TWO ELEMENTARY PROOFS OF KATSARAS' THEOREM ON P-ADIC COMPACTOIDS

by

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0. Introduction

The following 'convexification' of the notion of precompactness plays a central role in p-adic Functional Analysis. Let K be a nonarchimedean nontrivially valued field, and E a locally K-convex space. An absolutely convex subset A of E is called compactoid if for every (absolutely convex) neighbourhood U of 0 in E, there exists a finite subset \( S = \{x_1, \ldots, x_n\} \) of E such that \( A \subseteq \text{co}(S) + U \). Here \( \text{co}(S) \) denotes the absolute convex hull of S. Equivalently, we can say: for every absolutely convex neighbourhood U of 0, \( \pi_U(A) \) is contained in a finitely generated \( R \)-module; here \( R \) is the unit ball in K, and \( \pi_U \) is the canonical map \( E \to E/U \) in the category of \( R \)-modules. A natural question to ask is the following: can we choose \( S \) to be subset of \( A \)? Or, equivalently, is \( \pi_U(A) \) finitely generated as an \( R \)-module? The answer is affirmative if the valuation of K is discrete, because \( R \) is a noetherian ring in that case. If the valuation is dense, then we have an easy counterexample: take \( A = \{\lambda \in K : |\lambda| < 1\} \).

It is shown in [3] that, for E a Banach space, one may choose \( x_1, \ldots, x_n \) in \( \lambda A \), where \( \lambda \in K \), \( |\lambda| > 1 \). For locally convex E it is shown in [1] that it is possible to choose \( x_1, \ldots, x_n \) in the K-vector space generated by A, and in [2], [4] that \( x_1, \ldots, x_n \) may be chosen in \( \lambda A \). Yet, all these proofs are somewhat involved. In this note, both authors present a straightforward and elementary proof. We considered it worth while to publish our two proofs, since the statement is quite fundamental.
1. Proof by the Second Author

1.1. Lemma. Let $A$, $B$ be absolutely convex subsets of a $K$-vector space $E$. Suppose $A \subseteq B + \text{co}(x)$ for some $x \in E$. Let $\lambda \in K$, $0 < |\lambda| < 1$ if the valuation of $K$ is dense, $\lambda = 1$ otherwise. Then there exists an $a \in A$ such that $\lambda A \subseteq B + \text{co}(a)$.

**Proof.** The set $C \subseteq K$ defined by $C = \{ \mu \in K : |\mu| < 1, \mu x \in A + B \}$ is absolutely convex. It is not hard to see that there exists a $c \in C$ for which $\lambda C \subseteq \text{co}(c) \subseteq C$. As $c \in C$ there exists an $a \in A$ such that $cx \in a + B$. We claim that $\lambda A \subseteq B + \text{co}(a)$. Indeed, if $z \in A$ then $z = b + dx$ for some $b \in B$, $d \in C$ so we have $\lambda z = \lambda b + \lambda dx \in B + \text{co}(cx) \subseteq B + \text{co}(a + B) \subseteq B + \text{co}(a)$. \[\square\]

1.2. Lemma. Let $E$, $A$, $B$, $\lambda$ be as above. Suppose $A \subseteq B + \text{co}(x_1, \ldots, x_n)$ for some $x_1, \ldots, x_n \in E$. Then there exist $a_1, \ldots, a_n \in A$ such that $\lambda A \subseteq B + \text{co}(a_1, \ldots, a_n)$.

**Proof.** Choose $\lambda_1, \ldots, \lambda_n \in K$, $0 < |\lambda_i| < 1$ and $|\Pi_{i=1}^n \lambda_i| > |\lambda|$ if the valuation of $K$ is dense, $\lambda_i = 1$ for each $i$ otherwise. By applying Lemma 1.1 with $\lambda_1$ in place of $\lambda$ and $B + \text{co}(x_2, \ldots, x_n)$ in place of $B$ we find an $a_1 \in A$ such that $\lambda_1 A \subseteq B + \text{co}(a_1, x_2, \ldots, x_n)$. A second application of Lemma 1.1 with $\lambda_1 A$, $\lambda_2$, $B + \text{co}(a_1, x_3, \ldots, x_n)$ in place of $A$, $\lambda$, $B$ respectively yields an $a_2 \in \lambda_1 A \subseteq A$ for which $\lambda_1 \lambda_2 A \subseteq B + \text{co}(a_1, a_2, x_3, \ldots, x_n)$. Inductively we arrive at points $a_1, \ldots, a_n \in A$ such that $\lambda A \subseteq \lambda_1 \ldots \lambda_n A \subseteq B + \text{co}(a_1, \ldots, a_n)$. \[\square\]

1.3. Theorem (Katsaras). Let $A$ be an absolutely convex compactoid in a locally convex space over $K$. Let $\lambda \in K$, $|\lambda| > 1$ if the valuation of $K$ is dense, $\lambda = 1$ otherwise. Then for each absolutely convex neighbourhood $U$ of 0 in $E$ there exist $x_1, \ldots, x_n \in \lambda A$ such that $A \subseteq U + \text{co}(x_1, \ldots, x_n)$. 
Proof. \( \lambda^{-1}U \) is a zero neighbourhood. By definition there exist \( y_1, \ldots, y_n \) \( \in E \) such that \( A \subseteq \lambda^{-1}U + \text{co}(y_1, \ldots, y_n) \). By Lemma 1.2 we can find \( a_1, \ldots, a_n \) \( \in A \) such that \( \lambda^{-1}A \subseteq \lambda^{-1}U + \text{co}(a_1, \ldots, a_n) \), i.e. \( A \subseteq U + \text{co}(x_1, \ldots, x_n) \).

where, for each \( i \), \( x_i = \lambda a_i \in \lambda A \). \( \square \)

2. Proof by the First Author

In the introduction, we have seen that Theorem 1.3 is trivial if the valuation of \( K \) is discrete; so let us assume from now on that \( |K| \) is dense.

2.1. Lemma. Let \( A \) be an \( R \)-submodule of a finitely generated free \( R \)-module, and let \( \lambda \in R \) be such that \( |\lambda| < 1 \). Then we can find \( a_1, \ldots, a_n \in A \) such that \( \lambda A \subseteq Ra_1 + \ldots + Ra_n \).

Proof. \( A \subseteq R^n \subseteq K^n \). We furnish \( K^n \) with the usual supremum norm; it is well-known (cf. [3]) that every one dimensional subspace of \( K^n \) has an orthocomplement. Let us proceed using induction on \( n \). The case \( n = 1 \) is trivial.

Let \( m = \sup \{ \|x\| : x \in A \} \), and choose \( a_1 \in A \) such that \( \|a_1\| > \frac{1}{|\lambda'|} m \), where \( \lambda' \in K \) is such that \( |\lambda'|^2 < |\lambda| \). Let \( Q : K^n + Ka_1 \) be an orthoprojection, and take \( P = I - Q \). Then every \( x \in K^n \) may be written under the form \( x = \lambda(x)a_1 + Px \), where \( \|x\| = \max(\|\lambda(x)\|a_1\|, \|Px\|) \). If \( x \in A \), then

\[ |\lambda(x)\|a_1\| < \|x\| < m < |\lambda'|^{-1}\|a_1\|, \text{ so } |\lambda(x)| < |\lambda'|^{-1}. \]

Using the induction hypothesis, we find \( f_2, \ldots, f_n \in FA \) such that \( \lambda'PA \subseteq RF_2 + \ldots + Rf_n \). Lift \( f_1 \) to an element \( a_1 \in A \). Then, for \( i > 2 \), we have that \( a_i = f_i + \lambda_i a_1 \), where \( |\lambda_i| < |\lambda'|^{-1} \). We now have, for \( x \in A \):

\[ x = Qx + Px = \lambda(x)a_1 + \sum_{i=2}^{n} \mu_i f_i = (\lambda(x) - \sum_{i=2}^{n} \lambda_i \mu_i) a_1 + \sum_{i=2}^{n} \mu_i a_1, \]

where \( |\lambda(x)|, |\lambda_i|, |\mu_i| < |\lambda'|^{-1} \). This implies the result. \( \square \)

Proof of Theorem 1.3. Write \( \mu = \lambda^{-1} \), then \( |\mu| < 1 \). \( U \) is an absolutely convex neighbourhood of \( 0 \), so \( \pi_{\mu U}(A) \) is a submodule of a finitely generated \( R \)-module \( N \). So we have an epimorphism \( \phi : R^n + N \) in the category of
$R$-modules. By Lemma 2.1, we may find $a_1, \ldots, a_n \in \phi^{-1}(\mu \mathbb{U}(A))$ such that
\[\mu \phi^{-1}(\mu \mathbb{U}(A)) \subset R a_1 + \ldots + R a_n.\]
Choose $u_1, \ldots, u_n$ in $A$ such that $\mu \phi^{-1}(u_j) = \phi(a_j)$.
Then $\mu \phi^{-1}(\mu \mathbb{U}(A)) \subset R \phi(a_1) + \ldots + R \phi(a_n) = R \mu \phi^{-1}(u_1) + \ldots + R \mu \phi^{-1}(u_n)$, hence
\[\mu A \subset R u_1 + \ldots + R u_n + \mu U,\]
and, after multiplication by $\lambda$,
\[A \subset R A u_1 + \ldots + R A u_n + U,\] and this proves the theorem.

References