A COMPLEMENTARY VARIANT OF C-COMPACTNESS
IN $p$-ADIC FUNCTIONAL ANALYSIS

by

W.H. SCHIKHOF

Report 8647
October 1986

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ABSTRACT. An absolutely convex subset $A$ of a locally convex space $E$ over a nonarchimedean valued field is $c'$-compact in $E$ if each continuous seminorm on $E$, restricted to $A$, has a maximum. Various descriptions of $c'$-compactness are given revealing its close analogy to $c$-compactness (§ 4).

PRELIMINARIES. Throughout let $K$ be a nonarchimedean nontrivially valued field with valuation $|\cdot|$. For fundamentals on Banach spaces and locally convex spaces over $K$ we refer to [2], [5], [4]. For a subset $X$ of a locally convex space $E$ over $K$ we denote its absolutely convex hull by $\text{co } X$, its linear span by $[X]$. The closure of $X$ is $\overline{X}$. Instead of $\overline{\text{co } X}$ we write $\text{co } X$. A convex set is an (additive) coset of an absolutely convex set.
§ 1. ELEMENTARY PROPERTIES

In § 1 let $E$ be a locally convex space over $K$ and let $A \subset E$ be absolutely convex. The next proposition will justify the use of the term 'c'-compact' rather than 'c'-compact in $E$.

PROPOSITION 1.1.

$A$ is c'-compact in $[A]$ if and only if $A$ is c'-compact in $E$.

Proof.

Only the 'if' part needs a proof. Let $A$ be c'-compact in $E$, let $p$ be a continuous seminorm on $[A]$. There is a continuous seminorm $q$ on $E$ such that $p \leq q$ on $[A]$. The formula

$$\bar{p}(x) = \inf_{y \in [A]} \max(p(y), q(x-y))$$

defines a continuous seminorm $\bar{p}$ on $E$ whose restriction to $[A]$ is $p$. It follows that $\max \bar{p} = \max p$ exists so that $A$ is c'-compact in $[A]$.

PROPOSITION 1.2.

$A$ is c'-compact if and only if $\overline{A}$ is c'-compact.

Proof.

By the isosceles triangle principle

$$\{p(a) \in (0,\infty) : a \in A\} = \{p(a) \in (0,\infty) : a \in \overline{A}\}$$

for each continuous seminorm $p$.

We see that c'-compact sets need not be closed or complete. The next proposition furnishes examples of c'-compact sets.
PROPOSITION 1.3.

If $X \subset E$ is precompact then $\co X$ is $c'$-compact.

Proof.

Let $p$ be a continuous seminorm on $E$. If $z \in \co X$ then there exist $x_1, \ldots, x_n \in X$ and $\lambda_1, \ldots, \lambda_n \in K$ with $|\lambda_i| \leq 1$ for each $i \in \{1, \ldots, n\}$ such that $z = \sum_{i=1}^{n} \lambda_i x_i$. Then

$$p(z) \leq \max_i p(\lambda_i x_i) \leq \max_i p(x_i).$$

It follows that $\sup_{\co X} p = \sup_{X} p$. We complete the proof by showing that $\tilde{p}$ has a maximum on $X$. We may assume that $s := \sup_{X} p > 0$. Set

$$U := \{x \in E : p(x) < \frac{s}{2}\}.$$

By precompactness there exist $x_1, \ldots, x_n \in X$ such that

$$X \subset \bigcup_{i=1}^{n} (x_i + U).$$

For each $i \in \{1, \ldots, n\}$ we have either $p(x_i) < \frac{s}{2}$ (then $p < \frac{s}{2}$ on $x_i + U$) or $p(x_i) \geq \frac{s}{2}$ (then $p$ is constant on $x_i + U$). Hence,

$$\sup_{X} p = \max_{i \in X} p(x_i) = \max_{X} p.$$

PROPOSITION 1.4.

Let $F$ be a locally convex space over $K$, let $T : E \rightarrow F$ be a continuous linear map. If $A \subset E$ is $c'$-compact then so is $TA$.

Proof.

If $p$ is a continuous seminorm on $F$ then $p \circ T$ is a continuous seminorm on $E$. 
Remark.

If $K$ is spherically complete, $\{\lambda \in K : |\lambda| < 1\}$ is $c$-compact but not $c'$-compact if the valuation is dense. If $K$ is not spherically complete, $\{\lambda \in K : |\lambda| \leq 1\}$ is $c'$-compact (and complete) but not $c$-compact.
§ 2. C'-COMPACTNESS IN BANACH SPACES

Throughout § 2, E is a Banach space over K with norm \( \| \cdot \| \), and except for Corollary 2.8 A is a c'-compact subset of E. Our first goal is to show that \([A]\) is of countable type (Proposition 2.3).

**Lemma 2.1.**

Let \( p \) be a continuous seminorm on E, not vanishing on A. Then every \( p \)-orthogonal set in

\[ A_p := \{ x \in A : p(x) = \max_{A} p \} \]

is finite.

**Proof.**

We may assume \( \max p \neq 1 \). Let \( \{ e_i : i \in I \} \) be a maximal \( p \)-orthogonal set in \( A_p \). Suppose \( \exists N_1 \in I \); we derive a contradiction. \( p \) is a norm on \( D := [e_i : i \in I] \) and \( \{ e_i : i \in I \} \) is a \( p \)-orthonormal (algebraic) base for \( D \). Choose real numbers \( \rho_1, \rho_2, \ldots \) such that \( 0 < \rho_1 < \rho_2 < \ldots \) and \( \lim_{n \to \infty} \rho_n = 1 \), and consider the seminorm \( q \) on \( D \) defined by the formula

\[ q(\sum_{i} \lambda_i e_i) = \max_{i \in N} \max |\lambda_i| \rho_i \quad (\lambda_i \in K, \{ i : \lambda_i \neq 0 \} \text{ finite}). \]

As \( q \leq p \) on \( D \), \( q \) is continuous. Observe that \( q(e_i) < 1 \) for each \( i \in I \).

Set \( B := \text{co} \{ e_i : i \in I \} \). We have \( \sup_B q = 1 \), but \( q(b) < 1 \) for each \( b \in B \). The formula

\[ \overline{q}(x) = \inf_{d \in D} \max(q(d), p(x-d)) \]

defines a continuous seminorm \( \overline{q} \) on E extending \( q \) for which \( \overline{q} \leq p \). We
shall arrive at the desired contradiction by showing that \( \overline{q} \) does not have a maximum on \( A \). As \( B \subset A \) we have

\[
1 = \sup_B q = \sup_B \overline{q} \leq \sup_A \overline{q} \leq \sup_A p = 1,
\]

whence

\[
\sup_A \overline{q} = 1.
\]

Now we shall prove that \( \overline{q}(a) < 1 \) for each \( a \in A \). If \( a \in A \setminus p \) then \( \overline{q}(a) \leq p(a) < 1 \) so assume \( a \in A_p \). Then \( a \) is not \( p \)-orthogonal to \( D \) and there exists a finite set \( F \subset I \) and a map \( i \mapsto \lambda_i \in K \) \((i \in F)\) such that

\[
p(a - \sum_{i \in F} \lambda_i e_i) < p(a) = 1
\]

By the isosceles triangle principle, \( p(\sum_{i \in F} \lambda_i e_i) = p(a) = 1 \);

\( p \)-orthonormality yields \( \max_i |\lambda_i| = 1 \). Therefore,

\[
q(\sum_{i \in F} \lambda_i e_i) \leq \max_i |\lambda_i| \cdot q(e_i) < 1
\]

leading to

\[
\max (q(\sum_{i \in F} \lambda_i e_i), p(a - \sum_{i \in F} \lambda_i e_i)) < 1.
\]

Using the definition of \( \overline{q} \) we arrive at \( \overline{q}(a) < 1 \).

**COROLLARY 2.2.**

Let \( p \) be a continuous seminorm on \( E \), not vanishing on \( A \). There is a finite dimensional subspace \( D \) of \( E \) such that for the quotient seminorm \( q \) defined by

\[
q(x) = \inf \{ p(x-d) : d \in D \}
\]

we have
\[ \max q < \max p. \]

**Proof.**

Let \( \{e_1, \ldots, e_n\} \) be a maximal \( p \)-orthogonal set in \( A_p \) (Lemma 2.1) and set \( D := [e_1, \ldots, e_n] \). Suppose \( x \in A \) and \( q(x) = \max_p A \). Then \( x \in A_p \). For each \( d \in D \) we have

\[ p(x - d) \geq q(x) = \max_p A \]

so that \( x \) is \( p \)-orthogonal to \( [e_1, \ldots, e_n] \), a contradiction. Hence,

\[ q(x) < \max p \text{ for each } x \in A. \]

**PROPOSITION 2.3.**

\( [A] \) is of countable type.

**Proof.**

For each subspace \( D \) of countable type the formula

\[ p_D(x) = \inf \{ \|x - d\| : d \in D \} \]

defines a continuous seminorm \( p_D \) on \( E \). We set

\[ x_D := \max_{A} p_D \]

\[ R = \{ x_D : D \text{ is a subspace of countable type} \}, \]

Then \( R \subset [0, \infty) \). We have (i), (ii) below.

(i) **If** \( t_1, t_2, \ldots \) **are in** \( R \) **then there exists a** \( t \in R \) **with** \( t \leq \inf t_n \).

**Proof.** Let \( t_n = x_{D_n} \) \( (n \in \mathbb{N}) \) where each \( D_n \) is a subspace of countable type. Then \( D := [D_n : n \in \mathbb{N}] \) is of countable type. Obviously, \( p_D \leq p_{D_n} \) for each \( n \in \mathbb{N} \), so \( t := x_D \leq \inf x_{D_n} = \inf t_n \).

(ii) **If** \( t \in R, t > 0 \) **then there exists an** \( s \in R \) **with** \( s < t \).
Proof. Let \( t = r_D \) where \( D \) is a subspace of countable type. By Corollary 2.2 there is a finite dimensional space \( F \subset E \) such that

\[
\max_A q < \max_A p_D = r_D
\]

where

\[
q(x) = \inf \{ p_D(x-y) : y \in F \} \quad (x \in E)
\]

It is easily seen that

\[
q(x) = \inf \{ \|x-z\| : z \in F+D \} \quad (x \in E)
\]

i.e. \( q = p_{F+D} \). Now \( F+D \) is of countable type and (ii) is proved with \( s := r_{D+F} \).

From (i), (ii) above we obtain \( 0 \in R \). So there exists a subspace \( D \) of countable type with \( r_D = 0 \) i.e. \( p_D = 0 \) on \( A \) implying \( [A] \subset D \). It follows that \( [A] \) is of countable type.

Our next step is to prove that \( A \) is a compactoid (Corollary 2.6).

**Lemma 2.4.**

Every \( \| \|_{-} \) -orthogonal sequence in \( A \) tends to 0.

Proof.

We may assume \( E = [A] \). Then \( E \) is of countable type (Proposition 2.3). Suppose we had an orthogonal sequence \( e_1, e_2, \ldots \) in \( A \) with

\[
s := \inf \| e_n \| > 0.
\]

Set \( D := [e_1, e_2, \ldots] \). By [2], Theorem 3.16 (v) there exists a continuous linear projection \( P : E \to D \). By Proposition 1.4 \( PA \) is \( c' \)-compact. We have \( e_n \in PA \) for each \( n \in N \) and \( [PA] = D \).

We therefore may also assume that \( E = D \) i.e. that \( [A] = [e_1, e_2, \ldots] \).
For each \( n \in \mathbb{N} \) set

\[
D_n := [e_1, \ldots, e_n]
\]

\[
p_n(x) = \inf \{ \|x-d\| : d \in D_n \} \quad (x \in E)
\]

Each \( p_n \) is a continuous seminorm, \( p_1 \geq p_2 \geq \ldots \) and by assumption, \( \lim_{n \to \infty} p_n(x) = 0 \) for each \( x \in [e_1, e_2, \ldots] = E \). By orthogonality,

\[
p_n(e_n) = \|e_n\| \geq s \quad \text{for each } n \in \mathbb{N}.
\]

Let

\[
s_n := \max_{A} \max_{n} p_n \quad (n \in \mathbb{N})
\]

Then \( s_n \geq s \) for each \( n \). Choose real numbers \( \rho_1, \rho_2, \ldots \) such that

\[
0 < \rho_1 < \rho_2 < \ldots, \lim_{n \to \infty} \rho_n = 1.
\]

The formula

\[
p(x) = \sup_{n \in \mathbb{N}} s_n^{-1} p_n(x) \rho_n \quad (x \in E)
\]

defines a seminorm on \( E \). For each \( x \in E \) we have

\[
s_n^{-1} p_n(x) \rho_n \leq s_n^{-1} p_1(x) = s_n^{-1}\|x\| \quad (n \in \mathbb{N})
\]

so \( p \) is continuous. For each \( n \in \mathbb{N} \) we have

\[
\max_{x \in A} s_n^{-1} p_n(x) \rho_n = \rho_n
\]

yielding \( \sup_{A} p = 1 \).

By \( c' \)-compactness there is an \( a \in A \) with \( p(a) = 1 \). As, for each \( n \),

\[
s_n^{-1} p_n(a) \rho_n \leq \rho_n \neq 1 \quad \text{there must be infinitely many } n \in \mathbb{N} \text{ for which}
\]

\[
s_n^{-1} p_n(a) \rho_n \geq \frac{1}{2}. \quad \text{Then also } p_n(a) \rho_n \geq \frac{1}{2} \text{ for infinitely many } n \text{ which}
\]

is in conflict to \( \lim_{n \to \infty} p_n(a) = 0 \).

**PROPOSITION 2.5.**

Let \( E \) be of countable type. For each \( s \in (0,1) \) there exist a norm
on $E$ such that $s \|x\| < \|x\|$ for all $x \in E$ and for which $(E, \|\cdot\|')$ has an orthogonal base.

Proof.

The statement is an easy consequence of [2], Theorem 3.16 (ii).

COROLLARY 2.6.

$A$ is a compactoid.

Proof.

We may assume that $E$ is of countable type. By Proposition 2.5 we may even assume that $E$ has an orthogonal base. Now apply Lemma 2.4 and [2], Theorem 4.38, $(n) \Rightarrow (a)$.

THEOREM 2.7.

For each $s \in (0,1)$ there exists an $s$-orthogonal sequence $e_1, e_2, \ldots$ in $A$ for which $\lim_{n \to \infty} e_n = 0$ such that

$$\text{co} \{e_1, e_2, \ldots\} \subseteq A \subseteq \overline{\text{co}} \{e_1, e_2, \ldots\}.$$  

If, in addition, $[A]$ has an orthogonal base then the sequence $e_1, e_2, \ldots$ can be chosen to be orthogonal.

Proof.

We may assume that $[A] = E$. By Proposition 2.5 it suffices to prove only the second statement. We shall construct an orthogonal sequence $e_1, e_2, \ldots$ in $A$ such that for each $a \in A$ and $n \in \mathbb{N}$ there exists a $b \in \text{co} \{e_1, \ldots, e_n\}$ with $\|a - b\| \leq \|e_{n+1}\|$. (This proves the Theorem since, by Lemma 2.4, $\lim_{n \to \infty} \|e_n\| = 0$.) There is an $e_1 \in A$ with $\|e_1\| = \max \{\|x\| : x \in A\}$. By [2], Lemma 4.35, $E$ is the orthogonal
direct sum of $K_{e_1}$ and some subspace $D_1$. For each $a \in A$, $a = \lambda_1 e_1 + d_1$ $(\lambda_1 \in K_1$, $d_1 \in D)$, we have

$$\|a\| = \max (\|\lambda_1 e_1\|, \|d_1\|) \geq \|\lambda_1 e_1\| = \|\lambda_1\| e_1$$

so that $|\lambda_1| \leq 1$ (if $e_1 = 0$, choose $\lambda_1 = 0$). It follows that $d_1 \in D_1 \cap A$. Therefore, $A$ decomposes into an orthogonal sum of $K_{e_1}$ and $D_1 \cap A$, so $D_1 \cap A$ is $c'$-compact by Proposition 1.4. There exists an $e_2 \in D_1 \cap A$ with $\|e_2\| = \max \{\|x\| : x \in D_1 \cap A\}$. Then

$$\|a - \lambda_1 e_1\| \leq \|e_2\|.$$ 

In its turn, $D_1$ decomposes into an orthogonal sum of $K_{e_2}$ and a space $D_2$ such that $D_2 \cap A$ is $c'$-compact. Let $e_3 \in D_2 \cap A$ with $\|e_3\| = \max \{\|x\| : x \in D_2 \cap A\}$. Then $\|a - \lambda_1 e_1 - \lambda_2 e_2\| \leq \|e_3\|$ for some $\lambda_2 \in K$, $|\lambda_2| \leq 1$, etc.

**COROLLARY 2.8.** Let $A$ be an absolutely convex subset of a $K$-Banach space. The following statements (a)-(δ) are equivalent.

(a) $A$ is $c'$-compact.

(β) There exists a compact set $X$ with $\text{co} X \subset A \subset \text{co} X$.

(γ) There exists a sequence $e_1, e_2, \ldots$ in $A$ with $\lim_{n \to \infty} e_n = 0$ such that $\text{co} \{e_1, e_2, \ldots\} \subset A \subset \text{co} \{e_1, e_2, \ldots\}$.

(δ) For each $s \in (0,1)$ there exists an $s$-orthogonal sequence $e_1, e_2, \ldots$ in $A$ for which $\text{co} \{e_1, e_2, \ldots\} \subset A \subset \text{co} \{e_1, e_2, \ldots\}$.

If $K$ is spherically complete, (a)-(δ) are equivalent to:

(c) There exists an orthogonal sequence $e_1, e_2, \ldots$ in $A$ such that $\text{co} \{e_1, e_2, \ldots\} \subset A \subset \text{co} \{e_1, e_2, \ldots\}$.

**Proof.**

(c) $\Rightarrow$ (δ) $\Rightarrow$ (γ) $\Rightarrow$ (β) are obvious. (β) $\Rightarrow$ (α) follows from Propositions 1,2 and 1.3. The first part of Theorem 2.7 yields (α) $\Rightarrow$ (δ). If $K$ is spherically complete each Banach space of countable type has an
orthogonal base ([2], Lemma 5.5) and \((a) \Rightarrow (c)\) is a consequence of the second part of Theorem 2.7.
§ 3. OTHER CHARACTERIZATIONS OF $C'$-COMPACTNESS

In this section there is no need to restrict ourselves to Banach spaces so in § 3, let $E$ be a locally convex space over $K$.

DEFINITION 3.1.

An absolutely convex subset $A \subseteq E$ is a pure compactoid if for each neighbourhood $U$ of 0 there exist a finite set $F \subseteq A$ such that $A \subseteq U + \text{co} F$.

(The difference with the definition of 'ordinary' compactoidity ([2], p. 134) lies in the fact that we require $F \subseteq A$ rather than $F \subseteq E$.)

DEFINITION 3.2.

A function $\phi : E \to \mathbb{R}$ is convex if $n \in \mathbb{N}$, $x_1, \ldots, x_n \in E$, $\lambda_1, \ldots, \lambda_n \in K$, $|\lambda_i| \leq 1$ for each $i$, $\sum \lambda_i = 1$ imply

$$
\phi\left(\sum \lambda_i x_i\right) \leq \max_{i=1}^n |\lambda_i| \phi(x_i).
$$

(Example: $x \mapsto p(x-a)$ for $a \in E$ and a seminorm $p$.)

THEOREM 3.3.

For an absolutely convex $A \subseteq E$ the following statements are equivalent.

(a) $A$ is a pure compactoid.

(b) If $U$ is a covering of $A$ by (convex) open sets then there exist $n \in \mathbb{N}$ and $U_1, \ldots, U_n \in U$ such that $A \subseteq \text{co} \left( \bigcup_{i=1}^n U_i \right)$.

(c) Let $U_1 \subseteq U_2 \subseteq \ldots$ be open convex sets covering $A$. Then $A \subseteq U_n$ for some $n$.

(d) Each continuous convex function $\phi : E \to \mathbb{R}$, restricted to $A$, has a maximum.
(e) A is $c'$-compact.

Proof.

(a) ⇒ (β). There is a $U_0 \in U$ with $0 \in U_0$. There exist $x_1, \ldots, x_m \in A$ such that $A \subseteq U_0 + \text{co} \{x_1, \ldots, x_m\}$. Let $U_1, \ldots, U_m \in U$ be with $x_i \in U_i$ for each $i \in \{1, \ldots, m\}$. Then $A \subseteq U_0 + \text{co} \bigcup_{i=1}^{m} U_i \subseteq \text{co} \bigcup_{i=0}^{m} U_i$. To prove

$$0 \nsucceq 1 - 0 \nsucceq \cdots \nsucceq 0$$

we may assume that $0 \in U_1$ so that all $U_i$ are absolutely convex. By (β) there exists a finite set $F \subseteq N$ such that

$$A \subseteq \text{co} \bigcup_{i \in F} U_i.$$ 

(γ) ⇒ (β). Let $s = \sup \{\phi(x) : x \in A\}$ (possibly $s = \infty$). Suppose $s$ is not a value of $\phi|A$. Choose $s_1 < s_2 < \cdots$, $\lim_{n \to \infty} s_n = s$ and set

$$U_n := \{x \in E : \phi(x) < s_n\} \quad (n \in N)$$

Each $U_n$ is open. As $\phi$ is convex, $U_n$ is convex. Further we have

$$U_1 \subseteq U_2 \subseteq \cdots$$

and, by assumption, the $U_n$ cover $A$. By (γ), $A \subseteq U_n$ for some $n$ implying $\phi < s_n$ on $A$, a contradiction.

(δ) ⇒ (ε) is trivial.

Finally we prove (ε) ⇒ (α). First, assume that $E$ is a Banach space.

From Corollary 2.8 we obtain a compact set $X \subseteq E$ with $\text{co} X \subseteq A \subseteq \overline{\text{co} X}$. Let $U$ be an absolutely convex neighbourhood of $0$ in $E$. By compactness there exist $x_1, \ldots, x_n \in X$ such that $X \subseteq U \left(\bigcup_{i=1}^{n} x_i + U\right)$. Then

$$\text{co} X \subseteq U + \text{co} \{x_1, \ldots, x_n\}.$$ 

As the latter set is an open additive subgroup of $E$ it is also closed and we have

$$A \subseteq \overline{\text{co} X} \subseteq U + \text{co} \{x_1, \ldots, x_n\}.$$ 

Now let $E$ be a locally convex space, and let $U$ be an absolutely convex neighbourhood of $0$ in $E$. There is a continuous seminorm $p$ such that
\[ \{ x \in E : p(x) \leq 1 \} \subset U. \text{ Let } \pi_p : E \to E_p^\sim \text{ be the canonical quotient map,} \]

where \( E_p^\sim \) is the completion of the space \( E_p := E/\text{Ker}p \) with the norm induced by \( p \). By Proposition 1.4 the set \( \pi_p(A) \) is c'-'compact and we just have proved that it is a pure compactoid in \( E_p^\sim \), hence in \( E_p \).

Since \( \pi_p(U) \) is open in \( E_p \) there exists a finite set \( F \subset \pi_p(A) \) such that \( \pi_p(A) \subset \pi_p(U+\text{co } F) \). Choose a finite set \( F' \subset A \) such that \( \pi_p(F') = F \). We then have \( \pi_p(A) \subset \pi_p(U+\text{co } F') \) and, as \( \text{Ker } \pi_p \subset U \),

\[ A \subset U+\text{co } F'+\text{Ker } \pi_p \subset U+\text{co } F'. \]

**Remark.**

The following statements are easy to prove.

(i) If the valuation of \( K \) is discrete the properties (a) - (e) of Theorem 3.3 are equivalent to '\( A \) is a compactoid'.

(ii) If \( K \) is locally compact the properties (a) - (e) of Theorem 3.3 are equivalent to '\( A \) is precompact'.


§ 4. $C'$-COMPACTNESS VERSUS $C$-COMPACTNESS

First we extend Theorem 3.3 to convex sets. For a subset $X$ of a K-linear space, let $c(X)$ be its convex hull.

THEOREM 4.1.

Let $C$ be a convex subset of a locally convex space $E$ over $K$. The following are equivalent.

(a) For each neighbourhood of 0 in $E$ there exists a finite set $F \subseteq C$ with $C \subseteq \bigcup c(F)$.

(b) If $U$ is a covering of $C$ by (convex) open sets then there exist $n \in \mathbb{N}$ and $U_1, \ldots, U_n \in U$ such that $C \subseteq c(\bigcup_{i=1}^n U_i)$.

(c) Let $U_1 \subseteq U_2 \subseteq \cdots$ be open convex sets covering $C$. Then $C \subseteq U_n$ for some $n$.

(d) Each continuous convex function $\phi : E \to \mathbb{R}$, restricted to $C$, has a maximum.

(e) For each continuous seminorm $p$ and each $a \in C$, $\max\{p(x-a) : x \in C\}$ exists.

Proof.

Straightforward.

A convex set $C \subseteq E$ is $C'$-compact if it satisfies (a) - (e) of Theorem 4.1.

The following theorem explains the term 'complementary' in the title of this paper. (Compare Theorem 4.1.)

THEOREM 4.2.

Let $C$ be a convex subset of a Banach space $E$ over $K$. The following are
equivalent.

(*)' If $C$ is a collection of closed convex sets in $E$ such that $\cap C$ does not meet $C$ then there exist $n \in \mathbb{N}$ and $C_1, \ldots, C_n \in C$ such that $C \cap \cap_{i=1}^n C_i = \emptyset$.

\(\gamma\)' Let $C_1 \supset C_2 \supset \ldots$ be closed convex sets in $E$ such that $C \cap \cap_{n=1}^\infty C_n = \emptyset$. Then $C \cap C_n = \emptyset$ for some $n$.

\(\delta\)' Each continuous convex function $\phi : E \to \mathbb{R}$, restricted to $C$, has a minimum.

\(e\)' $C$ is $c$-compact.

Proof.

\(e\)' $\Rightarrow$ \(\beta\)' is just the definition of $c$-compactness ([4]), \(\beta\)' $\Rightarrow$ \(\gamma\)' is obvious. The proof of \(\gamma\)' $\Rightarrow$ \(\delta\)' runs similar to the one of \(\gamma\) $\Rightarrow$ \(\delta\) of Theorem 3.3. To prove \(\delta\)' $\Rightarrow$ \(e\)' it suffices, by [1] Theorem 6.15 \(\zeta\) $\Rightarrow$ \(a\), to show that $C$ is spherically complete relative to any norm $\| \| \|$ defining the topology of $E$. Thus, let $B_1 \supset B_2 \supset \ldots$ be balls in $C$ where

$$B_n = \{ x \in C : \| x - c_n \| \leq r_n \}$$

for some $c_1, c_2, \ldots$ in $C$ and $r_1 \geq r_2 \geq \ldots$. Let $(E^-, \| \|)$ be the spherical completion ([2], Theorem 4.43) of $(E, \| \|)$ and consider for each $n \in \mathbb{N}$

$$B_n^- := \{ x \in E^- : \| x - c_n \| \leq r_n \}$$

These $B_n^-$ form a nested sequence of balls in $E^-$ so there exists a $z \in \cap_n B_n^-$. The function $\phi : x \mapsto \| z - x \|$ (x $\in$ E) is convex and attains a minimum on $C$, say in $c \in C$. As $\phi(c_n) \leq r_n$ (n $\in$ $\mathbb{N}$) we have

$\phi(c) \leq \inf_n r_n$. For each $n \in \mathbb{N}$
\[ \| c - c_n \| \leq \max(\| c - z \|, \| z - c_n \|) \leq \max(\phi(c), \phi(c_n)) \leq r_n. \]

We see that \( c \in B_n \) for each \( n \) and it follows that \( A \) is spherically complete for \( \| \cdot \| \).

Remarks.

1. Theorem 4.2 is only of interest if \( K \) is spherically complete ([4], (2.1)). I do not know whether the properties (\( \beta \)' - (\( \varepsilon \)') are equivalent for a convex set \( C \) in a locally convex space \( E \). (Of course one has some obvious implications.)

2. The notion of \( c \)-compactness may be viewed as a 'convexification' of the intersection property for closed sets in a compact space, whereas \( c' \)-compactness can be seen as a 'convexification' of the 'open covering' definition of compactness. (Theorem 4.1 (\( \beta \)), Theorem 4.2 (\( \beta \)').

3. In a future paper [3] we shall discuss the relation between weak and strong \( c' \)-compactness.

REFERENCES


