THE CLOSED CONVEX HULL OF A COMPACT SET IN A NON-ARCHIMEDEAN LOCALLY CONVEX SPACE

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ABSTRACT. For a complete absolutely convex set $A$ in a locally convex space over a non-archimedean valued field $K$ it is proved that

(i) $A$ is the closed absolutely convex hull of a compact set if and only if $A$ is isomorphic to some power of $B(0,1) := \{ \lambda \in K : |\lambda| \leq 1 \}$,

(ii) if the valuation of $K$ is discrete and $A$ is a compactoid (equivalently, $A$ is c-compact and bounded) then $A$ is the closed absolutely convex hull of a compact set,

(iii) the conclusion of (ii) is also true for any $K$ if $A$ is a metrizable pure compactoid,

(iv) if $A$ is a compactoid it is isomorphic to a closed submodule of some power of $B(0,1)$.

These results extend those (for a locally compact base field) of Carpentier ([1], Propositions 72,73). Corollary 1.8 is a non-archimedean approach to the Krein-Milman Theorem.
Throughout $K$ is a non-archimedean nontrivially valued complete field with valuation $|\cdot|$. For fundamentals on locally convex spaces $E$ over $K$ (which we assume to be Hausdorff) we refer to [8], [7], [3], [4], [1]. A set $A \subset E$ is **absolutely convex** if it is a $B(0,1)$-module. If $F$ is a locally convex space over $K$ and $A \subset E$, $B \subset F$ are absolutely convex then $\phi : A \to B$ is **affine** if it is a homeomorphism of $B(0,1)$-modules. We shall write $A = B$ if there exists an affine homeomorphism of $A$ onto $B$. For a set $X \subset E$, let $\text{co} \ X$ be its absolutely convex hull and $\overline{\text{co}} \ X$ be its closure.

An absolutely convex set $A \subset E$ is **edged** if for each $x \in E$ the set $\{|\lambda| : \lambda x \in A\}$ is closed in $|K| := \{|\lambda| : \lambda \in K\}$ (or, equivalently, if $A = \{x \in [A] : p_A(x) \leq 1\}$, where $p_A$ is the Minkowski function, defined on the $K$-linear span $[A]$ of $A$ by the formula

$$p_A(x) = \inf \{|\lambda| : x \in \lambda A\}.$$  
It is easy to prove that if the valuation of $K$ is discrete each absolutely convex set is edged whereas, if the valuation of $K$ is dense, an absolutely convex $A \subset E$ is edged if and only if $\lambda x \in A$ for all $\lambda \in K$, $|\lambda| < 1$ implies $x \in A$.

For a subset $A$ of $E$, let $A^\circ := \{f \in E' : |f(x)| \leq 1 \text{ for all } x \in A\}$ (where $E'$ is the dual space of $E$) and let $A^{\circ \circ} := \{x \in E : |f(x)| \leq 1 \text{ for all } f \in A^\circ\}$. $A$ is a **polar set** if $A = A^{\circ \circ}$.

A set $A \subset E$ is (a) **compactoid** if for each neighbourhood $U$ of 0 there exist $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in E$ such that $A \subset U + \text{co}(x_1, \ldots, x_n)$; it is a **pure compactoid** if in the above we may choose $x_1, \ldots, x_n \in A$. If the valuation of $K$ is discrete each absolutely convex compactoid is pure (for example [4], Lemma 8.1), if the valuation of $K$ is dense $\{\lambda \in K : |\lambda| < 1\}$ is a compactoid in $E := K$ but not pure.
§ 1 COMPLETE COMPACTOIDS

LEMMA 1.1 Let $E$ be a locally convex space over $K$. Let $A \subset E$ be a complete, absolutely convex, edged, absorbing compactoid. Assume that a seminorm on $E$ is continuous if its restriction to $A$ is continuous.

(i) $E$ is of countable type ([4], Definition 4.3).

(ii) $A$ is a polar set.

(iii) $E'$ is a Banach space over $K$ with respect to the norm $\| \cdot \|_A$ defined by $\|f\|_A = \sup \{|f(x)| : x \in A\}$.

(iv) If $A = \overline{\text{co}} X$ for some compact set $X \subset A$ then $(E', \| \cdot \|_A)$ has an orthonormal base.

(v) The canonical map $E \to (E', \| \cdot \|_A')$ is a bijection.

Proof.

(i) [6], Proposition 4.3.

(ii) [4], Theorem 4.7.

(iii) As $A$ is absorbing $\| \cdot \|_A$ is a norm on $E'$. If $f_1, f_2, \ldots$ is a $\| \cdot \|_A$-Cauchy sequence in $E'$ then there is a linear $f : E \to K$ such that $f = \lim_{n \to \infty} f_n$ uniformly on $A$. Then $|f|$, restricted to $A$, is continuous. By assumption, $|f|$ is continuous. Hence, $f \in E'$ and $\lim_{n \to \infty} \|f-f_n\|_A = 0$.

(iv) Let $C(X \to K)$ be the Banach space of all continuous functions $X \to K$ with the supremum norm. For each $f \in E'$ we have

$$\|f\|_A = \sup_A |f| = \sup_{\text{co} X} |f| = \sup_X |f|$$

so that the map $T : (E', \| \cdot \|_A) \to C(X \to K)$ given by $Tf := f|_X$ is a linear isometry. By [3], Theorem 5.22, $C(X \to K)$ has an
orthonormal base. Then so has its closed subspace \( \text{Im } T \) by Gruson's Theorem ([3], Theorem 5.9) and has \( (E',\| \cdot \|_A) \).

(iv) Contained in the proof of [6], Theorem 3.2 (the metrizability condition is not needed for part (ii) of that proof).

**Lemma 1.2** Let \( E, A \) be as in Lemma 1.1. Suppose \( (E',\| \cdot \|_A) \) has an orthonormal base \( \{f_i : i \in I\} \). Then \( A \cong B(0,1)^I \).

**Proof.** The formula

\[
\Phi(x) = (f_i(x))_{i \in I} \quad (x \in E)
\]

defines a continuous linear map \( \Phi : E \to K^I \) (on \( K^I \) the product topology) sending \( A \) into \( B(0,1)^I \). We prove (i), (ii) below.

(i) \( \Phi|A \) is a homeomorphism into \( B(0,1)^I \). **Proof.** Let \( (x_j)_{j \in J} \) be a net in \( A \) for which \( \lim_j \Phi(x_j) = 0 \) i.e. \( \lim_j f_i(x_j) = 0 \) for all \( i \in I \). Then \( \lim_j g(x_j) = 0 \) for all \( g \) in a \( \| \cdot \|_A \)-dense subset \( H \) of \( E' \). Let \( f \in E', \varepsilon > 0 \). There is a \( g \in H \) with \( \| f-g \|_A < \varepsilon \). For large \( j \)

\[
|f(x_j)| \leq \max \{|f(x_j)-g(x_j)|, |g(x_j)|\} < \varepsilon
\]

so that \( \lim_j x_j = 0 \) weakly. But then \( \lim_j x_j = 0 \) for the initial topology of \( E \) ([4], Theorem 5.12).

(ii) \( \Phi \) maps \( A \) onto \( B(0,1)^I \). **Proof.** Let \( z := (z_i)_{i \in I} \in B(0,1)^I \). Define \( h \in (E',\| \cdot \|_A)' \) by

\[
h(f_i) = z_i \quad (i \in I)
\]

By Lemma 1.1 (v) there exists an \( x \in E \) with \( f(x) = h(f) \) for all \( f \in E' \) i.e. with \( \Phi(x) = z \). To prove that in fact \( x \in A^{\infty} = A \)
(Lemma 1.1 (ii)), let $f \in E'$, $f \in A^\circ$. Then $\|f\|_A \leq 1$. There exist
\[ \lambda_i \in K \text{ for which } f = \sum_{i \in I} \lambda_i f_i \text{ in the sense of } \| \cdot \|_A. \] By orthonormality
\[ \|f\|_A = \max |\lambda_i| \leq 1. \]
We see that $|f(x)| \leq \max |\lambda_i f_i(x)| = \max |\lambda_i z_i| \leq 1$. It follows that
\[ x \in A^\circ. \]

PROPOSITION 1.3 Let $X$ be a compact subset of a locally convex space $E$ over $K$. Then $\co X$ is edged.

Proof. We may assume that the valuation of $K$ is dense. Let $z \in E$,
\[ z \notin \co X. \] There is ([6], Proposition 4.2) a continuous seminorm $p$
with $p(z) = 1$ and $p < 1$ on $\co X$. By compactness, $s := \sup\{p(x) : x \in \co X\}$
\[ = \sup_X p = \max_X p < 1. \]
Hence, there is a $\lambda \in K$, $|\lambda| < 1$, such that
\[ p(\lambda z) > s \text{ i.e. } \lambda z \notin \co X. \]

THEOREM 1.4 Let $A$ be a complete absolutely convex compactoid in a
locally convex space $E$ over $K$. The following are equivalent.

(a) There is a compact set $X \subset A$ with $A = \co X$.
(b) $A = B(0,1)^I$ for some set $I$.

Proof. (a) $\Rightarrow$ (b). We may assume that $E = [A]$. If we replace the ini-
tial topology $\tau$ of $E$ by the stronger locally convex topology $\tau'$ gene-
rated by all seminorms $p$ on $E$ for which $p|A$ is $\tau$-continuous then
\[ \tau = \tau' \text{ on } A \text{ and } A \text{ is } \tau'-\text{complete and a } \tau'-\text{compactoid ([6], Proposition 4.5).} \]
Therefore, to prove (b), we may assume $\tau = \tau'$. Now apply Proposition
1.3, Lemma 1.1 (iv), Lemma 1.2.

(a) $\Rightarrow$ (b). Let $e_i \in B(0,1)^I$ ($i \in I$) be given by
It is easily seen that \( Y := \{0\} \cup \{e^j : i \in I\} \) is compact and that \( B(0,1)^I = \text{co} \ Y \).

**THEOREM 1.5** Let the valuation of \( K \) be discrete. Let \( A \) be a complete absolutely convex compactoid in a locally convex space \( E \) over \( K \). (Or, equivalently, let \( A \) be bounded, absolutely convex and \( c \)-compact ([6], Corollary 2.5).) Then there exists a compact set \( X \subset A \) with \( A = \text{co} \ X \).

**Proof.** For the same reasons as in the previous proof we may assume that \( E = [A] \) and that a seminorm \( p \) on \( E \) is continuous if \( p|A \) is continuous. By Lemma 1,1 (iii), \( (E', \| \cdot \|_A) \) is a Banach space. As the valuation is discrete we have

\[
\|f\|_A = \sup_{x \in A} |f(x)| \leq |K| \quad (f \in E')
\]

Then by [3], Theorem 5.16, \( (E', \| \cdot \|_A) \) has an orthonormal base. Now apply Lemma 1.2 and Theorem 1.4.

For general \( K \) not every edged complete absolutely convex compactoid is the closed convex hull of a compact set. In fact, if the valuation of \( K \) is dense and \( r \in (0,\infty) \setminus |K| \) then \( A := \{\lambda \in K : |\lambda| \leq r\} \) is edged but there is no compact set \( X \subset K \) for which \( A = \text{co} \ X \). Indeed, we have the following.

**PROPOSITION 1.6** Let \( X \) be a compact subset of a locally convex space \( E \) over \( K \). Then \( \text{co} \ X \) is a pure compactoid.

**Proof.** Let \( U \) be an absolutely convex neighbourhood of \( 0 \) in \( E \). By com-
pactness there exist \(x_1, \ldots, x_n \in X\) such that \(X \subseteq U(x_1 + U)\). Then
\[
\text{co } X \subseteq U + \text{co}\{x_1, \ldots, x_n\}.
\]
The set \(U + \text{co}\{x_1, \ldots, x_n\}\) is an open additive subgroup of \(E\), hence closed. It follows that \(\text{co } X \subseteq U + \text{co}\{x_1, \ldots, x_n\}\).

Note. One can prove that each closed pure absolutely convex compactoid is edged.

For metrizable pure compactoids we have the following version of Theorem 1.5 for general \(K\).

**Theorem 1.7** Let \(A \subseteq E\) be a complete absolutely convex pure compactoid that is metrizable. Then there is a sequence \(e_1, e_2, \ldots \) in \(A\) with
\[
\lim_{n \to \infty} e_n = 0 \quad \text{and} \quad A = \text{co}\{e_1, e_2, \ldots\}.
\]

**Proof.** The proof of [4], Proposition 8.2 applies with some minor modifications (as \(A\) is pure the finite sets \(F_1, F_2, \ldots\) constructed in that proof can be chosen in \(A\) rather than in \(\lambda A\)).

**Open Problem** Let \(A\) be complete absolutely convex pure compactoid in a locally convex space \(E\) over \(K\). Does it follow that \(A = \text{co } X\) for some compact \(X\) ?

The previous theory yields the following.

**Corollary 1.8** Let \(A\) be a complete subset of a locally convex space \(E\) over \(K\) such that \(A = \text{co } X\) for some compact set \(X\) (e.g. choose for \(A\) any complete absolutely convex compactoid if the valuation of \(K\) is discrete or any complete absolutely convex pure metrizable compactoid). Then there exists a linearly independent set.
\[
Y = \{e_i : i \in I\} \text{ in } A \text{ such that }
\]

(i) \(Y\) is discrete,

(ii) for each neighbourhood \(U\) of 0 the set \(\{i \in I : e_i \notin U\}\) is finite,

(iii) \(Y_0 := Y \cup \{0\}\) is compact,

(iv) \(A = \overline{\text{co}}\ Y = \overline{\text{co}}\ Y_0\),

(v) for each \((\lambda_i)_{i \in I} \in B(0,1)^I\), \(\sum_{i \in I} \lambda_i e_i\) converges and represents an element of \(A\),

(vi) each \(x \in A\) has a unique representation as a convergent sum \(x = \sum_{i \in I} \lambda_i e_i\) where \(\lambda_i \in B(0,1)\) for each \(i \in I\),

(vii) \(Y\) is a minimal element of \(\{Z \subset E : A = \overline{\text{co}}\ Z\}\)

(viii) \(Y_0\) is a minimal element of \(\{Z \subset E, Z \text{ is compact}, A = \overline{\text{co}}\ Z\}\),

(ix) \(Y\) is a \(p_A\)-orthonormal set.

Proof. By Theorem 1.4 we may assume \(A = B(0,1)^I\). Choose \(\{e_i : i \in I\}\) as in the second part of the proof of Theorem 1.4. We leave the details of checking (i)-(ix) to the reader.
THEOREM 2.1 Let $A$ be a compactoid in a locally convex space $E$ over $K$. Then there exists a locally convex space $F$ over $K$ containing $E$ as a subspace and a compact set $X \subset F$ such that $A \subset \text{co } X$.

Proof. For each continuous seminorm $p$ on $E$, let $E_p := E/\text{Kerp}$ with the norm induced by $p$. The natural maps $\pi_p : E \to E_p$ yield a linear homeomorphism

$$\pi : E \to F := \prod_{p \in \Gamma} E_p$$

where $\Gamma$ is the collection of continuous seminorms of $E$. For each $p \in \Gamma$ the set $\pi_p(A)$ is a (metrizable) compactoid in $E_p$. By [4], Proposition 8.2 there is a compact set $X_p \subset E_p$ such that $\pi_p(A) \subset \text{co } X_p$. Without loss we may assume that $0 \in X_p$. We have

$$\pi(A) \subset \prod_{p \in \Gamma} \pi(A) \subset \prod_{p \in \Gamma} \text{co } X_p$$

We claim that $\prod_{p \in \Gamma} \text{co } X_p \subset \text{co } \prod_{p \in \Gamma} X_p$. (Then the theorem is proved with $X := \prod_{p \in \Gamma} X_p$.) For $p \in \Gamma$ and $x \in X_p$ the element $f$ defined by

$$(*) \quad f(q) = \begin{cases} x & \text{if } q \in \Gamma, q = p \\ 0 & \text{if } q \in \Gamma, q \neq p \end{cases}$$

is in $\prod_{p \in \Gamma} X_p$. If $p \in \Gamma$ and $x \in \text{co } X_p$ then $f$, formally defined by $(*), p \in \Gamma$ is in $\text{co } \prod_{p \in \Gamma} X_p$. If $p_1, \ldots, p_n \in \Gamma$ and $x_p \in \text{co } X_p$ for $i \in \{1, \ldots, n\}$ the element $g$ defined by

$$(**) \quad g(q) = \begin{cases} x_{p_i} & \text{If } q \in \Gamma, i \in \{1, \ldots, n\}, q = p_i \\ 0 & \text{if } q \in \Gamma, \text{ otherwise} \end{cases}$$
is a finite sum of elements of $\bigcap_{p \in \Gamma} P X_p$, hence in $\bigcap_{p \in \Gamma} P X_p$. The elements of the type defined in (**) are dense in $\prod_{p \in \Gamma} \bigcap_{p \in \Gamma} X_p$.

**COROLLARY 2.2** Let $A$ be an absolutely convex compactoid in a locally convex space over $K$. Then $A$ is isomorphic to a submodule of some power of $B(0,1)$.

**Proof.** By the previous theorem, $A \subset \bigcap_{p \in \Gamma} P X$ for some compact $X \subset F$. We may suppose that $F$ is complete. Now apply Theorem 1.4.

**Note to Theorem 2.1.** It is too optimistic to hope that in Theorem 2.1 we may require that $F = E$ even when we allow $X$ to be precompact.

In fact, let $K$ be not locally compact, let $E = c_0$, with the weak topology. $A := \{ x \in E : \|x\| \leq 1 \}$ is a weak compactoid but according to [5], Proposition 3.3 there is no weakly precompact $X \subset E$ for which $A \subset \bigcap_{p \in \Gamma} X$. Observe that, if $K$ is not spherically complete, $A$ is even weakly complete ([4], Theorem 9.6), and that $A$ is pure if the valuation of $K$ is discrete.
REFERENCES


