TWO ELEMENTARY PROOFS OF KATSARAS' THEOREM ON P-ADIC COMPACTOIDS
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0. Introduction

The following 'convexification' of the notion of precompactness plays a central role in p-adic Functional Analysis. Let $K$ be a nonarchimedean nontrivially valued field, and $E$ a locally $K$-convex space. An absolutely convex subset $A$ of $E$ is called compactoid if for every (absolutely convex) neighbourhood $U$ of $0$ in $E$, there exists a finite subset $S = \{x_1, \ldots, x_n\}$ of $E$ such that $A \subseteq \text{co}(S) + U$. Here $\text{co}(S)$ denotes the absolute convex hull of $S$. Equivalently, we can say: for every absolutely convex neighbourhood $U$ of $0$, $\pi_U(A)$ is contained in a finitely generated $R$-module; here $R$ is the unit ball in $K$, and $\pi_U$ is the canonical map $E \rightarrow E/U$ in the category of $R$-modules. A natural question to ask is the following: can we choose $S$ to be subset of $A$? Or, equivalently, is $\pi_U(A)$ finitely generated as an $R$-module? The answer is affirmative if the valuation of $K$ is discrete, because $R$ is a noetherian ring in that case. If the valuation is dense, then we have an easy counterexample: take $A = \{\lambda \in K : |\lambda| < 1\}$.

It is shown in [3] that, for $E$ a Banach space, one may choose $x_1, \ldots, x_n$ in $\lambda A$, where $\lambda \in K$, $|\lambda| > 1$. For locally convex $E$ it is shown in [1] that it is possible to choose $x_1, \ldots, x_n$ in the $K$-vector space generated by $A$, and in [2], [4] that $x_1, \ldots, x_n$ may be chosen in $\lambda A$. Yet, all these proofs are somewhat involved. In this note, both authors present a straightforward and elementary proof. We considered it worth while to publish our two proofs, since the statement is quite fundamental.
1. Proof by the Second Author

1.1. Lemma. Let $A, B$ be absolutely convex subsets of a $K$-vector space $E$. Suppose $A \subseteq B + \text{co}(x)$ for some $x \in E$. Let $\lambda \in K$, $0 < |\lambda| < 1$ if the valuation of $K$ is dense, $\lambda = 1$ otherwise. Then there exists an $a \in A$ such that $\lambda A \subseteq B + \text{co}(a)$.

**Proof.** The set $C \subseteq K$ defined by $C = \{\mu \in K : |\mu| < 1, \mu x \in A + B\}$ is absolutely convex. It is not hard to see that there exists an $c \in C$ for which $\lambda c \subseteq c \subseteq \text{co}(c) \subseteq C$. As $c \in C$ there exists an $a \in A$ such that $cx \in a + B$. We claim that $\lambda A \subseteq B + \text{co}(a)$. Indeed, if $z \in A$ then $z = b + dx$ for some $b \in B, d \in C$ so we have $\lambda z = \lambda b + \lambda dx \in B + \text{co}(cx) \subseteq B + \text{co}(a + B) \subseteq B + \text{co}(a)$. □

1.2. Lemma. Let $E, A, B, \lambda$ be as above. Suppose $A \subseteq B + \text{co}(x_1, \ldots, x_n)$ for some $x_1, \ldots, x_n \in E$. Then there exist $a_1, \ldots, a_n \in A$ such that $\lambda A \subseteq B + \text{co}(a_1, \ldots, a_n)$.

**Proof.** Choose $\lambda_1, \ldots, \lambda_n \in K$, $0 < |\lambda_i| < 1$ and $|\prod_{i=1}^n \lambda_i| > |\lambda|$ if the valuation of $K$ is dense, $\lambda_i = 1$ for each $i$ otherwise. By applying Lemma 1.1 with $\lambda_i$ in place of $\lambda$ and $B + \text{co}(x_2, \ldots, x_n)$ in place of $B$ we find an $a_1 \in A$ such that $\lambda_1 A \subseteq B + \text{co}(a_1, x_2, \ldots, x_n)$. A second application of Lemma 1.1 with $\lambda_1 A, \lambda_2, B + \text{co}(a_1, x_3, \ldots, x_n)$ in place of $A, \lambda, B$ respectively yields an $a_2 \in \lambda_1 A \subseteq A$ for which $\lambda_1 \lambda_2 A \subseteq B + \text{co}(a_1, a_2, x_3, \ldots, x_n)$. Inductively we arrive at points $a_1, \ldots, a_n \in A$ such that $\lambda A \subseteq \lambda_1 \ldots \lambda_n A \subseteq B + \text{co}(a_1, \ldots, a_n)$. □

1.3. Theorem (Katsaras). Let $A$ be an absolutely convex compactoid in a locally convex space over $K$. Let $\lambda \in K$, $|\lambda| > 1$ if the valuation of $K$ is dense, $\lambda = 1$ otherwise. Then for each absolutely convex neighbourhood $U$ of $0$ in $E$ there exist $x_1, \ldots, x_n \in \lambda A$ such that $A \subseteq U + \text{co}(x_1, \ldots, x_n)$.
Proof. \( \lambda^{-1}U \) is a zero neighbourhood. By definition there exist \( y_1, \ldots, y_n \) \( \in E \) such that \( A \subseteq \lambda^{-1}U + co\{y_1, \ldots, y_n\} \). By Lemma 1.2 we can find \( a_1, \ldots, a_n \) \( \in A \) such that \( \lambda^{-1}A \subseteq \lambda^{-1}U + co\{a_1, \ldots, a_n\} \), i.e. \( A \subseteq U + co\{x_1, \ldots, x_n\} \), where, for each \( i \), \( x_i = \lambda a_i \in \lambda A \). □

2. Proof by the First Author

In the introduction, we have seen that Theorem 1.3 is trivial if the valuation of \( K \) is discrete; so let us assume from now on that \( |K| \) is dense.

2.1. Lemma. Let \( A \) be an \( R \)-submodule of a finitely generated free \( R \)-module, and let \( \lambda \in R \) be such that \( |\lambda| < 1 \). Then we can find \( a_1, \ldots, a_n \) \( \in A \) such that \( \lambda A \subseteq Ra_1 + \ldots + Ra_n \).

Proof. \( A \subseteq R^n \subseteq K^n \). We furnish \( K^n \) with the usual supremum norm; it is well-known (cf. [3]) that every one dimensional subspace of \( K^n \) has an orthocomplement. Let us proceed using induction on \( n \). The case \( n = 1 \) is trivial. Let \( m = \sup \{ \|x\| : x \in A \} \), and choose \( a_1 \in A \) such that \( \|a_1\| > \frac{1}{2} |\lambda'| m \), where \( \lambda' \in K \) is such that \( |\lambda'|^2 < |\lambda| \). Let \( Q : K^n + Ka_1 \to \) be an orthoprojection, and take \( P = I - Q \). Then every \( x \in K^n \) may be written under the form \( x = \lambda(x)a_1 + Px \), where \( \|x\| = \max(|\lambda(x)||a_1\|, \|Px\|) \). If \( x \in A \), then \( |\lambda(x)||a_1\| < \|x\| < m < |\lambda'|^{-1}\|a_1\| \), so \( |\lambda(x)| < |\lambda'|^{-1} \).

Using the induction hypothesis, we find \( f_2, \ldots, f_n \in FA \) such that \( \lambda'PA \subseteq Rf_2 + \ldots + Rf_n \). Lift \( f_1 \) to an element \( a_1 \in A \). Then, for \( i > 2 \), we have that \( a_i = f_i + \lambda a_i \), where \( |\lambda_i| < |\lambda'|^{-1} \). We now have, for \( x \in A : x = Qx + Px = \lambda(x)a_1 + \sum_{i=2}^{n} \mu_i f_i = (\lambda(x) - \sum_{i=2}^{n} \lambda \mu_i a_i) + \sum_{i=2}^{n} \mu_i a_i \), where \( |\lambda(x)|, |\lambda_i|, |\mu_i| < |\lambda'|^{-1} \). This implies the result. □

Proof of Theorem 1.3. Write \( \mu = \lambda^{-1} \), then \( |\mu| < 1 \). \( U \) is an absolutely convex neighbourhood of \( 0 \), so \( \pi U(A) \) is a submodule of a finitely generated \( R \)-module \( N \). So we have an epimorphism \( \phi : R^n + N \) in the category of
R-modules. By Lemma 2.1, we may find \( a_1, \ldots, a_n \subseteq \phi^{-1}(\pi_{\mu U}(A)) \) such that 
\[ \mu \phi^{-1}(\pi_{\mu U}(A)) \subseteq R a_1 + \cdots + R a_n. \]
Choose \( u_1, \ldots, u_n \) in \( A \) such that \( \pi_{\mu U}(u_1) = \phi(a_1) \).

Then \( \mu \pi_{\mu U}(A) \subseteq R \phi(a_1) + \cdots + R \phi(a_n) = R \pi_{\mu U}(u_1) + \cdots + R \pi_{\mu U}(u_n) \), hence 
\[ \mu A \subseteq R u_1 + \cdots + R u_n + \mu U, \]
and, after multiplication by \( \lambda \), 
\[ A \subseteq \lambda u_1 + \cdots + \lambda u_n + U, \] and this proves the theorem. \( \square \)

References