we shall also write \( a^n \). The continuous extension of \( n \mapsto f(0) + f(1) + \ldots + f(n-1) \) is called the indefinite sum of \( f \), denoted by \( Sf \). Observe that

\[
    S(f) - Sf = f - f(0)
\]

**Lemma 3.4.** Let \( \text{char } k = p \). The indefinite sum of a polycharacter \( \mathbb{Z}_p \to K \) is again a polycharacter.

**Proof.** We consider two cases.

(i) \( K \supseteq \mathbb{Q}_p \). It is not hard to see that the indefinite sum of a polynomial function is again a polynomial function. By linearity it therefore suffices to prove that for each \( j \in \{0,1,2,\ldots\} \) and each \( a \in K \) with \( 0 < |1-a| < 1 \) the function

\[
    S(f_j) = Sf_j = f_j - f(0)
\]
where $\omega^j$ is the polynomial $x \mapsto x^j$, is a polycharacter. We shall do this by proving the following statement (*) by induction on $j$.

There is a polynomial function $P_j$ of degree $\leq j$, whose coefficients are rational functions of $a$ and there is a rational function $Q_j$ of $a$ such that for all $n \in \mathbb{N}$ and $a \in K$ with $0 < |1-a| < 1$

\[ S(\omega^j a^*) = P_j(n)a^n + Q_j(a) \]

For the case $j=0$ observe that

\[ S(\omega^0 a^*)(n) = a^0 + a^1 + \ldots + a^{n-1} = \frac{1}{a-1}a^n + \frac{1}{1-a} \]

So, (*) holds with $P_0(n) = \frac{1}{a-1}$, $Q_0(a) = \frac{1}{1-a}$.

Now suppose we have (*) for some $j$:

\[ S(\omega^j a^*)(n) = \sum_{i=0}^{n-1} i^j a^i = P_j(n)a^n + Q_j(a) \quad (n \in \mathbb{N}) \]

Then

\[ S(\omega^{j+1} a^*)(n) = \sum_{i=0}^{n-1} i^{j+1} a^i = a \frac{d}{da} \sum_{i=0}^{n-1} i^j a^i \]

\[ = (a \frac{d}{da} P_j(n) + n P_j(n))a^n + a \frac{d}{da} Q_j(a) \]

So, if we take

\[ P_{j+1}(n) := a \frac{d}{da} P_j(n) + n P_j(n) \]

\[ Q_{j+1}(a) = a \frac{d}{da} Q_j(a) \]

then (*) holds for $j+1$ in place of $j$.

(ii) char$K$=p. First we prove that $Sf$ is a polycharacter for

\[ f = \xi_{p^mZ_p} \alpha \]

where $m \in \mathbb{N}$, where $\xi_{p^mZ_p}$ is the $K$-valued characteristic function of $p^mZ_p$ and where $\alpha$ is a character.

We have for $n \in \mathbb{N}$

\[ (Sf)(n) = \sum_{i=0}^{n-1} \xi_{p^mZ_p}(i)\alpha(i) = \sum_{j=0}^{p^m(n-1)} \alpha(p^n j) \]

If $\alpha(p^m) = 1_K$, the unit element of $K$, we obtain
\[(Sf)(n) = \left(\frac{n-1}{p^n}\right) + 1 \cdot k\]

and we see that \(Sf\) is a locally constant function.

If \(\alpha(p^n) \neq 1_k\) then \(\alpha(x) = a^x\) \((x \in \mathbb{Z}_p)\) where \(a \in K, 0 < |1_k - a| < 1\). We have, for \(n \in \mathbb{N}\)

\[\begin{align*}
(Sf)(n) &= \frac{\alpha(p^n)^{x-1}}{\alpha(p^n)-1_k} = \frac{a^{\frac{x-1}{p^n}}}{a^{p^n}-1_k}.
\end{align*}\]

It follows that \(Sf\) is a \(K\)-linear combination of a constant function and the function

\[x \mapsto a^{\frac{x-1}{p^n}} = a^{-x} \cdot a^{\frac{x-1}{p^n}}\]

which is the product of the character \(a^{-x}\) and a locally constant function (Lemma 3.3(iii)). Thus, we may conclude that \(S(\xi_{p^n}a)\) is a polycharacter.

By linearity of \(S\) and by the remark preceding this lemma the set of all polycharacters \(f\) for which \(Sf\) is also a polycharacter is a linear space, invariant under translations. A standard reasoning shows that the smallest translation invariant linear space containing all \(\xi_{p^n}a\) \((m \in \mathbb{N}, \alpha\) character\) is the set of all polycharacters which finishes the proof.

**Lemma 3.5.** Let \(\text{char } k = p\), let \(a \in K, |1 - a| < 1\). If \(f : \mathbb{Z}_p \rightarrow K\) is a polycharacter and if \(g\) is a continuous solution of

\[g(x + 1) - ag(x) = f(x) \quad (x \in \mathbb{Z}_p)\]

then \(g\) is a polycharacter.

**Proof.** Inductively we arrive easily at

\[g(n) = a^n g(0) + a^{n-1}S(a^{-x}f)(n) \quad (n \in \mathbb{N})\]

By continuity,

\[g = a^n g(0) + a^{n-1}S(a^{-x}f)\]

which is a polycharacter by Lemma 3.4.

Let \(L \) denote the operator \(BC(\mathbb{Z}_p \rightarrow K) \rightarrow BC(\mathbb{Z}_p \rightarrow K)\) sending \(f\) into \(f_1\) (recall that \(f_1(x) = f(x+1)\)).

**Lemma 3.6.** If, for some \(a \in K\), the operator \(L - af\) is not injective then \(|a-1| < 1\).

**Proof.** Let \(f \in BC(\mathbb{Z}_p \rightarrow K), f \neq 0\) be such that \(Lf - af = 0\). Then \(f(x+1) = af(x)\) for all \(x \in \mathbb{Z}_p\) so that \(f(n) = a^n f(0)\) for all \(n \in \mathbb{N}\). We have \(f(0) \neq 0\) and, by continuity of \(f\), the sequence \(x \mapsto a^n\) can be interpolated. By Lemma 3.3(i), \(|1-a| < 1\).
Proof of Theorem 3.1. Let $f$ be a representative function, $f \neq 0$; we shall prove that $f$ is a polycharacter. The sequence $f, Lf, L^2f, \ldots$ lies in a finite dimensional space so there is an $n \in \mathbb{N}$ such that $L^n f$ is a $K$-linear combination of $f, Lf, \ldots, L^{n-1}f$. We may choose $n$ minimal. In other words, we have a monic polynomial $P \in K[X]$ with $P(L)(f) = 0$ with minimal degree $n$. As $K$ is algebraically closed $P$ decomposes into linear factors $X - a_1, \ldots, X - a_n$ so we have

$$(L-a_1I)(L-a_2I) \ldots (L-a_nI)(f) = 0$$

The operators $L - a_i$ commute and $n$ is minimal so no $L - a_iI$ is injective. By Lemma 3.6, $|a_i - 1| < 1$ for $i \in \{1, \ldots, n\}$.

Lemma 3.5, applied for $a = a_1, g = (L-a_2I), \ldots (L-a_nI)f$ and $f = 0$ yields

$$(L-a_2I)(L-a_2I) \ldots (L-a_nI)(f) = g$$

where $g$ is a polycharacter. By repeated application of Lemma 3.5 we can remove all $L - a_iI$ obtaining that $f$ is a polycharacter.

Note. For results on closely related matters see [1].

References