WEAK C'-COMPACTNESS IN $p$-ADIC BANACH SPACES

by

W.H. SCHIKHOF

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DEPARTMENT OF MATHEMATICS
CATHOLIC UNIVERSITY
Toernooiveld
6525 ED Nijmegen
The Netherlands
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ABSTRACT. Let $X$ be a subset of a locally convex space $E$ of countable type over a nonarchimedean densely valued field $K$. If, for each continuous linear function $f : E \to K$, $\max_X |f|$ exists then $X$ is a compactoid in $E$. 
§ 1. PRELIMINARIES

Throughout, $K$ is a nonarchimedean nontrivially valued complete field with valuation $|\cdot|$. For fundamentals on Banach spaces and locally convex spaces over $K$ we refer to [3], [6], [4]. We recall a few definitions and facts and fix some notations. Let $E$ be a locally convex space over $K$. $E'$ is the linear space of all continuous linear functions $E \to K$. If $E = (E, \|\|)$ is a nonzero Banach space and $f \in E'$ then $\|f\| := \sup \{|f(x)|/\|x\| : x \in E, x \neq 0\}$. $E$ is a polar space if there exists a base of continuous seminorms $p$ for which $p = \sup \{|f| : f \in E', |f| \leq p\}$.

PROPOSITION 1.1.

Let $E$ be a Banach space over $K$ with a base. Then $E$ is a polar space.

Proof.

By [3], Corollary 3.7 there exists a norm $\|\|$ inducing the topology of $E$ for which $E$ has an orthogonal base $\{e_i : i \in I\}$. It is not hard to see that we even may assume that $\{e_i : i \in I\}$ is orthonormal. For each $i \in I$, let $f_i$ be the $i$-th coordinate function. Let $x \in E$. Then $x = \sum_{i \in I} f_i(x)e_i$ and $\|x\| = \max_{i \in I} |f_i(x)|$.

For a subset $X$ of a locally convex space $E$ over $K$ we denote by $\text{co} \ X$ its absolutely convex hull, by $[X]$ its $K$-linear span, by $\overline{X}$ its closure. Instead of $\text{co} \ X$ we write $\text{co} \ X$. For an absolutely convex $A \subset E$ the formula

$$p_A(x) = \inf \{|\lambda| : x \in \lambda A\}$$

defines, on $[A]$, the seminorm $p_A$ associated to $A$. 
PROPOSITION 1.2.

Let \( A \subseteq E \) be absolutely convex. Then

\[
\{ x \in [A] : p_A(x) < 1 \} \subseteq A \subseteq \{ x \in [A] : p_A(x) \leq 1 \}.
\]

Proof.

Straightforward.

An absolutely convex \( A \subseteq E \) is **edged** if for each \( x \in E \) the set

\[
\{ |\lambda| : \lambda x \in A \}
\]

is closed in \( |K| := \{ |\lambda| : \lambda \in K \} \). \( A \) is edged if and only if \( A = \{ x \in [A] : p_A(x) \leq 1 \} \). A subset \( X \) of \( E \) is a **compactoid** if for each neighbourhood \( U \) of \( 0 \) in \( E \) there exists a finite set \( F \subseteq E \) such that \( X \subseteq U + \text{co} F \). For some elementary properties of compactoids, see [2].
§ 2. BANACH SPACES WITH A BASE

In § 2 we assume that the valuation of \( K \) is dense.

**Lemma 2.1.**

The closed absolutely convex hull of an orthonormal set in a Banach space over \( K \) is edged.

**Proof.**

For an orthonormal set \( \{ e_i : i \in I \} \), set \( A := \text{co} \{ e_i : i \in I \} \), \( D := \{ e_i : i \in I \} \). Without any trouble one verifies that \( D = [A] \), \( p_A = \| \cdot \| \) on \( D \) and \( A = \{ x \in D : \| x \| \leq 1 \} \).

**Lemma 2.2.**

Let \( E \) be a Banach space of countable type over \( K \). Let \( A \) be an absolutely convex neighbourhood of 0 in \( E \) and suppose that, for each \( f \in E' \), the restriction of \( |f| \) to \( A \) has a maximum.

(i) \( A \) is bounded; \( p_A \) is a norm \( \| \cdot \| \) inducing the topology of \( E \).

(ii) Let \( \{ e_i : i \in I \} \) be a maximal \( \| \cdot \| \)-orthonormal set in \( A_m := \{ x \in A : \text{there is an } f \in E' \text{ with } |f(x)| = \max_{A} |f| \} \).

Then \( A = \text{co} \{ e_i : i \in I \} \).

**Proof.**

(i) \( A \) is weakly bounded hence bounded by [4], Corollary 7.7. The interior of \( A \) is not empty and \( A \) is an additive group so \( A \) is open (and closed) and (i) follows.

(ii) \( B := \text{co} \{ e_i : i \in I \} \) is contained in \( A \). Suppose \( B \neq A \); we shall prove
that \( \{e_i : i \in I \} \) is not maximal yielding a contradiction. The set \( B \) is closed and edged (Lemma 2.1) so by [4], Proposition 4.8 and 3.4, there exists an \( f \in E' \) with \( |f| \leq 1 \) on \( B \) and \( |f(y)| > 1 \) for some \( y \in A \). Then \( \max_\lambda |f| > 1 \) and after multiplying \( f \) by a suitable scalar we obtain \( A \) a \( g \in E' \) for which

\[
|g| < 1 \text{ on } B, \max_\lambda |g| = 1.
\]

From

\[(*) \quad \{x \in E : \|x\| < 1\} \subset A \subset \{x \in E : \|x\| \leq 1\}\]

(Proposition 1.2) it follows that \( \|g\| = 1 \). Choose an \( a \in A \) with \( |g(a)| = 1 \). We claim that \( \{a\} \cup \{e_i : i \in I\} \) is an orthonormal set in \( A_m \). In fact, we have \( a \in A_m \). By \((*)\), \( \|a\| \leq 1 \) but also

\[
1 = |g(a)| \leq \|g\| \|a\| = \|a\| \text{ so that } \|a\| = 1.\]

To prove orthogonality it suffices to show that for a \( K \)-linear combination \( z = \sum_{i \in F} \lambda_i e_i \) \((F \subset I, F \text{ finite})\) we have

\[
\|a-z\| \geq \|a\|.
\]

If \( \max |\lambda_i| > 1 \) this is an obvious consequence of the strong triangle inequality so assume \( \max |\lambda_i| \leq 1 \). Then \( z \in B \) so \( |g(z)| < 1 \) and

\[
\|a\| = 1 = |g(a)| = |g(a-z)| \leq \|g\| \|a-z\| = \|a-z\|
\]

which finishes the proof.

Remark.

The above proof works for a strongly polar ([4], Definition 3.5) Banach space \( E \), in particular for any Banach space over a spherically complete \( K \).

Two corollaries obtain. The first one we shall need later on. The
second one is rather surprising.

**PROPOSITION 2.3.**

Let $A$ be an absolutely convex subset of a finite dimensional space $E$ over $K$. The following are equivalent.

1. For each $f \in E'$, $\max_{A} |f|$ exists.
2. For each seminorm $p$ on $E$, $\max_{A} p$ exists.
3. There exists a finite set $F \subset A$ with $A = \text{co } F$.

**Proof.**

$(\gamma)$ $\Rightarrow$ $(\beta)$ is easy, $(\beta)$ $\Rightarrow$ $(\alpha)$ is trivial. To prove $(\alpha)$ $\Rightarrow$ $(\gamma)$ we may assume $[A] = E$. Then $A$ is open (with respect to the unique Banach space topology of $E$). Lemma 2.2 (ii) yields a (finite) set $F$ with $A = \text{co } F = \text{co } F$ (the second equality because each convex set in $E$ is closed).

**LEMMA 2.4.**

Let $E, A$ be as in Lemma 2.2. Then $E$ is finite dimensional.

**Proof.**

Suppose $E$ is not finite dimensional. From Lemma 2.2 we infer that $E$ has an orthonormal base $\{e_i : i \in \mathbb{N}\}$ (with respect to the norm $|| \cdot ||$) and that $A = \text{co } \{e_i : i \in \mathbb{N}\}$ is the 'closed' unit ball of $(E, || \cdot ||)$. Now choose $\tau_1, \tau_2, \ldots$ in $K$ such that $0 < |\tau_1| < |\tau_2| < \ldots$, $\lim_{n \to \infty} |\tau_n| = 1$ and define $f \in E'$ by the formula

$$f(\sum_{i \in \mathbb{N}} \lambda_i e_i) = \sum_{i \in \mathbb{N}} \lambda_i \tau_i$$

($\lambda_i \in K, \lim_{i \to \infty} \lambda_i = 0$).

We have $\sup_{A} |f| = 1$ but $|f(x)| < 1$ for each $x \in A$, a contradiction.

The proof of the next Lemma is left to the reader.
LEMMA 2.5.
Let $A$ be an absolutely convex subset of a locally convex space $E$ over $K$. Suppose max $|f|$ exists for each $f \in E'$.

(i) For each $f \in E'$, max $|f|$ exists.

(ii) If $T : E \to F$ is a continuous linear map into a locally convex space $F$ over $K$ then, for each $f \in F'$, max $|f|$ exists.

LEMMA 2.6.
Let $E$ be a Banach space over $K$ with a base. Let $A \subset E$ be absolutely convex and suppose that for each $f \in E'$ the restriction of $|f|$ to $A$ has a maximum. Then $A$ is a compactoid.

Proof.
We may assume ([3], Corollary 3.7) that $E$ has an orthogonal base.

Suppose $A$ is not a compactoid. By [3], Theorem 4.38, $(\xi) \Rightarrow (a)$ there exists an orthogonal sequence $e_1, e_2, \ldots$ in $A$ with inf $\|e_n\| > 0$. Set $D := [e_1, e_2, \ldots]$, and choose a linear continuous projection $P : E \to D$ ([3], Corollary 3.18). By Lemma 2.5, for each $f \in D'$ the restriction of $|f|$ to $PA$ has a maximum. Also $PA \supset \text{co} \{e_1, e_2, \ldots\}$ is open in $D$ and $D$ is infinite dimensional. But this is impossible (Lemma 2.4).

We now formulate the main Theorem of this section.

THEOREM 2.7.
Let $E$ be a Banach space with a base over (a densely valued) $K$. For a nonempty subset $X$ of $E$ the following are equivalent.

(a) For each $f \in E'$, max $|f|$ exists.

(\beta) For each weakly continuous seminorm $p$, max $p$ exists.

(\gamma) For each weak neighbourhood $U$ of $0$ there exists a finite set $P \subset X$
such that $X \subseteq U + \text{co } F$.

(δ) For each continuous seminorm $p$ on $E$, $\max_{X} p$ exists.

(ε) For each neighbourhood $U$ of $0$ there exists a finite set $F \subseteq X$ such that $X \subseteq U + \text{co } F$.

**Proof.**

Let (*) be any of the statements (α) - (ε). It is not hard to see that, for a nonempty set $Y \subseteq E$,

(*) holds for $X := Y \Rightarrow$ (*) holds for $X := \text{co } Y$

Therefore, to prove Theorem 2.7, we may assume that $X$ is absolutely convex. The equivalences (β) $\Rightarrow$ (γ) and (δ) $\Rightarrow$ (ε) are proved in [5], Theorem 3.3. Obviously, (δ) $\Rightarrow$ (β) $\Rightarrow$ (α). We shall prove (α) $\Rightarrow$ (β) and (γ) $\Rightarrow$ (ε). Suppose (α) and let $p$ be a weakly continuous seminorm on $E$. Then $\text{Ker } p$ has finite codimension. Let $\pi_p : E \rightarrow E/\text{Ker } p$ be the quotient map and let $\overline{p}$ be the norm on $E/\text{Ker } p$ induced by $p$. For each $f \in (E/\text{Ker } p)'$, $\max |f|$ exists (Lemma 2.5). Then also (Proposition 2.3)

$$\max_{\pi_p(X)} \overline{p}$$

exists. But $\{p(a) : a \in X\} = \{\overline{p}(x) : x \in \pi_p(X)\}$ and therefore

$$\max_{X} p$$

exists and we have (β). Now suppose (γ). Lemma 2.6 tells us that $X$ is a compactoid. $E$ is a polar space (Proposition 1.1) so by [4], Theorem 5.12, the weak topology and the norm topology coincide on $X$. Let $U$ be an absolutely convex neighbourhood of $0$. There is a weak neighbourhood $V$ of $0$ with $V \cap X \subseteq U \cap X$. By (γ) there is a finite set $F \subseteq X$ such that $X \subseteq V + \text{co } F$. Then also $X \subseteq (V \cap X) + \text{co } F \subseteq U + \text{co } F$ and (ε) is proved.
Remark.
In the terminology of [4], Theorem 2.7 entails that weak c'-compactness implies c'-compactness. Compare [1], Proposition 3a, where it is proved that weak c-compactness implies c-compactness.

PROBLEM.
Are (a) - (e) equivalent for a nonempty subset X of a strongly polar ([4], Definition 3.5) Banach space E (in particular, an arbitrary Banach space E over a spherically complete K)? The following example shows that just 'E is a polar space' is not enough.

EXAMPLE.
Let K be not spherically complete, let $A = \{x \in \ell^\infty : \|x\| \leq 1\}$. Then for each $f \in (\ell^\infty)'$, $\max_A |f|$ exists although A is not a compactoid. [Since $(\ell^\infty)' \cong c_0$ ([3], Theorem 4.17) an $f \in (\ell^\infty)'$ has the form

$$x \mapsto \sum_{i \in \mathbb{N}} x_i a_i \quad (x \in \ell^\infty)$$

for some $a \in c_0$.]
§ 3. LOCALLY CONVEX SPACES OF COUNTABLE TYPE

Also in § 3 we assume that the valuation of \( K \) is dense. We shall extend the results of § 2 to locally convex spaces of countable type (i.e. for each continuous seminorm \( p \) the normed space \( E_p := E/\text{Ker} \ p \) with the norm induced by \( p \) is of countable type, see [4], Definition 4.3).

First we extend Lemma 2.6.

**Lemma 3.1.**

Let \( E \) be a locally convex space of countable type over \( K \). Let \( A \subset E \) be absolutely convex and suppose that for each \( f \in E' \) the restriction of \( f \) to \( A \) has a maximum. Then \( A \) is a compactoid.

**Proof.**

Let \( U \) be a neighbourhood of 0 in \( E \). There is a continuous seminorm \( p \) such that \( \{ x \in E : p(x) \leq 1 \} \subset U \). Let \( E_p^- \) be the completion of \( E_p \) (see above). The canonical map \( \pi_p : E \to E_p^- \subset E^-_p \) is continuous. By Lemma 2.5, for each \( f \in E^-_p \) the restriction of \( |f| \) to \( \pi_p(A) \) has a maximum. Now \( E^-_p \) is of countable type and therefore has a base ([3], Theorem 3.16) so we may apply Lemma 2.6 and conclude that \( \pi_p(A) \) is a compactoid in \( E^-_p \), hence in \( E_p \). Since \( \pi_p(U) \) is open in \( E_p \) there is a finite set \( F \subset E \) such that \( \pi_p(A) \subset \pi_p(U) + \text{co} \pi_p F \). We have

\[
A \subset U + \text{co} \ F + \text{Ker} \ p \subset U + \text{co} \ F.
\]

**Theorem 3.2.**

Let \( X \) be a nonempty subset of a locally convex space \( E \) of countable type over \( K \). The statements \((a) \sim (e)\) of Theorem 2.7 are equivalent.
Proof.
The proof of Theorem 2.7 applies with two modifications in the proof of $(y) \Rightarrow (e)$. The compactoidity of $X$ follows from Lemma 3.1 (rather than Lemma 2.6) and the fact that $E$ is a polar space is proved in [4], Theorem 4.4.
§ 4. DISCRETELY VALUED BASE FIELDS

One may wonder what happens to our results if we allow the valuation of \( K \) to be discrete. The next two Properties show the deviation from the previous theory.

**PROPOSITION 4.1.**

Let \( A \) be an absolutely convex subset of a locally convex space \( E \) over a discretely valued \( K \). The following are equivalent.

(a) \( A \) is a compactoid.

(b) Each continuous seminorm \( E \to [0,\infty] \) has a maximum on \( A \).

**Proof.**

[5], Remark following Theorem 3.3.

**PROPOSITION 4.2.**

Let \( A,E,K \) be as above. The following are equivalent.

(a) \( A \) is a compactoid for the weak topology.

(b) Each weakly continuous seminorm on \( E \) has a maximum in \( A \).

(c) For each \( f \in E' \), \( \max_{A} |f| \) exists.

(d) \( A \) is bounded.

**Proof.**

For (a) \( \Leftrightarrow \) (b) see [5], Remark following Theorem 3.3. (a) \( \Leftrightarrow \) (d) is proved in [4]. (b) \( \Rightarrow \) (c) is obvious. (c) implies weak boundedness of \( A \), hence boundedness ([4], Corollary 7.7), so that we have (c) \( \Rightarrow \) (d).
REFERENCES


