TWO ELEMENTARY PROOFS OF KATSARAS' THEOREM ON P-ADIC COMPACTOIDS

by

S. Caenepeel, W.H. Schikhof

0. Introduction

The following 'convexification' of the notion of precompactness plays a central role in p-adic Functional Analysis. Let $K$ be a nonarchimedean nontrivially valued field, and $E$ a locally $K$-convex space. An absolutely convex subset $A$ of $E$ is called compactoid if for every (absolutely convex) neighbourhood $U$ of 0 in $E$, there exists a finite subset $S = \{x_1, \ldots, x_n\}$ of $E$ such that $A \subseteq \text{co}(S) + U$. Here $\text{co}(S)$ denotes the absolute convex hull of $S$. Equivalently, we can say: for every absolutely convex neighbourhood $U$ of 0, $\pi_U(A)$ is contained in a finitely generated $R$-module; here $R$ is the unit ball in $K$, and $\pi_U$ is the canonical map $E \rightarrow E/U$ in the category of $R$-modules. A natural question to ask is the following: can we choose $S$ to be subset of $A$? Or, equivalently, is $\pi_U(A)$ finitely generated as an $R$-module? The answer is affirmative if the valuation of $K$ is discrete, because $R$ is a noetherian ring in that case. If the valuation is dense, then we have an easy counterexample: take $A = \{\lambda \in K : |\lambda| < 1\}$.

It is shown in [3] that, for $E$ a Banach space, one may choose $x_1, \ldots, x_n$ in $\lambda A$, where $\lambda \in K$, $|\lambda| > 1$. For locally convex $E$ it is shown in [1] that it is possible to choose $x_1, \ldots, x_n$ in the $K$-vector space generated by $A$, and in [2], [4] that $x_1, \ldots, x_n$ may be chosen in $\lambda A$. Yet, all these proofs are somewhat involved. In this note, both authors present a straightforward and elementary proof. We considered it worth while to publish our two proofs, since the statement is quite fundamental.
1. Proof by the Second Author

1.1. Lemma. Let $A$, $B$ be absolutely convex subsets of a $K$-vector space $E$. Suppose $A \subseteq B + \text{co}\{x\}$ for some $x \in E$. Let $\lambda \in K$, $0 < |\lambda| < 1$ if the valuation of $K$ is dense, $\lambda = 1$ otherwise. Then there exists an $a \in A$ such that $\lambda A \subseteq B + \text{co}\{a\}$.

Proof. The set $C \subseteq K$ defined by $C = \{\mu \in K : |\mu| < 1, \mu x \in A + B\}$ is absolutely convex. It is not hard to see that there exists a $c \in C$ for which $\lambda c \subseteq \text{co}\{c\} \subseteq C$. As $c \in C$ there exists an $a \in A$ such that $cx \in a + B$. We claim that $\lambda A \subseteq B + \text{co}\{a\}$. Indeed, if $z \in A$ then $z = b + dx$ for some $b \in B$, $d \in C$ so we have $\lambda z = \lambda b + \lambda dx \in B + \text{co}(cx) \subseteq B + \text{co}(a + B) \subseteq B + \text{co}\{a\}$. □

1.2. Lemma. Let $E$, $A$, $B$, $\lambda$ be as above. Suppose $A \subseteq B + \text{co}\{x_1, \ldots, x_n\}$ for some $x_1, \ldots, x_n \in E$. Then there exist $a_1, \ldots, a_n \in A$ such that $\lambda A \subseteq B + \text{co}\{a_1, \ldots, a_n\}$.

Proof. Choose $\lambda_1, \ldots, \lambda_n \in K$, $0 < |\lambda_i| < 1$ and $|\prod_{i=1}^n \lambda_i| > |\lambda|$ if the valuation of $K$ is dense, $\lambda_i = 1$ for each $i$ otherwise. By applying Lemma 1.1 with $\lambda_i$ in place of $\lambda$ and $B + \text{co}\{x_2, \ldots, x_n\}$ in place of $B$ we find an $a_i \in A$ such that $\lambda_i A \subseteq B + \text{co}\{a_1, x_2, \ldots, x_n\}$. A second application of Lemma 1.1 with $\lambda_1 A$, $\lambda_2$ $B + \text{co}\{a_1, x_3, \ldots, x_n\}$ in place of $A$, $\lambda$, $B$ respectively yields an $a_2 \in \lambda_1 A \subseteq A$ for which $\lambda_1 \lambda_2 A \subseteq B + \text{co}(a_1, a_2, x_3, \ldots, x_n)$. Inductively we arrive at points $a_1, \ldots, a_n \in A$ such that $\lambda A \subseteq a_1, \ldots, a_n A \subseteq B + \text{co}(a_1, \ldots, a_n)$. □

1.3. Theorem (Katsaras). Let $A$ be an absolutely convex compactoid in a locally convex space over $K$. Let $\lambda \in K$, $|\lambda| > 1$ if the valuation of $K$ is dense, $\lambda = 1$ otherwise. Then for each absolutely convex neighbourhood $U$ of $0$ in $E$ there exist $x_1, \ldots, x_n \in \lambda A$ such that $A \subseteq U + \text{co}(x_1, \ldots, x_n)$.
Proof. $\lambda^{-1}U$ is a zero neighbourhood. By definition there exist $y_1, \ldots, y_n$ $\in E$ such that $A \subset \lambda^{-1}U + \text{co}(y_1, \ldots, y_n)$. By Lemma 1.2 we can find $a_1, \ldots, a_n$ $\in A$ such that $\lambda^{-1}A \subset \lambda^{-1}U + \text{co}(a_1, \ldots, a_n)$, i.e. $A \subset U + \text{co}(x_1, \ldots, x_n)$, where, for each $i$, $x_i = \lambda a_i \in \lambda A$. □

2. Proof by the First Author

In the introduction, we have seen that Theorem 1.3 is trivial if the valuation of $K$ is discrete; so let us assume from now on that $|K|$ is dense.

2.1. Lemma. Let $A$ be an $R$-submodule of a finitely generated free $R$-module, and let $\lambda \in R$ be such that $|\lambda| < 1$. Then we can find $a_1, \ldots, a_n \in A$ such that $\lambda A \subset Ra_1 + \ldots + Ra_n$.

Proof. $A \subset R^n \subset K^n$. We furnish $K^n$ with the usual supremum norm; it is well-known (cf. [3]) that every one-dimensional subspace of $K^n$ has an orthocomplement. Let us proceed using induction on $n$. The case $n = 1$ is trivial.

Let $m = \sup \{ \|x\| : x \in A \}$, and choose $a_1 \in A$ such that $\|a_1\| > \frac{1}{2} |\lambda'| m$, where $\lambda' \in K$ is such that $|\lambda'|^2 < |\lambda|$. Let $Q : K^n + Ka_1 \rightarrow$ be an orthoprojection, and take $P = I - Q$. Then every $x \in K^n$ may be written under the form $x = \lambda(x)a_1 + Px$, where $\|x\| = \max(|\lambda(x)\|a_1\|, \|Px\|)$. If $x \in A$, then $|\lambda(x)\|a_1\| < \|x\| < m < |\lambda'|^{-1}\|a_1\|$, so $|\lambda(x)| < |\lambda'|^{-1}$.

Using the induction hypothesis, we find $f_2, \ldots, f_n \in PA$ such that $\lambda'PA \subset Rf_2 + \ldots + Rf_n$. Lift $f_i$ to an element $e_i \in A$. Then, for $i > 2$, we have that $a_i = f_i + \lambda' e_i$, where $|\lambda_1| < |\lambda'|^{-1}$. We now have, for $x \in A$:

$x = Qx + Px = \lambda(x)a_1 + \sum_{i=2}^n \mu_i f_i = (\lambda(x) - \sum_{i=2}^n \lambda_1 \mu_1) a_1 + \sum_{i=2}^n \mu_i e_i$, where $|\lambda(x)|, |\lambda_1|, |\mu_1| < |\lambda'|^{-1}$. This implies the result. □

Proof of Theorem 1.3. Write $\mu = \lambda^{-1}$, then $|\mu| < 1$. $U$ is an absolutely convex neighbourhood of $0$, so $\pi_\mu(U)$ is a submodule of a finitely generated $R$-module $N$. So we have an epimorphism $\phi : R^n + N$ in the category of
R-modules. By Lemma 2.1, we may find $a_1, \ldots, a_n \subseteq \phi^{-1}(\Pi_{\mu U}(A))$ such that 
$\mu \phi^{-1}(\Pi_{\mu U}(A)) \subseteq R a_1 \oplus \cdots \oplus R a_n$. Choose $u_1, \ldots, u_n$ in $A$ such that $\phi_{\mu U}(u_j) = \phi(a_j)$.

Then $\phi_{\mu U}(A) \subseteq R \phi(a_1) + \cdots + R \phi(a_n) = R \phi_{\mu U}(u_1) + \cdots + R \phi_{\mu U}(u_n)$, hence $\mu A \subseteq R u_1 + \cdots + R u_n + \mu U$, and, after multiplication by $\lambda$,

$A \subseteq R u_1 + \cdots + R u_n + U$, and this proves the theorem. \(\square\)

References