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TWO ELEMENTARY PROOFS OF KATSARAS' THEOREM ON P-ADIC COMPACTOIDS

by

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0. Introduction

The following 'convexification' of the notion of precompactness plays a central role in p-adic Functional Analysis. Let \( K \) be a nonarchimedean nontrivially valued field, and \( E \) a locally \( K \)-convex space. An absolutely convex subset \( A \) of \( E \) is called compactoid if for every (absolutely convex) neighbourhood \( U \) of \( 0 \) in \( E \), there exists a finite subset \( S = \{x_1, \ldots, x_n\} \) of \( E \) such that \( A \subseteq \text{co}(S) + U \), where \( \text{co}(S) \) denotes the absolute convex hull of \( S \). Equivalently, we can say: for every absolutely convex neighbourhood \( U \) of \( 0 \), \( \pi_U(A) \) is contained in a finitely generated \( R \)-module; here \( R \) is the unit ball in \( K \), and \( \pi_U \) is the canonical map \( E \to E/U \) in the category of \( R \)-modules. A natural question to ask is the following: can we choose \( S \) to be subset of \( A \)? Or, equivalently, is \( \pi_U(A) \) finitely generated as an \( R \)-module? The answer is affirmative if the valuation of \( K \) is discrete, because \( R \) is a noetherian ring in that case. If the valuation is dense, then we have an easy counterexample: take \( A = \{\lambda \in K : |\lambda| < 1\} \).

It is shown in [3] that, for \( E \) a Banach space, one may choose \( x_1, \ldots, x_n \) in \( \lambda A \), where \( \lambda \in K \), \( |\lambda| > 1 \). For locally convex \( E \) it is shown in [1] that it is possible to choose \( x_1, \ldots, x_n \) in the \( K \)-vector space generated by \( A \), and in [2], [4] that \( x_1, \ldots, x_n \) may be chosen in \( \lambda A \). Yet, all these proofs are somewhat involved. In this note, both authors present a straightforward and elementary proof. We considered it worth while to publish our two proofs, since the statement is quite fundamental.
1. Proof by the Second Author

1.1. Lemma. Let A, B be absolutely convex subsets of a K-vector space \( E \).
Suppose \( A \subset B + \text{co}(x) \) for some \( x \in E \). Let \( \lambda \in K \), \( 0 < |\lambda| < 1 \) if the valuation of \( K \) is dense, \( \lambda = 1 \) otherwise. Then there exists an \( a \in A \) such that 
\( \lambda A \subset B + \text{co}(a) \).

Proof. The set \( C \subset K \) defined by \( C = \{ \mu \in K : |\mu| < 1, \mu x \in A + B \} \) is absolutely convex. It is not hard to see that there exists a \( c \in C \) for which 
\( \lambda c \subset \text{co}(c) \subset C \). As \( c \in C \) there exists an \( a \in A \) such that \( cx \in a + B \). We claim that 
\( \lambda A \subset B + \text{co}(a) \). Indeed, if \( z \in A \) then \( z = b + dx \) for some \( b \in B, d \in C \) so we have 
\( \lambda z = \lambda b + \lambda dx \in B + \text{co}(cx) \subset B + \text{co}(a + B) \subset B + \text{co}(a) \). \( \square \)

1.2. Lemma. Let \( E, A, B, \lambda \) be as above. Suppose \( A \subset B + \text{co}(x_1, \ldots, x_n) \) for some \( x_1, \ldots, x_n \in E \). Then there exist \( a_1, \ldots, a_n \in A \) such that 
\( \lambda A \subset B + \text{co}(a_1, \ldots, a_n) \).

Proof. Choose \( \lambda_1, \ldots, \lambda_n \in K \), \( 0 < |\lambda_i| < 1 \) and \( |\prod_{i=1}^n \lambda_i| > |\lambda| \) if the valuation of \( K \) is dense, \( \lambda_i = 1 \) for each \( i \) otherwise. By applying Lemma 1.1 with \( \lambda_i \) in place of \( \lambda \) and \( B + \text{co}(x_2, \ldots, x_n) \) in place of \( B \) we find an 
\( a_1 \in A \) such that 
\( \lambda_1 A \subset B + \text{co}(a_1, x_2, \ldots, x_n) \).

A second application of Lemma 1.1 with \( \lambda_1 A, \lambda_2, B + \text{co}(a_1, x_3, \ldots, x_n) \) in place of \( A, \lambda, B \) respectively yields an \( a_2 \in \lambda_1 A \subset A \) for which 
\( \lambda_1 \lambda_2 A \subset B + \text{co}(a_1, a_2, x_3, \ldots, x_n) \). Inductively we arrive at points \( a_1, \ldots, a_n \in A \) such that 
\( \lambda A \subset \lambda_1 \ldots \lambda_n A \subset B + \text{co}(a_1, \ldots, a_n) \). \( \square \)

1.3. Theorem (Katsaras). Let \( A \) be an absolutely convex compactoid in a locally convex space over \( K \). Let \( \lambda \in K, |\lambda| > 1 \) if the valuation of \( K \) is dense, \( \lambda = 1 \) otherwise. Then for each absolutely convex neighbourhood \( U \) of 0 in \( E \) there exist \( x_1, \ldots, x_n \in \lambda A \) such that 
\( A \subset U + \text{co}(x_1, \ldots, x_n) \).
Proof. \( \lambda^{-1}U \) is a zero neighbourhood. By definition there exist \( y_1, \ldots, y_n \in E \) such that \( A \subseteq \lambda^{-1}U + \text{co}(y_1, \ldots, y_n) \). By Lemma 1.2 we can find \( a_1, \ldots, a_n \in A \) such that \( \lambda^{-1}A \subseteq \lambda^{-1}U + \text{co}(a_1, \ldots, a_n) \), i.e. \( A \subseteq U + \text{co}(x_1, \ldots, x_n) \), where, for each \( i \), \( x_i = \lambda a_i \in \lambda A \). □

2. Proof by the First Author

In the introduction, we have seen that Theorem 1.3 is trivial if the valuation of \( K \) is discrete; so let us assume from now on that \( |K| \) is dense.

2.1. Lemma. Let \( A \) be an \( R \)-submodule of a finitely generated free \( R \)-module, and let \( \lambda \in R \) be such that \( |\lambda| < 1 \). Then we can find \( a_1, \ldots, a_n \in A \) such that \( \lambda A \subseteq Ra_1 + \cdots + Ra_n \).

Proof. \( A \subseteq R^n \subseteq K^n \). We furnish \( K^n \) with the usual supremum norm; it is well-known (cf. [3]) that every one dimensional subspace of \( K^n \) has an orthocomplement. Let us proceed using induction on \( n \). The case \( n = 1 \) is trivial.

Let \( m = \sup \{ \| x \| : x \in A \} \), and choose \( a_1 \in A \) such that \( \| a_1 \| > \frac{1}{2} \lambda' m \), where \( \lambda' \in K \) is such that \( |\lambda'|^2 < |\lambda| \). Let \( Q : K^n + K a_1 \) be an orthoprojection, and take \( P = I - Q \). Then every \( x \in K^n \) may be written under the form \( x = \lambda(x)a_1 + Px \), where \( \| x \| = \max(\| \lambda(x) \| a_1 \|, \| Px \|) \). If \( x \in A \), then \( |\lambda(x)||a_1|| < m < |\lambda'|^{-1} \| a_1 \|, \) so \( |\lambda(x)| < |\lambda'|^{-1} \).

Using the induction hypothesis, we find \( f_2, \ldots, f_n \in FA \) such that \( \lambda'PA \subseteq Rf_2 + \cdots + Rf_n \). Lift \( f_i \) to an element \( a_i \in A \). Then, for \( i > 1 \), we have that \( a_i = f_i + \lambda a_i \), where \( |\lambda a_i| < |\lambda'|^{-1} \). We now have, for \( x \in A \):

\[
x = Qx + Px = \lambda(x)a_1 + \sum_{i=2}^{n} \mu f_i = (\lambda(x) - \sum_{i=1}^{n} \lambda i \mu_i) a_1 + \sum_{i=2}^{n} \mu_i a_i ,
\]

where \( |\lambda(x)|, |\lambda a_i|, |\mu_i| < |\lambda'|^{-1} \). This implies the result. □

Proof of Theorem 1.3. Write \( \mu = \lambda^{-1} \), then \( |\mu| < 1 \). \( U \) is an absolutely convex neighbourhood of \( 0 \), so \( \pi_{\mu U}(A) \) is a submodule of a finitely generated \( R \)-module \( N \). So we have an epimorphism \( \phi : R^n + N \) in the category of
R-modules. By Lemma 2.1, we may find $a_1, \ldots, a_n \subseteq \phi^{-1}(\mu \mathcal{U}(A))$ such that

$$\mu \phi^{-1}(\mu \mathcal{U}(A)) \subseteq R a_1 + \ldots + R a_n.$$ Choose $u_1, \ldots, u_n$ in $A$ such that $\pi_{\mu \mathcal{U}}(u_i) = \phi(a_i)$.

Then $\mu \pi_{\mu \mathcal{U}}(A) \subseteq R \phi(a_1) + \ldots + R \phi(a_n) = R \pi_{\mu \mathcal{U}}(u_1) + \ldots + R \pi_{\mu \mathcal{U}}(u_n)$, hence

$$\mu A \subseteq R u_1 + \ldots + R u_n + \mu U,$$

and, after multiplication by $\lambda$,

$$A \subseteq R \lambda u_1 + \ldots + R \lambda u_n + U,$$ and this proves the theorem. \qed

References


