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A COMPLEMENTARY VARIANT OF C-COMPACTNESS
IN $p$-ADIC FUNCTIONAL ANALYSIS

by

W.H. SCHIKHOF

Report 8647
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ABSTRACT. An absolutely convex subset $A$ of a locally convex space $E$ over a nonarchimedean valued field is $c'$-compact in $E$ if each continuous seminorm on $E$, restricted to $A$, has a maximum. Various descriptions of $c'$-compactness are given revealing its close analogy to $c$-compactness (§ 4).

PRELIMINARIES. Throughout let $K$ be a nonarchimedean nontrivially valued field with valuation $|\cdot|$. For fundamentals on Banach spaces and locally convex spaces over $K$ we refer to [2], [5], [4]. For a subset $X$ of a locally convex space $E$ over $K$ we denote its absolutely convex hull by $\text{co } X$, its linear span by $[X]$. The closure of $X$ is $\overline{X}$. Instead of $\overline{\text{co } X}$ we write $\overline{\text{co } X}$. A convex set is an (additive) coset of an absolutely convex set.
§ 1. ELEMENTARY PROPERTIES

In § 1 let $E$ be a locally convex space over $K$ and let $A \subset E$ be absolutely convex. The next proposition will justify the use of the term 'c'-compact' rather than 'c'-compact in $E$.

**PROPOSITION 1.1.**

$A$ is c'-compact in $[A]$ if and only if $A$ is c'-compact in $E$.

**Proof.**

Only the 'if' part needs a proof. Let $A$ be c'-compact in $E$, let $p$ be a continuous seminorm on $[A]$. There is a continuous seminorm $q$ on $E$ such that $p \leq q$ on $[A]$. The formula

$$
\tilde{p}(x) = \inf_{y \in [A]} \max(p(y), q(x-y))
$$

defines a continuous seminorm $\tilde{p}$ on $E$ whose restriction to $[A]$ is $p$. It follows that $\max_A \tilde{p} = \max_A p$ exists so that $A$ is c'-compact in $[A]$.

**PROPOSITION 1.2.**

$A$ is c'-compact if and only if $\overline{A}$ is c'-compact.

**Proof.**

By the isosceles triangle principle

$$
\{p(a) \in (0,\infty) : a \in A\} = \{p(a) \in (0,\infty) : a \in \overline{A}\}
$$

for each continuous seminorm $p$.

We see that c'-compact sets need not be closed or complete. The next proposition furnishes examples of c'-compact sets.
PROPOSITION 1.3.

If \( X \subset E \) is precompact then \( \text{co} \ X \) is \( c' \)-compact.

Proof.

Let \( p \) be a continuous seminorm on \( E \). If \( z \in \text{co} \ X \) then there exist \( x_1, \ldots, x_n \in X \) and \( \lambda_1, \ldots, \lambda_n \in K \) with \( |\lambda_i| \leq 1 \) for each \( i \in \{1, \ldots, n\} \) such that \( z = \sum_{i=1}^{n} \lambda_i x_i \). Then

\[
p(z) \leq \max_{1 \leq i \leq n} p(\lambda_i x_i) \leq \max_p(x_i).
\]

It follows that \( \sup_{\text{co} \ X} p = \sup_{X} p \). We complete the proof by showing that \( \hat{p} \) has a maximum on \( X \). We may assume that \( s := \sup_{X} p > 0 \). Set

\[
U := \{x \in E : p(x) < \frac{s}{2}s\}.
\]

By precompactness there exist \( x_1, \ldots, x_n \in X \) such that

\[
X \subset \bigcup_{i=1}^{n} (x_i + U).
\]

For each \( i \in \{1, \ldots, n\} \) we have either \( p(x_i) < \frac{s}{2}s \) (then \( p < \frac{s}{2}s \) on \( x_i + U \)) or \( p(x_i) \geq \frac{s}{2}s \) (then \( p \) is constant on \( x_i + U \)). Hence,

\[
\sup_{X} p = \max_{i} p(x_i) = \max_{X} p.
\]

PROPOSITION 1.4.

Let \( F \) be a locally convex space over \( K \), let \( T : E \rightarrow F \) be a continuous linear map. If \( A \subset E \) is \( c' \)-compact then so is \( TA \).

Proof.

If \( p \) is a continuous seminorm on \( F \) then \( p \circ T \) is a continuous seminorm on \( E \).
Remark.

If $K$ is spherically complete, $\{\lambda \in K : |\lambda| < 1\}$ is $c$-compact but not $c'$-compact if the valuation is dense. If $K$ is not spherically complete, $\{\lambda \in K : |\lambda| \leq 1\}$ is $c'$-compact (and complete) but not $c$-compact.
§ 2. C'-COMPACTNESS IN BANACH SPACES

Throughout § 2, $E$ is a Banach space over $K$ with norm $\| \cdot \|$, and except for Corollary 2.8 $A$ is a $c'$-compact subset of $E$. Our first goal is to show that $[A]$ is of countable type (Proposition 2.3).

**Lemma 2.1.**

Let $p$ be a continuous seminorm on $E$, not vanishing on $A$. Then every $p$-orthogonal set in

$$ A_p := \{ x \in A : p(x) = \max_{A} p \} $$

is finite.

**Proof.**

We may assume $\max_{A} p = 1$. Let $\{ e_i : i \in I \}$ be a maximal $p$-orthogonal set in $A$. Suppose $\exists N \in I$; we derive a contradiction. $p$ is a norm on $D := [e_i : i \in I]$ and $\{ e_i : i \in I \}$ is a $p$-orthonormal (algebraic) base for $D$. Choose real numbers $\rho_1, \rho_2, \ldots$ such that $0 < \rho_1 < \rho_2 < \ldots$ and $\lim_{n \to \infty} \rho_n = 1$, and consider the seminorm $q$ on $D$ defined by the formula

$$ q(\sum_{i} \lambda_i e_i) = \max_{i \in I} |\lambda_i| \rho_i \quad (\lambda_i \in K, \{ i : \lambda_i \neq 0 \} \text{ finite}). $$

As $q \leq p$ on $D$, $q$ is continuous. Observe that $q(e_i) < 1$ for each $i \in I$.

Set $B := \text{co} \{ e_i : i \in I \}$. We have $\sup_{B} q = 1$, but $q(b) < 1$ for each $b \in B$. The formula

$$ \underline{q}(x) = \inf_{d \in D} \max(q(d), p(x-d)) $$

defines a continuous seminorm $\underline{q}$ on $E$ extending $q$ for which $\underline{q} \leq p$. We
shall arrive at the desired contradiction by showing that $\bar{q}$ does not have a maximum on $A$. As $B \subseteq A$ we have

$$1 = \sup_{B} q = \sup_{B} \bar{q} \leq \sup_{A} \bar{q} \leq \sup_{A} p = 1,$$

whence

$$\sup_{A} \bar{q} = 1.$$ 

Now we shall prove that $\bar{q}(a) < 1$ for each $a \in A$. If $a \in A \setminus \overline{p}$ then $\bar{q}(a) \leq p(a) < 1$ so assume $a \in A$. Then $a$ is not $p$-orthogonal to $D$ and there exists a finite set $F \subseteq I$ and a map $i \mapsto \lambda_{i} \in K$ ($i \in F$) such that

$$p(a - \sum_{i \in F} \lambda_{i} e_{i}) < p(a) = 1.$$ 

By the isosceles triangle principle, $p(\sum_{i \in F} \lambda_{i} e_{i}) = p(a) = 1$; $p$-orthonormality yields $\max_{i \in F} |\lambda_{i}| = 1$. Therefore,

$$q(\sum_{i \in F} \lambda_{i} e_{i}) \leq \max_{i \in F} |\lambda_{i}| q(e_{i}) < 1$$

leading to

$$\max(q(\sum_{i \in F} \lambda_{i} e_{i}), p(a - \sum_{i \in F} \lambda_{i} e_{i})) < 1.$$ 

Using the definition of $\bar{q}$ we arrive at $\bar{q}(a) < 1$.

**COROLLARY 2.2.**

Let $p$ be a continuous seminorm on $E$, not vanishing on $A$. There is a finite dimensional subspace $D$ of $E$ such that for the quotient seminorm $q$ defined by

$$q(x) = \inf \{p(x-d) : d \in D\}$$

we have
\[ \max q < \max p. \]

**Proof.**

Let \( \{e_1, \ldots, e_n\} \) be a maximal \( p \)-orthogonal set in \( A \) (Lemma 2.1) and \( p \)-orthogonal set in \( A \) (Lemma 2.1) and set \( D := [e_1, \ldots, e_n] \). Suppose \( x \in A \) and \( q(x) = \max p. \) Then \( x \in A \). For each \( d \in D \) we have

\[ p(x-d) \geq q(x) = \max p \geq p(x) \]

so that \( x \) is \( p \)-orthogonal to \( [e_1, \ldots, e_n] \), a contradiction. Hence,

\[ q(x) < \max p \text{ for each } x \in A. \]

**PROPOSITION 2.3.**

\([A] \text{ is of countable type}.\)

**Proof.**

For each subspace \( D \) of countable type the formula

\[ p_D(x) = \inf \{ \|x-d\| : d \in D \} \]

defines a continuous seminorm \( p_D \) on \( E \). We set

\[ x_D := \max_{A} p_D \]

\[ R = \{ x_D : D \text{ is a subspace of countable type} \}, \]

Then \( R \subseteq [0, \infty) \). We have (i), (ii) below.

(i) **If** \( t_1, t_2, \ldots \text{ are in } R \) **then there exists a** \( t \in R \text{ with } t \leq \inf t_n. \)

**Proof.** Let \( t_n = x_{D_n} \) (\( n \in \mathbb{N} \)) where each \( D_n \) is a subspace of countable type. Then \( D := [D_n : n \in \mathbb{N}] \) is of countable type. Obviously, \( p_D \leq p_{D_n} \) for each \( n \in \mathbb{N} \), so \( t := x_D = \inf x_{D_n} = \inf t_n. \)

(ii) **If** \( t \in R, t > 0 \text{ then there exists an } s \in R \text{ with } s < t. \)
Proof. Let $t = r_D$ where $D$ is a subspace of countable type. By Corollary 2.2 there is a finite dimensional space $F \subseteq E$ such that

$$\max_{A} q < \max_{A} p_D = r_D$$

where

$$q(x) = \inf \{ p_D(x-y) : y \in F \} \quad (x \in E)$$

It is easily seen that

$$q(x) = \inf \{ \|x-z\| : z \in F+D \} \quad (x \in E)$$

i.e. $q = p_{F+D}$. Now $F+D$ is of countable type and (ii) is proved with $s := r_{D+F}$. From (i), (ii) above we obtain $0 \in R$. So there exists a subspace $D$ of countable type with $r_D = 0$ i.e. $p_D = 0$ on $A$ implying $[A] \subseteq D$. It follows that $[A]$ is of countable type.

Our next step is to prove that $A$ is a compactoid (Corollary 2.6).

**Lemma 2.4.**

Every $\| \| -$orthogonal sequence in $A$ tends to 0.

Proof.

We may assume $E = \overline{[A]}$. Then $E$ is of countable type (Proposition 2.3).

Suppose we had an orthogonal sequence $e_1, e_2, \ldots$ in $A$ with $s := \inf \|e_n\| > 0$. Set $D := \overline{[e_1, e_2, \ldots]}$. By [2], Theorem 3.16 (v) there exists a continuous linear projection $P : E \to D$. By Proposition 1.4 $PA$ is $c'$-compact. We have $e_n \in PA$ for each $n \in \mathbb{N}$ and $\overline{[PA]} = D$.

We therefore may also assume that $E = D$ i.e. that $\overline{[A]} = \overline{[e_1, e_2, \ldots]}$. 

For each \( n \in \mathbb{N} \) set

\[
D_n := [e_1, \ldots, e_n]
\]

\[
p_n(x) = \inf \{ ||x-d|| : d \in D_n \} \quad (x \in E)
\]

Each \( p_n \) is a continuous seminorm, \( p_1 \geq p_2 \geq \ldots \) and by assumption,

\[
\lim_{n \to \infty} p_n(x) = 0 \quad \text{for each } x \in [e_1, e_2, \ldots] = E.
\]

By orthogonality,

\[
p_n(e_n) = ||e_n|| \geq s \quad \text{for each } n \in \mathbb{N}.
\]

Let

\[
s_n := \max_{\lambda} p_n \quad (n \in \mathbb{N})
\]

Then \( s_n \geq s \) for each \( n \). Choose real numbers \( \rho_1, \rho_2, \ldots \) such that

\[
0 < \rho_1 < \rho_2 < \ldots, \quad \lim_{n \to \infty} \rho_n = 1.
\]

The formula

\[
p(x) = \sup_{n \in \mathbb{N}} s_n^{-1} p_n(x) \rho_n \quad (x \in E)
\]

defines a seminorm on \( E \). For each \( x \in E \) we have

\[
s_n^{-1} p_n(x) \rho_n \leq s_n^{-1} p_1(x) = s_n^{-1} ||x|| \quad (n \in \mathbb{N})
\]

so \( p \) is continuous. For each \( n \in \mathbb{N} \) we have

\[
\max_{x \in A} s_n^{-1} p_n(x) \rho_n = \rho_n
\]

yielding \( \sup_{A} p = 1 \).

By \( c' \)-compactness there is an \( a \in A \) with \( p(a) = 1 \). As, for each \( n \),

\[
s_n^{-1} p_n(a) \rho_n \leq \rho_n \neq 1 \quad \text{there must be infinitely many } n \in \mathbb{N} \text{ for which}
\]

\[
s_n^{-1} p_n(a) \rho_n \geq \frac{1}{2}.
\]

Then also \( p_n(a) \rho_n \geq \frac{1}{2} s \) for infinitely many \( n \) which is in conflict to \( \lim_{n \to \infty} p_n(a) = 0 \).

**PROPOSITION 2.5.**

Let \( E \) be of countable type. For each \( s \in (0, 1) \) there exist a norm
||' on E such that s ||x'|| \leq ||x|| \leq ||x||' for all x \in E and for which (E, ||'||) has an orthogonal base.

Proof.
The statement is an easy consequence of [2], Theorem 3.16 (ii).

COROLLARY 2.6.
A is a compactoid.

Proof.
We may assume that E is of countable type. By Proposition 2.5 we may even assume that E has an orthogonal base. Now apply Lemma 2.4 and [2], Theorem 4.38, \( (\xi) \Rightarrow (\alpha) \).

THEOREM 2.7.
For each \( s \in (0,1) \) there exists an \( s \)-orthogonal sequence \( e_1, e_2, \ldots \) in A for which \( \lim_{n \to \infty} e_n = 0 \) such that

$$\text{co} \{e_1, e_2, \ldots\} \subset A \subset \overline{\text{co} \{e_1, e_2, \ldots\}}.$$  

If, in addition, \( [A] \) has an orthogonal base then the sequence \( e_1, e_2, \ldots \) can be chosen to be orthogonal.

Proof.
We may assume that \( [A] = E \). By Proposition 2.5 it suffices to prove only the second statement. We shall construct an orthogonal sequence \( e_1, e_2, \ldots \) in A such that for each \( a \in A \) and \( n \in \mathbb{N} \) there exists a \( b \in \text{co} \{e_1, \ldots, e_n\} \) with \( ||a-b|| \leq ||e_{n+1}|| \). (This proves the Theorem since, by Lemma 2.4, \( \lim_{n \to \infty} ||e_n|| = 0 \).) There is an \( e_1 \in A \) with

$$||e_1|| = \max \{||x|| : x \in A\}. \text{By [2], Lemma 4.35}, E \text{ is the orthogonal}$$
direct sum of $K_e^1$ and some subspace $D_1$. For each $a \in A$, $a = \lambda_1 e_1 + d_1$

$(\lambda_1 \in K_1, d_1 \in D)$, we have

$$\|a\| = \max (\|\lambda_1 e_1\|, \|d_1\|) \geq \|\lambda_1 e_1\| = \|\lambda_1\| e_1$$

so that $|\lambda_1| \leq 1$ (if $e_1 = 0$, choose $\lambda_1 = 0$). It follows that $d_1 \in D_1 \cap A$. Therefore, $A$ decomposes into an orthogonal sum of $K_e^1$ and $D_1 \cap A$, so $D_1 \cap A$ is $c'$-compact by Proposition 1.4. There exists an $e_2 \in D_1 \cap A$ with $\|e_2\| = \max \{\|x\| : x \in D_1 \cap A\}$. Then

$$\|a - \lambda_1 e_1\| \leq \|e_2\|.$$ In its turn, $D_1$ decomposes into an orthogonal sum of $K_e^2$ and a space $D_2$ such that $D_2 \cap A$ is $c'$-compact. Let $e_3 \in D_2 \cap A$ with $\|e_3\| = \max \{\|x\| : x \in D_2 \cap A\}$. Then $\|a - \lambda_1 e_1 - \lambda_2 e_2\| \leq \|e_3\|$ for some $\lambda_2 \in K$, $|\lambda_2| \leq 1$, etc.

**Corollary 2.8.** Let $A$ be an absolutely convex subset of a $K$-Banach space. The following statements (a)–(δ) are equivalent.

(a) $A$ is $c'$-compact.

(b) There exists a compact set $X$ with $co X \subseteq A \subseteq co X$.

(γ) There exists a sequence $e_1, e_2, \ldots$ in $A$ with $\lim_{n \to \infty} e_n = 0$ such that $co \{e_1, e_2, \ldots\} \subseteq A \subseteq co \{e_1, e_2, \ldots\}$.

(δ) For each $s \in (0, 1)$ there exists an $s$-orthogonal sequence $e_1, e_2, \ldots$ in $A$ for which $co \{e_1, e_2, \ldots\} \subseteq A \subseteq co \{e_1, e_2, \ldots\}$.

If $K$ is spherically complete, (a)–(δ) are equivalent to:

(c) There exists an orthogonal sequence $e_1, e_2, \ldots$ in $A$ such that $co \{e_1, e_2, \ldots\} \subseteq A \subseteq co \{e_1, e_2, \ldots\}$.

**Proof.**

(c) $\Rightarrow$ (δ) $\Rightarrow$ (γ) $\Rightarrow$ (β) are obvious. (β) $\Rightarrow$ (α) follows from Propositions 1, 2 and 1.3. The first part of Theorem 2.7 yields (α) $\Rightarrow$ (δ). If $K$ is spherically complete each Banach space of countable type has an
orthogonal base ([2], Lemma 5.5) and \((a) \Rightarrow (c)\) is a consequence of
the second part of Theorem 2.7.
§ 3. OTHER CHARACTERIZATIONS OF C'-COMPACTNESS

In this section there is no need to restrict ourselves to Banach spaces so in § 3, let E be a locally convex space over K.

DEFINITION 3.1.
An absolutely convex subset $A \subset E$ is a pure compactoid if for each neighbourhood $U$ of 0 there exist a finite set $F \subset A$ such that $A \subset U + \text{co } F$.

(The difference with the definition of 'ordinary' compactoidity ([2], p. 134) lies in the fact that we require $F \subset A$ rather than $F \subset E$.)

DEFINITION 3.2.
A function $\phi : E \rightarrow \mathbb{R}$ is convex if $n \in \mathbb{N}$, $x_1, \ldots, x_n \in E$, $\lambda_1, \ldots, \lambda_n \in K$, $|\lambda_i| \leq 1$ for each $i$, $\sum \lambda_i = 1$ imply

$$\phi(\sum \lambda_i x_i) \leq \max_{i=1}^n |\lambda_i| \phi(x_i).$$

(Example: $x \mapsto p(x-a)$ for $a \in E$ and a seminorm $p$.)

THEOREM 3.3.
For an absolutely convex $A \subset E$ the following statements are equivalent.

(a) $A$ is a pure compactoid.

(β) If $U$ is a covering of $A$ by (convex) open sets then there exist $n \in \mathbb{N}$ and $U_1, \ldots, U_n \in U$ such that $A \subset \text{co } (\bigcup_{i=1}^n U_i)$.

(γ) Let $U_1 \subset U_2 \subset \ldots$ be open convex sets covering $A$. Then $A \subset U_n$ for some $n$.

(δ) Each continuous convex function $\phi : E \rightarrow \mathbb{R}$, restricted to $A$, has a maximum.
(e) A is $c^\prime$-compact.

Proof.

(a) $\Rightarrow$ (b). There is a $U_0 \in U$ with $0 \in U_0$. There exist $x_1, \ldots, x_m \in A$ such that $A \subseteq U_0 + \text{co}\{x_1, \ldots, x_m\}$. Let $U_1, \ldots, U_m \subseteq U$ be with $x_i \in U_i$ for each $i \in \{1, \ldots, m\}$. Then $A \subseteq U_0 + \text{co}(U_1 \cup \ldots \cup U_m) \subseteq \text{co}(U_1 \cup \ldots \cup U_m)$. To prove

$\Rightarrow$ (b) we may assume that $0 \in U_1$ so that all $U_i$ are absolutely convex. By (b) there exists a finite set $F \subseteq \mathbb{N}$ such that $A \subseteq \text{co}(U_{i_1} \cap \ldots \cap U_{i_k})$. But then $A \subseteq \text{co}(U_n) = U_n$ where $n = \max F$.

(γ) $\Rightarrow$ (b). Let $s = \sup \phi$ (possibly $s = \infty$). Suppose $s$ is not a value of $\phi|_A$. Choose $s_1 < s_2 < \ldots$, $\lim_{n \to \infty} s_n = s$ and set

$U_n := \{x \in E : \phi(x) < s_n\}$

(n $\in \mathbb{N}$)

Each $U_n$ is open. As $\phi$ is convex, $U_n$ is convex. Further we have $U_1 \subseteq U_2 \subseteq \ldots$ and, by assumption, the $U_n$ cover $A$. By (γ), $A \subseteq U_n$ for some $n$ implying $\phi < s_n$ on $A$, a contradiction.

(δ) $\Rightarrow$ (e) is trivial.

Finally we prove (e) $\Rightarrow$ (a). First, assume that $E$ is a Banach space.

From Corollary 2.8 we obtain a compact set $X \subseteq E$ with $\text{co} X \subseteq A \subseteq \overline{\text{co} X}$.

Let $U$ be an absolutely convex neighbourhood of $0$ in $E$. By compactness there exist $x_1, \ldots, x_n \in X$ such that $X \subseteq \bigcup_{i=1}^{n} (x_i + U)$. Then

$X \subseteq U + \text{co}\{x_1, \ldots, x_n\}$. As the latter set is an open additive subgroup of $E$ it is also closed and we have

$A \subseteq \overline{\text{co} X} = U + \text{co}\{x_1, \ldots, x_n\}$.

Now let $E$ be a locally convex space, and let $U$ be an absolutely convex neighbourhood of $0$ in $E$. There is a continuous seminorm $p$ such that
{x ∈ E : p(x) ≤ 1} ⊂ U. Let π_p : E → E_p be the canonical quotient map, where E_p is the completion of the space E_p := E/Ker p with the norm induced by p. By Proposition 1.4 the set π_p(A) is c'-compact and we just have proved that it is a pure compactoid in E_p^\sim, hence in E_p.

Since π_p(U) is open in E_p there exists a finite set F ⊂ π_p(A) such that π_p(A) ⊂ π_p(U) + co F. Choose a finite set F' ⊂ A such that π_p(F') = F. We then have π_p(A) ⊂ π_p(U + co F') and, as Ker π_p ⊂ U,

A ⊂ U + co F' + Ker π_p ⊂ U + co F'.

Remark.

The following statements are easy to prove.

(i) If the valuation of K is discrete the properties (α) - (ε) of Theorem 3.3 are equivalent to 'A is a compactoid'.

(ii) If K is locally compact the properties (α) - (ε) of Theorem 3.3 are equivalent to 'A is precompact'.
§ 4. C'-COMPACTNESS VERSUS C-COMPACTNESS

First we extend Theorem 3.3 to convex sets. For a subset X of a K-linear space, let c(X) be its convex hull.

THEOREM 4.1.

Let C be a convex subset of a locally convex space E over K. The following are equivalent.

(a) For each neighbourhood of 0 in E there exists a finite set F ⊂ C with C ⊂ U+c(F).

(b) If U is a covering of C by (convex) open sets then there exist \( n \in \mathbb{N} \) and \( U_1, \ldots, U_n \in U \) such that \( C \subset c(\bigcup_{i=1}^{n} U_i) \).

(c) Let \( U_1 \subset U_2 \subset \ldots \) be open convex sets covering C. Then \( C \subset U_n \) for some n.

(d) Each continuous convex function \( \phi : E \to \mathbb{R} \), restricted to C, has a maximum.

(e) For each continuous seminorm p and each \( a \in C \), \( \max \{ p(x-a) : x \in C \} \) exists.

Proof.

Straightforward.

A convex set \( C \subset E \) is \( c'\)-compact if it satisfies (a) - (e) of Theorem 4.1.

The following theorem explains the term 'complementary' in the title of this paper. (Compare Theorem 4.1.)

THEOREM 4.2.

Let C be a convex subset of a Banach space E over K. The following are
equivalent.

(β)' If $C$ is a collection of closed convex sets in $E$ such that $\cap C$

--- does not meet $C$ then there exist $n \in \mathbb{N}$ and $C_1, \ldots, C_n \in C$ such

--- that $C \cap \cap_{i=1}^n C_n = \emptyset$.

(γ)' Let $C_1 \supset C_2 \supset \ldots$ be closed convex sets in $E$ such that

--- $C \cap \cap_{n=1}^\infty C_n = \emptyset$. Then $C \cap C_n = \emptyset$ for some $n$.

(δ)' Each continuous convex function $\phi : E \to \mathbb{R}$, restricted to $C$, has

--- a minimum.

(ε)' $C$ is $c$-compact.

Proof.

(ε)' $\Rightarrow$ (β)' is just the definition of $c$-compactness ([4]), (β)' $\Rightarrow$ (γ)'

--- is obvious. The proof of (γ)' $\Rightarrow$ (δ)' runs similar to the one of

--- (γ) $\Rightarrow$ (δ) of Theorem 3.3. To prove (δ)' $\Rightarrow$ (ε)' it suffices, by [1]

--- Theorem 6.15 (ζ)' $\Rightarrow$ (α), to show that $C$ is spherically complete

--- relative to any norm $\| \|$ defining the topology of $E$. Thus, let

--- $B_1 \supset B_2 \supset \ldots$ be balls in $C$ where

--- $B_n = \{ x \in C : \| x - c_n \| \leq r_n \}$ (n $\in \mathbb{N}$)

--- for some $c_1, c_2, \ldots$ in $C$ and $r_1 \geq r_2 \geq \ldots$. Let $(E^-, \|\|)$ be the

--- spherical completion ([2], Theorem 4.43) of $(E, \|\|)$ and consider

--- for each $n \in \mathbb{N}$

--- $B_n^- := \{ x \in E^- : \| x - c_n \| \leq r_n \}$

--- These $B_n^-$ form a nested sequence of balls in $E^-$ so there exists a

--- $z \in \cap B_n^-$. The function $\phi : x \mapsto \| z - x \| (x \in E)$ is convex and attains

--- a minimum on $C$, say in $c \in C$. As $\phi(c_n) \leq r_n$ (n $\in \mathbb{N}$) we have

--- $\phi(c) \leq \inf_{n} r_n$. For each $n \in \mathbb{N}$
\[ \|c_{-n}\| \leq \max(\|c-z\|, \|z_{-n}\|) \leq \max(i_{c}(z), i_{c}(c)) \leq \varepsilon_{n}. \]

We see that \( c \in B_{n} \) for each \( n \) and it follows that \( A \) is spherically complete for \( \| \| \).

Remarks.

(1) Theorem 4.2 is only of interest if \( K \) is spherically complete ([4], (2.1)). I do not know whether the properties \((\beta)' - (\varepsilon)'\) are equivalent for a convex set \( C \) in a locally convex space \( E \). Of course one has some obvious implications.

(2) The notion of \( c \)-compactness may be viewed as a 'convexification' of the intersection property for closed sets in a compact space, whereas \( c' \)-compactness can be seen as a 'convexification' of the 'open covering' definition of compactness. (Theorem 4.1 (\( \beta \)), Theorem 4.2 (\( \beta' \)).)

(3) In a future paper [3] we shall discuss the relation between weak and strong \( c' \)-compactness.

REFERENCES


