THE CLOSED CONVEX HULL OF A COMPACT SET IN A NON-ARCHIMEDEAN LOCALLY CONVEX SPACE

by

W.H. SCHIKHOF

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DEPARTMENT OF MATHEMATICS
CATHOLIC UNIVERSITY
Toernooiveld
6525 ED Nijmegen
The Netherlands
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ABSTRACT. For a complete absolutely convex set \( A \) in a locally convex space over a non-archimedean valued field \( K \) it is proved that

(i) \( A \) is the closed absolutely convex hull of a compact set if and only if \( A \) is isomorphic to some power of \( B(0,1) := \{ \lambda \in K : |\lambda| \leq 1 \} \),

(ii) if the valuation of \( K \) is discrete and \( A \) is a compactoid (equivalently; \( A \) is c-compact and bounded) then \( A \) is the closed absolutely convex hull of a compact set,

(iii) the conclusion of (ii) is also true for any \( K \) if \( A \) is a metrizable pure compactoid,

(iv) if \( A \) is a compactoid it is isomorphic to a closed submodule of some power of \( B(0,1) \).

These results extend those (for a locally compact base field) of Carpentier ([1], Propositions 72,73). Corollary 1.8 is a non-archimedean approach to the Krein-Milman Theorem.
PRELIMINARIES. Throughout \( K \) is a non-archimedean nontrivially valued complete field with valuation \(| \cdot |\). For fundamentals on locally convex spaces \( E \) over \( K \) (which we assume to be Hausdorff) we refer to [8], [7], [3], [4], [1]. A set \( A \subset E \) is absolutely convex if it is a \( B(0,1) \)-module. If \( F \) is a locally convex space over \( K \) and \( A \subset E, B \subset F \) are absolutely convex then \( \phi : A \to B \) is affine if it is a homeomorphism of \( B(0,1) \)-modules. We shall write \( A = B \) if there exists an affine homeomorphism of \( A \) onto \( B \). For a set \( X \subset E \), let \( \overline{co} \ X \) be its absolutely convex hull and \( co \ X \) be its closure.

An absolutely convex set \( A \subset E \) is edged if for each \( x \in E \) the set \( \{ \lambda \mid \lambda x \in A \} \) is closed in \(|K| := \{ \lambda \mid \lambda \in K \} \) (or, equivalently, if \( A = \{ x \in [A] : \rho_A(x) \leq 1 \} \), where \( \rho_A \) is the Minkowski function, defined on the \( K \)-linear span \([A]\) of \( A \) by the formula

\[
\rho_A(x) = \inf \{ |\lambda| : x \in \lambda A \} .
\]

It is easy to prove that if the valuation of \( K \) is discrete each absolutely convex set is edged whereas, if the valuation of \( K \) is dense, an absolutely convex \( A \subset E \) is edged if and only if \( \lambda x \in A \) for all \( \lambda \in K, |\lambda| < 1 \) implies \( x \in A \).

For a subset \( A \) of \( E \), let \( A^\circ := \{ f \in E' : |f(x)| \leq 1 \text{ for all } x \in A \} \) (where \( E' \) is the dual space of \( E \)) and let \( A^{\circ\circ} := \{ x \in E : |f(x)| \leq 1 \text{ for all } f \in A^\circ \} \). \( A \) is a polar set if \( A = A^{\circ\circ} \).

A set \( A \subset E \) is (a) compactoid if for each neighbourhood \( U \) of \( 0 \) there exist \( n \in \mathbb{N} \) and \( x_1, \ldots, x_n \in E \) such that \( A \subset U + co(x_1, \ldots, x_n) \); it is a pure compactoid if in the above we may choose \( x_1, \ldots, x_n \in A \). If the valuation of \( K \) is discrete each absolutely convex compactoid is pure (for example [4], Lemma 8.1), if the valuation of \( K \) is dense \( \{ \lambda \in K : |\lambda| < 1 \} \) is a compactoid in \( E := K \) but not pure.
Lemma 1.1 Let $E$ be a locally convex space over $K$. Let $A \subset E$ be a complete, absolutely convex, absorbing compactoid. Assume that a seminorm on $E$ is continuous if its restriction to $A$ is continuous.

(i) $E$ is of countable type ([4], Definition 4.3).
(ii) $A$ is a polar set.
(iii) $E'$ is a Banach space over $K$ with respect to the norm $\| \cdot \|_A$ defined by $\|f\|_A = \sup \{ |f(x)| : x \in A \}$.
(iv) If $A = \text{co} X$ for some compact set $X \subset A$ then $(E', \| \cdot \|_A)$ has an orthonormal base.
(v) The canonical map $E \to (E', \| \cdot \|_A)'$ is a bijection.

Proof.

(i) [6], Proposition 4.3.
(ii) [4], Theorem 4.7.
(iii) As $A$ is absorbing $\| \cdot \|_A$ is a norm on $E$. If $f_1, f_2, \ldots$ is a $\| \cdot \|_A$-Cauchy sequence in $E'$ then there is a linear $f : E \to K$ such that $f = \lim_{n \to \infty} f_n$ uniformly on $A$. Then $|f|$, restricted to $A$, is continuous. By assumption, $|f|$ is continuous. Hence, $f \in E'$ and $\lim_{n \to \infty} \|f-f_n\|_A = 0$.
(iv) Let $C(X \to K)$ be the Banach space of all continuous functions $X \to K$ with the supremum norm. For each $f \in E'$ we have
\[
\|f\|_A = \sup_A |f| = \sup_{\text{co} X} |f| = \sup_X |f|
\]
so that the map $T : (E', \| \cdot \|_A) \to C(X \to K)$ given by $Tf := f|_X$ is a linear isometry. By [3], Theorem 5.22, $C(X \to K)$ has an
orthonormal base. Then so has its closed subspace \( \text{Im } T \) by Gruson's Theorem ([3], Theorem 5.9) and has \((\mathcal{E}', || \cdot ||_A)\).

(iv) Contained in the proof of [6], Theorem 3.2 (the metrizability condition is not needed for part (ii) of that proof).

**Lemma 1.2** Let \( \mathcal{E}, \mathcal{A}, || \cdot ||_A \) be as in Lemma 1.1. Suppose \((\mathcal{E}', || \cdot ||_A)\) has an orthonormal base \( \{ f_i : i \in I \} \). Then \( \mathcal{A} = B(0, 1)^{I} \).

**Proof.** The formula

\[
\phi(x) = (f_i(x))_{i \in I} \quad (x \in \mathcal{E})
\]

defines a continuous linear map \( \phi : \mathcal{E} \to \mathcal{K}^I \) (on \( \mathcal{K}^I \) the product topology) sending \( \mathcal{A} \) into \( B(0, 1)^{I} \). We prove (i), (ii) below.

(i) \( \phi |_{\mathcal{A}} \) is a homeomorphism into \( B(0, 1)^{I} \). Proof. Let \( (x_j)_{j \in J} \) be a net in \( \mathcal{A} \) for which \( \lim_j \phi(x_j) = 0 \) i.e. \( \lim_j f_i(x_j) = 0 \) for all \( i \in I \). Then \( \lim_j g(x_j) = 0 \) for all \( g \) in a \( || \cdot ||_A \) dense subset \( \mathcal{H} \) of \( \mathcal{E}' \). Let \( f \in \mathcal{E}' \), \( \varepsilon > 0 \). There is a \( g \in \mathcal{H} \) with \( ||f-g||_A < \varepsilon \).

For large \( j \)

\[
|f(x_j)| \leq \max \{ |f(x_J) - g(x_j)|, |g(x_j)| \} < \varepsilon
\]

so that \( \lim_j x_j = 0 \) weakly. But then \( \lim_j x_J = 0 \) for the initial topology of \( \mathcal{E} \) ([4], Theorem 5.12).

(ii) \( \phi \) maps \( \mathcal{A} \) onto \( B(0, 1)^{I} \). Proof. Let \( z := (z_i)_{i \in I} \in B(0, 1)^{I} \). Define \( h \in (\mathcal{E}', || \cdot ||_A)' \) by

\[
h(f_i) = z_i \quad (i \in I)
\]

By Lemma 1.1 (v) there exists an \( x \in \mathcal{E} \) with \( f(x) = h(f) \) for all \( f \in \mathcal{E}' \) i.e. with \( \phi(x) = z \). To prove that in fact \( x \in \mathcal{A}^{oo} = \mathcal{A} \)
Lemma 1.1 (ii)), let \( f \in E', f \in A^\circ \). Then \( \|f\|_A \leq 1 \). There exist \( \lambda_i \in K \) for which \( f = \sum_{i \in I} \lambda_i f_i \) in the sense of \( \| \cdot \|_A \). By orthonormality
\[
\|f\|_A = \max |\lambda_i| \leq 1.
\]
We see that \( |f(x)| \leq \max \{ |\lambda_i f_i(x)| : i \in I \} = \max \{ |\lambda_i z_i| : i \in I \} \leq 1 \). It follows that
\( x \in A^\circ \).

PROPOSITION 1.3 Let \( X \) be a compact subset of a locally convex space \( E \) over \( K \). Then \( \text{co} \ X \) is edged.

**Proof.** We may assume that the valuation of \( K \) is dense. Let \( z \in E \), \( z \notin \text{co} \ X \). There is ([6], Proposition 4.2) a continuous seminorm \( p \)

with \( p(z) = 1 \) and \( p < 1 \) on \( \overline{\text{co}} \ X \). By compactness, \( s := \sup \{ p(x) : x \in \overline{\text{co}} \ X \} \)

\( = \sup_X p = \max_X p < 1 \). Hence, there is a \( \lambda \in K \), \( |\lambda| < 1 \), such that

\( p(\lambda z) > s \) i.e. \( \lambda z \notin \text{co} \ X \).

THEOREM 1.4 Let \( A \) be a complete absolutely convex compactoid in a locally convex space \( E \) over \( K \). The following are equivalent.

(a) There is a compact set \( X \subset A \) with \( A = \text{co} \ X \).

(b) \( A = B(0,1)^I \) for some set \( I \).

**Proof.** (a) \( \Rightarrow \) (β). We may assume that \( E = [A] \). If we replace the initial topology \( \tau \) of \( E \) by the stronger locally convex topology \( \tau' \) generated by all seminorms \( p \) on \( E \) for which \( p|A \) is \( \tau \)-continuous then

\( \tau = \tau' \) on \( A \) and \( A \) is \( \tau' \)-complete and a \( \tau' \)-compactoid ([6], Proposition 4.5). Therefore, to prove (β), we may assume \( \tau = \tau' \). Now apply Proposition 1.3, Lemma 1.1 (iv), Lemma 1.2.

(a) \( \Rightarrow \) (β). Let \( e_i \in B(0,1)^I \) (\( i \in I \)) be given by
It is easily seen that \( Y = \{0\} \cup \{e^i : i \in I\} \) is compact and that 
\( B(0,1)^I = \overline{\text{co} \ Y} \).

**Theorem 1.5** Let the valuation of \( K \) be discrete. Let \( A \) be a complete absolutely convex compactoid in a locally convex space \( E \) over \( K \). (Or, equivalently, let \( A \) be bounded, absolutely convex and \( c \)-compact ([6], Corollary 2.5).) Then there exists a compact set \( X \subseteq A \) with \( A = \overline{\text{co} \ X} \).

**Proof.** For the same reasons as in the previous proof we may assume that \( E = [A] \) and that a seminorm \( p \) on \( E \) is continuous if \( p|A \) is continuous. By Lemma 1.1 (iii), \( (E', \|\cdot\|_A) \) is a Banach space. As the valuation is discrete we have

\[
\|f\|_A = \sup_{x \in A} |f(x)| \leq |K| (f \in E')
\]

Then by [3], Theorem 5.16, \( (E', \|\cdot\|_A) \) has an orthonormal base. Now apply Lemma 1.2 and Theorem 1.4.

For general \( K \) not every edged complete absolutely convex compactoid is the closed convex hull of a compact set. In fact, if the valuation of \( K \) is dense and \( r \in (0,\infty) \setminus |K| \) then \( A := \{\lambda \in K : |\lambda| \leq r\} \) is edged but there is no compact set \( X \subseteq K \) for which \( A = \overline{\text{co} \ X} \). Indeed, we have the following.

**Proposition 1.6** Let \( X \) be a compact subset of a locally convex space \( E \) over \( K \). Then \( \overline{\text{co} \ X} \) is a pure compactoid.

**Proof.** Let \( U \) be an absolutely convex neighbourhood of \( 0 \) in \( E \). By com-
pactness there exist \( x_1, \ldots, x_n \in X \) such that \( X \subseteq \bigcup_{i=1}^{i} U(x_i + U). \) Then \( \co X \subseteq U + \co \{ x_1, \ldots, x_n \}. \) The set \( U + \co \{ x_1, \ldots, x_n \} \) is an open additive subgroup of \( E, \) hence closed. It follows that \( \co X \subseteq U + \co \{ x_1, \ldots, x_n \}. \)

Note. One can prove that each closed pure absolutely convex compactoid is edged.

For metrizable pure compactoids we have the following version of Theorem 1.5 for general \( K. \)

**THEOREM 1.7** Let \( A \subseteq E \) be a complete absolutely convex pure compactoid that is metrizable. Then there is a sequence \( \varepsilon_1, \varepsilon_2, \ldots \) in \( A \) with \( \lim_{n \to \infty} \varepsilon_n = 0 \) and \( A = \co \{ \varepsilon_1, \varepsilon_2, \ldots \}. \)

**Proof.** The proof of [4], Proposition 8.2 applies with some minor modifications (as \( A \) is pure the finite sets \( F_1, F_2, \ldots \) constructed in that proof can be chosen in \( A \) rather than in \( \lambda A \)).

**OPEN PROBLEM** Let \( A \) be complete absolutely convex pure compactoid in a locally convex space \( E \) over \( K. \) Does it follow that \( A = \co X \) for some compact \( X ? \)

The previous theory yields the following.

**COROLLARY 1.8** Let \( A \) be a complete subset of a locally convex space \( E \) over \( K \) such that \( A = \co X \) for some compact set \( X \) (e.g. choose for \( A \) any complete absolutely convex compactoid if the valuation of \( K \) is discrete or any complete absolutely convex pure metrizable compactoid). Then there exists a linearly independent set.
\[ Y = \{ e_i : i \in I \} \] in \( A \) such that

(i) \( Y \) is discrete,

(ii) for each neighbourhood \( U \) of \( 0 \) the set \( \{ i \in I : e_i \notin U \} \) is finite,

(iii) \( Y_0 := Y \cup \{ 0 \} \) is compact,

(iv) \( A = \overline{\text{co} \ Y} = \overline{\text{co} \ Y_0} \),

(v) for each \( (\lambda_i)_{i \in I} \in B(0,1)^I \), \( \sum_{i \in I} \lambda_i e_i \) converges and represents an element of \( A \),

(vi) each \( x \in A \) has a unique representation as a convergent sum \( x = \sum_{i \in I} \lambda_i e_i \) where \( \lambda_i \in B(0,1) \) for each \( i \in I \),

(vii) \( Y \) is a minimal element of \( \{ Z \subset E : A = \overline{\text{co} \ Z} \} \)

(viii) \( Y_0 \) is a minimal element of \( \{ Z \subset E, Z \text{ is compact}, A = \overline{\text{co} \ Z} \} \),

(ix) \( Y \) is a \( p_A \)-orthonormal set.

\[ \text{Proof. By Theorem 1.4 we may assume } A = B(0,1)^I. \text{ Choose } \{ e_i : i \in I \} \]

as in the second part of the proof of Theorem 1.4. We leave the details of checking (i)-(ix) to the reader.
§2 GENERAL COMPACTOIDS

THEOREM 2.1 Let A be a compactoid in a locally convex space E over K. Then there exists a locally convex space F over K containing E as a subspace and a compact set $X \subset F$ such that $A \subset \text{co } X$.

Proof. For each continuous seminorm $p$ on $E$, let $E_p := E/\text{Kerp}$ with the norm induced by $p$. The natural maps $\pi_p : E \to E_p$ yield a linear homeomorphic embedding

$$\pi : E \to F := \prod_{p \in \Gamma} E_p$$

where $\Gamma$ is the collection of continuous seminorms of $E$. For each $p \in \Gamma$ the set $\pi_p(A)$ is a (metrizable) compactoid in $E_p$. By [4], Proposition 8.2 there is a compact set $X_p \subset E_p$ such that $\pi_p(A) \subset \text{co } X_p$. Without loss we may assume that $0 \in X_p$. We have

$$\pi(A) \subset \prod_{p \in \Gamma} \pi(A) \subset \prod_{p \in \Gamma} \text{co } X_p$$

We claim that $\prod_{p \in \Gamma} \text{co } X_p \subset \text{co } \prod_{p \in \Gamma} X_p$. (Then the theorem is proved with $X := \prod_{p \in \Gamma} X_p$.)

For $p \in \Gamma$ and $x \in X_p$ the element $f$ defined by

$$(*) \quad f(q) = \begin{cases} x & \text{if } q \in \Gamma, q = p \\ 0 & \text{if } q \notin \Gamma, q \neq p \end{cases}$$

is in $\prod_{p \in \Gamma} X_p$. If $p \in \Gamma$ and $x \in \text{co } X_p$ then $f$, formally defined by $(*)$, is in $\text{co } \prod_{p \in \Gamma} X_p$. If $p_1, \ldots, p_n \in \Gamma$ and $x_{p_i} \in \text{co } X_{p_i}$ for $i \in \{1, \ldots, n\}$ the element $g$ defined by

$$(***) \quad g(q) = \begin{cases} x_{p_i} & \text{if } q \in \Gamma, i \in \{1, \ldots, n\}, q = p_i \\ 0 & \text{if } q \notin \Gamma, \text{ otherwise} \end{cases}$$
is a finite sum of elements of \( \overline{\text{co}} \prod_{p \in \Gamma} X_p \), hence in \( \overline{\text{co}} \prod_{p \in \Gamma} X_p \). The elements of the type defined in (**) are dense in \( \prod_{p \in \Gamma} \overline{\text{co}} X_p \).

**COROLLARY 2.2** Let \( A \) be an absolutely convex compactoid in a locally convex space over \( K \). Then \( A \) is isomorphic to a submodule of some power of \( B(0,1) \).

**Proof.** By the previous theorem, \( A \subseteq \overline{\text{co}} X \) for some compact \( X \subseteq F \). We may suppose that \( F \) is complete. Now apply Theorem 1.4.

**Note to Theorem 2.1.** It is too optimistic to hope that in Theorem 2.1 we may require that \( F = E \) even when we allow \( X \) to be precompact.

In fact, let \( K \) be not locally compact, let \( E = c_0 \), with the weak topology. \( A := \{ x \in E : \| x \| \leq 1 \} \) is a weak compactoid but according to [5], Proposition 3.3 there is no weakly precompact \( X \subseteq E \) for which \( A \subseteq \overline{\text{co}} X \). Observe that, if \( K \) is not spherically complete, \( A \) is even weakly complete ([4], Theorem 9.6), and that \( A \) is pure if the valuation of \( K \) is discrete.
REFERENCES


