THE CLOSED CONVEX HULL OF A COMPACT SET
IN A NON-ARCHIMEDEAN LOCALLY CONVEX SPACE

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ABSTRACT. For a complete absolutely convex set \( A \) in a locally convex space over a non-archimedean valued field \( K \) it is proved that

(i) \( A \) is the closed absolutely convex hull of a compact set if and only if \( A \) is isomorphic to some power of

\[
B(0,1) := \{ \lambda \in K : |\lambda| \leq 1 \},
\]

(ii) if the valuation of \( K \) is discrete and \( A \) is a compactoid (equivalently; \( A \) is c-compact and bounded) then \( A \) is the closed absolutely convex hull of a compact set,

(iii) the conclusion of (ii) is also true for any \( K \) if \( A \) is a metrizable pure compactoid,

(iv) if \( A \) is a compactoid it is isomorphic to a closed submodule of some power of \( B(0,1) \).

These results extend those (for a locally compact base field) of Carpentier ([1], Propositions 72,73). Corollary 1.8 is a non-archimedean approach to the Krein-Milman Theorem.
PRELIMINARIES. Throughout $K$ is a non-archimedean nontrivially valued complete field with valuation $|\cdot|$. For fundamentals on locally convex spaces $E$ over $K$ (which we assume to be Hausdorff) we refer to [8], [7], [3], [4], [1]. A set $A \subseteq E$ is **absolutely convex** if it is a $B(0,1)$-module. If $F$ is a locally convex space over $K$ and $A \subseteq E$, $B \subseteq F$ are absolutely convex then $\phi : A \to B$ is **affine** if it is a homeomorphism of $B(0,1)$-modules. We shall write $A = B$ if there exists an affine homeomorphism of $A$ onto $B$. For a set $X \subseteq E$, let $\text{co} X$ be its absolutely convex hull and $\overline{\text{co}} X$ be its closure.

An absolutely convex set $A \subseteq E$ is **edged** if for each $x \in E$ the set $
{\lambda : \lambda x \in A}$ is closed in $|\cdot| := \{\lambda : \lambda \in K\}$ (or, equivalently, if $A = \{x \in [A] : p_A(x) \leq 1\}$, where $p_A$ is the Minkowski function, defined on the $K$-linear span $[A]$ of $A$ by the formula $p_A(x) = \inf \{|\lambda| : x \in \lambda A\}$). It is easy to prove that if the valuation of $K$ is discrete each absolutely convex set is edged whereas, if the valuation of $K$ is dense, an absolutely convex $A \subseteq E$ is edged if and only if $\lambda x \in A$ for all $\lambda \in K$, $|\lambda| < 1$ implies $x \in A$.

For a subset $A$ of $E$, let $A^o := \{f \in E' : |f(x)| \leq 1 \text{ for all } x \in A\}$ (where $E'$ is the dual space of $E$) and let $A^{oo} := \{x \in E : |f(x)| \leq 1 \text{ for all } f \in A\}$. $A$ is a **polar set** if $A = A^{oo}$.

A set $A \subseteq E$ is (a) **compactoid** if for each neighbourhood $U$ of 0 there exist $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in E$ such that $A \subseteq U + \text{co}(x_1, \ldots, x_n)$; it is a **pure compactoid** if in the above we may choose $x_1, \ldots, x_n \in A$. If the valuation of $K$ is discrete each absolutely convex compactoid is pure (for example [4], Lemma 8.1), if the valuation of $K$ is dense $\{\lambda \in K : |\lambda| < 1\}$ is a compactoid in $E := K$ but not pure.
§ 1 COMPLETE COMPACTOIDS

LEMMA 1.1 Let $E$ be a locally convex space over $K$. Let $A \subset E$ be a complete, absolutely convex, edged, absorbing compactoid. Assume that a seminorm on $E$ is continuous if its restriction to $A$ is continuous.

(i) $E$ is of countable type ([4], Definition 4.3).

(ii) $A$ is a polar set.

(iii) $E'$ is a Banach space over $K$ with respect to the norm $\| \|_A$ defined by $\| f \|_A = \sup \{ |f(x)| : x \in A \}$.

(iv) If $A = \co X$ for some compact set $X \subset A$ then $(E', \| \|_A)$ has an orthonormal base.

(v) The canonical map $E \to (E', \| \|_A)'$ is a bijection.

Proof.

(i) [6], Proposition 4.3.

(ii) [4], Theorem 4.7.

(iii) As $A$ is absorbing $\| \|_A$ is a norm on $E'$. If $f_1, f_2, \ldots$ is a $\| \|_A$-Cauchy sequence in $E'$ then there is a linear $f : E \to K$ such that $f = \lim_{n \to \infty} f_n$ uniformly on $A$. Then $|f|$, restricted to $A$, is continuous. By assumption, $|f|$ is continuous. Hence, $f \in E'$ and $\lim_{n \to \infty} \| f - f_n \|_A = 0$.

(iv) Let $C(X \to K)$ be the Banach space of all continuous functions: $X \to K$ with the supremum norm. For each $f \in E'$ we have

$$\| f \|_A = \sup_A |f| = \sup_{\co X} |f| = \sup_X |f|$$

so that the map $T : (E', \| \|_A) \to C(X \to K)$ given by $Tf := f|_X$ is a linear isometry. By [3], Theorem 5.22, $C(X \to K)$ has an
orthonormal base. Then so has its closed subspace $\operatorname{Im} T$ by Gruson's Theorem ([3], Theorem 5.9) and has $(E', \| \cdot \|_A)$.

(iv) Contained in the proof of [6], Theorem 3.2 (the metrizability condition is not needed for part (ii) of that proof).

**Lemma 1.2** Let $E$, $A$, be as in Lemma 1.1. Suppose $(E', \| \cdot \|_A)$ has an orthonormal base $\{f_i : i \in I\}$. Then $A \approx B(0,1)^I$.

**Proof.** The formula

$$\Phi(x) = (f_i(x))_{i \in I} \quad (x \in E)$$

defines a continuous linear map $\Phi : E \to K^I$ (on $K^I$ the product topology) sending $A$ into $B(0,1)^I$. We prove (i), (ii) below.

(i) $\Phi|A$ is a homeomorphism into $B(0,1)^I$. Proof. Let $(x_j)_j \in J$ be a net in $A$ for which $\lim_j \Phi(x_j) = 0$ i.e. $\lim_j f_i(x_j) = 0$ for all $i \in I$. Then $\lim_j g(x_j) = 0$ for all $g$ in a $A$ dense subset $H$ of $E'$. Let $f \in E'$, $\varepsilon > 0$. There is a $g \in H$ with $\|f - g\|_A < \varepsilon$.

For large $j$

$$|f(x_j)| \leq \max \{|f(x_j) - g(x_j)|, |g(x_j)|\} < \varepsilon$$

so that $\lim_j x_j = 0$ weakly. But then $\lim_j x_j = 0$ for the initial topology of $E$ ([4], Theorem 5.12).

(ii) $\Phi$ maps $A$ onto $B(0,1)^I$. Proof. Let $z := (z_i)_{i \in I} \in B(0,1)^I$. Define $h \in (E', \| \cdot \|_A)'$ by

$$h(f_i) = z_i \quad (i \in I)$$

By Lemma 1.1 (v) there exists an $x \in E$ with $f(x) = h(f)$ for all $f \in E'$ i.e. with $\Phi(x) = z$. To prove that in fact $x \in A^{\infty} = A$
(Lemma 1.1 (ii)), let \( f \in E' \), \( f \in A^0 \). Then \( \|f\|_A \leq 1 \). There exist \( \lambda_i \in \mathbb{K} \) for which \( f = \sum \lambda_i f_i \) in the sense of \( \| \|_A \). By orthonormality

\[
\|f\|_A = \max \{\lambda_i\} \leq 1.
\]

We see that \( |f(x)| \leq \max \{\lambda_i f_i(x)\} = \max \{\lambda_i z_i\} \leq 1 \). It follows that \( x \in A^0 \).

**PROPOSITION 1.3** Let \( X \) be a compact subset of a locally convex space \( E \) over \( \mathbb{K} \). Then \( \overline{co X} \) is edged.

**Proof.** We may assume that the valuation of \( \mathbb{K} \) is dense. Let \( z \in E \), \( z \notin \overline{co X} \). There is ([6], Proposition 4.2) a continuous seminorm \( p \) with \( p(z) = 1 \) and \( p < 1 \) on \( \overline{co X} \). By compactness, \( s := \sup \{p(x) : x \in \overline{co X}\} = \sup_X p = \max_X p < 1 \). Hence, there is a \( \lambda \in \mathbb{K} \), \( |\lambda| < 1 \), such that \( p(\lambda z) > s \) i.e. \( \lambda z \notin \overline{co X} \).

**THEOREM 1.4** Let \( A \) be a complete absolutely convex compactoid in a locally convex space \( E \) over \( \mathbb{K} \). The following are equivalent.

(a) There is a compact set \( X \subset A \) with \( A = \overline{co X} \).

(b) \( A = \overline{B(0,1)^I} \) for some set \( I \).

**Proof.** (a) \( \Rightarrow \) (b). We may assume that \( E = [A] \). If we replace the initial topology \( \tau \) of \( E \) by the stronger locally convex topology \( \tau' \) generated by all seminorms \( p \) on \( E \) for which \( p|A \) is \( \tau \)-continuous then \( \tau = \tau' \) on \( A \) and \( A \) is \( \tau' \)-complete and a \( \tau' \)-compactoid ([6], Proposition 4.5). Therefore, to prove (b), we may assume \( \tau = \tau' \). Now apply Proposition 1.3, Lemma 1.1 (iv), Lemma 1.2.

(a) \( \Rightarrow \) (b). Let \( e_i \in \overline{B(0,1)^I} \) (\( i \in I \)) be given by
It is easily seen that $Y := \{0\} \cup \{e_i : i \in I\}$ is compact and that $B(0,1)^I = \text{co } Y$.

**Theorem 1.5** Let the valuation of $K$ be discrete. Let $A$ be a complete absolutely convex compactoid in a locally convex space $E$ over $K$. (Or, equivalently, let $A$ be bounded, absolutely convex and $c$-compact ([6], Corollary 2.5).) Then there exists a compact set $X \subset A$ with $A = \text{co } X$.

**Proof.** For the same reasons as in the previous proof we may assume that $E = [A]$ and that a seminorm $p$ on $E$ is continuous if $p|A$ is continuous. By Lemma 1.1 (iii), $(E', \| \cdot \|_A)$ is a Banach space. As the valuation is discrete we have

$$\|f\|_A = \sup_{x \in A} |f(x)| \leq |x|_K \quad (f \in E')$$

Then by [3], Theorem 5.16, $(E', \| \cdot \|_A)$ has an orthonormal base. Now apply Lemma 1.2 and Theorem 1.4.

For general $K$ not every edged complete absolutely convex compactoid is the closed convex hull of a compact set. In fact, if the valuation of $K$ is dense and $r \in (0,\infty)\setminus\{1\}$ then $A := \{\lambda \in K : |\lambda| \leq r\}$ is edged but there is no compact set $X \subset K$ for which $A = \text{co } X$. Indeed, we have the following.

**Proposition 1.6** Let $X$ be a compact subset of a locally convex space $E$ over $K$. Then $\text{co } X$ is a pure compactoid.

**Proof.** Let $U$ be an absolutely convex neighbourhood of $0$ in $E$. By com-
pactness there exist $x_1, \ldots, x_n \in X$ such that $X \subseteq U(x_1 + U)$. Then $\text{co } X \subseteq U + \text{co}\{x_1, \ldots, x_n\}$. The set $U + \text{co}\{x_1, \ldots, x_n\}$ is an open additive subgroup of $E$, hence closed. It follows that $\text{co } X \subseteq U + \text{co}\{x_1, \ldots, x_n\}$.

Note. One can prove that each closed pure absolutely convex compactoid is edged.

For metrizable pure compactoids we have the following version of Theorem 1.5 for general $K$.

**Theorem 1.7** Let $A \subseteq E$ be a complete absolutely convex pure compactoid that is metrizable. Then there is a sequence $e_1, e_2, \ldots$ in $A$ with $\lim_{n \to \infty} e_n = 0$ and $A = \text{co}\{e_1, e_2, \ldots\}$.

**Proof.** The proof of [4], Proposition 8.2 applies with some minor modifications (as $A$ is pure the finite sets $F_1, F_2, \ldots$ constructed in that proof can be chosen in $A$ rather than in $\lambda A$).

**Open Problem** Let $A$ be complete absolutely convex pure compactoid in a locally convex space $E$ over $K$. Does it follow that $A = \text{co } X$ for some compact $X$?

The previous theory yields the following.

**Corollary 1.8** Let $A$ be a complete subset of a locally convex space $E$ over $K$ such that $A = \text{co } X$ for some compact set $X$ (e.g. choose for $A$ any complete absolutely convex compactoid if the valuation of $K$ is discrete or any complete absolutely convex pure metrizable compactoid). Then there exists a linearly independent set.
\[
Y = \{e_i : i \in I\} \text{ in } A \text{ such that }
\]

(i) \(Y\) is discrete,

(ii) for each neighbourhood \(U\) of \(0\) the set \(\{i \in I : e_i \notin U\}\) is finite,

(iii) \(Y_0 := Y \cup \{0\}\) is compact,

(iv) \(A = \overline{\text{co } Y} = \overline{\text{co } Y_0}\),

(v) for each \((\lambda_i)_{i \in I} \in B(0,1)^I\), \(\sum_{i \in I} \lambda_i e_i\) converges and represents an element of \(A\),

(vi) each \(x \in A\) has a unique representation as a convergent sum
\[
x = \sum_{i \in I} \lambda_i e_i \text{ where } \lambda_i \in B(0,1) \text{ for each } i \in I,
\]

(vii) \(Y\) is a minimal element of \(\{Z \subset E : A = \overline{\text{co } Z}\}\)

(viii) \(Y_0\) is a minimal element of \(\{Z \subset E, Z \text{ is compact, } A = \overline{\text{co } Z}\}\),

(ix) \(Y\) is a \(p_A\)-orthonormal set.

Proof. By Theorem 1.4 we may assume \(A = B(0,1)^I\). Choose \(\{e_i : i \in I\}\) as in the second part of the proof of Theorem 1.4. We leave the details of checking (i)-(ix) to the reader.
§ 2 GENERAL COMPACTOIDS

THEOREM 2.1 Let \( A \) be a compactoid in a locally convex space \( E \) over \( K \). Then there exists a locally convex space \( F \) over \( K \) containing \( E \) as a subspace and a compact set \( X \subset F \) such that \( A \subset \text{co} \ X \).

Proof. For each continuous seminorm \( p \) on \( E \), let \( E_p := E / \text{Kerp} \) with the norm induced by \( p \). The natural maps \( \pi_p : E \to E_p \) yield a linear homeomorphic embedding

\[
\pi : E \to F := \prod_{p \in \Gamma} E_p
\]

where \( \Gamma \) is the collection of continuous seminorms of \( E \). For each \( p \in \Gamma \) the set \( \pi_p(A) \) is a (metrizable) compactoid in \( E_p \). By [4], Proposition 8.2 there is a compact set \( X_p \subset E_p \) such that \( \pi_p(A) \subset \text{co} X_p \). Without loss we may assume that \( 0 \in X_p \). We have

\[
\pi(A) \subset \prod_{p \in \Gamma} \pi(A) \subset \prod_{p \in \Gamma} \text{co} X_p
\]

We claim that \( \prod_{p \in \Gamma} \text{co} X_p \subset \text{co} \prod_{p \in \Gamma} X_p \). (Then the theorem is proved with \( X := \prod_{p \in \Gamma} X_p \).) For \( p \in \Gamma \) and \( x \in X \) the element \( f \) defined by

\[
f(q) = \begin{cases} x & \text{if } q \in \Gamma, q = p \\ 0 & \text{if } q \in \Gamma, q \neq p \end{cases}
\]

is in \( \prod_{p \in \Gamma} X_p \). If \( p \in \Gamma \) and \( x \in \text{co} X_p \) then \( f \), formally defined by \((*)\),

is in \( \text{co} \prod_{p \in \Gamma} X_p \). If \( p_1, \ldots, p_n \in \Gamma \) and \( x_{p_i} \in \text{co} X_{p_i} \) for \( i \in \{1, \ldots, n\} \) the element \( g \) defined by

\[
g(q) = \begin{cases} x_{p_i} & \text{if } q \in \Gamma, i \in \{1, \ldots, n\}, q = p_i \\ 0 & \text{if } q \in \Gamma, \text{otherwise} \end{cases}
\]
is a finite sum of elements of $\bigcap_{p \in \Gamma} X_p$, hence in $\bigcap_{p \in \Gamma} X_p$. The elements of the type defined in (***) are dense in $\prod_{p \in \Gamma} \overline{co} X_p$.

**COROLLARY 2.2** Let $A$ be an absolutely convex compactoid in a locally convex space over $K$. Then $A$ is isomorphic to a submodule of some power of $B(0,1)$.

**Proof.** By the previous theorem, $A \subseteq \overline{co} X$ for some compact $X \subseteq F$. We may suppose that $F$ is complete. Now apply Theorem 1.4.

**Note to Theorem 2.1.** It is too optimistic to hope that in Theorem 2.1 we may require that $F = E$ even when we allow $X$ to be precompact. In fact, let $K$ be not locally compact, let $E = c_0$, with the weak topology. $A := \{x \in E : \|x\| \leq 1\}$ is a weak compactoid but according to [5], Proposition 3.3 there is no weakly precompact $X \subseteq E$ for which $A \subseteq \overline{co} X$. Observe that, if $K$ is not spherically complete, $A$ is even weakly complete ([4], Theorem 9.6), and that $A$ is pure if the valuation of $K$ is discrete.
REFERENCES


