COMPACT-LIKE SETS IN NON-ARCHIMEDEAN FUNCTIONAL ANALYSIS
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Abstract

A survey of properties of convex compact-like sets in a locally convex or Banach space over a non-
archimedean valued field is given. A similar outline appeared in [3] but recently new results have been
obtained.

Notations, Preliminaries

Throughout this note $K$ is a non-archimedean nontrivially valued complete field with valuation $| |$ and
valuation ring $B(0,1) := \{ \lambda \in K : |\lambda| < 1 \}$.
Let $E$ be a $K$-vector space. A subset $A$ of $E$ is absolutely convex if it is a $B(0,1)$-module. For a set
$X \subseteq E$ its absolutely convex hull $\text{co} X$ is the smallest absolutely convex set containing $X$; $[X]$ is the $K$
-vector space generated by $X$. A subset of $E$ is convex if it is either empty or an additive coset of an
absolutely convex set. An $x \in E$ is a convex combination of $x_1, \ldots, x_n \in E$ if $x = \sum_{i=1}^n \lambda_i x_i$, where
$\lambda_i \in B(0,1)$ for each $i \in \{1, \ldots, n\}$ and $\sum_{i=1}^n \lambda_i = 1$. Thus, a subset of $E$ is convex if and only if it is closed
for forming convex combinations.
A function $\phi : E \to R$ is a convex function if for each $n \in \mathbb{N}$, each $x_1, \ldots, x_n \in E$ and each convex
combination $\sum_{i=1}^n \lambda_i x_i$

$$\phi(\sum_{i=1}^n \lambda_i x_i) \leq \max_{i \in \{1, \ldots, n\}} |\lambda_i| \phi(x_i)$$
For each (non-archimedean) seminorm \( p : E \to [0,\infty) \) and \( a \in E \) the map \( x \mapsto p(x-a) \ (x \in E) \) is convex.

From now on in this paper \( E \) is a Hausdorff locally convex space over \( K \). Then an absolutely convex subset of \( E \) inherits the structure of a topological \( B(0,1) \)-module. The (topological) dual space of \( E \) is denoted \( E' \). The weak topology \( o(E,E') \) on \( E \) is the weakest (locally convex) topology on \( E \) for which each \( f \in E' \) is continuous.

The closure of a set \( S \subseteq E \) is \( \overline{S} \). Instead of \( \overline{\overline{X}} \) we write \( \overline{\overline{X}} \).

Introduction

In non-archimedean Functional Analysis the classical notion of (pre)compactness looses some of its importance since, if \( K \) is not locally compact, convex (pre)compact sets are trivial. To overcome this difficulty the idea is to 'convexify' the definition of (pre)compactness. This can be carried out in various ways leading to different generalizations. In §§1-2 we discuss the well known concepts of compactoidity and \( c \)-compactness. In §§3-4 we consider a few alternatives. Finally, in §5, we compare compact-like sets for the weak and initial topology.

[For fundamentals on \( p \)-adic Banach and locally convex spaces we refer to [2], [22], [13]. For applications of compact-like sets in \( p \)-adic analysis, see for example [3],[4],[7],[8],[9],[13],[22].]

§1 COMPACTOIDS

The following 'convexification' of the notion of precompactness is due to L. Gruson and M. van der Put [6].

Definition. An absolutely convex \( A \subseteq E \) is \( (a) \) compactoid if for each neighbourhood \( U \) of 0 in \( E \) there exists a finite set \( F \subseteq E \) such that \( A \subseteq U + \text{co} \ F \).

The decency of this generalization is indicated by the next two elementary propositions.

Proposition 1. ([15]) If \( K \) is locally compact then an absolutely convex set is a compactoid if and only if it is precompact.
Proposition 2.

(i) Absolutely convex subsets of a compactoid are compactoid.
(ii) The image of a compactoid under a continuous linear map is a compactoid.
(iii) The closure of a compactoid is a compactoid.
(iv) Products of compactoids are compactoid [4].
(v) If $X \subseteq E$ is precompact then $\text{co } X$ is a compactoid.
(vi) A compactoid is bounded.
(vii) If $A$ is a compactoid then $[A]$ is of countable type [13] (Def.4.3.).

The following characterization of compactoidity is less elementary.

Theorem 3 ([14],[16]). Let $A \subseteq E$ be absolutely convex. The statements (a), (β), (γ) below are equivalent.

(a) $A$ is a compactoid.
(β) As a topological $B(0,1)$-module $A$ is isomorphic to a submodule of $B(0,1)^I$ for some set $I$.
(γ) $A$ is bounded. If $\tau'$ is a locally convex topology on $E$, weaker than the initial topology $\tau$, and if there exists a $\tau$-neighbourhood base of 0 in $E$ consisting of absolutely convex $\tau'$-closed sets, then $\tau = \tau'$ on $A$.

Two corollaries. 1. (From (a) $\iff$ (β)) Compactoidity of an absolutely convex $A$ depends only on the structure of $A$ as a topological $B(0,1)$-module and therefore does not depend on the space $E$ in which it is embedded. (Observe that the definition of compactoidity involves $E$.)
2. (From (a) $\iff$ (γ)) Let $K$ be spherically (= maximally) complete (or, more generally, let $E$ be a polar space ([13], Def.3.5)). Then, on compactoids, the initial topology and the weak topology coincide.

§1½ C-COMPACT SETS

We consider the 'convexified' version of compactness introduced by T.A Springer [21].

Definition. An absolutely convex $A \subseteq E$ is c-compact if for any family $\zeta$, consisting of nonempty relatively closed convex subsets $A$, $\zeta$ closed for finite intersections, we have $\bigcap \zeta \neq \emptyset$.

This concept is not useful for all $K$:
Proposition 4 ([21],[22]). If $K$ is not spherically complete the only $c$-compact subset of $E$ is $(0)$.

Proposition 5 ([21],[22]). If $K$ is spherically complete then $K$ is $c$-compact.

Proposition 6 ([2]). If $K$ is locally compact then an absolutely convex set is compact if and only if it is bounded and $c$-compact.

Remarks
1. Although $c$-compact sets may be unbounded (Proposition 5) they behave in a very compact-like manner (Proposition 7, Theorem 8).
2. However, Proposition 6 invites us to consider bounded $c$-compact sets. We shall do so in §2. A second reason may be the observation that, on spaces of linear operators, the topology of uniform convergence on $c$-compact sets is useful only when we add a boundedness assumption.

Proposition 7.
(i) A $c$-compact set is complete [2].
(ii) A $c$-compact set is a Baire space [19].
(iii) Absolutely convex closed subsets of $c$-compact sets are $c$-compact [21].
(iv) The image of a $c$-compact set under a continuous linear map is $c$-compact [21].
(v) Products of $c$-compact sets are $c$-compact [21].
(vi) If $A_1, A_2 \subset E$ are $c$-compact then so is $A_1 + A_2$ (from (iv), (v)).
(vii) If $X \subset E$ is closed and $A \subset E$ is $c$-compact then $X + A$ is closed [19].

We also have a characterization in the spirit of Theorem 3.

Theorem 8 ([5],[14],[16],[19]). Let $K$ be spherically complete, let $A \subset E$ be absolutely convex. The following are equivalent.
(a) $A$ is $c$-compact.
(b) As a topological $B(0,1)$-module $A$ is isomorphic to a closed submodule of $K^l$ for some set $l$.
(c) $A$ is closed. If $\tau'$ is a Hausdorff locally convex topology on $E$, weaker than the initial topology $\tau$, then $\tau = \tau'$ on $A$. 

§2 COMPLETE COMPACTOIDS

The next theorem is an easy corollary of Theorems 8, 3 and Proposition 7.

Theorem 9. Let $K$ be spherically complete, let $A \subseteq E$ be absolutely convex. The following are equivalent.

(a) $A$ is bounded and $c$-compact.

(b) $A$ is a complete compactoid.

(γ) As a topological $B(0,1)$-module $A$ is isomorphic to a closed submodule of $B(0,1)^{I}$ some set $I$.

(δ) $A$ is closed and bounded. If $\tau'$ is a Hausdorff locally convex topology on $E$, weaker than the initial topology $\tau$, then $\tau = \tau'$ on $A$.

Now let $K$ be not spherically complete. Property (α) of above reduces to $A = (0)$ but there do exist nontrivial $A$ for which (β)-(δ) are true. This leads to a natural question, answered by Theorem 10.

Theorem 10 ([14],[16]). Let $K$ be not spherically complete, let $A \subseteq E$ be absolutely convex. With (β), (γ), (δ) as in Theorem 9 we have

(i) (δ) $\Rightarrow$ (β) $\Rightarrow$ (γ).

(ii) if, in addition, $A$ is metrizable then also (β) $\Rightarrow$ (δ).

Remarks

1. A nonmetrizable counterexample to (β) $\Rightarrow$ (δ) is given in [14], Example 3.1.

2. Although the properties (β), (γ), (δ) seem to be natural translations of 'convex-compact' for the nonspherically complete case, we do not have, for sets satisfying (β), (γ), (δ), a theorem like Proposition 7 even in the case where $E$ is a Banach space:

(a) ([11], Compare Proposition 7(iv)) Let $K$ be not spherically complete, let $A$ be an absolutely convex subset of a Banach space $E$. Suppose that for each $K$-Banach space $F$ and for each continuous linear map $T : E \rightarrow F$ the set $TA$ is closed in $F$. Then $A$ is finite dimensional (i.e. $\dim|A| < \infty$).

(b) ([11], Compare Proposition 7(iv)) If $K$ is not spherically complete there exists a closed absolutely convex compactoid $A \subseteq c_{0}$ and an $a \in c_{0}$ such that $A + co\{a\}$ is not closed.

3. On the other hand, the following is easily deduced from Theorem 10. If $A, B$ are closed absolutely convex compactoids in a Fréchet space and if $A \cap B = (0)$ then $A + B$ is closed.
\section*{§2.5 COMPLETE LOCAL COMPACTOIDS}

To obtain a version of Theorem 9 in which the boundedness condition is dropped (see the remarks following Proposition 5) we define local compactoidity.

\textbf{Definition.} An absolutely convex $A \subset E$ is a \textit{local compactoid} if for every neighbourhood $U$ of 0 in $E$ there exists a finite dimensional $K$-linear subspace $D$ of $E$ such that $A \subset U + D$.

\textbf{Theorem 11 (\cite{19}).} Let $K$ be spherically complete. An absolutely convex subset of $E$ is $c$-compact if and only if it is a complete local compactoid.

\section*{§3 C'-COMPACT SETS}

In §§3-4 we shall consider 'convexifications' of (pre)compactness other than 'compactoid' or 'c-compact'. If we 'convexify' the 'open covering' version of compactness (rather than the intersection property of closed sets) we obtain the following.

\textbf{Definition.} An absolutely convex set $A \subset E$ is \textit{c'-compact} if for each covering \{$U_i : i \in I$\} of $A$ by (convex) open sets $U_i$ there exists a finite set $F \subset I$ such that $A \subset \text{co}(\bigcup_{i \in F} U_i)$.

Further, if we read 'precompactness' of $X \subset E$ as

"For each neighbourhood $U$ of 0 there exists a finite set $F \subset X$ such that $X \subset F + U$ "

and convexify it we obtain (the crucial phrase below being $F \subset A$):

\textbf{Definition.} An absolutely convex set $A \subset E$ is a \textit{pure compactoid} if for each neighbourhood $U$ of 0 there exists a finite set $F \subset A$ such that $A \subset U + \text{co} F$.

\textbf{Theorem 12 (\cite{17}).} For an absolutely convex $A \subset E$ the following are equivalent

(a) $A$ is c'-compact.

(b) $A$ is a pure compactoid.

(c) For each absolutely convex neighbourhood $U$ of 0 the $B(0,1)$-module $A/A \cap U$ is finitely generated.
(d) If $U_1 \subset U_2 \subset U_3 \subset \ldots$ are open convex sets covering $A$ then $A \subset U_n$ for some $n$.

(e) Each continuous seminorm $E \to [0, \infty)$, restricted to $A$, has a maximum.

(f) Each continuous convex function $E \to \mathbb{R}$, restricted to $A$, has a maximum.

Remarks

1. Each pure compactoid is a compactoid. If the valuation of $K$ is dense $\{x \in K : |x| < 1\}$ is a compactoid but not pure. In this context Katsaras' Theorem [7] is worth mentioning. If $A$ is a compactoid, if $U$ is a neighbourhood of 0, and if $\lambda \in K, |\lambda| > 1$ then there exists a finite set $F \subset \lambda A$ such that $A \subset U + \co F$.

2. If $K$ is spherically complete $\{x \in K : |x| < 1\}$ is c-compact but not c'-compact if the valuation is dense. If $K$ is not spherically complete $\{x \in K : |x| \leq 1\}$ is c'-compact (and complete) but not c-compact. Yet the next theorem reveals a striking analogy between the two classes.

**Theorem 13 ([17]).** Let $E$ be a Banach space and let $A \subset E$ be absolutely convex. The following are equivalent.

(a) $A$ is c-compact.

(b) If $C_1 \supset C_2 \supset \ldots$ are closed convex sets such that $\bigcap C_n$ does not meet $A$ then $A \cap C_n = \emptyset$ for some $n$.

(c) Each continuous convex function $E \to \mathbb{R}$, restricted to $A$, has a minimum.

(For $A$ an arbitrary locally convex space we have (a) $\implies$ (b) $\implies$ (c) but I do not know whether (c) $\implies$ (a) is true.)

**Definition.** An absolutely convex set $A \subset E$ is of finite type if for each neighbourhood $U$ of 0 in $E$ there exists a bounded finite dimensional set $G \subset A$ such that $A \subset U + G$.

Each pure compactoid is of finite type. The converse is not true. Not every compactoid is of finite type! [20].

**Proposition 14 ([20]).** A bounded c-compact set is a compactoid of finite type.

See [20] for further results.
§4 THE CLOSED CONVEX HULL OF A COMPACT SET

Let \( A \) be a compactoid in \( E \). Does there exist a precompact set \( X \subset E \) such that \( A \subset \overline{X} \)? Yes, if \( A \) is metrizable ([6],[13]); no in general [15]. But we do have the following.

**Proposition 15** ([16]). Let \( A \subset E \) be a compactoid. There exists a locally convex space \( F \supseteq E \) and a compact \( X \subset F \) such that \( A \subset \overline{X} \).

Of particular interest (Krein-Milman theorem) are the compactoids of the form \( \overline{X} \) where \( X \) is compact. Let us say that an absolutely convex \( A \subset E \) is a KM-compactoid if it is complete and there exists a compact set \( X \subset E \) such that \( A = \overline{X} \).

**Theorem 16** ([16]).
(i) If the valuation of \( K \) is discrete each complete compactoid is a KM-compactoid.
(ii) Each metrizable pure complete compactoid is a KM-compactoid.
(iii) (Compare Theorems 9 & 10) An absolutely convex \( A \subset E \) is a KM-compactoid if and only if \( A \) is isomorphic to \( B(0,1)^I \) for some set \( I \).

As a corollary one obtains easily the following p-adic version of the Krein-Milman Theorem which generalizes a result of [1].

**Theorem 17** ([16]). Let \( A \subset E \) be a KM-compactoid. There exists a linearly independent set \( Y := \{ e_i : i \in I \} \) in \( A \) such that

(i) \( Y \) is discrete,
(ii) \( Y_0 := Y \cup \{ 0 \} \) is compact,
(iii) \( A = \overline{Y} = \overline{Y_0} \),
(iv) \( Y \) is a minimal element of \( \{ Z \subset E : A = \overline{Z} \} \),
(v) \( Y_0 \) is a minimal element of \( \{ Z \subset E, Z \text{ is compact}, A = \overline{Z} \} \),
(vi) for each \( (\lambda_i)_{i \in I} \in B(0,1)^I \) the series \( \sum_{i \in I} \lambda_i e_i \) converges and represents an element of \( A \),
(vii) each \( x \in A \) has a unique representation as a convergent sum \( x = \sum_{i \in I} \lambda_i e_i \) where \( \lambda_i \in B(0,1) \) for each \( i \in I \).
§5 WEAK AND STRONG COMPACTOIDITY

For simplicity we assume in §5 that $K$ is spherically complete (for results for nonspherically complete $K$, see [15], [18]). By now the following proposition is well-known.

Proposition 18. Let $A \subset E$ be absolutely convex.
(i) $A$ is closed $\iff A$ is weakly closed [22].
(ii) $A$ is bounded $\iff A$ is weakly bounded $\iff A$ is a weak compactoid [22].
(iii) $A$ is $c$-compact $\iff A$ is weakly $c$-compact [2].

So, in general, 'compactoidity' is not the same as 'weak compactoidity'. Surprisingly, the notions 'compact', 'precompact', 'c'-compact' act quite differently.

Proposition 19 ([10],[15]). A subset of $E$ is compact if and only if it is weakly compact. (Corollary. Each weakly convergent sequence is convergent.)

Proposition 20 ([15]). Let $X$ be a subset of $E$.
(i) If $K$ is not locally compact then $X$ is precompact $\iff X$ is weakly precompact.
(ii) If $K$ is locally compact then $X$ is bounded $\iff X$ is weakly precompact.

Proposition 21 ([18]). Let $A \subset E$ be absolutely convex.
(i) If $K$ has a discrete valuation then $A$ is bounded $\iff A$ is weakly $c$'-compact.
(ii) If $K$ has a dense valuation and $E$ is either a Banach space with a base or a locally convex space of countable type then $A$ is $c$'-compact $\iff A$ is weakly $c$'-compact.

REFERENCES


