TWO ELEMENTARY PROOFS OF KATSARAS' THEOREM ON P-ADIC COMPACTOIDS

by

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0. Introduction

The following 'convexification' of the notion of precompactness plays a central role in p-adic Functional Analysis. Let $K$ be a nonarchimedean nontrivially valued field, and $E$ a locally $K$-convex space. An absolutely convex subset $A$ of $E$ is called compactoid if for every (absolutely convex) neighbourhood $U$ of $0$ in $E$, there exists a finite subset $S = \{ x_1, \ldots, x_n \}$ of $E$ such that $A \subseteq \text{co}(S) + U$. Here $\text{co}(S)$ denotes the absolute convex hull of $S$. Equivalently, we can say: for every absolutely convex neighbourhood $U$ of $0$, $\pi_U(A)$ is contained in a finitely generated $R$-module; here $R$ is the unit ball in $K$, and $\pi_U$ is the canonical map $E + E/U$ in the category of $R$-modules. A natural question to ask is the following: can we choose $S$ to be subset of $A$? Or, equivalently, is $\pi_U(A)$ finitely generated as an $R$-module? The answer is affirmative if the valuation of $K$ is discrete, because $R$ is a noetherian ring in that case. If the valuation is dense, then we have an easy counterexample: take $A = \{ \lambda \in K : |\lambda| < 1 \}$.

It is shown in [3] that, for $E$ a Banach space, one may choose $x_1, \ldots, x_n$ in $\lambda A$, where $\lambda \in K$, $|\lambda| > 1$. For locally convex $E$ it is shown in [1] that it is possible to choose $x_1, \ldots, x_n$ in the $K$-vector space generated by $A$, and in [2], [4] that $x_1, \ldots, x_n$ may be chosen in $\lambda A$. Yet, all these proofs are somewhat involved. In this note, both authors present a straightforward and elementary proof. We considered it worth while to publish our two proofs, since the statement is quite fundamental.
I. Proof by the Second Author

1.1. Lemma. Let \( A, B \) be absolutely convex subsets of a \( K \)-vector space \( E \). Suppose \( A \subset B + \operatorname{co}(x) \) for some \( x \in E \). Let \( \lambda \in K, \, 0 < |\lambda| < 1 \) if the valuation of \( K \) is dense, \( \lambda = 1 \) otherwise. Then there exists an \( a \in A \) such that 
\[ \lambda A \subset B + \operatorname{co}(a). \]

Proof. The set \( C \subset K \) defined by \( C = \{ \mu \in K : |\mu| < 1, \, \mu x \in A + B \} \) is absolutely convex. It is not hard to see that there exists a \( c \in C \) for which
\[ \lambda C \subset \operatorname{co}(c). \]
As \( c \in C \) there exists an \( a \in A \) such that \( cx \in a + B \). We claim that \( \lambda A \subset B + \operatorname{co}(a) \). Indeed, if \( z \in A \) then \( z = b + dx \) for some \( b \in B, d \in C \) so we have \( \lambda z = \lambda b + \lambda dx \in B + \operatorname{co}(cx) \subset B + \operatorname{co}(a) \subset B + \operatorname{co}(a) \). □

1.2. Lemma. Let \( E, A, B, \lambda \) be as above. Suppose \( A \subset B + \operatorname{co}(x_1, \ldots, x_n) \) for some \( x_1, \ldots, x_n \in E \). Then there exist \( a_1, \ldots, a_n \in A \) such that \( \lambda A \subset B + \operatorname{co}(a_1, \ldots, a_n) \).

Proof. Choose \( \lambda_1, \ldots, \lambda_n \in K, \, 0 < |\lambda_i| < 1 \) and \( |\prod_{i=1}^n \lambda_i| > |\lambda| \) if the valuation of \( K \) is dense, \( \lambda_i = 1 \) for each \( i \) otherwise. By applying Lemma 1.1 with \( \lambda_i \) in place of \( \lambda \) and \( B + \operatorname{co}(x_2, \ldots, x_n) \) in place of \( B \) we find an \( a_1 \in A \) such that 
\[ \lambda_1 A \subset B + \operatorname{co}(a_1, x_2, \ldots, x_n). \]
A second application of Lemma 1.1 with \( \lambda_1 A, \lambda_2, B + \operatorname{co}(a_1, x_3, \ldots, x_n) \) in place of \( A, \lambda, B \) respectively yields an \( a_2 \in \lambda_1 A \subset A \) for which
\[ \lambda_1 \lambda_2 A \subset B + \operatorname{co}(a_1, a_2, x_3, \ldots, x_n). \]
Inductively we arrive at points \( a_1, \ldots, a_n \in A \) such that 
\[ \lambda A \subset \lambda_1 \ldots \lambda_n A \subset B + \operatorname{co}(a_1, \ldots, a_n). \] □

1.3. Theorem (Katsaras). Let \( A \) be an absolutely convex compactoid in a locally convex space over \( K \). Let \( \lambda \in K, \, |\lambda| > 1 \) if the valuation of \( K \) is dense, \( \lambda = 1 \) otherwise. Then for each absolutely convex neighbourhood \( U \) of \( 0 \) in \( E \) there exist \( x_1, \ldots, x_n \in \lambda A \) such that 
\[ A \subset U + \operatorname{co}(x_1, \ldots, x_n). \]
Proof. $\lambda^{-1}U$ is a zero neighbourhood. By definition there exist $y_1, \ldots, y_n \in E$ such that $A \subseteq \lambda^{-1}U + \text{co}(y_1, \ldots, y_n)$. By Lemma 1.2 we can find $a_1, \ldots, a_n \in A$ such that $\lambda^{-1}A \subseteq \lambda^{-1}U + \text{co}(a_1, \ldots, a_n)$, i.e. $A \subseteq U + \text{co}(x_1, \ldots, x_n)$, where, for each $i$, $x_i = \lambda a_i \in \lambda A$. □

2. Proof by the First Author

In the introduction, we have seen that Theorem 1.3 is trivial if the valuation of $K$ is discrete; so let us assume from now on that $|K|$ is dense.

2.1. Lemma. Let $A$ be an $R$-submodule of a finitely generated free $R$-module, and let $\lambda \in R$ be such that $|\lambda| < 1$. Then we can find $a_1, \ldots, a_n \in A$ such that $\lambda A \subseteq Ra_1 + \cdots + Ra_n$.

Proof. $A \subseteq R^n \subseteq K^n$. We furnish $K^n$ with the usual supremum norm; it is well-known (cf. [3]) that every one dimensional subspace of $K^n$ has an orthocomplement. Let us proceed using induction on $n$. The case $n = 1$ is trivial.

Let $m = \sup \{ \|x\| : x \in A \}$, and choose $a_1 \in A$ such that $\|a_1\| > \frac{1}{\lambda'}m$, where $\lambda' \in K$ is such that $|\lambda'|^2 < |\lambda|$. Let $Q : K^n + Ka_1 \to$ be an orthoprojection, and take $P = I - Q$. Then every $x \in K^n$ may be written under the form $x = \lambda(x)a_1 + Px$, where $\|x\| = \max (|\lambda(x)||a_1||, \|Px\|)$. If $x \in A$, then $|\lambda(x)||a_1|| < \|x\| < m < |\lambda'|^{-1}\|a_1||$, so $|\lambda(x)| < |\lambda'|^{-1}$.

Using the induction hypothesis, we find $f_2, \ldots, f_n \in FA$ such that $\lambda'PA \subseteq Rf_2 + \cdots + Rf_n$. Lift $f_i$ to an element $a_i \in A$. Then, for $i > 2$, we have that $a_i = f_i + \lambda_ia_i$, where $|\lambda_i| < |\lambda'|^{-1}$. We now have, for $x \in A$:

$x = Qx + Px = \lambda(x)a_1 + \sum_{i=2}^{n} \mu_i f_i = (\lambda(x) - \sum_{i=2}^{n} \lambda_1^i \mu_i)a_1 + \sum_{i=2}^{n} \mu_i a_1$, where $|\lambda(x)|, |\lambda_i|, |\mu_i| < |\lambda'|^{-1}$. This implies the result. □

Proof of Theorem 1.3. Write $\mu = \lambda^{-1}$, then $|\mu| < 1$. $U$ is an absolutely convex neighbourhood of $0$, so $\pi_{\mu U}(A)$ is a submodule of a finitely generated $R$-module $N$. So we have an epimorphism $\phi : R^n + N$ in the category of
R-modules. By Lemma 2.1, we may find \( a_1, \ldots, a_n \subseteq \phi^{-1}(\mu_\mathcal{U}(A)) \) such that

\[ \mu^{-1}(\pi_\mathcal{U}(A)) \subseteq R_{a_1} + \ldots + R_{a_n}. \]

Choose \( u_1, \ldots, u_n \) in \( A \) such that \( \pi_\mathcal{U}(u_i) = \phi(a_i) \).

Then \( \mu_\mathcal{U}(A) \subseteq R\phi(a_1) + \ldots + R\phi(a_n) = R\pi_\mathcal{U}(u_1) + \ldots + R\pi_\mathcal{U}(u_n) \), hence \( \mu A \subseteq Ru_1 + \ldots + Ru_n + \mu U \), and, after multiplication by \( \lambda \),

\[ A \subseteq R\lambda u_1 + \ldots + R\lambda u_n + U, \]

and this proves the theorem. \( \square \)

References


