ULTRAMETRIC COMPACTOIDS OF FINITE TYPE

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Throughout, let $K$ be a nonarchimedean nontrivially valued complete field with valuation $| |$.

§ 0. PRELIMINARIES, NOTATIONS

For fundamentals on Banach spaces, locally convex spaces over $K$ we refer to [1], [6], [3].

Let $E$ be a $K$-vector space. A subset $A$ of $E$ is absolutely convex if it is a submodule of $E$ considered as a module over $\{ \lambda \in K : |\lambda| \leq 1 \}$. A nonempty set is convex if it is an additive coset of an absolutely convex set. For $X \subset E$ let $\text{co } X$ be its absolutely convex hull (= the module generated by $X$), let $[X]$ be its $K$-linear span. $X$ is a finite dimensional set if $\dim [X] < \infty$.

Let $E$ be a locally convex space over $K$. The closure of $X \subset E$ is denoted $\overline{X}$. Instead of $\text{co } X$ we write $\overline{X}$. The dual space $E'$ of $E$ is the K-linear space of all continuous linear maps $E \to K$. The weak topology
on $E$ is the weakest locally convex topology on $E$ for which all elements of $E'$ are continuous. For a normed space $E = (E, ||\cdot||)$ and a nonempty bounded subset $X$ of $E$ we write
\[
\text{diam } X := \sup \{ ||x-y|| : x \in X, y \in X \}.
\]
§ 1. COMPACTOIDS OF FINITE TYPE

We recall the definition of compactoidity ([1], p. 134).

DEFINITION 1.1. An absolutely convex subset $A$ of a locally convex space $E$ over $K$ is (a) \textit{compactoid} if for each neighbourhood $U$ of $0$ in $E$ there exist $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in E$ such that $A \subset U + \text{co}\{x_1, \ldots, x_n\}$.

The purpose of this paper is to study the impact of the following innocent-looking modification of Definition 1.1.

DEFINITION 1.2. An absolutely convex subset $A$ of a locally convex space $E$ over $K$ is a \textit{compactoid of finite type} if for each neighbourhood $U$ of $0$ in $E$ there exists a bounded finite dimensional (absolutely convex) set $F \subset A$ such that $A \subset U + F$.

Two remarks.

(i) A compactoid of finite type is, indeed, a compactoid in the sense of Definition 1.1. (One verifies easily that a bounded finite dimensional set lies in the absolutely convex hull of some finite set.)

(ii) Suppose the valuation of $K$ is discrete. It is not hard to prove ([3], Lemma 8.1) that in this case we can without harm replace the expression $'x_1, \ldots, x_n \in E'$ in Definition 1.1 by $'x_1, \ldots, x_n \in A'$ Thus, each compactoid is automatically of finite type. Therefore

FROM NOW ON IN THIS PAPER WE ASSUME THAT THE VALUATION OF $K$ IS DENSE.

For the construction of a compactoid that is not of finite type we shall use the following simple lemma.
LEMMA 1.3. Let \( x_1, x_2, \ldots \) be linearly independent elements of a \( K \)-vector space. Let \( F \) be a finite dimensional absolutely convex subset of \( \text{co}(x_1, x_2, \ldots) \). Then \( F \subset \text{co}(x_1, \ldots, x_n) \) for some \( n \).

Proof. \([F]\) is a finite dimensional subspace of \([x_1, x_2, \ldots]\) so \( F \subset [x_1, \ldots, x_n] \) for some \( n \). From linear independence it follows easily that \( \text{co}(x_1, x_2, \ldots) \cap [x_1, \ldots, x_n] = \text{co}(x_1, \ldots, x_n) \).

EXAMPLE 1.4. There exists a compactoid \( A \) in \( c_0 \) that is not of finite type.

Proof. Choose a two-sided sequence \((\lambda_n)_{n \in \mathbb{Z}}\) in \( K \) such that for each \( n \in \mathbb{Z} \), \( \lim_{n \to \infty} \lambda_n = 1 \), \( \lim_{n \to -\infty} \lambda_n = 0 \). Let \( e_1, e_2, \ldots \) be the standard unit vectors in \( c_0 \), define

\[
x_n := \lambda_n e_1 + \lambda_{-n} e_{n+1}
\]

and set

\[
A := \text{co}(x_1, x_2, \ldots).
\]

\( A \) is a compactoid since \( A \subset \{ x \in c_0 : \|x\| \leq |\lambda_n| \} + \text{co}(e_1, \ldots, e_n) \) for each \( n \in \mathbb{N} \). Now let \( F \) be a finite dimensional absolutely convex subset of \( A \); we shall prove that \( A \) is not contained in \( U+F \) where

\[
U := \{ x \in c_0 : \|x\| \leq |\lambda_1| \}. \quad \text{Lemma 1.3 } (x_1, x_2, \ldots \text{ are linearly independent})
\]

yields \( F \subset \text{co}(x_1, \ldots, x_n) \) for some \( n \). For each \( i \in \{1, \ldots, n\} \)

\[
\|x_i\| = \|\lambda_i e_1 + \lambda_{-i} e_{i+1}\| = |\lambda_i| \leq |\lambda_n|
\]

We see that the norm function is bounded by \( |\lambda_n| \) on \( \text{co}(x_1, \ldots, x_n) \), so certainly on \( F \), hence also on \( U+F \). Consequently, \( x_{n+1} \in A \setminus U+F \).

The set \( A \) we just have constructed is not closed. (It is not hard to see that its closure is of finite type.) This is not completely accidental:
THEOREM 1.5. Let $K$ be spherically (= maximally) complete. Let $A$ be a complete absolutely convex compactoid in a Hausdorff locally convex space over $K$. Then $A$ is of finite type.

Proof. $A$ is bounded and $c$-compact ([4], Proposition 2.2). Now apply [4], Proposition 2.3.

PROBLEM. Let $A$ be an absolutely convex complete compactoid in a Banach space $E$ over a nonspherically complete $K$. Does it follow that $A$ is of finite type? (By [1] 4.5 (viii) it suffices to consider $E = c_0$.)

Surprisingly we have

EXAMPLE 1.6. Let $K$ be not spherically complete. The unit ball $\{x \in c_0 : \|x\| \leq 1\}$ is, for the weak topology on $c_0$, a complete compactoid but not of finite type.

Proof. From the reflexivity of $c_0$ ([1], Theorem 4.17) it follows that the weak topology is quasicomplete ([3], Theorem 9.6). As the unit ball is weakly bounded and weakly closed it is weakly complete. The other statements follow from the following example (where $K$ is allowed to be spherically complete):

EXAMPLE 1.7. The unit ball of $c_0$ is a weak compactoid but not of finite type.

Proof. Weak compactoidity of $A := \{x \in c_0 : \|x\| \leq 1\}$ follows almost immediately from the fact that each weak neighbourhood of 0 contains a $K$-linear space with finite codimension. Now choose $\tau_1, \tau_2, \ldots \in K$ such that $0 < |\tau_1| < |\tau_2| < \ldots$, $\lim_{n \to \infty} |\tau_n| = 1$. There is a (unique) $f \in c_0^*$ for which $f(e_n) = \tau_n$ for each $n \in \mathbb{N}$. Then $|f(x)| < 1$ for each $x \in A$. The set

$U := \{x \in c_0 : |f(x)| \leq \frac{1}{2}\}$
that $A$ is not contained in $U+F$. By Gruson's Theorem ([1], Theorem 5.9) the (finite dimensional) space $[F]$ has an orthonormal base $z_1, \ldots, z_n$.

From $F \subset A \cap [F] \subset \text{co}(z_1, \ldots, z_n)$ we obtain

$$\sup_{F} |f| \leq \max_{1 \leq i \leq n} |f(z_i)| < 1.$$

It follows that

$$\sup_{U+F} |f| < 1.$$

But

$$\sup_{A} |f| \geq \sup_{n} |f(e_n)| = \lim_{n \to \infty} |\tau_n| = 1$$

and therefore $A \not\subset U+F$. 

is a weak neighbourhood of 0. Let $F \subset A$ be finite dimensional; we prove
In § 2, $E$ and $E_i$ are (Hausdorff) locally convex spaces over $K$.

PROPOSITION 2.1. If $A \subset E$ is a compactoid of finite type then so is $\overline{A}$.

Proof. Let $U$ be an absolutely convex neighbourhood of 0 in $E$. There is a finite dimensional bounded $F \subset A$ with $A \subset U+F$. Now $U+F$, and also its complement, is a union of cosets of the open additive group $U$. So $U+F$ is closed and $\overline{A} \subset U+F$.

PROPOSITION 2.2. Let $T$ be a continuous linear map of $E_1$ into $E_2$. If $A \subset E_1$ is a compactoid of finite type then so is $TA \subset E_2$.

Proof. Let $U$ be an absolutely convex neighbourhood of 0 in $E_2$. There is a finite dimensional bounded $F \subset A$ with $A \subset T^{-1}(U)+F$. Then $TF \subset TA$ is finite dimensional, bounded and $TA \subset U+TF$.

PROPOSITION 2.3. Let $A_i \subset E_i$ ($i \in I$) be compactoids of finite type. Then $A := \cap_{i \in I} A_i$ is a compactoid of finite type in $E := \cap_{i \in I} E_i$.

Proof. Let $U$ be a neighbourhood of 0 in $E$; we construct a finite dimensional $F \subset A$ with $A \subset U+F$. We may assume that $U = \cap_{i \in I} U_i$ where, for each $i$, $U_i$ is an absolutely convex neighbourhood of 0 in $E_i$ and where $U_i = E_i$ except for $i$ in some finite set $J \subset I$. For each $j \in J$, choose a finite dimensional $F_j \subset A_j$ such that $A_j \subset U_j+F_j$. If $i \in I \setminus J$, choose $F_i := (0) \subset A_i$. The set

$$F := \cap_{i \in I} F_i$$

is finite dimensional and $F \subset A$. To prove $A \subset U+F$, let $a = (a_i)_{i \in I} \in \cap_{i \in I} A_i$. If $j \in J$, choose $u_j \in U_j$, $f_j \in F_j$ such that

$$a_j = u_j + f_j.$$
If \( i \in I \setminus J \), take \( u_i := a_i \) and \( f_i := 0 \). We obtain a decomposition

\[
a = u + f
\]

where \( u = \{u_i\}_{i \in I} \in U \) and \( f = \{f_i\}_{i \in I} \in F \).

Absolutely convex subsets of a compactoid of finite type may fail to be of finite type (Example 1.4: \( \overline{A} \) is of finite type, \( A \) is not). In fact, each compactoid is a subset of some compactoid of finite type. ([5], Theorem 2.1. Observe that \( \overline{coX} \) (\( X \) compact) is of finite type.) However, we do have results for special subsets (Proposition 2.4 and 2.5).

**Proposition 2.4.** Let \( A \subset E \) be a compactoid of finite type. Then so is \( A^i := \bigcup_{|\lambda| < 1} \lambda A \).

**Proof.** An obvious verification yields

\[
(S + T)^i = S^i + T^i
\]

for absolutely convex \( S, T \subset E \). Now let \( U \) be an absolutely convex neighbourhood of 0 in \( E \). There is a bounded absolutely convex finite dimensional \( F \subset A \) with \( A \subset U + F \). Then \( A^i \subset (U + F)^i = U^i + F^i \subset U + F^i \) and \( F^i \subset A^i \).

**Proposition 2.5.** Let \( A \subset E \) be a compactoid of finite type, let \( U \subset E \) be an absolutely convex neighbourhood of 0. Then \( A \cap U \) is of finite type.

**Proof.** Let \( V \) be an absolutely convex neighbourhood of 0 in \( E \). To prove that \( A \cap U \subset V + F \) for some finite dimensional \( F \subset A \cap U \) we may assume \( V \subset U \). There is a finite dimensional bounded absolutely convex \( G \subset A \) for which \( A \subset V + G \). Set \( F := G \cap U \). If \( x \in A \cap U \) then \( x = v + g \) for some \( v \in V \subset U \) and \( g \in G \). Then \( g = x - v \in U \). Hence, \( A \cap U \subset V + F \).
Proposition 2.4 leads to a 'dual' question. For an absolutely convex set \( A^e := \bigcap \lambda A \). If \( A \) is a compactoid of finite type, does it follow that \( A^e \) is of finite type? This question is more difficult than the one for \( A^i \). I can answer it only for spherically complete \( K \). (Proposition 2.7.)

**Lemma 2.6.** Let \( S, T \subseteq E \) be absolutely convex. Suppose \( S \) is closed and \( T \) is c-compact. Then \((S+T)^e = S^e + T^e\).

**Proof.** We may assume \( S = S^e, T = T^e \); we prove that \((S+T)^e \subseteq S+T\). Let \( z \in (S+T)^e \). For each \( \lambda \in K, |\lambda| > 1 \) we have \( z \in \lambda(S+T) \) i.e. \( z-\lambda S \) meets \( \lambda T \). So, for each \( \lambda \in K, |\lambda| > 1 \) the convex set

\[
V_\lambda := (z-\lambda S) \cap \lambda T
\]

is a nonempty closed subset of \( \lambda T \), hence c-compact. If \( 1 < |\lambda| < |\mu| \) then \( V_\lambda \subseteq V_\mu \). By c-compactness there is a \( t \in \bigcap V_\lambda \). Then

\[
t \in \bigcap \lambda T = T^e = T \quad \text{and} \quad t \in z-\lambda S \text{ for all } \lambda \in K, |\lambda| > 1, \text{ i.e.}
\]

\[
z-t \in S^e = S. \quad \text{It follows that } z \in S+T.
\]

**Proposition 2.7.** Let \( K \) be spherically complete. If \( A \subseteq E \) is a compactoid of finite type then so is \( A^e := \bigcap \lambda A \) where \( |\lambda| > 1 \).

**Proof.** Let \( U \) be an absolutely convex neighbourhood of \( 0 \) in \( E \), let \( \lambda \in K, 0 < |\lambda| < 1 \). There is a finite dimensional absolutely convex set \( F \subseteq A \) such that \( A \subseteq \lambda U + F \). Now \( \lambda U \) is closed and \( F \) is c-compact (each convex subset of \( K^n \) is closed hence c-compact) so that we may apply the previous Lemma. We find

\[
A^e \subseteq (\lambda U + F)^e = (\lambda U)^e + F^e \subseteq U + F^e
\]
which proves Proposition 2.7.

PROBLEM. Is the statement about $A$ in Proposition 2.7 true if $K$ is not spherically complete? (See [2], Example 5.4 for the difficulties one encounters with the identity $(S+T)^e = S^e + T^e$.)

PROPOSITION 2.8. Let $A \subseteq E$ be a compactoid of finite type. For each continuous seminorm $p$ on $E$ there exists a finite dimensional set $F \subseteq A$ for which

$$\sup_{A} p = \sup_{F} p.$$

Proof. We may suppose $s := \sup_{A} p > 0$. Set $U := \{x \in E : p(x) \leq \frac{1}{2}s\}$. There is a finite dimensional absolutely convex set $F \subseteq A$ for which $A \subseteq U + F$. Then $A = A \cap U + F$. Now $p \leq \frac{1}{2}s$ on $U \cap A$. It follows easily (strong triangle inequality) that $\sup_{A} p = \sup_{F} p$.

LEMMA 2.9. Let $K$ be spherically complete, let $F \neq \{0\}$ be a finite dimensional absolutely convex subset of some $K$-vector space. Then there are onedimensional absolutely convex sets $F_1, \ldots, F_n$ for some $n \in \mathbb{N}$ such that $F = \bigoplus_{i=1}^{n} F_i$.

Proof. [2], Corollary 2.13 (i).

Remark. Let $K$ be not spherically complete. The unit ball of $K_0^2$ (see [1], p. 68) is twodimensional but indecomposable.

COROLLARY 2.10. Let $K$ be spherically complete, let $A \subseteq E$ be a compactoid of finite type. For each continuous seminorm $p$ on $E$ there exists an $x \in E$ such that

$$\sup_{A} p = \sup_{Kx \cap A} p.$$
Proof. We may assume $p \neq 0$. By Proposition 2.8, $\sup_{A} p = \sup_{F} p$ for some finite dimensional absolutely convex $F \subset A$. By the Lemma,

$$F = \sum_{i=1}^{n} F_{i}$$

for some onedimensional $F_{1}, \ldots, F_{n}$. The strong triangle inequality yields $\sup_{F} p = \max_{i=1}^{n} \sup_{F_{i}} p = \sup_{F_{j}} p$ for some $j$. There is an $x \in E$ and an absolutely convex set $C \subset K$ such that $F_{j} = Cx$. Then

$$Kx \cap A \supset Cx$$

and we have

$$\sup_{F} p = \sup_{C} p \leq \sup_{Kx \cap A} p \leq \sup_{A} p = \sup_{F} p.$$
§ 3. COMPACTOIDS OF FINITE TYPE IN NORMED SPACES

THEOREM 3.1. Let $E$ be a normed space over $K$, let $A \subseteq E$ be absolutely convex. The following are equivalent.

(a) $A$ is a compactoid of finite type.

(b) There exist bounded finite dimensional absolutely convex sets $F_0, F_1, \ldots$ such that $\lim_{n \to \infty} \text{diam } F_n = 0$ and $\sum F_n \subseteq A \subseteq \sum F_n$.

Proof.

(a) $\Rightarrow$ (b). For $n \in \mathbb{N}$, set $U_n := \{x \in E : \|x\| < 2^{-n}\}$. There is a bounded finite dimensional absolutely convex $F_0 \subseteq A$ such that $A \subseteq U_1 + F_0$. Then $A = A \cap U_1 + F_0$. By Proposition 2.5, $A \cap U_1$ is a compactoid of finite type. So there exists a finite dimensional absolutely convex $F_1 \subseteq A \cap U_1$ such that $A \cap U_1 \subseteq U_2 + F_1$. We have $\text{diam } F_1 \leq 2^{-1}$ and $A = A \cap U_2 + F_1 + F_0$.

Inductively we find bounded absolutely convex finite dimensional sets $F_0, F_1, \ldots$ with $\text{diam } F_n \leq 2^{-n}$ and

$$A = A \cap U_n + \sum_{i=1}^{n-1} F_i$$

for each $n \in \{1, 2, \ldots\}$. As $\text{diam } A \cap U_n \leq 2^{-n}$, (b) follows.

(b) $\Rightarrow$ (a). Let $\varepsilon > 0$. There is an $N \in \mathbb{N}$ such that $\text{diam } F_n \leq \varepsilon$ for $n \geq N$. We see that

$$\sum F_i \subseteq \{x \in E : \|x\| \leq \varepsilon\} + \sum_{i=0}^{n-1} F_i.$$

The set at the right hand side is closed so that

$$A \subseteq \{x \in E : \|x\| \leq \varepsilon\} + \sum_{i=0}^{n-1} F_i$$

and we have (a).
Two remarks.

(i) For spherically complete $K$ we have a more refined version of Theorem 3.1, namely we may replace (β) by:

(β)' There exist $e_1, e_2, \ldots$ in $E$ and absolutely convex sets $C_1, C_2, \ldots \subset K$ such that $C_n e_n$ is bounded for each $n$, 

$$\lim_{n \to \infty} (\text{diam } C_n) \| e_n \| = 0$$

and

$$\sum_n C_n e_n \subset A \subset \overline{\sum_n C_n e_n}.$$ 

The proof is obvious (Lemma 2.9).

(ii) It is not hard to generalize Theorem 3.1 to the case of a metrizable absolutely convex set $A$ in a locally convex space $E$. We leave the details to the reader.
§ 4. WEAK COMPACTOIDS OF FINITE TYPE

The main goal in this section is the following theorem, which is a generalization of Example 1.7. Recall that the valuation of $K$ is dense.

**THEOREM 4.1.** Let $E$ be a Banach space with an orthogonal base over a spherically complete $K$. If $A \subseteq E$ is a compactoid of finite type for the weak topology then $A$ is a compactoid (of finite type) for the norm topology.

This Theorem is in contrast to

**PROPOSITION 4.2.** Let $E$ be a Banach space over a spherically complete $K$. An absolutely convex $A \subseteq E$ is a compactoid for the weak topology if and only if $A$ is bounded for the norm topology.

**Proof.** A weak compactoid is weakly bounded, hence bounded by [6], Théorème 4.21. Conversely, if $A$ is norm bounded, let $U$ be a weak neighbourhood of $0$ in $E$. There is a closed $K$-subspace $D$ of $E$ of finite codimension with $D \subseteq U$. Let $\pi : E \to E/D$ be the quotient map. $\pi(A)$ is bounded in the finite dimensional space $E/D$, so a compactoid. As $\pi(u)$ is open in $E/D$ there exist $x_1, \ldots, x_n \in E/D$ such that $\pi(A) \subseteq \pi(U) + \text{co}\{x_1, \ldots, x_n\}$. Let $y_1, \ldots, y_n \in E$ be such that $\pi(y_i) = x_i$ for each $i \in \{1, \ldots, n\}$. It is easy to see that $A \subseteq U + \text{co}\{y_1, \ldots, y_n\}$.

Theorem 4.1 and Proposition 4.2 indicate that, for a compactoid, to require it to be of finite type may be a severe restriction!

**PROBLEM.** Does the conclusion of Theorem 4.1 hold for arbitrary Banach spaces over a spherically complete $K$ or for decent Banach spaces
(e.g. $c_0$) over a nonspherically complete $K$?

The proof of Theorem 4.1 requires several preparations. From now on in § 4 we assume that $K$ is spherically complete.

For an element $f$ of the dual $E'$ of a Banach space $E$ over $K$, let $\|f\|$ denote the operator norm of $f$ defined by the usual formula

$$\|f\| = \inf \{M : |f(x)| \leq M\|x\| \text{ for all } x \in E\}.$$

**Lemma 4.3.** Let $E$ be a Banach space over $K$. Then

$$\|f\| = \sup \{|f(x)| : \|x\| \leq 1\} \quad (f \in E').$$

**Proof.** We may assume dim $E \geq 1$. Set $\|f\|_0 := \sup \{|f(x)| : \|x\| \leq 1\}$.

If $x \in E$, $0 < \|x\| \leq 1$ then $|f(x)| \leq \frac{|f(x)|}{\|x\|} \leq \|f\|$ so that $\|f\|_0 \leq \|f\|$.

Conversely, let $x \in E$, $x \neq 0$. As the valuation of $K$ is dense we can find $\lambda_1, \lambda_2, \ldots \in K$ such that $|\lambda_1| < |\lambda_2| < \ldots$, $\lim_{n \to \infty} |\lambda_n| = \|x\|^{-1}$.

Then $\|\lambda_n x\| \leq 1$ whence $|f(\lambda_n x)| \leq \|f\|_0$ for each $n$. We have

$$\frac{|f(x)|}{\|x\|} = \frac{|f(\lambda_n x)|}{\|\lambda_n x\|} \leq \frac{\|f\|_0}{\|\lambda_n x\|}$$

which, after taking limits becomes $\frac{|f(x)|}{\|x\|} \leq \|f\|_0$ and we obtain $\|f\| \leq \|f\|_0$.

**Lemma 4.4.** Let $\{D_i : i \in I\}$ be an orthogonal system of closed subspaces ([1], p. 166) in a Banach space $E$ over $K$. For each $i \in I$, let $S_i \subseteq D_i$ be closed, absolutely convex, $[S_i] = D_i$, $S_i^e = S_i$ (see the remark following Proposition 2.5). Then $(ES_i)^e = ES_i$.

**Proof.** Let $x \in (ES_i)^e$. Then $x \in \overline{ES_i}$ so $x$ has a unique decomposition
x = \Sigma d_i where d_i \in D_i = [S_i] for each i \in I. For any \lambda \in K,
0 < |\lambda| < 1, \lambda x \in \overline{LS_i} so \lambda x = Es_i, where s_i \in S_i for each i. But also
\lambda x = \Sigma \lambda d_i. By the uniqueness of the decomposition of \lambda x we have
\lambda d_i = s_i i.e. d_i = \lambda^{-1} s_i for each i. This goes for any \lambda \in K,
0 < |\lambda| < 1 and we find d_i \in S_i^e = S_i for each i. It follows that
x \in \overline{LS_i}.

**LEMMA 4.5.** Let S be a closed absolutely convex subset of a Banach space
E over K for which S = S^e. If x \in E \setminus S there exists an f \in E' with
|\lambda| \leq 1 on S, |f(x)| > 1.

**Proof.** [6], Théorème 4.7a.

**LEMMA 4.6.** Let E be a Banach space over K such that each maximal
orthogonal system of vectors in E is an orthogonal base. Then E is
finite dimensional.

**Proof.** [1], Theorem 5.16 (a) \iff (b). (Recall that the valuation of K is
dense.)

**LEMMA 4.7.** Let A be an absolutely convex subset of a K-vector space.
For the seminorm p_A associated to A defined on [A] by the formula
\[ p_A(x) = \inf \{ |\lambda| : \lambda \in K, x \in \lambda A \} \quad (x \in [A]) \]
we have
\[ \{ x \in [A] : p_A(x) < 1 \} \subset A \subset \{ x \in [A] : p_A(x) \leq 1 \}. \]

Further, A = A^e if and only if
\[ A = \{ x \in [A] : p_A(x) \leq 1 \}. \]

**Proof.** Left to the reader.
PROPOSITION 4.8. (Compare Example 1.7.) Let $E$ be a Banach space over $K$ and suppose there exists an open absolutely convex set $A \subseteq E$ that is a compactoid of finite type for the weak topology. Then $E$ is finite dimensional.

Proof. By Proposition 2.7 we may assume that $A = A^e$. $A$ is bounded by Proposition 4.2 and therefore the seminorm associated to $A$ is norm $\| \|$ inducing the topology of $E$. Lemma 4.7 yields

$$A = \{ x \in E : \| x \| \leq 1 \}.$$ 

Now let $\{ e_i : i \in I \}$ be a maximal orthogonal system in $E$, it suffices to prove that it is an orthogonal base for $E$ (Lemma 4.6). For each $i \in I$ set $C_i = \{ \lambda \in K : \lambda e_i \in A \}$. Then $C_i = \{ \lambda \in K : |\lambda| \leq \| e_i \|^{-1} \}$ so that $(C_i e_i)^e = C_i e_i$. Set

$$B := \bigoplus_{i \in I} C_i e_i.$$ 

Obviously, $B \subseteq A$. It remains to prove that $B = A$. Suppose $B \neq A$. Lemma 4.4 tells us that $B = B^e$ so, by Lemma 4.5, there exists an $f \in E'$ such that $|f| \leq 1$ on $B$ and $\sup_{A} |f| > 1$. According to Corollary 2.10 and Lemma 4.3 there exists an $x \in E$ such that

$$\| f \| = \sup_{A} |f| = \sup_{A \cap B} |f| > 1.$$ 

Without harm (suitable scalar multiplication) we may assume $x \in A$ and $|f(x)| > 1$. Now $x \neq e_i$ for each $i \in I$ and, by maximality, the system $\{ x \} \cup \{ e_i : i \in I \}$ is not orthogonal. So there exists a (finite) $K$-linear combination $z = \sum \lambda_i e_i$ for which $\| x - z \| < \| x \|$. Then

$$\max_i \| \lambda_i e_i \| = \| z \| = \| x \| \leq 1,$$

so that $\lambda_i \in C_i$ for each $i$ whence $z \in B$ and $|f(z)| < 1$. We get

$$|f(x)| = |f(x-z)| \leq \| f \| \| x-z \| < \| f \| \| x \|.$$
Now let $C := \{ \lambda \in K : \lambda x \in A \}$. We have $\lambda \in C \Rightarrow \|\lambda x\| \leq 1$. Hence,

$$(\text{diam } C) \|x\| \leq 1.$$ 

On the other hand we obtain, using (*),

$$\|f\| = \sup_{\lambda \in C} |f(\lambda x)| = (\text{diam } C) |f(x)| < (\text{diam } C) \|f\| \|x\|$$

yielding

$$(\text{diam } C) \|x\| > 1,$$

a contradiction.

**Proof of Theorem 4.1.** Suppose $A$ is not a compactoid for the norm topology. Then, by [1], Theorem 4.38, (η) ⇒ (a) there exists an orthogonal sequence $e_1, e_2, \ldots$ in $A$ such that $\inf_n \|e_n\| > 0$. Set $D := \{e_1, e_2, \ldots\}$. By [1], Corollary 3.18 there exists a continuous linear projection $P$ of $E$ onto $D$. Since $e_n \in PA$ for each $n$ and $\inf_n \|e_n\| > 0$ we have that $PA$ is open in $D$. But ($P$ is also weakly continuous) the weak closure of $PA$ is a compactoid of finite type in $D$ for the weak topology of $D$ (Propositions 2.2 and 2.1). This is impossible by Proposition 4.8. Thus, $A$ is a compactoid for the norm topology. To see that $A$ is also of finite type it suffices to observe that the weak and norm topology coincide on $A$ ([3], Theorem 5.12).
REFERENCES


