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Two Elementary Proofs of Katsaras' Theorem on \( \text{p-adic Compactoids} \)

by

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0. Introduction

The following 'convexification' of the notion of precompactness plays a central role in \( \text{p-adic Functional Analysis}. \) Let \( K \) be a nonarchimedean nontrivially valued field, and \( E \) a locally \( K \)-convex space. An absolutely convex subset \( A \) of \( E \) is called \textit{compactoid} if for every (absolutely convex) neighbourhood \( U \) of 0 in \( E \), there exists a finite subset \( S = \{x_1, \ldots, x_n\} \) of \( E \) such that \( A \subseteq \text{co}(S) + U \). Here \( \text{co}(S) \) denotes the absolute convex hull of \( S \). Equivalently, we can say: for every absolutely convex neighbourhood \( U \) of 0, \( \pi_U(A) \) is contained in a finitely generated \( \mathbb{R} \)-module; here \( \mathbb{R} \) is the unit ball in \( K \), and \( \pi_U \) is the canonical map \( E \to E/U \) in the category of \( \mathbb{R} \)-modules. A natural question to ask is the following: can we choose \( S \) to be subset of \( A \)? Or, equivalently, is \( \pi_U(A) \) finitely generated as an \( \mathbb{R} \)-module? The answer is affirmative if the valuation of \( K \) is discrete, because \( \mathbb{R} \) is a noetherian ring in that case. If the valuation is dense, then we have an easy counterexample: take \( A = \{\lambda \in K : |\lambda| < 1\} \).

It is shown in [3] that, for \( E \) a Banach space, one may choose \( x_1, \ldots, x_n \) in \( \lambda A \), where \( \lambda \in K, |\lambda| > 1 \). For locally convex \( E \) it is shown in [1] that it is possible to choose \( x_1, \ldots, x_n \) in the \( K \)-vector space generated by \( A \), and in [2], [4] that \( x_1, \ldots, x_n \) may be chosen in \( \lambda A \). Yet, all these proofs are somewhat involved. In this note, both authors present a straightforward and elementary proof. We considered it worth while to publish our two proofs, since the statement is quite fundamental.
1. Proof by the Second Author

1.1. Lemma. Let $A$, $B$ be absolutely convex subsets of a $K$-vector space $E$. Suppose $A \subseteq B + \operatorname{co}(x)$ for some $x \in E$. Let $\lambda \in K$, $0 < |\lambda| < 1$ if the valuation of $K$ is dense, $\lambda = 1$ otherwise. Then there exists an $a \in A$ such that $\lambda A \subseteq B + \operatorname{co}(a)$.

Proof. The set $C \subseteq K$ defined by $C = \{\mu \in K : |\mu| < 1, \mu x \in A + B\}$ is absolutely convex. It is not hard to see that there exists a $c \in C$ for which $\lambda C \subseteq \operatorname{co}(c) \subseteq C$. As $c \in C$ there exists an $a \in A$ such that $cx \in a + B$. We claim that $\lambda A \subseteq B + \operatorname{co}(a)$. Indeed, if $z \in A$ then $z = b + dx$ for some $b \in B$, $d \in C$ so we have $\lambda z = \lambda b + \lambda dx \in B + \operatorname{co}(cx) \subseteq \operatorname{co}(a + B) \subseteq B + \operatorname{co}(a)$. □

1.2. Lemma. Let $E$, $A$, $B$, $\lambda$ be as above. Suppose $A \subseteq B + \operatorname{co}(x_1, \ldots, x_n)$ for some $x_1, \ldots, x_n \in E$. Then there exist $a_1, \ldots, a_n \in A$ such that $\lambda A \subseteq B + \operatorname{co}(a_1, \ldots, a_n)$.

Proof. Choose $\lambda_1, \ldots, \lambda_n \in K$, $0 < |\lambda_i| < 1$ and $|\prod_{i=1}^n \lambda_i| > |\lambda|$ if the valuation of $K$ is dense, $\lambda_i = 1$ for each $i$ otherwise. By applying Lemma 1.1 with $\lambda_i$ in place of $\lambda$ and $B + \operatorname{co}(x_2, \ldots, x_n)$ in place of $B$ we find an $a_1 \in A$ such that $\lambda_1 A \subseteq B + \operatorname{co}(a_1, x_2, \ldots, x_n)$. A second application of Lemma 1.1 with $\lambda_1 A$, $\lambda_2$, $B + \operatorname{co}(a_1, x_3, \ldots, x_n)$ in place of $A$, $\lambda$, $B$ respectively yields an $a_2 \in \lambda_1 A \subseteq A$ for which $\lambda_1 \lambda_2 A \subseteq B + \operatorname{co}(a_1, a_2, x_3, \ldots, x_n)$. Inductively we arrive at points $a_1, \ldots, a_n \in A$ such that $\lambda A \subseteq \lambda_1 \ldots \lambda_n A \subseteq B + \operatorname{co}(a_1, \ldots, a_n)$. □

1.3. Theorem (Katsaras). Let $A$ be an absolutely convex compactoid in a locally convex space over $K$. Let $\lambda \in K$, $|\lambda| > 1$ if the valuation of $K$ is dense, $\lambda = 1$ otherwise. Then for each absolutely convex neighbourhood $U$ of $0$ in $E$ there exist $x_1, \ldots, x_n \in \lambda A$ such that $A \subseteq U + \operatorname{co}(x_1, \ldots, x_n)$. 
Proof. $\lambda^{-1}U$ is a zero neighbourhood. By definition there exist $y_1,\ldots, y_n \in E$ such that $A \subset \lambda^{-1}U \cup \text{co}(y_1,\ldots, y_n)$. By Lemma 1.2 we can find $a_1,\ldots, a_n \in A$ such that $\lambda^{-1}A \subset \lambda^{-1}U \cup \text{co}(a_1,\ldots, a_n)$, i.e. $A \subset U \cup \text{co}(x_1,\ldots, x_n)$, where, for each $i$, $x_i = \lambda a_i \in \lambda A$. □

2. Proof by the First Author

In the introduction, we have seen that Theorem 1.3 is trivial if the valuation of $K$ is discrete; so let us assume from now on that $|K|$ is dense.

2.1. Lemma. Let $A$ be an $R$-submodule of a finitely generated free $R$-module, and let $\lambda \in R$ be such that $|\lambda| < 1$. Then we can find $a_1,\ldots, a_n \in A$ such that $\lambda A \subset Ra_1 + \ldots + Ra_n$.

Proof. $A \subset R^n \subset K^n$. We furnish $K^n$ with the usual supremum norm; it is well-known (cf. [3]) that every one dimensional subspace of $K^n$ has an orthocomplement. Let us proceed using induction on $n$. The case $n = 1$ is trivial.

Let $m = \sup \{ \|x\| : x \in A \}$, and choose $a_1 \in A$ such that $\|a_1\| > \frac{1}{|\lambda'|} m$, where $\lambda' \in K$ is such that $|\lambda'|^2 < |\lambda|$. Let $Q : K^n \rightarrow Ka_1$ be an orthoprojection, and take $P = I - Q$. Then every $x \in K^n$ may be written under the form $x = \lambda(x)a_1 + Px$, where $\|x\| = \max(|\lambda(x)|\|a_1\|, \|Px\|)$. If $x \in A$, then $|\lambda(x)|\|a_1\| < \|x\| < m < |\lambda'|^{-1}\|a_1\|$, so $|\lambda(x)| < |\lambda'|^{-1}$.

Using the induction hypothesis, we find $f_2,\ldots, f_n \in PA$ such that $\lambda'PA \subset Rf_2 + \ldots + Rf_n$. Lift $f_i$ to an element $a_i \in A$. Then, for $i > 2$, we have that $a_i = f_i + \lambda_i a_i$, where $|\lambda_i| < |\lambda'|^{-1}$. We now have, for $x \in A$:

$x = Qx + Px = \lambda(x)a_1 + \sum_{i=2}^{n} \mu_i f_i = (\lambda(x) - \sum_{i=2}^{n} \lambda_i \mu_i) a_1 + \sum_{i=2}^{n} \lambda_i \mu_i a_i$, where $|\lambda(x)|, |\lambda_i|, |\mu_i| < |\lambda'|^{-1}$. This implies the result. □

Proof of Theorem 1.3. Write $\mu = \lambda^{-1}$, then $|\mu| < 1$. $U$ is an absolutely convex neighbourhood of 0, so $\pi_{\mu U}(A)$ is a submodule of a finitely generated $R$-module $N$. So we have an epimorphism $\phi : R^n + N$ in the category of
R-modules. By Lemma 2.1, we may find $a_1, \ldots, a_n \subseteq \phi^{-1}(\pi_{\mu U}(A))$ such that 
\[ \mu \phi^{-1}(\pi_{\mu U}(A)) \subseteq R a_1 + \ldots + R a_n. \]
Choose $u_1, \ldots, u_n$ in $A$ such that $\pi_{\mu U}(u_i) = \phi(a_i)$. Then 
\[ \mu A \subseteq R u_1 + \ldots + R u_n + \mu U, \]
and, after multiplication by $\lambda$, 
\[ A \subseteq \lambda u_1 + \ldots + \lambda u_n + U, \]
and this proves the theorem. \(\square\)

References