THE CLOSED CONVEX HULL OF A COMPACT SET
IN A NON-ARCHIMEDEAN LOCALLY CONVEX SPACE

by

W.H. SCHIKHOF

Report 8646
October 1986

DEPARTMENT OF MATHEMATICS
CATHOLIC UNIVERSITY
Toernooiveld
6525 ED Nijmegen
The Netherlands
THE CLOSED CONVEX HULL OF A COMPACT SET

IN A NON-ARCHIMEDEAN LOCALLY CONVEX SPACE

by

W.H. Schikhof

ABSTRACT. For a complete absolutely convex set $A$ in a locally convex space over a non-archimedean valued field $K$ it is proved that

(i) $A$ is the closed absolutely convex hull of a compact set if and only if $A$ is isomorphic to some power of $B(0,1) := \{ \lambda \in K : |\lambda| \leq 1 \}$,

(ii) if the valuation of $K$ is discrete and $A$ is a compactoid (equivalently; $A$ is c-compact and bounded) then $A$ is the closed absolutely convex hull of a compact set,

(iii) the conclusion of (ii) is also true for any $K$ if $A$ is a metrizable pure compactoid,

(iv) if $A$ is a compactoid it is isomorphic to a closed submodule of some power of $B(0,1)$.

These results extend those (for a locally compact base field) of Carpentier ([1], Propositions 72,73). Corollary 1.8 is a non-archimedean approach to the Krein-Milman Theorem.
PRELIMINARIES. Throughout $K$ is a non-archimedean nontrivially valued complete field with valuation $|\cdot|$. For fundamentals on locally convex spaces $E$ over $K$ (which we assume to be Hausdorff) we refer to [8], [7], [3], [4], [1]. A set $A \subset E$ is absolutely convex if it is a $B(0,1)$-module. If $F$ is a locally convex space over $K$ and $A \subset E$, $B \subset F$ are absolutely convex then $\phi : A \to B$ is affine if it is a homeomorphism of $B(0,1)$-modules. We shall write $A = B$ if there exists an affine homeomorphism of $A$ onto $B$. For a set $X \subset E$, let $\text{co} \ X$ be its absolutely convex hull and $\overline{\text{co}} \ X$ be its closure.

An absolutely convex set $A \subset E$ is edged if for each $x \in E$ the set 
$$\{ |\lambda| : \lambda x \in A \}$$

is closed in $|K| := \{ |\lambda| : \lambda \in K \}$ (or, equivalently, if $A = \{ x \in [A] : \varrho_A(x) \leq 1 \}$, where $\varrho_A$ is the Minkowski function, defined on the $K$-linear span $[A]$ of $A$ by the formula

$$\varrho_A(x) = \inf \{ |\lambda| : x \in \lambda A \}.$$  

It is easy to prove that if the valuation of $K$ is discrete each absolutely convex set is edged whereas, if the valuation of $K$ is dense, an absolutely convex $A \subset E$ is edged if and only if $\lambda x \in A$ for all $\lambda \in K$, $|\lambda| < 1$ implies $x \in A$.

For a subset $A$ of $E$, let $A^0 := \{ f \in E' : |f(x)| \leq 1 \text{ for all } x \in A \}$ (where $E'$ is the dual space of $E$) and let $A^{\infty} := \{ x \in E : |f(x)| \leq 1 \text{ for all } f \in A \}$. $A$ is a polar set if $A = A^{\infty}$.

A set $A \subset E$ is (a) compactoid if for each neighbourhood $U$ of $0$ there exist $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in E$ such that $A \subset U + \text{co}(x_1, \ldots, x_n)$; it is a pure compactoid if in the above we may choose $x_1, \ldots, x_n \in A$. If the valuation of $K$ is discrete each absolutely convex compactoid is pure (for example [4], Lemma 8.1), if the valuation of $K$ is dense

$$\{ \lambda \in K : |\lambda| < 1 \}$$

is a compactoid in $E := K$ but not pure.
§ 1 COMPLETE COMPACTOIDS

LEMMA 1.1 Let $E$ be a locally convex space over $K$. Let $A \subset E$ be a complete, absolutely convex, edged, absorbing compactoid. Assume that a seminorm on $E$ is continuous if its restriction to $A$ is continuous.

(i) $E$ is of countable type ([4], Definition 4.3).

(ii) $A$ is a polar set.

(iii) $E'$ is a Banach space over $K$ with respect to the norm $\| \cdot \|_A$ defined by $\| f \|_A = \sup \{ |f(x)| : x \in A \}$.

(iv) If $A = \text{co } X$ for some compact set $X \subset A$ then $(E', \| \cdot \|_A)$ has an orthonormal base.

(v) The canonical map $E \to (E', \| \cdot \|_A)'$ is a bijection.

Proof.

(i) [6], Proposition 4.3.

(ii) [4], Theorem 4.7.

(iii) As $A$ is absorbing $\| \cdot \|_A$ is a norm on $E'$. If $f_1, f_2, \ldots$ is a $\| \cdot \|_A$-Cauchy sequence in $E'$ then there is a linear $f : E \to K$ such that $f = \lim_{n \to \infty} f_n$ uniformly on $A$. Then $|f|$, restricted to $A$, is continuous. By assumption, $|f|$ is continuous. Hence, $f \in E'$ and $\lim_{n \to \infty} \| f - f_n \|_A = 0$.

(iv) Let $C(X \to K)$ be the Banach space of all continuous functions $X \to K$ with the supremum norm. For each $f \in E'$ we have

$$\| f \|_A = \sup_{A} |f| = \sup_{\text{co } X} |f| = \sup_{X} |f|$$

so that the map $T : (E', \| \cdot \|_A) \to C(X \to K)$ given by $Tf := f|_X$ is a linear isometry. By [3], Theorem 5.22, $C(X \to K)$ has an
orthonormal base. Then so has its closed subspace \( \text{Im } T \) by Gruson's Theorem ([3], Theorem 5.9) and has \((E',|| ||_A)\).

(iv) Contained in the proof of [6], Theorem 3.2 (the metrizability condition is not needed for part (ii) of that proof).

**Lemma 1.2** Let \( E, A || ||_A \) be as in Lemma 1.1. Suppose \((E',|| ||_A)\) has an orthonormal base \( \{f_i : i \in I\} \). Then \( A \cong B(0,1)^I \).

**Proof.** The formula

\[
\Phi(x) = (f_i(x))_i \in I \quad (x \in E)
\]

defines a continuous linear map \( \Phi : E \to K^I \) (on \( K^I \) the product topology) sending \( A \) into \( B(0,1)^I \). We prove (i), (ii) below.

(i) \( \Phi|A \) is a homeomorphism into \( B(0,1)^I \). **Proof.** Let \( (x_j)_j \in J \) be a net in \( A \) for which \( \lim_J \Phi(x_j) = 0 \) i.e. \( \lim_J f_i(x_j) = 0 \) for all \( i \in I \). Then \( \lim_J g(x_j) = 0 \) for all \( g \) in a \( || ||_A \)-dense subset \( H \) of \( E' \). Let \( f \in E', \varepsilon > 0 \). There is a \( g \in H \) with \( ||f-g||_A < \varepsilon \). For large \( j \)

\[
|f(x_j)| \leq \max \{|f(x_j) - g(x_j)|, |g(x_j)|\} < \varepsilon
\]

so that \( \lim_J x_j = 0 \) weakly. But then \( \lim_J x_j = 0 \) for the initial topology of \( E \) ([4], Theorem 5.12).

(ii) \( \Phi \) maps \( A \) onto \( B(0,1)^I \). **Proof.** Let \( z := (z_i)_i \in I \in B(0,1)^I \). Define \( h \in (E',|| ||_A)' \) by

\[
h(f_i) = z_i \quad (i \in I)
\]

By Lemma 1.1 (v) there exists an \( x \in E \) with \( f(x) = h(f) \) for all \( f \in E' \) i.e. with \( \Phi(x) = z \). To prove that in fact \( x \in A^{\circ\circ} = A \).
(Lemma 1.1 (ii)), let \( f \in E', f \in A^0 \). Then \( \|f\|_A \leq 1 \). There exist \( \lambda_i \in K \) for which \( f = \sum \lambda_i f_i \) in the sense of \( \|f\|_A \). By orthonormality

\[
\|f\|_A = \max |\lambda_i| \leq 1.
\]

We see that \( |f(x)| \leq \max_i |\lambda_i f_i(x)| = \max_i |\lambda_i z_i| \leq 1 \). It follows that \( x \in A^0 \).

**PROPOSITION 1.3** Let \( X \) be a compact subset of a locally convex space \( E \) over \( K \). Then \( \text{co } X \) is edged.

**Proof.** We may assume that the valuation of \( K \) is dense. Let \( z \in E, z \notin \text{co } X \). There is ([6], Proposition 4.2) a continuous seminorm \( p \) with \( p(z) = 1 \) and \( p < 1 \) on \( \text{co } X \). By compactness, \( s := \sup \{p(x) : x \in \text{co } X\} \),

\[
\sup X p = \max p < 1.
\]

Hence, there is a \( \lambda \in K, |\lambda| < 1 \), such that \( p(\lambda z) > s \) i.e. \( \lambda z \notin \text{co } X \).

**THEOREM 1.4** Let \( A \) be a complete absolutely convex compactoid in a locally convex space \( E \) over \( K \). The following are equivalent.

(a) There is a compact set \( X \subset A \) with \( A = \text{co } X \).

(\( \beta \)) \( A = B(0,1)^I \) for some set \( I \).

**Proof.** (a) \( \Rightarrow \) (\( \beta \)). We may assume that \( E = [A] \). If we replace the initial topology \( \tau \) of \( E \) by the stronger locally convex topology \( \tau' \) generated by all seminorms \( p \) on \( E \) for which \( p|A \) is \( \tau \)-continuous then \( \tau = \tau' \) on \( A \) and \( A \) is \( \tau' \)-complete and a \( \tau' \)-compactoid ([6], Proposition 4.5). Therefore, to prove (\( \beta \)), we may assume \( \tau = \tau' \). Now apply Proposition 1.3, Lemma 1.1 (iv), Lemma 1.2.

(a) \( \Rightarrow \) (\( \beta \)). Let \( e_i \in B(0,1)^I \) (\( i \in I \)) be given by
It is easily seen that \( Y := \{0\} \cup \{e_i : i \in I\} \) is compact and that \( B(0,1) = \text{co} \ Y \).

**Theorem 1.5** Let the valuation of \( K \) be discrete. Let \( A \) be a complete absolutely convex compactoid in a locally convex space \( E \) over \( K \). (Or, equivalently, let \( A \) be bounded, absolutely convex and \( c \)-compact ([6], Corollary 2.5).) Then there exists a compact set \( X \subset A \) with \( A = \text{co} X \).

**Proof.** For the same reasons as in the previous proof we may assume that \( E = [A] \) and that a seminorm \( p \) on \( E \) is continuous if \( p|A \) is continuous. By Lemma 1.1 (iii), \((E', \|\|_A)\) is a Banach space. As the valuation is discrete we have

\[
\|f\|_A = \sup_{x \in A} |f(x)| \in |K| \quad (f \in E')
\]

Then by [3], Theorem 5.16, \((E', \|\|_A)\) has an orthonormal base. Now apply Lemma 1.2 and Theorem 1.4.

For general \( K \) not every edged complete absolutely convex compactoid is the closed convex hull of a compact set. In fact, if the valuation of \( K \) is dense and \( r \in (0,\infty) \setminus |K| \) then \( A := \{\lambda \in K : |\lambda| \leq r\} \) is edged but there is no compact set \( X \subset K \) for which \( A = \text{co} X \). Indeed, we have the following.

**Proposition 1.6** Let \( X \) be a compact subset of a locally convex space \( E \) over \( K \). Then \( \text{co} X \) is a pure compactoid.

**Proof.** Let \( U \) be an absolutely convex neighbourhood of \( 0 \) in \( E \). By com-
pactness there exist $x_1, \ldots, x_n \in X$ such that $X \subseteq U(x_1 + U)$. Then $\text{co } X \subseteq U + \text{co}(x_1, \ldots, x_n)$. The set $U + \text{co}(x_1, \ldots, x_n)$ is an open additive subgroup of $E$, hence closed. It follows that $\text{co } X \subseteq U + \text{co}(x_1, \ldots, x_n)$.

Note. One can prove that each closed pure absolutely convex compactoid is edged.

For metrizable pure compactoids we have the following version of Theorem 1.5 for general $K$.

**THEOREM 1.7** Let $A \subseteq E$ be a complete absolutely convex pure compactoid that is metrizable. Then there is a sequence $e_1, e_2, \ldots$ in $A$ with $\lim_{n \to \infty} e_n = 0$ and $A = \text{co}(e_1, e_2, \ldots)$.

**Proof.** The proof of [4], Proposition 8.2 applies with some minor modifications (as $A$ is pure the finite sets $F_1, F_2, \ldots$ constructed in that proof can be chosen in $A$ rather than in $\lambda A$).

**OPEN PROBLEM** Let $A$ be complete absolutely convex pure compactoid in a locally convex space $E$ over $K$. Does it follow that $A = \text{co } X$ for some compact $X$?

The previous theory yields the following.

**COROLLARY 1.8** Let $A$ be a complete subset of a locally convex space $E$ over $K$ such that $A = \text{co } X$ for some compact set $X$ (e.g. choose for $A$ any complete absolutely convex compactoid if the valuation of $K$ is discrete or any complete absolutely convex pure metrizable compactoid). Then there exists a linearly independent set
\[ Y = \{ e_i : i \in I \} \text{ in } A \text{ such that} \]

(i) \( Y \) is discrete,
(ii) for each neighbourhood \( U \) of 0 the set \( \{ i \in I : e_i \notin U \} \) is finite,
(iii) \( Y_0 := Y \cup \{ 0 \} \) is compact,
(iv) \( A = \text{co} \ Y = \text{co} \ Y_0 \),
(v) for each \( (\lambda_i)_{i \in I} \in B(0,1)^I \), \( \sum_{i \in I} \lambda_i e_i \) converges and represents an element of \( A \),
(vi) each \( x \in A \) has a unique representation as a convergent sum \( x = \sum_{i \in I} \lambda_i e_i \) where \( \lambda_i \in B(0,1) \) for each \( i \in I \),
(vii) \( Y \) is a minimal element of \{ \( Z \subset E : A = \overline{\text{co} \ Z} \) \}
(viii) \( Y_0 \) is a minimal element of \{ \( Z \subset E, Z \) is compact, \( A = \overline{\text{co} \ Z} \) \},
(ix) \( Y \) is a \( p_A \)-orthonormal set.

Proof. By Theorem 1.4 we may assume \( A = B(0,1)^I \). Choose \( \{ e_i : i \in I \} \) as in the second part of the proof of Theorem 1.4. We leave the details of checking (i)-(ix) to the reader.
§ 2 GENERAL COMPACTOIDS

THEOREM 2.1 Let $A$ be a compactoid in a locally convex space $E$ over $K$. Then there exists a locally convex space $F$ over $K$ containing $E$ as a subspace and a compact set $X \subset F$ such that $A \subset \overline{\text{co } X}$.

Proof. For each continuous seminorm $p$ on $E$, let $E_p := E/\text{Kerp}$ with the norm induced by $p$. The natural maps $\pi_p : E \to E_p$ yield a linear homeomorphic embedding

$$\pi : E \to F := \prod_{p \in \Gamma} E_p$$

where $\Gamma$ is the collection of continuous seminorms of $E$. For each $p \in \Gamma$ the set $\pi_p(A)$ is a (metrizable) compactoid in $E_p$. By [4], Proposition 8.2 there is a compact set $X_p \subset E_p$ such that $\pi_p(A) \subset \overline{\text{co } X_p}$. Without loss we may assume that $0 \in X_p$. We have

$$\pi(A) \subset \prod_{p \in \Gamma} \pi(A) \subset \prod_{p \in \Gamma} \overline{\text{co } X_p}.$$

We claim that $\prod_{p \in \Gamma} \overline{\text{co } X_p} \subset \overline{\text{co } \prod_{p \in \Gamma} X_p}$. (Then the theorem is proved with $X := \prod_{p \in \Gamma} X_p$.) For $p \in \Gamma$ and $x \in X_p$ the element $f$ defined by

$$(*) \quad f(q) = \begin{cases} x & \text{if } q \in \Gamma, q = p \\ 0 & \text{if } q \in \Gamma, q \neq p \end{cases}$$

is in $\prod_{p \in \Gamma} X_p$. If $p \in \Gamma$ and $x \in \overline{\text{co } X_p}$ then $f$, formally defined by $(*)$, $p \in \Gamma$, is in $\overline{\text{co } \prod_{p \in \Gamma} X_p}$. If $p_1, \ldots, p_n \in \Gamma$ and $x_{p_i} \in \overline{\text{co } X_{p_i}}$ for $i \in \{1, \ldots, n\}$ the element $g$ defined by

$$(**) \quad g(q) = \begin{cases} x_{p_i} & \text{if } q \in \Gamma, i \in \{1, \ldots, n\}, q = p_i \\ 0 & \text{if } q \in \Gamma, \text{ otherwise} \end{cases}$$
is a finite sum of elements of $\overline{\co} \cap X_p$, hence in $\overline{\co} \cap X$. The elements of the type defined in (**) are dense in $\prod_{p \in \Gamma} \overline{\co} X_p$.

COROLLARY 2.2 Let $A$ be an absolutely convex compactoid in a locally convex space over $K$. Then $A$ is isomorphic to a submodule of some power of $B(0,1)$.

**Proof.** By the previous theorem, $A \subset \overline{\co} X$ for some compact $X \subset F$. We may suppose that $F$ is complete. Now apply Theorem 1.4.

**Note to Theorem 2.1.** It is too optimistic to hope that in Theorem 2.1 we may require that $F = E$ even when we allow $X$ to be precompact. In fact, let $K$ be not locally compact, let $E = c_0$, with the weak topology. $A := \{x \in E : \|x\| \leq 1\}$ is a weak compactoid but according to [5], Proposition 3.3 there is no weakly precompact $X \subset E$ for which $A \subset \overline{\co} X$. Observe that, if $K$ is not spherically complete, $A$ is even weakly complete ([4], Theorem 9.6), and that $A$ is pure if the valuation of $K$ is discrete.
REFERENCES


