ON WEAKLY PRECOMPACT SETS IN NON-ARCHIMEDEAN BANACH SPACES

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ABSTRACT. For members of a large class of locally convex spaces over a non-archimedean valued field it is proved that weakly precompact subsets are precompact (Theorem 3.8). Also, the answer to a question of Gruson and van der Put ([2], Problem following 5.8) is given.

INTRODUCTION. In [3] Monna proved that in a Banach space over a spherically complete field each weakly convergent sequence is norm convergent. In [5] van Rooij considered a larger class of Banach spaces (thus, including also certain spaces over nonspherically complete fields) and also strengthened Monna's conclusion by showing that each weakly compact set is compact. (In the same spirit is [1], Proposition 3a, stating that, in a locally convex space over a spherically complete field, every weakly c-compact set is c-compact.) All these properties indicate the deviation from the classical theory of spaces over $\mathbb{R}$ or $\mathbb{C}$.

In this note we shall compare 'weak precompactness' to 'precompactness'. This time, the results depend heavily on whether the scalar field is locally compact or not (Theorems 2.2 and 3.8(i)).
§ 1. PRELIMINARIES

For terms that are unexplained here we refer to [4], [7].

Throughout K is a non-archimedean nontrivially valued complete field with valuation \(|\cdot|\). We set \(|K| := \{|\lambda| : \lambda \in K\}\), \(|K| := \text{the closure of } |K| \text{ in } \mathbb{R}\). \(B(0,1) := \{\lambda \in K : |\lambda| \leq 1\}\).

A subset A of a K-vector space E is absolutely convex if \(x, y \in A, \lambda, \mu \in B(0,1)\) implies \(\lambda x + \mu y \in A\). Such an absolutely convex set is edged if, for each \(x \in E\), the set \(\{|\lambda| : \lambda \in K, \lambda x \in A\}\) is closed in \(|K|\). For a set \(X \subset E\), \(\text{co } X\) is the smallest absolutely convex set containing \(X\), \([X]\) is the K-linear span of \(X\). If \(E\) carries a topology, the closure of a set \(S \subset E\) is denoted \(\overline{S}\). Instead of \(\text{co } X\) we shall write \(\overline{co } X\). For an absolutely convex set \(A \subset E\) the usual formula 
\[p_A(x) = \inf\{|\lambda| : \lambda \in K, x \in \lambda A\}\] defines a (non-archimedean) seminorm \(p_A\) on \([A]\), the seminorm associated to \(A\). We have \(p_A(x) \in |K|\) for all \(x \in [A]\).

A (non-archimedean) Banach space \(E\) over \(K\) is of countable type ([4], p.66) if there exists a countable set \(X \subset E\) with \([X] = E\).

PROPOSITION 1.1 Let \(E\) be a Banach space of countable type over \(K\).

(i) There is an equivalent norm in \(E\) for which \(E\) has an orthogonal base.

(ii) For every closed linear subspace \(D\) of \(E\) there exists a continuous linear projection of \(E\) onto \(D\).

Proof. [4], Theorem 3.16 (ii), (v).
A locally convex space over $K$ is a $K$-vector space $E$ with a topology induced by (non-archimedean) seminorms in the usual way ([7]). Throughout we assume that $E$ is Hausdorff. Its dual space $E'$ is the space of all continuous linear functions $E \to K$.

A subset $X$ of a locally convex space $E$ over $K$ is (a) compactoid ([4], p. 134) if for every neighbourhood $U$ of 0 in $E$ there exist $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in E$ such that $X \subset U + \text{co}(x_1, \ldots, x_n)$. If $X$ is a compactoid then so are $\text{co} X$, $\text{co} X$ and are subsets of $X$. Precompact subsets of $E$ are compactoid.

**Proposition 1.2** Let $E$ be a Banach space over $K$, let $A \subset E$ be bounded and absolutely convex.

(i) If each countable subset of $A$ is a compactoid then so is $A$.

(ii) Suppose $E$ has an orthogonal base. Then $A$ is a compactoid if and only if each orthogonal sequence in $A$ tends to 0.

**Proof.** [4], Theorems 4.37 and 4.38.

Let $E$ be a locally convex space over $K$. For each continuous seminorm $p$, let $E^\sim_p$ be the completion of the normed space $E^\sim_p := E / \ker p$ with the quotient norm induced by $p$. The canonical maps $\pi_p : E \to E^\sim_p$ (where $p$ runs through the collection of all continuous seminorms) induce a map

$$\pi : E \to \prod_{\pi} E^\sim_p$$

which is (since $E$ is Hausdorff) a linear homeomorphism into.

A locally convex space $E$ over $K$ is of countable type ([6], Definition...
4.3) if each $E_p$ is of countable type. (This definition is less restrictive than: $\overline{[X]} = E$ for some $X \subset E$, $X$ is countable'.)

PROPOSITION 1.3 Let $E$ be either a locally convex space over a spherically complete $K$ or a locally convex space of countable type over $K$.

(i) If $B \subset E$ is closed, absolutely convex and edged and if $x \in E \setminus B$ then there exists an $f \in E'$ for which $|f(B)| \leq 1$, $|f(x)| > 1$.

(ii) The weak topology on $E$ is Hausdorff.

(iii) Let $D$ be a linear subspace of $E$. Any $f \in D'$ has an extension in $E'$.

(iv) For any $X \subset E$,

$X$ is bounded $\Leftrightarrow X$ is weakly bounded $\Leftrightarrow X$ is a weak compactoid.

(v) Let $X \subset E$ be a compactoid. Then, on $X$, the topology of $E$ and the weak topology coincide. $X$ is closed if and only if $X$ is weakly closed.

Proof.

(i) $[6]$, Theorem 4.7 (Theorem 4.4).

(ii) Follows from (i).

(iii) $[6]$, Theorem 4.2.

(iv) $[6]$, Corollary 7.7 and the observation that $\dim E_p < \infty$ for each weakly continuous seminorm $p$.

§ 2 LOCALLY COMPACT SCALAR FIELDS

Throughout § 2, K is locally compact.

PROPOSITION 2.1 Each compactoid in a locally convex space over K is precompact.

Proof. Let A be a compactoid in a locally convex space E over K, let U be an absolutely convex neighbourhood of 0 in E. By compactoidity there exist $x_1,\ldots,x_n \in E$ such that $A \subseteq U + \text{co}(x_1,\ldots,x_n)$. The formula

$$\left(\lambda_1,\ldots,\lambda_n\right) \mapsto \sum_{i=1}^{n} \lambda_i x_i$$

defines a continuous map $K^n \to E$ sending $B(0,1)^n$ onto $\text{co}(x_1,\ldots,x_n)$. Hence, the latter set is compact so there exists a finite set $F \subseteq E$ such that $\text{co}(x_1,\ldots,x_n) \subseteq F + U$. Then also $A \subseteq F + U$. For each $x \in F$ for which $(x+U) \cap A \neq \emptyset$, choose an $x' \in (x+U) \cap A$. Then $x'+U = x+U$. These $x'$ form a finite set $F' \subseteq A$ for which

$$A \subseteq F'+U.$$

It follows that $A$ is precompact.

THEOREM 2.2 A subset of a locally convex space over K is weakly precompact if and only if it is bounded.

Proof. If $X$ is weakly precompact then $X$ is weakly bounded, hence bounded (Proposition 1.3 (iv)). If, conversely, $X$ is bounded then $X$ is weakly bounded, hence weakly compactoid (Proposition 1.3 (iv)). By Proposition 2.1, $X$ is weakly precompact.
We see that weakly precompact sets are in general not precompact for the initial topology. The picture changes, however, if we consider compactness.

**Theorem 2.3** A subset of a locally convex space over \( K \) is compact if and only if it is weakly compact.

**Proof.** For Banach spaces this is proved in [5], Theorem 5.2. Now let \( A \) be a weakly compact subset of a locally convex space \( E \). The canonical injection

\[
E \rightarrow \Pi \bigoplus_{p} E^*_{p}
\]

maps \( A \) homeomorphically into a subset of \( \Pi \bigoplus_{p} \pi_p(A) \). The latter set is compact since each \( \pi_p(A) \) is weakly compact, hence compact, in \( E_p \). Thus, \( A \) is precompact, hence a compactoid. By Proposition 1.3 (v), the weak topology and the initial topology coincide on \( A \). It follows that \( A \) is compact.
Throughout § 3, K is not locally compact.

The key lemma is the following.

**LEMMA 3.1** In a Banach space E of countable type over K no infinite orthonormal set is weakly precompact.

**Proof.** As K is not locally compact we can choose $\rho_1, \rho_2, \ldots \in B(0,1)$ for which $\inf_{n \neq m} |\rho_n - \rho_m| > \frac{1}{2}$. Let $e_1, e_2, \ldots$ be an infinite orthonormal system in E. The formula

$$f(\sum_{i=1}^{n} \lambda_i e_i) = \sum_{i=1}^{n} \lambda_i \rho_i \quad (n \in \mathbb{N}, \lambda_1, \ldots, \lambda_n \in K)$$

defines a continuous linear K-valued function $f$ on $[e_1, e_2, \ldots]$. By Proposition 1.3 (iii), $f$ extends to an element $g$ of $E'$. The set

$$U := \{x \in E : |g(x)| \leq \frac{1}{2}\}$$

is a weak neighbourhood of 0. We claim that $e_1 + U, e_2 + U, \ldots$ are pairwise disjoint (which implies that $\{e_1, e_2, \ldots\}$ is not weakly precompact). In fact, let $n, m \in \mathbb{N}$ and $(e_n + U) \cap (e_m + U) \neq \emptyset$. Then $e_n - e_m \in U$ and

$$|\rho_n - \rho_m| = |f(e_n) - f(e_m)| = |g(e_n - e_m)| \leq \frac{1}{2}.$$

It follows that $n = m$.

Our next step is to prove that weakly precompact sets in E are norm compactoid (Proposition 3.4).

**Remark.** For a subset $X$ of a Banach space (of countable type) over K, consider the following statements.
(a) For each $c \in (0,1)$, each $c$-orthogonal sequence in $X$ tends to 0.

(b) $X$ is a compactoid.

It is a simple consequence of Lemma 3.1 that (a) holds for a weakly precompact $X$. Also we have (a) $\Rightarrow$ (b) for any bounded absolutely convex $X$ ([4], Theorem 4.37). Annoyingly, I have not been able to extend (a) $\Rightarrow$ (b) to arbitrary subsets $X$.

**Lemma 3.2** Let $X$ be a nonempty precompact subset of a locally convex space $E$ over $K$. For each continuous seminorm $p$

$$\max_{X} p$$

exists.

**Proof.** Set $p(X) := \{p(x) : x \in X\}$. Let $a \in X$, $p(a) > 0$; we prove that

$$\{x \in p(x) : x > p(a)\}$$

is finite. The set

$$U := \{x \in E : p(x) \leq p(a)\}$$

is an open absolutely convex neighbourhood of 0 in $E$. By precompactness there exist $x_1,\ldots,x_n \in X$ such that

$$X \subset \bigcup_{i=1}^{n} (x_i + U).$$

For each $i \in \{1,\ldots,n\}$ we have either $p(x_i) \leq p(a)$ (then $p \leq p(a)$ on $x_i + U$) or $p(x_i) > p(a)$ (then $p$ is constant on $x_i + U$).

**Proposition 3.3** Let $A$ be an open absolutely convex subset of an infinite dimensional Banach space of countable type over $K$. Then there is no weakly precompact set $X \subset E$ for which $A = \text{co} \ X$.

**Proof.** Suppose $A = \text{co} \ X$ for some weakly precompact $X$. $X$ is weakly
bounded hence bounded (Proposition 1.3 (iv)) so $A$ is bounded. Therefore the seminorm associated to $A$ is a norm $|| \cdot ||$ inducing the topology of $E$. We have

\[ ||x|| \in [K] \text{ for all } x \in E \]

\[ \{x \in E : ||x|| < 1\} \subseteq A \subseteq \{x \in E : ||x|| \leq 1\}. \]

Let $\{e_i : i \in I\}$ be a (possibly empty) maximal $|| \cdot ||$-orthonormal set in $X$ and consider

\[ B := \text{co} \{e_i : i \in I\}. \]

Then $B \subseteq A$, $B$ is absolutely convex, closed and edged. Suppose $B \neq A$. Then by Proposition 1.3 (i) there exists an $f \in E'$ with $\sup_B |f| \leq 1$ and $\sup_A |f| > 1$. We have

\[ \sup_A |f| = \sup_{\text{co} X} |f| = \sup_X |f| \]

so that, by Lemma 3.2,

\[ \max_X |f| = \max_A |f| \]

exists. After multiplying $f$ by a suitable scalar we therefore obtain a $g \in E'$ for which

\[ \max_X |g| = \max_A |g| = 1, \quad |g(b)| < 1 \text{ for each } b \in B. \]

Observe that

\[ ||g|| := \sup_{x \neq 0} |g(x)| / ||x|| = \max_A |g| = 1, \]

where for the second equality one may use the first line of ($\ast$). By ($\ast$) there exists an $a \in X$ with $|g(a)| = 1$. From $a \in A$ and

\[ 1 = |g(a)| \leq ||g|| \cdot ||a|| = ||a|| \]
we obtain $||a|| = 1$. For each $s \in \text{co} \{e_i : i \in I\}$ we have $|g(s)| < 1$ and from

$$||a-s|| = ||g|| \cdot ||a-s|| \geq |g(a)-g(s)| = |g(a)| = 1 = ||a||$$

it follows directly that \(\{a\} \cup \{e_i : i \in I\}\) is an orthonormal set in \(X\), a contradiction. Thus, \(B = A\). But then \(\{e_i : e \in I\}\) is an orthonormal base of \(E\) and, therefore, an infinite orthonormal subset of \(X\) yielding, again, a contradiction (Lemma 3.1), completing the proof of Proposition 3.3.

**Note.** We now are in a position to give a (negative) answer to the question raised by Gruson and van der Put in [2], Problem following 5.8. Let \(A\) be a compactoid in a locally convex space \(E\) over \(K\). Does there exist a compact \(X \subset E\) with \(A \subset \text{co} X\)? In fact, let \(E\) be the space \(c_0\) with the weak topology, let \(A := \{x \in c_0 : ||x|| \leq 1\}\). Then \(A\) is a compactoid for the weak topology. If \(A \subset \text{co} X\) then \(\text{co} X\) is open in the norm topology. \(X\) can not be weakly precompact according to Proposition 3.3.

**PROPOSITION 3.4** In a Banach space \(E\) of countable type over \(K\) each weakly precompact set is a normcompactoid.

**Proof.** By Proposition 1.1 (i) we may assume that \(E\) has an orthogonal base. Let \(X \subset E\) be weakly precompact, not a normcompactoid. Then, as \(X\), hence \(\text{co} X\), is bounded we can find an orthogonal sequence \(x_1, x_2, \ldots\) in \(\text{co} X\) for which \(\inf_n ||x_n|| > 0\) (Proposition 1.2 (ii)). Let \(D := [x_1, x_2, \ldots]\) and let \(P\) be a continuous linear projection of \(E\) onto \(D\) (Proposition 1.1 (ii)). Consider

\[A := P(\text{co} X).\]
Then $A \supset \text{co} \{x_1, x_2, \ldots\}$ so that $A$ is open in $D$. Since $P(\text{co} \ X) = \text{co} \ PX$ we have $A = \text{co} \ PX$. A standard argument yields weak precompactness of $PX$ in $D$. We see that $A$ is an open subset of $D$ which is the closed absolutely convex hull of a weakly precompact set in $D$. But this is impossible according to Proposition 3.3.

We now generalize Proposition 3.4 to members of a large class of locally convex spaces (Proposition 3.7).

**Lemma 3.5** Let $X$ be a subset of a Banach space over $K$. Suppose each countable subset of $X$ is a compactoid. Then $X$ itself is compactoid.

**Proof.** Each countable subset of $X$ is bounded. Then $X$ and $\text{co} \ X$ are bounded. By Proposition 1.2 (i) it suffices to prove that each countable set $T \subset \text{co} \ X$ is a compactoid. For each $t \in T$ there exists a finite set $F_t \subset X$ such that $t \in \text{co} \ F_t$. By assumption $F := \cup_{t \in T} F_t$ is a compactoid, hence so is $\text{co} \ F$ and is its subset $T$.

**Proposition 3.6** Let $E$ be a Banach space over a spherically complete $K$. Then each weakly precompact set in $E$ is a normcompactoid.

**Proof.** Let $X \subset E$ be weakly precompact, let $Y$ be a countable subset of $X$. We prove (Lemma 3.5) that $Y$ is a normcompactoid. Set $D := \overline{\{Y\}}$. It follows from Proposition 1.3 (iii) that the inclusion map $D \hookrightarrow E$ is a homeomorphism into with respect to the weak topologies. Then $Y$ is weakly precompact in $D$ and, by Proposition 3.4, a normcompactoid in $D$, hence in $E$. 
Proposition 3.7 Let $E$ be either a locally convex space over a spherically complete $K$ or a locally convex space of countable type over $K$. Then each weakly precompact set is a compactoid.

**Proof.** Let $X \subset E$ be weakly precompact. For each continuous seminorm $p$ the set $\pi_p(X)$ is weakly precompact in $E$. Either from Proposition 3.6 or from Proposition 3.4 we infer that $\pi_p(X)$ is a normcompactoid in $E$. Then, almost by definition, $X$ is a compactoid in $E$.

The previous efforts lead to the following result. (Recall that $K$ is not locally compact.)

**Theorem 3.8** Let $E$ be either a locally convex space over a spherically complete $K$ or a locally convex space of countable type over $K$.

(i) Each weakly precompact set in $E$ is precompact (compare Theorem 2.2).

(ii) Each weakly compact set in $E$ is compact (compare Theorem 2.3).

(iii) On (weakly) precompact sets the topology of $E$ and the weak topology coincide.

(iv) The closure of a (weakly) precompact set in $E$ is identical to its weak closure.

**Proof.** Let $X \subset E$ be weakly precompact. By Proposition 3.7, $X$ is a compactoid. Now Proposition 1.3 (v) yields (ii), (iii) and (iv). To prove (i), let $U$ be a neighbourhood of $0$ in $E$. Since the topology of $E$ and the weak topology coincide on $\text{co } X$ there is a weak neighbourhood $V$ of $0$ in $E$ with $V \cap \text{co } X \subset U \cap \text{co } X$. By weak precompactness there exist $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$ such that $X \subset \bigcup_{i=1}^{n} (x_i + V)$. For each $i \in \{1, \ldots, n\}$,
\[(x_i^* + V) \cap X \subset (x_i^* + V) \cap \text{co } X \subset x_i^* + V \cap \text{co } X \subset x_i^* + U.\]

It follows that

\[X \subset \bigcup_{i=1}^{n} (x_i^* + U)\]

and (i) is proved.

Remark 1. (Extension of Theorem 3.8) It is not hard to extend Theorem 3.8 to all locally convex spaces \(E\) for which there exists a base \(P\) of continuous seminorms such that, for each \(p \in P\), the Banach space \(E_p\) has property (*) ([5], p. 57). Observe that such a Banach space is polar in the sense of [6], Definition 3.5, as ([5], Theorem 5.2) \(x \mapsto \sup \{|f(x)| / \|x\| : f \in E', f \neq 0\}\) is a polar norm equivalent to the initial norm on \(E\). In particular, the conclusions of Theorem 3.8 hold for any Banach space \(E\) over \(K\) having a base (e.g. \(C(X \to K)\) where \(X\) is compact, nonmetrizable, \(K\) not spherically complete).

Remark 2. (A weakly precompact set that is not precompact for the norm topology) Let \(K\) be not spherically complete, let \(l^\infty\) be the Banach space consisting of all bounded sequences in \(K\), with the supremum norm. Then \(X := \{e_1, e_2, \ldots\}\) (where \(e_1, e_2, \ldots\) are the standard unit vectors) is not precompact for the norm topology but is weakly precompact. (From \((l^\infty)' = c_0\) ([4], Theorem 4.17) it follows easily that \(\lim_{n \to \infty} e_n = 0\) weakly.) Observe that yet the weak and norm topologies coincide on \(X\)!
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