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THE CLOSED CONVEX HULL OF A COMPACT SET
IN A NON-ARCHIMEDEAN LOCALLY CONVEX SPACE

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ABSTRACT. For a complete absolutely convex set $A$ in a locally convex space over a non-archimedean valued field $K$ it is proved that

(i) $A$ is the closed absolutely convex hull of a compact set if and only if $A$ is isomorphic to some power of $B(0,1) := \{ \lambda \in K : |\lambda| \leq 1 \}$,

(ii) if the valuation of $K$ is discrete and $A$ is a compactoid (equivalently; $A$ is $c$-compact and bounded) then $A$ is the closed absolutely convex hull of a compact set,

(iii) the conclusion of (ii) is also true for any $K$ if $A$ is a metrizable pure compactoid,

(iv) if $A$ is a compactoid it is isomorphic to a closed submodule of some power of $B(0,1)$.

These results extend those (for a locally compact base field) of Carpentier ([1], Propositions 72,73). Corollary 1.8 is a non-archimedean approach to the Krein-Milman Theorem.
PRELIMINARIES. Throughout $K$ is a non-archimedean nontrivially valued complete field with valuation $|\cdot|$. For fundamentals on locally convex spaces $E$ over $K$ (which we assume to be Hausdorff) we refer to [83], [83], [33], [43], [13]. A set $A \subseteq E$ is absolutely convex if it is a $\mathcal{B}(0,1)$-module. If $F$ is a locally convex space over $K$ and $A \subseteq E$, $B \subseteq F$ are absolutely convex then $\phi : A \rightarrow B$ is affine if it is a homeomorphism of $\mathcal{B}(0,1)$-modules. We shall write $A = B$ if there exists an affine homeomorphism of $A$ onto $B$. For a set $X \subseteq E$, let $\overline{co} X$ be its absolutely convex hull and $co X$ be its closure.

An absolutely convex set $A \subseteq E$ is edged if for each $x \in E$ the set \[ \{ |\lambda| : \lambda x \in A \} \] is closed in $|\cdot| := \{ |\lambda| : \lambda \in K \}$ (or, equivalently, if $A = \{ x \in [A] : p_A(x) \leq 1 \}$, where $p_A$ is the Minkowski function, defined on the $K$-linear span $[A]$ of $A$ by the formula \[ p_A(x) = \inf \{ |\lambda| : x \in \lambda A \} \]. It is easy to prove that if the valuation of $K$ is discrete each absolutely convex set is edged whereas, if the valuation of $K$ is dense, an absolutely convex $A \subseteq E$ is edged if and only if $\lambda x \in A$ for all $\lambda \in K$, $|\lambda| < 1$ implies $x \in A$.

For a subset $A$ of $E$, let $A^\circ := \{ f \in E^\prime : |f(x)| \leq 1 \text{ for all } x \in A \}$ (where $E^\prime$ is the dual space of $E$) and let $A^{\circ \circ} := \{ x \in E : |f(x)| \leq 1 \text{ for all } f \in A^\circ \}$. $A$ is a polar set if $A = A^{\circ \circ}$.

A set $A \subseteq E$ is (a) compactoid if for each neighbourhood $U$ of 0 there exist $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in E$ such that $A \subseteq U + \text{co}(x_1, \ldots, x_n)$; it is a pure compactoid if in the above we may choose $x_1, \ldots, x_n \in A$. If the valuation of $K$ is discrete each absolutely convex compactoid is pure (for example [4], Lemma 8.1), if the valuation of $K$ is dense \[ \{ \lambda \in K : |\lambda| < 1 \} \] is a compactoid in $E := K$ but not pure.
§ 1 COMPLETE COMPACTOIDS

LEMMA 1.1 Let E be a locally convex space over K. Let A ⊂ E be a complete, absolutely convex, edged, absorbing compactoid. Assume that a seminorm on E is continuous if its restriction to A is continuous.

(i) E is of countable type ([4], Definition 4.3).

(ii) A is a polar set.

(iii) E' is a Banach space over K with respect to the norm ∥∥_A defined by ∥∥_A := sup { |f(x)| : x ∈ A}.

(iv) If A = ∪_X for some compact set X ⊂ A then (E', ∥∥_A) has an orthonormal base.

(v) The canonical map E → (E', ∥∥_A)' is a bijection.

Proof.

(i) [6], Proposition 4.3.

(ii) [4], Theorem 4.7.

(iii) As A is absorbing ∥∥_A is a norm on E'. If f_1, f_2, ... is a ∥∥_A -Cauchy sequence in E' then there is a linear f : E → K such that f = lim f_n uniformly on A. Then |f|, restricted to A, is continuous. By assumption, |f| is continuous. Hence, f ∈ E' and lim ∥f-f_n∥_A = 0.

(iv) Let C(X → K) be the Banach space of all continuous functions: X → K with the supremum norm. For each f ∈ E' we have

∥∥_A = sup_ A |f| = sup_co X |f| = sup_X |f|

so that the map T : (E', ∥∥_A) → C(X → K) given by Tf := f|X is a linear isometry. By [3], Theorem 5.22, C(X → K) has an
orthonormal base. Then so has its closed subspace \( \text{Im } T \) by Gruson's Theorem ([3], Theorem 5.9) and has \( (E', \| \cdot \|_A) \).

(iv) Contained in the proof of [6], Theorem 3.2 (the metrizability condition is not needed for part (ii) of that proof).

**Lemma 1.2** Let \( E, A, \| \cdot \|_A \) be as in Lemma 1.1. Suppose \( (E', \| \cdot \|_A) \) has an orthonormal base \( \{f_i : i \in I\} \). Then \( A \approx B(0,1)^I \).

**Proof.** The formula

\[
\phi(x) = (f_i(x))_{i \in I} \quad (x \in E)
\]

defines a continuous linear map \( \phi : E \to K^I \) (on \( K^I \) the product topology) sending \( A \) into \( B(0,1)^I \). We prove (i), (ii) below.

(i) \( \phi|A \) is a homeomorphism into \( B(0,1)^I \). **Proof.** Let \( (x_j)_{j \in J} \) be a net in \( A \) for which \( \lim_{j} \phi(x_j) = 0 \) i.e. \( \lim_{j} f_i(x_j) = 0 \) for all \( i \in I \). Then \( \lim_{j} g(x_j) = 0 \) for all \( g \) in a \( \| \cdot \|_A \) dense subset \( \mathcal{H} \) of \( E' \). Let \( f \in E' \), \( \varepsilon > 0 \). There is a \( g \in \mathcal{H} \) with \( \|f-g\|_A < \varepsilon \).

For large \( j \)

\[
|f(x_j)| \leq \max \{|f(x_j) - g(x_j)|, |g(x_j)|\} < \varepsilon
\]

so that \( \lim_{j} x_j = 0 \) weakly. But then \( \lim_{j} x_j = 0 \) for the initial topology of \( E \) ([4], Theorem 5.12).

(ii) \( \phi \) maps \( A \) onto \( B(0,1)^I \). **Proof.** Let \( z := (z_i)_{i \in I} \in B(0,1)^I \). Define \( h \in (E', \| \cdot \|_A)' \) by

\[
h(f_i) = z_i \quad (i \in I)
\]

By Lemma 1.1 (v) there exists an \( x \in E \) with \( f(x) = h(f) \) for all \( f \in E' \) i.e. with \( \phi(x) = z \). To prove that in fact \( x \in A^\circ = A \)
(Lemma 1.1 (ii)), let $f \in E'$, $f \in A^\circ$. Then $\|f\|_A \leq 1$. There exist $\lambda_i \in K$ for which $f = \sum_{i \in I} \lambda_i f_i$ in the sense of $\|\|_A$. By orthonormality

$$\|f\|_A = \max \|\lambda_i\| \leq 1.$$ 

We see that $|f(x)| \leq \max \|\lambda_i f_i(x)\| = \max \|\lambda_i z_i\| \leq 1$. It follows that $x \in A^\circ$.

**PROPOSITION 1.3** Let $X$ be a compact subset of a locally convex space $E$ over $K$. Then $\overline{co} X$ is edged.

**Proof.** We may assume that the valuation of $K$ is dense. Let $z \in E$, $z \notin \overline{co} X$. There is ([6], Proposition 4.2) a continuous seminorm $p$ with $p(z) = 1$ and $p < 1$ on $\overline{co} X$. By compactness, $s := \sup\{p(x) : x \in \overline{co} X\}$

$$\sup p = \max p < 1.$$ 

Hence, there is a $\lambda \in K$, $|\lambda| < 1$, such that $p(\lambda z) > s$, i.e. $\lambda z \notin \overline{co} X$.

**THEOREM 1.4** Let $A$ be a complete absolutely convex compactoid in a locally convex space $E$ over $K$. The following are equivalent.

(a) There is a compact set $X \subset A$ with $A = \overline{co} X$.

(b) $A = B(0, 1)^I$ for some set $I$.

**Proof.** (a) $\Rightarrow$ (b). We may assume that $E = [A]$. If we replace the initial topology $\tau$ of $E$ by the stronger locally convex topology $\tau'$ generated by all seminorms $p$ on $E$ for which $p|A$ is $\tau$-continuous then $\tau = \tau'$ on $A$ and $A$ is $\tau'$-complete and a $\tau'$-compactoid ([6], Proposition 4.5). Therefore, to prove (b), we may assume $\tau = \tau'$. Now apply Proposition 1.3, Lemma 1.1 (iv), Lemma 1.2.

(a) $\Rightarrow$ (b). Let $e_i \in B(0, 1)^I$ ($i \in I$) be given by
It is easily seen that \( Y := \{0\} \cup \{e_i : i \in I\} \) is compact and that \( B(0,1) = \overline{\text{co} \ Y} \).

**THEOREM 1.5** Let the valuation of \( K \) be discrete. Let \( A \) be a complete absolutely convex compactoid in a locally convex space \( E \) over \( K \). (Or, equivalently, let \( A \) be bounded, absolutely convex and \( c \)-compact ([6], Corollary 2.5).) Then there exists a compact set \( X \subseteq A \) with \( A = \text{co} \ X \).

**Proof.** For the same reasons as in the previous proof we may assume that \( E = \mathbb{A} \) and that a seminorm \( p \) on \( E \) is continuous if \( p|A \) is continuous. By Lemma 1.1 (iii), \( (E', \|\cdot\|_A) \) is a Banach space. As the valuation is discrete we have

\[
\|f\|_A = \sup_{x \in A} |f(x)| \leq K (f \in E')
\]

Then by [3], Theorem 5.16, \( (E', \|\cdot\|_A) \) has an orthonormal base. Now apply Lemma 1.2 and Theorem 1.4.

For general \( K \) not every edged complete absolutely convex compactoid is the closed convex hull of a compact set. In fact, if the valuation of \( K \) is dense and \( r \in (0,\infty) \backslash \mathbb{K} \) then \( A := \{\lambda \in K : |\lambda| \leq r\} \) is edged but there is no compact set \( X \subseteq K \) for which \( A = \text{co} \ X \). Indeed, we have the following.

**PROPOSITION 1.6** Let \( X \) be a compact subset of a locally convex space \( E \) over \( K \). Then \( \text{co} \ X \) is a pure compactoid.

**Proof.** Let \( U \) be an absolutely convex neighbourhood of \( 0 \) in \( E \). By com-
pactness there exist $x_1, \ldots, x_n \in X$ such that $X \subseteq U(x_1 + U)$. Then
$\text{co } X \subseteq U + \text{co}\{x_1, \ldots, x_n\}$. The set $U + \text{co}\{x_1, \ldots, x_n\}$ is an open additive subgroup of $E$, hence closed. It follows that $\text{co } X \subseteq U + \text{co}\{x_1, \ldots, x_n\}$.

Note. One can prove that each closed pure absolutely convex compactoid is edged.

For metrizable pure compactoids we have the following version of Theorem 1.5 for general $K$.

**THEOREM 1.7** Let $A \subseteq E$ be a complete absolutely convex pure compactoid that is metrizable. Then there is a sequence $e_1, e_2, \ldots$ in $A$ with
$$\lim_{n \to \infty} e_n = 0 \quad \text{and } A = \text{co}\{e_1, e_2, \ldots\}.$$

**Proof.** The proof of [4], Proposition 8.2 applies with some minor modifications (as $A$ is pure the finite sets $F_1, F_2, \ldots$ constructed in that proof can be chosen in $A$ rather than in $\lambda A$).

**OPEN PROBLEM** Let $A$ be complete absolutely convex pure compactoid in a locally convex space $E$ over $K$. Does it follow that $A = \text{co } X$ for some compact $X$?

The previous theory yields the following.

**COROLLARY 1.8** Let $A$ be a complete subset of a locally convex space $E$ over $K$ such that $A = \text{co } X$ for some compact set $X$ (e.g. choose for $A$ any complete absolutely convex compactoid if the valuation of $K$ is discrete or any complete absolutely convex pure metrizable compactoid). Then there exists a linearly independent set.
\( Y = \{e_i : i \in I\} \) in \( A \) such that

(i) \( Y \) is discrete,

(ii) for each neighbourhood \( U \) of 0 the set \( \{i \in I : e_i \notin U\} \) is finite,

(iii) \( Y_0 := Y \cup \{0\} \) is compact,

(iv) \( A = \overline{\text{co}} Y = \overline{\text{co}} Y_0 \),

(v) for each \( (\lambda_i)_{i \in I} \in B(0,1)^I \), \( \sum_{i \in I} \lambda_i e_i \) converges and represents an element of \( A \),

(vi) each \( x \in A \) has a unique representation as a convergent sum

\[
x = \sum_{i \in I} \lambda_i e_i \text{ where } \lambda_i \in B(0,1) \text{ for each } i \in I,
\]

(vii) \( Y \) is a minimal element of \( \{Z \subset E : A = \overline{\text{co}} Z\} \)

(viii) \( Y_0 \) is a minimal element of \( \{Z \subset E, \text{ Z is compact, } A = \overline{\text{co}} Z\} \),

(ix) \( Y \) is a \( p_A \)-orthonormal set.

Proof. By Theorem 1.4 we may assume \( A = B(0,1)^I \). Choose \( \{e_i : i \in I\} \) as in the second part of the proof of Theorem 1.4. We leave the details of checking (i)-(ix) to the reader.
**THEOREM 2.1** Let $A$ be a compactoid in a locally convex space $E$ over $K$. Then there exists a locally convex space $F$ over $K$ containing $E$ as a subspace and a compact set $X \subseteq F$ such that $A \subseteq \text{co } X$.

**Proof.** For each continuous seminorm $p$ on $E$, let $E_p := E/\text{Ker} p$ with the norm induced by $p$. The natural maps $\pi_p : E \to E_p$ yield a linear homeomorphic embedding

$$
\pi : E \to F := \prod_{p \in \Gamma} E_p
$$

where $\Gamma$ is the collection of continuous seminorms of $E$. For each $p \in \Gamma$ the set $\pi_p(A)$ is a (metrizable) compactoid in $E_p$. By [4], Proposition 8.2 there is a compact set $X_p \subseteq E_p$ such that $\pi_p(A) \subseteq \text{co } X_p$. Without loss we may assume that $0 \in X_p$. We have

$$
\pi(A) \subseteq \prod_{p \in \Gamma} \pi(A) \subseteq \prod_{p \in \Gamma} \text{co } X_p
$$

We claim that $\prod_{p \in \Gamma} \text{co } X_p \subseteq \text{co } \prod_{p \in \Gamma} X_p$. (Then the theorem is proved with $X := \prod_{p \in \Gamma} X_p$.) For $p \in \Gamma$ and $x \in X_p$ the element $f$ defined by

$$
(*) \quad f(q) = \begin{cases} 
    x & \text{if } q \in \Gamma, \quad q = p \\
    0 & \text{if } q \in \Gamma, \quad q \neq p
\end{cases}
$$

is in $\prod_{p \in \Gamma} X_p$. If $p \in \Gamma$ and $x \in \text{co } X_p$ then $f$, formally defined by $(*)$, is in $\text{co } \prod_{p \in \Gamma} X_p$. If $p_1, \ldots, p_n \in \Gamma$ and $x \in \text{co } X_p$ for $i \in \{1, \ldots, n\}$ the element $g$ defined by

$$
(**) \quad g(q) = \begin{cases} 
    x_{p_i} & \text{if } q \in \Gamma, \quad i \in \{1, \ldots, n\}, \quad q = p_i \\
    0 & \text{if } q \in \Gamma, \quad \text{otherwise}
\end{cases}
$$

is in $\text{co } \prod_{p \in \Gamma} X_p$. If $p_1, \ldots, p_n \in \Gamma$ and $x \in \text{co } X_{p_i}$ for $i \in \{1, \ldots, n\}$ the element $g$ defined by

$$
(**) \quad g(q) = \begin{cases} 
    x_{p_i} & \text{if } q \in \Gamma, \quad i \in \{1, \ldots, n\}, \quad q = p_i \\
    0 & \text{if } q \in \Gamma, \quad \text{otherwise}
\end{cases}
$$
is a finite sum of elements of \( \bigoplus_{p \in \Gamma} X_p \), hence in \( \bigoplus_{p \in \Gamma} X_p \). The elements of the type defined in (***) are dense in \( \prod_{p \in \Gamma} X_p \).

**COROLLARY 2.2** Let \( A \) be an absolutely convex compactoid in a locally convex space over \( K \). Then \( A \) is isomorphic to a submodule of some power of \( B(0,1) \).

**Proof.** By the previous theorem, \( A \subseteq \bigoplus X \) for some compact \( X \subseteq F \). We may suppose that \( F \) is complete. Now apply Theorem 1.4.

**Note to Theorem 2.1.** It is too optimistic to hope that in Theorem 2.1 we may require that \( F = E \) even when we allow \( X \) to be precompact. In fact, let \( K \) be not locally compact, let \( E = c_0 \), with the weak topology. \( A := \{ x \in E : \|x\| \leq 1 \} \) is a weak compactoid but according to [5], Proposition 3.3 there is no weakly precompact \( X \subseteq E \) for which \( A \subseteq \bigoplus X \). Observe that, if \( K \) is not spherically complete, \( A \) is even weakly complete ([4], Theorem 9.6), and that \( A \) is pure if the valuation of \( K \) is discrete.
REFERENCES


