DUALITY OF PROJECTIVE LIMIT SPACES AND INDUCTIVE LIMIT SPACES OVER A NONSPHERICALLY COMPLETE NONARCHIMEDEAN FIELD

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(Received October 14, 1985)

Abstract. A duality theorem of projective and inductive limit spaces over a nonspherically complete valued field is obtained under a certain condition, and topologies of spaces of locally analytic functions are studied.

Introduction. Morita obtained in [5] a duality theorem of projective limit spaces and inductive limit spaces over a spherically complete nonarchimedean valued field, and Schikhof studied in [8] locally convex spaces over a nonspherically complete nonarchimedean valued field. In this paper, we use the results of [8] and study the duality of such spaces over a nonspherically complete nonarchimedean valued field.

The duality theorem of [5] was obtained as a generalization of the results of Komatsu [3] by Morita using the theory of van Tiel [10] about locally convex spaces over a spherically complete nonarchimedean valued field. There the following two facts are used essentially: (i) The Mackey topology is the topology of uniform convergence on weakly e-compact sets; (ii) Any absolutely convex weakly e-compact set is strongly closed. Though we can generalize the notion of e-compactness to our case, it is difficult to obtain good analogues of these two facts over a nonspherically complete valued field. Hence we restrict our attention to a more restricted class than in [5], and prove a duality theorem by making use of van der Put’s duality theorem of sequence spaces $c_0 = \{(a_n, a_2, a_3, \cdots) \in K^n; |a_m| \cdot 0 (m \cdots)\}$ and $l^\infty = \{(b_1, b_2, b_3, \cdots) \in K^n; \sup |b_m| < \infty\}$ over a nonspherically complete valued field $K$.

We prove a general duality theorem over such a field in Section 1, and apply the theorem to some examples in Section 2. In particular, we generalize the results of Morita [6] to any complete nonarchimedean valued field, and give a positive answer to a question of P. Robba.

We use the notation and terminology of Schikhof [8] throughout this paper.
1. Duality theorem.

1.1. Let $K$ be a complete field with a nontrivial nonarchimedean valuation $| |$. We assume in Section 1 that $K$ is not spherically complete. For each positive integer $m$, let $(X_m, | |_m)$ and $(Y_m, | |_m)$ be Banach spaces over $K$. We assume that $X_m$ is of countable type. Hence $X_m$ is reflexive (cf. e.g. van Rooij [7, Corollary 4.18]). Let

$$( , )_m: X_m \times Y_m \to K$$

be a nondegenerate bicontinuous $K$-bilinear form such that $X_m$ and $Y_m$ become mutually dual locally convex spaces with respect to $( , )_m$. Let $\{u_{m,n}: X_n \to X_m \ (m < n)\}$ be a projective system, and $\{v_{n,m}: Y_m \to Y_n \ (m < n)\}$ be an inductive system such that (i) the $u_{m,n}$'s are $K$-linear continuous maps, (ii) the $v_{n,m}$'s are $K$-linear continuous injective maps, and (iii) the equality $(u_{m,n}(x_m), y_m)_m = (x_n, v_{n,m}(y_m))_n$ holds for any $x_n \in X_n$ and $y_m \in Y_m$. Let $(X, u_m)$ be the locally convex projective limit of $\{X_m, u_{m,n}\}$ and let $(Y, v_m)$ be the locally convex inductive limit of $\{Y_m, v_{n,m}\}$. We assume further that (iv) the projection map $u_m: X \to X_m$ has a dense image for each $m$.

By definition, any element $x$ of the projective limit $X$ can be written as $x = (x_m)$ with $x_m \in X_m$ satisfying $u_{m,n}(x_n) = x_m$ for any $m$ and $n$ with $m < n$, and any element $y$ of the inductive limit space $Y$ can be written as $y = v_m(y_m)$ with some $y_m \in Y_m$. By our assumption (iii), the equality $(u_{m,n}(x_m), y_m)_m = (x_n, v_{n,m}(y_m))_n$ holds for any $m < n$. Hence $(u_m(x), y_m)_m$ does not depend on a special choice of $m$ with $y = v_m(y_m) \in v_m(Y_m)$, and we can define a pairing

$$( , )_m: X \times Y \to K$$

by $(x, y) = (u_m(x), y_m)_m$ with such a $y_m \in Y_m$. It is easy to see that this pairing $( , )_m$ is $K$-bilinear. Since the projection map $u_m: X \to X_m$ is continuous, our pairing $( , )_m$ is bicontinuous on $X \times Y_m$ for each $m$. Hence, by the universal mapping property of the inductive limit topology, $( , )$ is bicontinuous on $X \times Y$.

Let $x = (x_m)$ be a nonzero element of $X$. Then $x_m \neq 0$ for some $m$. Since $( , )_m$ is nondegenerate, $(x_m, y_m)_m \neq 0$ for some $y_m \in Y_m$. Hence $(x, y) = (x_m, y_m)_m \neq 0$ for some $y = v_m(y_m) \in v_m(Y_m) \subset Y$. Let $y = v_m(y)$ $(y_m \in Y_m)$ be a nonzero element of $Y$. Then $\{x_m \in X_m; (x_m, y_m)_m \neq 0\}$ is a non-empty open subset of $X_m$. Since the image of the projection map $u_m: X \to X_m$ is dense, there is an element $x = (x_m) \in X$ such that $(x, y) = (x_m, y_m)_m \neq 0$. Therefore our pairing $( , )$ is nondegenerate. Hence we have proved the following:
Proposition 1. Let \( X = \text{proj lim } X_n \) and \( Y = \text{ind lim } Y_n \) be as before. Then we have a nondegenerate bicontinuous \( K \)-bilinear form
\[
(\, , \,) : X \times Y \to K.
\]

1.2.

Lemma 1. Let \( E \) and \( F \) be locally convex \( K \)-vector spaces, and let \( (\, , \,) : E \times F \to K \) be a nondegenerate bicontinuous \( K \)-bilinear form. Let \( \sigma(E, F) \) be the weakest locally convex topology on \( E \) such that \( E \ni e \mapsto (e, f) \in K \) is continuous for each \( f \in F \). Then for any continuous \( K \)-linear form \( L : E \to K \) with respect to \( \sigma(E, F) \), there exists an element \( f \in F \) such that
\[
L(e) = (e, f)
\]
holds for any \( e \in E \). In particular, \( (E, \sigma(E, F))' = F \).

Proof. Let \( L : E \to K \) be as in the lemma. Since \( L \) is continuous, there are a finite number of elements \( f_1, \ldots, f_n \) in \( F \) such that for all \( e \in E \)
\[
|L(e)| \leq \sup_{1 \leq i \leq n} |(e, f_i)|.
\]
Let \( E^* = \{ e \in E ; (e, f_i) = 0 \text{ for any } i = 1, \ldots, n \} \). Then \( E^* \) is contained in the kernel of \( L \), and \( L \) factors through \( E/E^* \). Since \( (\, , \,) \) induces a nondegenerate \( K \)-bilinear form on \((E/E^*) \times (Kf_1 + \cdots + Kf_n)\), the algebraic dual of \( E/E^* \) can be identified with \( Kf_1 + \cdots + Kf_n \). Hence there are \( a_1, \ldots, a_n \in K \) such that
\[
L(e) = \left( e, \sum_{i=1}^n a_i f_i \right)
\]
holds for any \( e \in E \). Then \( f = \sum a_i f_i \) satisfies the condition of the lemma.

We apply this lemma to our case. Let \( E = X \) and \( F = Y \). Then our bilinear form \( (\, , \,) \) satisfies the condition of the lemma. Let \( \sigma(X, Y) \) (resp. \( \sigma(Y, X) \)) be the weakest locally convex topology on \( X \) (resp. on \( Y \)) such that \( X \ni x \mapsto (x, y) \in K \) is continuous for each \( y \in Y \) (resp. \( Y \ni y \mapsto (x, y) \in K \) is continuous for each \( x \in X \)). Then it follows from Lemma 1 that \( (X, \sigma(X, Y))' = Y \) and \( (Y, \sigma(Y, X))' = X \).

Let \( \tau(X) \) be the projective limit topology on \( X \), and let \( \tau(Y) \) be the inductive limit topology of \( Y \). Since our pairing is bicontinuous, \( \sigma(X, Y) \leq \tau(X) \) and \( \sigma(Y, X) \leq \tau(Y) \). Hence we have
\[
Y = (X, \sigma(X, Y))' \subset (X, \tau(X))' \quad \text{and} \quad X = (Y, \sigma(Y, X))' \subset (Y, \tau(Y))'.
\]
Let $f: X \to K$ be a $K$-linear continuous map with respect to $\tau(X)$. Then $f^{-1}(\{x \in X; |f(x)| < 1\})$ is open in $X$. It follows from the definition of the projective limit topology that there exist a positive integer $m$ and a positive number $\varepsilon$ such that $f^{-1}(\{x \in X; |f(x)| < 1\}) \supset \{x \in X; |u_m(x)|_m < \varepsilon\}$. This shows that $u_m(X) \ni u_m(x) \mapsto f(x) \in K$ is continuous. Since $u_m(X)$ is dense in $X_m$, this map can be extended to a continuous $K$-linear map $f_m: X_m \to K$. Since $X_m$ and $Y_m$ are mutually dual with respect to $(\ , \ )_m$, there is an element $y_m \in Y_m$ such that $(x_m, y_m)_m = (x, y)$ holds for any $x = (x_m) \in X$ with $y = u_m(y_m) \in u_m(Y_m) \subset Y$. Therefore $(X, \tau(X))' = Y$.

Let $g: Y \to K$ be a continuous $K$-linear map with respect to $\tau(Y)$. Since the natural injection $v_m: Y_m \to Y$ is continuous, $g$ induces a continuous map $g_m: Y_m \to K$ for each $m$. Since $X_m$ is the dual of $Y_m$, there is a unique element $x_m \in X_m$ such that $g_m(y_m) = (x_m, y_m)_m$ holds for any $y_m \in Y_m$. If $n > m$, then $(x_n, y_m)_m = g_m(y_m) = (v_m(y_m))_m = (v_n, v_m(y_m))_n = (u_{m,n}(x_m), y_m)_n$ holds for any $y_m \in Y_m$. Since the pairing $(\ , \ )_m$ is nondegenerate, $u_{m,n}(x_n) = x_m$ holds for $n > m$. Hence $x = (x_m)$ is an element of $\prod X_m = X$ such that $g(y) = (x, y)$ holds for any $y \in \text{ind lim } Y_m = Y$. Hence $(Y, \tau(Y))' = X$. Therefore we have proved the following:

**Proposition 2.** We have $(X, \tau(X))' = Y$ and $(Y, \tau(Y))' = X$ as sets.

1.3. Since each $X_m$ is a Banach space of countable type, it follows from Schikhof [8, 4.12] that $X = \text{proj lim } X_m$ is a Fréchet space of countable type. Since $K$ is not spherically complete, it follows from [8, Corollary 9.8] that $X$ is reflexive. Hence we have the following:

**Proposition 3.** $X$ is a Fréchet space of countable type. In particular, $X$ is reflexive.

Let $y$ be a nonzero element of $Y$. Since the pairing $(\ , \): X \times Y \to K$ is nondegenerate, there is an element $x$ of $X$ such that $(x, y) \neq 0$. Then $|(x, y)| = 0$. Since $p_x(y) = |(x, y)|$ is a continuous seminorm for $\sigma(Y, X)$, it follows that $(Y, \sigma(Y, X))$ is Hausdorff. Since $\tau(Y)$ is stronger than $\sigma(Y, X)$, $(Y, \tau(Y))$ is also Hausdorff. Hence we have proved the following:

**Proposition 4.** $(Y, \tau(Y))$ is a Hausdorff space.

**Remark.** If the maps $u_m,n$'s are compact maps, then we can show that $X = \text{proj lim } X_m$ is a nuclear Montel space. In general, since each $Y_m$ is barrelled, $Y = \text{ind lim } Y_m$ is also barrelled.

Now we can prove the following key lemma:
Proposition 5. The strong topology \( b(Y, X) \) on \((X, \tau(X))' = Y \) and the inductive limit topology \( \tau(Y) \) of \( Y \) coincide.

Proof. Since any bounded set of \((X, \tau(X))\) is contained in a bounded set of the form \( B = \{x = (x_m) \in X; |x_m| \leq M_m \} \) with a sequence \((M_m)\) of positive numbers, the subsets of \( Y \) of the form \( U_B = \{y \in Y; |(x, y)| \leq 1 \text{ for all } x \in B \} \) make a fundamental system of neighbourhoods of \( 0 \in Y \) with respect to \( b(Y, X) \). Since the pairing \((,)_m: X_m \times Y_m \to K\) makes \( X_m \) and \( Y_m \) into mutually dual Banach spaces, for any positive number \( M_m \), there is a positive number \( N_m \) such that, if \( y_m \in Y_m \) satisfies \( |y_m| \leq N_m \), then the condition \( |(x_m, y_m)| \leq 1\) is satisfied for any \( x_m \in X_m \) with \( |x_m| \leq M_m \). Then \( y = v_m(y_m) \in v_m(Y_m) \) is contained in \( U_B \) if \( |y_m| \leq N_m \). Hence \( U_B \) contains
\[
\bigcup_{m \geq 1} v_m(\{y_m \in Y_m; |y_m| \leq N_m\})
\]
Since the subsets of \( Y \) of this form make a fundamental system of neighbourhoods of \( 0 \in Y \) with respect to \( \tau(Y) \), we have \( b(Y, X) \leq \tau(Y) \).

Since we can prove the opposite inequality \( \tau(Y) \leq b(Y, X) \) in the same way, the strong topology \( b(Y, X) \) and the inductive limit topology \( \tau(Y) \) of \( Y \) coincide.

Since \((X, \tau(X))\) is reflexive, the following corollary follows from Proposition 5.

Corollary. \((Y, \tau(Y))\) is reflexive, and the strong dual space of \((Y, \tau(Y))\) is isomorphic to \((X, \tau(X))\).

Since \(X\) is a Fréchet space, \(X\) is bornologic (cf. proofs of van Tiel [10, Théorèmes 3.17 and 4.30]). It follows from Schikhof [8, Proposition 6.8] that \((Y, \tau(Y)) \simeq ((X, \tau(X))', b(Y, X))\) is complete. Therefore we have proved the following theorem:

Theorem 1. Let \(X = \text{proj lim} X_m\), \(Y = \text{ind lim} Y_m\) and \((,): X \times Y \to K\) be as in 1.1. Then \(X\) is a Fréchet space of countable type, \(Y\) is Hausdorff and complete and the pairing \((,\) makes \(X\) and \(Y\) into mutually dual locally convex spaces over \(K\).

2. Examples.

2.1. Let \(k\) be an algebraically closed field with a nontrivial non-archimedean complete valuation \(||\). Let \(\mathbb{P}^1(k) = k \cup \{\infty\}\) be the one-dimensional projective space over \(k\), let \(K\) be a complete subfield of \(k\), and let \(C\) be a compact subset of \(K\). Put \(V = P^1(k)\). Let \(\{r_n\}_{n=1}^{\infty}\) be a
strictly decreasing sequence in \(|K^*|\) such that \(r_n \to 0\) \((n \to \infty)\). Then, for each \(n\), \(C\) is covered by a finite number of open balls of the form

\[ C_{n,i} = \{ z \in k; |z - c_{n,i}| < r_n \} \quad (c_{n,i} \in C). \]

We assume that (i) \(C\) is covered by \(C_{n,i}\) \((i = 1, \ldots, l_n)\) and (ii) the \(C_{n,i}\)'s are mutually disjoint. Put

\[ C_n = \prod_{i=1}^{l_n} C_{n,i}. \]

Then \(C = \cap C_n\).

Let \(f\) be a \(k\)-valued function on \(V - C = \{ z \in V; z \notin C \}\). Then \(f\) is called a \(K\)-analytic function on \(V - C\) if and only if the restriction of \(f\) to each \(V - C_n\) is given by a convergent series of the form

\[ f_n(z) = a_{k \infty} + \sum_{i=1}^{l_n} \sum_{m=1}^{\infty} a_m^{(i)}(z - c_{n,i})^m \]

with \(a_{k \infty} \in K\) and \(|a_m^{(i)}| r_n^m \to 0\) \((m \to -\infty)\) (cf. Morita [4], Gerritzen-van der Put [1], etc.). Let \(\mathcal{O}_K(V - C_n)\) be the space of all functions \(f_n; V - C_n \to k\) of this form. Then the equality

\[ \max_{z \in V - C_n} (|a_{k \infty}|, \max_{z \in V - C_n} |a_m^{(i)}| r_n^m) = \max_{z \in V - C_n} |f_n(z)| \]

holds. If we define a norm \(|f_n|_n\) by this formula, then the \(K\)-vector space \(\mathcal{O}_K(V - C)\) becomes a complete Banach space with this norm. Further, we can identify the quotient space \(\mathcal{O}_K(V - C_n)/K (K = \{ f_n(z) = a_{k \infty}; a_{k \infty} \in K \})\) with the subspace \(\{ \sum_{i=1}^{l_n} \sum_{m=1}^{\infty} a_m^{(i)}(z - c_{n,i})^m; a_m^{(i)} \in K, |a_m^{(i)}| r_n^m \to 0\} (m \to -\infty)\) of \(\mathcal{O}_K(V - C_n)\). Since the set of all finite sums of this form is dense in \(\mathcal{O}_K(V - C_n)/K\), \(\mathcal{O}_K(V - C_n)/K\) is a Banach space of countable type.

Let \(\mathcal{O}_K(V - C)\) be the set of all \(K\)-analytic functions on \(V - C\), and put \(\mathcal{B}_K(C) = \mathcal{O}_K(V - C)/K\). Then \(\mathcal{B}_K(C)\) can be identified with the locally convex projective limit of the \(\mathcal{O}_K(V - C_n)/K (n < l)\) with respect to the restriction maps. Obviously the restriction maps \(\mathcal{O}_K(V - C)/K \to \mathcal{O}_K(V - C_n)/K (n < l)\) are \(K\)-linear and continuous. Since any finite sum of the form \(\sum_{i=1}^{l_n} \sum_{m=1}^{\infty} a_m^{(i)}(z - c_{n,i})^m\) \((a_m^{(i)} \in K)\) is in \(\mathcal{O}_K(V - C)/K = \mathcal{B}_K(C)\), the image of the projection map \(\mathcal{B}_K(C) \to \mathcal{O}_K(V - C_n)/K\) is dense for each \(n\).

Put

\[ \mathcal{O}_b(K)(C_n) = \left\{ g(z) = \sum_{i=1}^{l_n} \sum_{m=0}^{\infty} b_m^{(i)}(z - c_{n,i})^m; b_m^{(i)} \in K, \sup_{0 \leq m < \infty} |b_m^{(i)}| r_n^m < +\infty \right\}. \]

Then \(\mathcal{O}_b(K)(C_n)\) becomes a Banach space with

\[ |g(z)|_n = \sup_{i \leq l_n, 0 \leq m < \infty} |b_m^{(i)}| r_n^m. \]
If $n < l$ and $|c_{n,i} - c_{i,i}| < r_n$, then $\sum_{m=1}^{+\infty} b_m^{(i)}(z - c_{n,i})^m$ can be written in the form $\sum_{m=1}^{+\infty} b_m^{(i)}(z - c_{i,i})^m$ with $b_m^{(i)} \in K$, and we have $\sup_{m=1}^{+\infty} |b_m^{(i)}| r_n^m = \sup_{m=1}^{+\infty} |b_m^{(i)}| r_n^m$. Hence we have an injective $K$-linear continuous map $u_{i,n}: \mathcal{O}_{b,K}(C_n) \rightarrow \mathcal{O}_{b,K}(C_l)$ ($n < l$). Let $\mathcal{A}_K(C)$ be the locally convex inductive limit space of the Banach spaces $\mathcal{O}_K(C_n)$.

For any

$$f(z) = \sum_{i=1}^{l_n} \sum_{m=1}^{+\infty} a_m^{(i)}(z - c_{n,i})^m \in \mathcal{O}_K(V - C_n)/K$$

and

$$g(z) = \sum_{i=1}^{l_n} \sum_{m=1}^{+\infty} b_m^{(i)}(z - c_{n,i})^m \in \mathcal{O}_{b,K}(C_n),$$

we define

$$(f(z), g(z))_n = \sum_{i=1}^{l_n} \sum_{m=1}^{+\infty} a_m^{(i)} b_m^{(i)}.$$ 

Since $|a_m^{(i)}| r_n^m \rightarrow 0 (m \rightarrow \infty)$, and since the $|b_m^{(i)}| r_n^m$'s are bounded, this pairing $(,)_n$ is a well defined bicontinuous $K$-bilinear nondegenerate pairing. If $n < l$ and $f(z) \in \mathcal{O}_K(V - C_l)/K$, then $u_{n,i}f(z) \in \mathcal{O}_K(V - C_n)/K$ and $f(z)g(z)$ is a $K$-analytic function on $C_n - C_l$. Since $(u_{n,i}f(z), g(z))_n$ can be regarded as the sum of residues of $f(z)g(z)$ in $C_l$, it is equal to $(f(z), v_{l,n}g(z))_n$.

For any $f(z) \in \mathcal{B}(C)$ and $g(z) \in \mathcal{A}_K(C)$, we choose a positive integer $n$ and a unique element $g_n(z)$ of $\mathcal{O}_{b,K}(C_n)$ such that $g(z) = v_n(g_n(z))$, and we define

$$(f(z), g(z)) = (u_n(f(z)), g_n(z))_n.$$ 

Then it follows from the arguments in 1.1 that this pairing $(,)_n$ is well defined, bicontinuous, $K$-bilinear and nondegenerate.

Now we have the following theorem:

**Theorem 2.** Let $C$, $\mathcal{B}_K(C)$, $\mathcal{A}_K(C)$ and

$$(,) : \mathcal{B}_K(C) \times \mathcal{A}_K(C) \rightarrow K$$

be as before. Then $\mathcal{B}_K(C)$ is a Fréchet space of countable type, $\mathcal{A}_K(C)$ is a complete Hausdorff space, $(,)$ is a bicontinuous $K$-bilinear nondegenerate pairing, and $\mathcal{B}_K(C)$ and $\mathcal{A}_K(C)$ become mutually dual locally convex spaces with respect to $(,)$. 

**Proof.** Let $r_n = |d| (d \in K)$. Then the mapping

$$\mathcal{O}_K(V - C_n)/K \ni \sum_{i=1}^{l_n} \sum_{m=1}^{+\infty} a_m^{(i)}(z - c_{n,i})^m$$

$$\mapsto (a_1^{(i)}d^{-1}, a_2^{(i)}d^{-2}, \ldots, a_m^{(i)}d^{-m}, \ldots) \in c_0$$

and
are $K$-linear isometries of Banach spaces, and compatible with the pairing $(\cdot,\cdot)$ and the standard pairing $\langle \cdot,\cdot \rangle$ of $c_0$ and $l^\infty$ up to a constant factor $d^{-1}$, where

$$c_0 = \{(a_1, a_2, a_3, \ldots); a_1, a_2, a_3, \ldots \in K, |a_m| \to 0 (m \to \infty)\},$$

$$l^\infty = \{(b_1, b_2, b_3, \ldots); b_1, b_2, b_3, \ldots \in K, \sup_{1 \leq m < \infty} |b_m| < \infty\},$$

$$\langle (a_1, a_2, a_3, \ldots), (b_1, b_2, b_3, \ldots) \rangle = \sum_{m=1}^{\infty} a_m b_m.$$
2.3. Let $K$ be a field with a complete nontrivial nonarchimedean valuation $| |$. Let \( \{ r_n \}_{n=1}^{\infty} \) be a strictly increasing sequence in $|K^*|$ such that $r_n \to 1$ ($n \to \infty$). Let $W$ be the $K$-vector space consisting of all Laurent series $\sum_{m=-\infty}^{+\infty} a_m z^m$ ($a_m \in K$) such that $|a_m| r^m \to 0$ ($m \to +\infty$) for any $r$ with $0 < r < 1$, and $|a_m| r^m \to 0$ ($m \to -\infty$) for some $r$ with $0 < r < 1$. Then $W$ is the direct sum $W_1 \oplus W_2$ of two subspaces:

$W_1 = \left\{ \sum_{m=0}^{+\infty} a_m z^m ; a_m \in K, |a_m| r^m \to 0 \ (m \to +\infty) \right\}$

$W_2 = \left\{ \sum_{m=-\infty}^{-1} b_m z^m ; b_m \in K, |b_m| r^m \to 0 \ (m \to -\infty) \right\}$.

Put

$W_{1,n} = \left\{ \sum_{m=0}^{+\infty} a_m z^m ; a_m \in K, |a_m| r^m_n \to 0 \ (m \to +\infty) \right\}$

and

$W_{2,n} = \left\{ \sum_{m=-\infty}^{-1} b_m z^m ; b_m \in K, \sup_{m \in \mathbb{N}} |b_m| r^m_n < \infty \right\}$.

Then they become Banach spaces with the following norms:

$\left| \sum_{m=0}^{+\infty} a_m z^m \right|_{1,n} = \sup_{m \in \mathbb{N}} |a_m| r^m_n$ and

$\left| \sum_{m=-\infty}^{-1} b_m z^m \right|_{2,n} = \sup_{m \in \mathbb{N}} |b_m| r^m_n$.

Put

$$\left( \sum_{m=0}^{+\infty} a_m z^m, \sum_{m=-\infty}^{-1} b_m z^m \right) = \sum_{m+n=0} a_m b_n.$$

Let $d$ be an element of $K$ with $|d| = r^*_n$. Then

$$\sum_{m=0}^{+\infty} a_m z^m \mapsto (a_0, a_1 d^1, a_2 d^2, \ldots)$$

and

$$\sum_{m=-\infty}^{-1} b_m z^m \mapsto (b_{-1} d^{-1}, b_{-2} d^{-2}, b_{-3} d^{-3}, \ldots)$$

induce isometries $W_{1,n} \sim \ell^\infty$ and $W_{2,n} \sim \ell^\infty$, preserving the pairings up to a constant factor. Hence $W_{1,n}$ is of countable type, and $W_{1,n}$ and $W_{2,n}$ become mutually dual Banach spaces with respect to $(,)_n$. Further $W_1$ and $W_2$ can be identified with $\text{proj lim } W_{1,n}$ and $\text{ind lim } W_{2,n}$ with respect to the natural maps $\imath_{n,1} : \sum a_m z^m \mapsto \sum a_m z^m \ (n < l)$ and $\imath_{1,n} : \sum b_m z^m \mapsto \sum b_m z^m \ (n < l)$. Let $\tau_1$ and $\tau_2$ be the projective limit topology of $W_1$ and the
inductive limit topology of $W_2$. By Morita [5, Lemma 3.5], the $v_{i,n}$'s are c-compact maps and the projective system $\{W_{i,n}, u_{n,i}\}$ can be replaced by a cofinal system $\{W_{i,m}, u_{n,i}\}$ so that the resulting maps $u_{n,i}$ are also c-compact if $K$ is spherically complete (cf. the arguments in [5, 3–3–3–3–4]). Since $W_i$ contains all finite sums of the form $\sum_{m=0}^{\infty} a_m z^m$, the image of the projection map $u_n: W_i \rightarrow W_{i,n}$ is dense for each $n$. Hence it follows from Theorem 1 of this paper and theorems in Morita [5, 3–1] that (i) $W_i$ is a Fréchet space of countable type, (ii) $W_i$ is a complete Hausdorff space, and (iii) the pairing

$$\left( \sum_{m=0}^{\infty} a_m z^m, \sum_{m=0}^{\infty} b_m z^m \right) = \sum_{m+n=0}^{\infty} a_m b_n$$

makes $W_i$ and $W_2$ into mutually dual spaces. Hence the direct sum $W = W_i \oplus W_2$ is a complete Hausdorff space, and the inner product

$$\left( \sum_{m=0}^{\infty} a_m z^m, \sum_{m=0}^{\infty} b_m z^m \right) = \sum_{m+n=0}^{\infty} a_m b_n$$

makes $W$ into a self dual space. This selfduality of $W$ was conjectured by P. Robba.

**References**


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