ULTRAMETRIC COMPACTOIDS OF FINITE TYPE

by

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Throughout, let $K$ be a nonarchimedean nontrivially valued complete field with valuation $| |$.

§ 0. PRELIMINARIES, NOTATIONS

For fundamentals on Banach spaces, locally convex spaces over $K$ we refer to [1], [6], [3].

Let $E$ be a $K$-vector space. A subset $A$ of $E$ is absolutely convex if it is a submodule of $E$ considered as a module over $\{ \lambda \in K : |\lambda| \leq 1 \}$. A nonempty set is convex if it is an additive coset of an absolutely convex set. For $X \subseteq E$ let $\text{co } X$ be its absolutely convex hull (= the module generated by $X$), let $[X]$ be its $K$-linear span. $X$ is a finite dimensional set if $\dim [X] < \infty$.

Let $E$ be a locally convex space over $K$. The closure of $X \subseteq E$ is denoted $\overline{X}$. Instead of $\text{co } X$ we write $\overline{X}$. The dual space $E'$ of $E$ is the $K$-linear space of all continuous linear maps $E \rightarrow K$. The weak topology
on $E$ is the weakest locally convex topology on $E$ for which all elements of $E'$ are continuous. For a normed space $E = (E, \| \|)$ and a nonempty bounded subset $X$ of $E$ we write

$$\text{diam } X := \sup \{ \|x-y\| : x \in X, y \in X \}.$$
§ 1. COMPACTOIDS OF FINITE TYPE

We recall the definition of compactoidity ([1], p. 134).

DEFINITION 1.1. An absolutely convex subset $A$ of a locally convex space $E$ over $K$ is (a) compactoid if for each neighbourhood $U$ of 0 in $E$ there exist $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in E$ such that $A \subset U + \text{co}\{x_1, \ldots, x_n\}$.

The purpose of this paper is to study the impact of the following innocent-looking modification of Definition 1.1.

DEFINITION 1.2. An absolutely convex subset $A$ of a locally convex space $E$ over $K$ is a compactoid of finite type if for each neighbourhood $U$ of 0 in $E$ there exists a bounded finite dimensional (absolutely convex) set $F \subset A$ such that $A \subset U + F$.

Two remarks.

(i) A compactoid of finite type is, indeed, a compactoid in the sense of Definition 1.1. (One verifies easily that a bounded finite dimensional set lies in the absolutely convex hull of some finite set.)

(ii) Suppose the valuation of $K$ is discrete. It is not hard to prove ([3], Lemma 8.1) that in this case we can without harm replace the expression '$x_1, \ldots, x_n \in E$' in Definition 1.1 by '$x_1, \ldots, x_n \in A$'. Thus, each compactoid is automatically of finite type. Therefore

FROM NOW ON IN THIS PAPER WE ASSUME THAT THE VALUATION OF $K$ IS DENSE.

For the construction of a compactoid that is not of finite type we shall use the following simple lemma.
LEMMA 1.3. Let \( x_1, x_2, \ldots \) be linearly independent elements of a K-vector space. Let \( F \) be a finite dimensional absolutely convex subset of \( \text{co}(x_1, x_2, \ldots) \). Then \( F \subset \text{co}(x_1, \ldots, x_n) \) for some \( n \).

Proof. \([F]\) is a finite dimensional subspace of \([x_1, x_2, \ldots]\) so \( F \subset [x_1, \ldots, x_n] \) for some \( n \). From linear independence it follows easily that \( \text{co}(x_1, x_2, \ldots) \cap [x_1, \ldots, x_n] = \text{co}(x_1, \ldots, x_n) \).

EXAMPLE 1.4. There exists a compactoid \( A \) in \( c_0 \) that is not of finite type.

Proof. Choose a two-sided sequence \((\lambda_n)\) in \( K \) such that \( |\lambda_n| < |\lambda_{n+1}| \) for each \( n \in \mathbb{Z} \), \( \lim_{n \to \infty} |\lambda_n| = 1 \), \( \lim_{n \to -\infty} |\lambda_n| = 0 \). Let \( e_1, e_2, \ldots \) be the standard unit vectors in \( c_0 \), define

\[
x_n := \lambda_n e_1 + \lambda_{-n} e_{n+1}
\]

and set

\[
A := \text{co}(x_1, x_2, \ldots).
\]

\( A \) is a compactoid since \( A \subset \{ x \in c_0 : \|x\| \leq |\lambda_{-n}| \} + \text{co}(e_1, \ldots, e_n) \) for each \( n \in \mathbb{N} \). Now let \( F \) be a finite dimensional absolutely convex subset of \( A \); we shall prove that \( A \) is not contained in \( U + F \) where

\[
U := \{ x \in c_0 : \|x\| \leq |\lambda_1| \}.
\]

Lemma 1.3 \((x_1, x_2, \ldots \) are linearly independent\) yields \( F \subset \text{co}(x_1, \ldots, x_n) \) for some \( n \). For each \( i \in \{1, \ldots, n\} \)

\[
\|x_i\| = \|\lambda_i e_1 + \lambda_{-i} e_{i+1}\| = |\lambda_i| \leq |\lambda_n|
\]

We see that the norm function is bounded by \( |\lambda_n| \) on \( \text{co}(x_1, \ldots, x_n) \), so certainly on \( F \), hence also on \( U + F \). Consequently, \( x_{n+1} \in A \setminus U + F \).

The set \( A \) we just have constructed is not closed. (It is not hard to see that its closure is of finite type.) This is not completely accidental:
THEOREM 1.5. Let $K$ be spherically (= maximally) complete. Let $A$ be a complete absolutely convex compactoid in a Hausdorff locally convex space over $K$. Then $A$ is of finite type.

Proof. $A$ is bounded and $c$-compact ([4], Proposition 2.2). Now apply [4], Proposition 2.3.

PROBLEM. Let $A$ be an absolutely convex complete compactoid in a Banach space $E$ over a nonspherically complete $K$. Does it follow that $A$ is of finite type? (By [1] 4.5 (viii) it suffices to consider $E = c_0$.)

Surprisingly we have

EXAMPLE 1.6. Let $K$ be not spherically complete. The unit ball
\[ \{ x \in c_0 : \| x \| \leq 1 \} \] is, for the weak topology on $c_0$, a complete compactoid but not of finite type.

Proof. From the reflexivity of $c_0$ ([1], Theorem 4.17) it follows that the weak topology is quasicomplete ([3], Theorem 9.6). As the unit ball is weakly bounded and weakly closed it is weakly complete. The other statements follow from the following example (where $K$ is allowed to be spherically complete):

EXAMPLE 1.7. The unit ball of $c_0$ is a weak compactoid but not of finite type.

Proof. Weak compactoidity of $A := \{ x \in c_0 : \| x \| \leq 1 \}$ follows almost immediately from the fact that each weak neighbourhood of 0 contains a $K$-linear space with finite codimension. Now choose $\tau_1, \tau_2, \ldots \in K$ such that $0 < |\tau_1| < |\tau_2| < \ldots$, $\lim_{n \to \infty} |\tau_n| = 1$. There is a (unique) $f \in c_0'$ for which $f(e_n) = \tau_n$ for each $n \in \mathbb{N}$. Then $|f(x)| < 1$ for each $x \in A$. The set
\[ U := \{ x \in c_0 : |f(x)| \leq \xi \} \]
is a weak neighbourhood of 0. Let $F \subset A$ be finite dimensional; we prove that $A$ is not contained in $U+F$. By Gruson's Theorem ([1], Theorem 5.9) the (finite dimensional) space $[F]$ has an orthonormal base $z_1, \ldots, z_n$.

From $F \subset A \cap [F] \subset \text{co}\{z_1, \ldots, z_n\}$ we obtain

$$\sup_{F} |f| \leq \max_{1 \leq i \leq n} |f(z_i)| < 1.$$  

It follows that

$$\sup_{U+F} |f| < 1.$$  

But

$$\sup_A |f| \geq \sup_{n} |f(e_n)| = \lim_{n \to \infty} |\tau_n| = 1$$

and therefore $A \not\subset U+F$. 

§ 2. SOME GENERAL PROPERTIES

In § 2, \( E \) and \( E_i \) are (Hausdorff) locally convex spaces over \( K \).

PROPOSITION 2.1. If \( A \subset E \) is a compactoid of finite type then so is \( \overline{A} \).

Proof. Let \( U \) be an absolutely convex neighbourhood of 0 in \( E \). There is a finite dimensional bounded \( F \subset A \) with \( A \subset U + F \). Now \( U + F \), and also its complement, is a union of cosets of the open additive group \( U \). So \( U + F \) is closed and \( \overline{A} \subset U + F \).

PROPOSITION 2.2. Let \( T \) be a continuous linear map of \( E_i \) into \( E_2 \). If \( A \subset E_i \) is a compactoid of finite type then so is \( TA \subset E_2 \).

Proof. Let \( U \) be an absolutely convex neighbourhood of 0 in \( E_2 \). There is a finite dimensional bounded \( F \subset A \) with \( A \subset T^{-1}(U) + F \). Then \( TF \subset TA \) is finite dimensional, bounded and \( TA \subset U + TF \).

PROPOSITION 2.3. Let \( A_i \subset E_i \) \( (i \in I) \) be compactoids of finite type. Then \( A := \bigcap_i A_i \) is a compactoid of finite type in \( E := \prod_i E_i \).

Proof. Let \( U \) be a neighbourhood of 0 in \( E \); we construct a finite dimensional \( F \subset A \) with \( A \subset U + F \). We may assume that \( U = \bigcap_i U_i \) where, for each \( i \), \( U_i \) is an absolutely convex neighbourhood of 0 in \( E_i \) and where \( U_i = E_i \) except for \( i \) in some finite set \( J \subset I \). For each \( j \in J \), choose a finite dimensional \( F_j \subset A_j \) such that \( A_j \subset U_j + F_j \). If \( i \in I \setminus J \), choose \( F_i := (0) \subset A_i \). The set

\[
F := \bigcap_i F_i
\]

is finite dimensional and \( F \subset A \). To prove \( A \subset U + F \), let \( a = (a_i) \in I \in A \).

If \( j \in J \), choose \( u_j \in U_j \), \( f_j \in F_j \) such that

\[
a_j = u_j + f_j.
\]
If \( i \in I \setminus J \), take \( u_i := a_i \) and \( f_i := 0 \). We obtain a decomposition

\[
a = u + f
\]

where \( u = \{u_i\}_{i \in I} \leq U \) and \( f = \{f_i\}_{i \in I} \leq F \).

Absolutely convex subsets of a compactoid of finite type may fail to be
of finite type (Example 1.4: \( \overline{A} \) is of finite type, \( A \) is not). In fact,
each compactoid is a subset of some compactoid of finite type. ([5],
Theorem 2.1. Observe that \( \overline{co \ X (X \ compact)} \) is of finite type.) However,
we do have results for special subsets (Proposition 2.4 and 2.5).

**PROPOSITION 2.4.** Let \( A \subset E \) be a compactoid of finite type. Then so is
\[
A^i := \bigcup_{\lambda \in \lambda} \lambda A,
\]
where \( |\lambda| < 1 \).

**Proof.** An obvious verification yields

\[
(S + T)^i = S^i + T^i
\]

for absolutely convex \( S, T \subset E \). Now let \( U \) be an absolutely convex
neighbourhood of 0 in \( E \). There is a bounded absolutely convex finite
dimensional \( F \subset A \) with \( A \leq U + F \). Then \( A^i \leq (U + F)^i = U^i + F^i \leq U + F^i \) and
\( F^i \leq A^i \).

**PROPOSITION 2.5.** Let \( A \subset E \) be a compactoid of finite type, let \( U \subset E \) be
an absolutely convex neighbourhood of 0. Then \( A \cap U \) is of finite type.

**Proof.** Let \( V \) be an absolutely convex neighbourhood of 0 in \( E \). To prove
that \( A \cap U \subset V + F \) for some finite dimensional \( F \subset A \cap U \) we may assume
\( V \subset U \). There is a finite dimensional bounded absolutely convex \( G \subset A \)
for which \( A \subset V + G \). Set \( F : G \cap U \). If \( x \in A \cap U \) then \( x = v + g \) for some
\( v \in V \subset U \) and \( g \in G \). Then \( g = x - v \in U \). Hence, \( A \cap U \subset V + F \).
Proposition 2.4 leads to a 'dual' question. For an absolutely convex $A$ set $A^e := \bigcap_{|\lambda| > 1} \lambda A$. If $A$ is a compactoid of finite type, does it follow that $A^e$ is of finite type? This question is more difficult than the one for $A$. I can answer it only for spherically complete $K$. (Proposition 2.7.)

**Lemma 2.6.** Let $S, T \subset E$ be absolutely convex. Suppose $S$ is closed and $T$ is c-compact. Then $(S+T)^e = S^e + T^e$.

**Proof.** We may assume $S = S^e$, $T = T^e$; we prove that $(S+T)^e \subset S+T$. Let $z \in (S+T)^e$. For each $\lambda \in K$, $|\lambda| > 1$ we have $z \in \lambda (S+T)$ i.e. $z-\lambda S$ meets $\lambda T$. So, for each $\lambda \in K$, $|\lambda| > 1$ the convex set

$$V_\lambda := (z-\lambda S) \cap \lambda T$$

is a nonempty closed subset of $\lambda T$, hence c-compact. If $1 < |\lambda| < |\mu|$ then $V_\lambda \subset V_\mu$. By c-compactness there is a $t \in \bigcap_{|\lambda| > 1} V_\lambda$. Then $t \in \bigcap_{|\lambda| > 1} \lambda T = T^e = T$ and $t \in z-\lambda S$ for all $\lambda \in K$, $|\lambda| > 1$, i.e. $z-t \in S^e = S$. It follows that $z \in S+T$.

**Proposition 2.7.** Let $K$ be spherically complete. If $A \subset E$ is a compactoid of finite type then so is $A^e := \bigcap_{|\lambda| > 1} \lambda A$.

**Proof.** Let $U$ be an absolutely convex neighbourhood of 0 in $E$, let $\lambda \in K$, $0 < |\lambda| < 1$. There is a finite dimensional absolutely convex set $F \subset A$ such that $A \subset \lambda U + F$. Now $\lambda U$ is closed and $F$ is c-compact (each convex subset of $K^n$ is closed hence c-compact) so that we may apply the previous Lemma. We find

$$A^e \subset (\lambda U + F)^e = (\lambda U)^e + F^e \subset U + F^e$$
which proves Proposition 2.7.

PROBLEM. Is the statement about $A$ in Proposition 2.7 true if $K$ is not spherically complete? (See [2], Example 5.4 for the difficulties one encounters with the identity $(S+T)^e = S^e + T^e$.)

PROPOSITION 2.8. Let $A \subseteq E$ be a compactoid of finite type. For each continuous seminorm $p$ on $E$ there exists a finite dimensional set $F \subseteq A$ for which $\sup_{A} p = \sup_{F} p$.

Proof. We may suppose $s := \sup_{A} p > 0$. Set $U := \{x \in E : p(x) \leq \frac{1}{2}s\}$. There is a finite dimensional absolutely convex set $F \subseteq A$ for which $A \subseteq U+F$. Then $A = A \cap U + F$. Now $p \leq \frac{1}{2}s$ on $U \cap A$. It follows easily (strong triangle inequality) that $\sup_{A} p = \sup_{F} p$.

LEMMA 2.9. Let $K$ be spherically complete, let $F \neq (0)$ be a finite dimensional absolutely convex subset of some $K$-vector space. Then there are onedimensional absolutely convex sets $F_1, \ldots, F_n$ for some $n \in \mathbb{N}$ such that $F = \bigoplus_{i=1}^{n} F_i$.

Proof. [2], Corollary 2.13 (i).

Remark. Let $K$ be not spherically complete. The unit ball of $K^2_0$ (see [1], p. 68) is twodimensional but indecomposable.

COROLLARY 2.10. Let $K$ be spherically complete, let $A \subseteq E$ be a compactoid of finite type. For each continuous seminorm $p$ on $E$ there exists an $x \in E$ such that $\sup_{A} p = \sup_{Kx \cap A} p$. 
Proof. We may assume $p \neq 0$. By Proposition 2.8, $\sup_{A} p = \sup_{F} p$ for some finite dimensional absolutely convex $F \subset A$. By the Lemma,

$$F = \sum_{i=1}^{n} F_{i}$$

for some onedimensional $F_{1}, \ldots, F_{n}$. The strong triangle inequality yields $\sup_{F} p = \max_{i} \sup_{F_{i}} p = \sup_{F_{j}} p$ for some $j$. There is an $x \in E$ and an absolutely convex set $C \subset K$ such that $F_{j} = Cx$. Then $Kx \cap A \supset Cx$ and we have

$$\sup_{F} p = \sup_{F} p \leq \sup_{Kx} p \leq \sup_{Kx \cap A} p = \sup_{A} p.$$

$$F \quad Cx \quad Kx \cap A \quad A \quad F$$
§ 3. COMPACTOIDS OF FINITE TYPE IN NORMED SPACES

THEOREM 3.1. Let $E$ be a normed space over $K$, let $A \subseteq E$ be absolutely convex. The following are equivalent.

(a) $A$ is a compactoid of finite type.

(β) There exist bounded finite dimensional absolutely convex sets $F_0, F_1, \ldots$ such that $\lim_{n \to \infty} \text{diam } F_n = 0$ and

$$\sum F_n \subseteq A \subseteq \sum F_n$$

Proof.

(a) $\Rightarrow$ (β). For $n \in \mathbb{N}$, set $U_n := \{ x \in E : \|x\| \leq 2^{-n} \}$. There is a bounded finite dimensional absolutely convex $F_0 \subset A$ such that $A \subseteq U_1 + F_0$. Then $A = A \cap U_1 + F_0$. By Proposition 2.5, $A \cap U_1$ is a compactoid of finite type. So there exists a finite dimensional absolutely convex $F_1 \subset A \cap U_1$ such that $A \cap U_1 \subseteq U_2 + F_1$ i.e.

$$A \cap U_1 = A \cap U_2 + F_1$$

We have $\text{diam } F_1 \leq 2^{-1}$ and $A = A \cap U_2 + F_1 + F_0$.

Inductively we find bounded absolutely convex finite dimensional sets $F_0, F_1, \ldots$ with $\text{diam } F_n \leq 2^{-n}$ and

$$A = A \cap U_n + \sum_{i=1}^{n-1} F_i$$

for each $n \in \{1, 2, \ldots\}$. As $\text{diam } A \cap U_n \leq 2^{-n}$, (β) follows.

(β) $\Rightarrow$ (a). Let $\varepsilon > 0$. There is an $N \in \mathbb{N}$ such that $\text{diam } F_n \leq \varepsilon$ for $n \geq N$. We see that

$$\sum F_i \subseteq \{ x \in E : \|x\| \leq \varepsilon \} + \sum_{i=0}^{n-1} F_i$$

The set at the right hand side is closed so that

$$A \subseteq \{ x \in E : \|x\| \leq \varepsilon \} + \sum_{i=0}^{n-1} F_i$$

and we have (a).
Two remarks.

(i) For spherically complete $K$ we have a more refined version of Theorem 3.1, namely we may replace (β) by:

(β)' There exist $e_1, e_2, \ldots$ in $E$ and absolutely convex sets $C_1, C_2, \ldots \subseteq K$ such that $C_n e_n$ is bounded for each $n$,

\[ \lim_{n \to \infty} (\text{diam } C_n) \|e_n\| = 0 \text{ and} \]

\[ \sum_{n} C_n e_n \subseteq A \subseteq \sum_{n} C_n e_n. \]

The proof is obvious (Lemma 2.9).

(ii) It is not hard to generalize Theorem 3.1 to the case of a metrizable absolutely convex set $A$ in a locally convex space $E$. We leave the details to the reader.
§ 4. WEAK COMPACTOIDS OF FINITE TYPE

The main goal in this section is the following theorem, which is a generalization of Example 1.7. Recall that the valuation of $K$ is dense.

**THEOREM 4.1.** Let $E$ be a Banach space with an orthogonal base over a spherically complete $K$. If $A \subset E$ is a compactoid of finite type for the weak topology then $A$ is a compactoid (of finite type) for the norm topology.

This Theorem is in contrast to

**PROPOSITION 4.2.** Let $E$ be a Banach space over a spherically complete $K$. An absolutely convex $A \subset E$ is a compactoid for the weak topology if and only if $A$ is bounded for the norm topology.

**Proof.** A weak compactoid is weakly bounded, hence bounded by [6], Théorème 4.21. Conversely, if $A$ is norm bounded, let $U$ be a weak neighbourhood of $0$ in $E$. There is a closed $K$-subspace $D$ of $E$ of finite codimension with $D \subset U$. Let $\pi : E \to E/D$ be the quotient map. $\pi(A)$ is bounded in the finite dimensional space $E/D$, so a compactoid. As $\pi(U)$ is open in $E/D$ there exist $x_1, \ldots, x_n \in E/D$ such that $\pi(A) \subset \pi(U) + \text{co}\{x_1, \ldots, x_n\}$. Let $y_1, \ldots, y_n \in E$ be such that $\pi(y_i) = x_i$ for each $i \in \{1, \ldots, n\}$. It is easy to see that $A \subset U + \text{co}\{y_1, \ldots, y_n\}$.

Theorem 4.1 and Proposition 4.2 indicate that, for a compactoid, to require it to be of finite type may be a severe restriction!

**PROBLEM.** Does the conclusion of Theorem 4.1 hold for arbitrary Banach spaces over a spherically complete $K$ or for decent Banach spaces
(e.g. \( c_0 \)) over a nonspherically complete \( K \)?

The proof of Theorem 4.1 requires several preparations. From now on in § 4 we assume that \( K \) is spherically complete.

For an element \( f \) of the dual \( E' \) of a Banach space \( E \) over \( K \), let \( \| f \| \)
 denote the operator norm of \( f \) defined by the usual formula

\[ \| f \| = \inf \{ M : |f(x)| \leq M \| x \| \text{ for all } x \in E \}. \]

**Lemma 4.3.** Let \( E \) be a Banach space over \( K \). Then

\[ \| f \| = \sup \{ |f(x)| : \| x \| \leq 1 \} \quad (f \in E'). \]

**Proof.** We may assume \( \dim E \geq 1 \). Set \( \| f \|_0 := \sup \{ |f(x)| : \| x \| \leq 1 \} \).

If \( x \in E, 0 < \| x \| \leq 1 \) then \( |f(x)| \leq \frac{|f(x)|}{\| x \|} \leq \| f \| \) so that \( \| f \|_0 \leq \| f \| \).

Conversely, let \( x \in E, x \neq 0 \). As the valuation of \( K \) is dense we can find \( \lambda_1, \lambda_2, \ldots \in K \) such that \( |\lambda_1| < |\lambda_2| < \ldots, \lim_{n \to \infty} |\lambda_n| = \| x \|^{-1} \).

Then \( \| \lambda_n x \| \leq 1 \) whence \( |f(\lambda_n x)| \leq \| f \|_0 \) for each \( n \). We have

\[ \frac{|f(x)|}{\| x \|} = \frac{|f(\lambda_n x)|}{\| \lambda_n x \|} \leq \frac{\| f \|_0}{\| \lambda_n x \|} \]

which, after taking limits becomes \( \frac{|f(x)|}{\| x \|} \leq \| f \|_0 \) and we obtain \( \| f \| \leq \| f \|_0 \).

**Lemma 4.4.** Let \( \{ D_i : i \in I \} \) be an orthogonal system of closed subspaces ([1], p. 166) in a Banach space \( E \) over \( K \). For each \( i \in I \), let \( S_i \subseteq D_i \)
 be closed, absolutely convex, \( [S_i] = D_i, S_i^e = S_i \) (see the remark following Proposition 2.5). Then \( (ES_i)^e = ES_i \).

**Proof.** Let \( x \in (ES_i)^e \). Then \( x \in \overline{D_i} \) so \( x \) has a unique decomposition...
\[ x = \sum d_i \text{ where } d_i \in D_i = [S_i] \text{ for each } i \in I. \text{ For any } \lambda \in K, \]
\[ 0 < |\lambda| < 1, \lambda x \in \overline{LS_i} \text{ so } \lambda x = \sum \lambda s_i, \text{ where } s_i \in S_i \text{ for each } i. \text{ But also } \]
\[ \lambda x = \sum \lambda d_i. \text{ By the uniqueness of the decomposition of } \lambda x \text{ we have } \]
\[ \lambda d_i = s_i \text{ i.e. } d_i \in \lambda^{-1} S_i \text{ for each } i. \text{ This goes for any } \lambda \in K, \]
\[ 0 < |\lambda| < 1 \text{ and we find } d_i \in S^e_i = S_i \text{ for each } i. \text{ It follows that } \]
\[ x \in \overline{LS_i}. \]

**Lemma 4.5.** Let \( S \) be a closed absolutely convex subset of a Banach space \( E \) over \( K \) for which \( S = S^e \). If \( x \in E \setminus S \) there exists an \( f \in E' \) with
\[ |\lambda| \leq 1 \text{ on } S, |f(x)| > 1. \]
**Proof.** [6], Théorème 4.7a.

**Lemma 4.6.** Let \( E \) be a Banach space over \( K \) such that each maximal orthogonal system of vectors in \( E \) is an orthogonal base. Then \( E \) is finite dimensional.
**Proof.** [1], Theorem 5.16 (a) \( \Rightarrow \) (b). (Recall that the valuation of \( K \) is dense.)

**Lemma 4.7.** Let \( \Lambda \) be an absolutely convex subset of a \( K \)-vector space.
For the seminorm \( p_\Lambda \) associated to \( \Lambda \) defined on \( [\Lambda] \) by the formula
\[ p_\Lambda(x) = \inf \{ |\lambda| : \lambda \in K, x \in \lambda \Lambda \} \quad (x \in [\Lambda]) \]
we have
\[ \{x \in [\Lambda] : p_\Lambda(x) < 1\} \subset \Lambda \subset \{x \in [\Lambda] : p_\Lambda(x) \leq 1\}. \]
Further, \( \Lambda = \Lambda^e \) if and only if
\[ \Lambda = \{x \in [\Lambda] : p_\Lambda(x) \leq 1\}. \]
**Proof.** Left to the reader.
PROPOSITION 4.8. (Compare Example 1.7.) Let $E$ be a Banach space over $K$ and suppose there exists an open absolutely convex set $A \subset E$ that is a compactoid of finite type for the weak topology. Then $E$ is finite dimensional.

Proof. By Proposition 2.7 we may assume that $A = A^E$. $A$ is bounded by Proposition 4.2 and therefore the seminorm associated to $A$ is norm $|| \cdot ||$ inducing the topology of $E$. Lemma 4.7 yields

$$A = \{ x \in E : ||x|| \leq 1 \}.$$

Now let $\{ e_i : i \in I \}$ be a maximal orthogonal system in $E$, it suffices to prove that it is an orthogonal base for $E$ (Lemma 4.6). For each $i \in I$ set $C_i := \{ \lambda \in K : \lambda e_i \in A \}$. Then $C_i = \{ \lambda \in K : |\lambda| \leq ||e_i||^{-1} \}$ so that $(C_i e_i)^e = C_i e_i$. Set

$$B := \bigoplus_{i \in I} C_i e_i.$$

Obviously, $B \subset A$. It remains to prove that $B = A$. Suppose $B \neq A$. Lemma 4.4 tells us that $B = B^e$ so, by Lemma 4.5, there exists an $f \in E'$ such that $|f| \leq 1$ on $B$ and $\sup_{A} |f| > 1$. According to Corollary 2.10 and Lemma 4.3 there exists an $x \in E$ such that

$$||f|| = \sup_{A} |f| = \sup_{Kx \cap A} |f| > 1.$$

Without harm (suitable scalar multiplication) we may assume $x \in A$ and $|f(x)| > 1$. Now $x \neq e_i$ for each $i \in I$ and, by maximality, the system $\{x\} \cup \{ e_i : i \in I \}$ is not orthogonal. So there exists a (finite) $K$-linear combination $z = \sum \lambda_i e_i$ for which $||x-z|| < ||x||$. Then max $||\lambda_i e_i|| = ||z|| = ||x|| \leq 1$, so that $\lambda_i \in C_i$ for each $i$ whence $z \in B$ and $|f(z)| < 1$. We get

$$(*) \quad |f(x)| = |f(x-z)| \leq ||f|| \ ||x-z|| < ||f|| \ ||x||.$$
Now let \( C := \{ \lambda \in K : \lambda x \in A \} \). We have \( \lambda \in C \Rightarrow \|\lambda x\| \leq 1 \). Hence,
\[
(diam C) \|x\| \leq 1.
\]
On the other hand we obtain, using (*),
\[
\|f\| = \sup_{\lambda \in C} |f(\lambda x)| = (diam C) |f(x)| < (diam C) \|f\| \|x\|
\]
yielding
\[
(diam C) \|x\| > 1,
\]
a contradiction.

Proof of Theorem 4.1. Suppose \( A \) is not a compactoid for the norm topology. Then, by [1], Theorem 4.38, \((\eta) \Rightarrow (a)\) there exists an orthogonal sequence \( e_1, e_2, \ldots \) in \( A \) such that \( \inf_{n} \|e_n\| > 0 \). Set \( D := \{e_1, e_2, \ldots \} \). By [1], Corollary 3.18 there exists a continuous linear projection \( P \) of \( E \) onto \( D \). Since \( e_n \in PA \) for each \( n \) and
\[
\inf_{n} \|e_n\| > 0 \text{ we have that } \overline{PA} \text{ is open in } D. \text{ But (}P \text{ is also weakly continuous) the weak closure of } \overline{PA} \text{ is a compactoid of finite type in } D \text{ for the weak topology of } D \text{ (Propositions 2.2 and 2.1). This is impossible by Proposition 4.8. Thus, } A \text{ is a compactoid for the norm topology. To see that } A \text{ is also of finite type it suffices to observe that the weak and norm topology coincide on } A \text{ ([3], Theorem 5.12).}
REFERENCES


