SOME PROPERTIES OF C-COMPACT SETS

IN p-ADIC SPACES

by

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In this note we shall prove some properties of c-compact sets that may or may not be part of the 'folklore'. The concept of c-compactness, introduced by Springer [7], takes over the role played by convex-compact sets in Functional Analysis over \( \mathbb{R} \) or \( \mathbb{C} \) (or, any locally compact valued field).

Throughout, let \( K \) be a nonarchimedean nontrivially valued field with valuation \( | \cdot | \). We assume \( K \) to be maximally (= spherically) complete. A subset \( A \) of a \( K \)-linear space \( E \) is absolutely convex if it is a submodule of \( E \), considered as a module over the valuation ring \( \{ \lambda \in K : |\lambda| \leq 1 \} \). A set \( C \subset E \) is convex if it is either empty or an additive coset of an absolutely convex set. For a set \( X \subset E \) we denote by \( \text{co} \ X \) its absolutely convex hull, by \( [X] \) its \( K \)-linear span.

From now on in this paper \( E \) is a locally convex space over \( K \) ([8]). We assume \( E \) to be Hausdorff.
\section{Definition and First Properties}

**Definition 1.1.** ([7]) Let \( C \subset \mathbb{R} \) be a nonempty convex set. A convex filter on \( C \) is a filter of subsets of \( C \) that has a basis consisting of convex sets. \( C \) is \textit{c-compact} if each convex filter on \( C \) has a cluster point in \( C \).

In other words, \( C \) is c-compact if and only if the following is true. Let \( C \) be a family of nonempty relatively closed convex subsets of \( C \) such that \( C_1, C_2 \in C \) implies \( C_1 \cap C_2 \in C \). Then \( \cap \in C \neq \emptyset \).

We quote the following properties, proved in [7].

**Proposition 1.2.**

(i) \( K \) is c-compact.

(ii) A c-compact set is complete.

(iii) A nonempty closed convex subset of a c-compact set is c-compact.

(iv) Let \( (E_i)_{i \in I} \) be a family of Hausdorff locally convex spaces over \( K \). Suppose, for each \( i \), \( C_i \) is c-compact in \( E_i \). Then \( \bigcap_{i \in I} C_i \) is c-compact in \( \bigcap_{i \in I} E_i \).

(v) The image of a c-compact set under a continuous linear map is c-compact.

In [1] we find the following.

**Proposition 1.3.**

(i) If \( K \) is locally compact then a bounded nonempty convex set \( C \subset \mathbb{R} \) is c-compact if and only if it is convex and compact.

(ii) \( E \) is c-compact if and only if \( E \) is linearly homeomorphic to a
In § 3 (Theorem 3.3) we shall characterize arbitrary c-compact sets in the spirit of Proposition 1.3 (ii). But we conclude this first section with two statements that have nothing to do with the sequel. I just want to get rid of them.

**Proposition 1.4.**

A c-compact set is a Baire space.

**Proof.**

Let $U_1, U_2, \ldots$ be (relatively) open dense subsets of a c-compact set $C \subset E$. We prove that $\cap U_n \neq \emptyset$. There exists a nonempty open convex subset $B_1 \subset U_1$. As $U_2$ is dense we can find a nonempty open convex set $B_2 \subset B_1 \cap U_2$. Continuing this way we find nonempty open convex sets $B_1 \supset B_2 \supset \ldots$ with $B_n \subset \cap U_i$ for each $n$. The open sets $B_n$ are cosets of an additive group, hence closed. By c-compactness, $\cap B_n \neq \emptyset$. It follows that $\cap U_n \neq \emptyset$.

**Proposition 1.5.**

Let $X \subset E$ be closed, let $C \subset E$ be c-compact. Then $X+C$ is closed.

**Proof.**

Let $z \in \overline{X+C}$ (the closure of $X+C$), let $U$ be the collection of all absolutely convex neighbourhoods of 0. For each $U \in \mathcal{U}$ the set $z+U$ intersects $X+C$ so

$$C_U := \{ c \in C : z-c \in X+U \}$$

is not empty. $X+U$ is a union of cosets of $U$, so is its complement.
Therefore, $X+U$ is closed and $C_U$ is closed in $C$. Further we have

\[ C_U \cap C_V \supseteq C_{U \cap V} \quad (U, V \in U) \]

By $c$-compactness there exists a $c \in C$ such that

\[ z-c \in \bigcap_{U \in U} (X+U) = \overline{X} = X \]

i.e., $z \in X+c \subseteq X+C$.

**Remark.**

If the base field is not spherically complete there exist a complete absolutely convex compactoid $C \subseteq C_0$ and an element $a \in C_0$ such that $C+co\{a\}$ is not closed ([3], 6.25).
§ 2. LOCAL COMPACTOIDITY

DEFINITION 2.1. ([3], (6.7)) A subset \( X \) of \( E \) is a local compactoid if for each neighbourhood \( U \) of 0 in \( E \) there exists a finite dimensional K-linear subspace \( D \) of \( E \) such that \( X \subset U+D \).

PROPOSITION 2.2.

Let \( A \) be an absolutely convex subset of \( E \). \( A \) is c-compact if and only if \( A \) is a complete local compactoid.

Proof.

For \( E \) a Banach space this is proved in [3], 6.15. Now let \( E \) be a locally convex.

(i) Assume \( A \) is c-compact. By Proposition 1.2 (ii), \( A \) is complete. To prove local compactoidity let \( U \) be an absolutely convex neighbourhood of 0 in \( E \). There is a continuous seminorm \( p \) such that \( \{ x \in E : p(x) \leq 1 \} \subset U \). Let \( \pi_p : E \to E \) be the quotient map where \( E \) is the canonically normed space \( E/\text{Ker} p \). Now \( \pi_p(A) \) is c-compact (Proposition 1.2 (v)) so by the above it is a local compactoid in the completion \( E_p^\infty \) of \( E_p \). By Corollary 6.15 of [3] we have \( \pi_p(A) = R+T \) where \( R \) is a compactoid and \( T \) a finite dimensional subspace of \( E_p^\infty \). Then \( T \subset E_p^\infty \). Now \( \pi_p(U) \) is open in \( E_p \) and by Katsaras' Theorem ([5], Lemma 8.1) there exist \( x_1, \ldots, x_n \in [R] \) such that \( R \subset \pi_p(U) + \text{co}(x_1, \ldots, x_n) \). Combining our knowledge on \( R \) and \( T \) we find a finite dimensional space \( F \subset [\pi_p(A)] \) such that \( \pi_p(A) \subset \pi_p(U)+F \). Choose a finite dimensional space \( D \subset [A] \) such that \( \pi_p(D) = F \). Then

\[
A \subset U + D + \text{Ker} \pi_p \subset U + D.
\]

(ii) Let \( A \) be a complete local compactoid. Let \( \Gamma \) be the collection of all continuous seminorms on \( E \). For each \( p \in \Gamma \) we have that \( \pi_p(A) \), and also \( \pi_p(A) \), is a local compactoid in \( E_p^\infty \).
As $E^{-}$ is a Banach space we know that $\prod_{\mathcal{P}} \pi_\mathcal{P}(A)$ is $c$-compact. Then also $A_0 := \prod_{\mathcal{P}} \pi_\mathcal{P}(A)$ is a $c$-compact subset of $\prod_{\mathcal{P}} E^{-}$ (Proposition 1.1 (iv)). The canonical map $E \rightarrow \prod_{\mathcal{P}} E^{-}$ sends $A$ homeomorphically and linearly into $A_0$. Its image is closed in $A_0$ because $A$ is complete. Then $A$ is $c$-compact (Proposition 1.2 (iii)).

The following Proposition may look innocent.

PROPOSITION 2.3.

Let $A \subset E$ be absolutely convex and $c$-compact. For each neighbourhood $U$ of $0$ there exists a finite dimensional absolutely convex set $F \subset A$ such that $A = U + F$.

(The crucial part is the phrase 'F $\subset A$'.) For the proof we use a lemma.

LEMMA 2.4.

Let $A, U$ be absolutely convex subsets of $E$, where $U$ is closed, $A$ is $c$-compact. Let $x \in E$ be such that $A \subset U + Kx$. Then there exists an $y \in E$ and an absolutely convex $C \subset K$ such that $Cy \subset A$ and $A \subset U + Cy$.

Proof.

Let $C := \{c \in K : (U + cx) \cap A \neq \emptyset\}$. We have $A \subset U + Cx$, $C = \{c \in K : cx \in A + U\}$, so $C$ is absolutely convex. If $C = (0)$ then $A \subset U$ and we choose $y := 0$.

So assume $C \neq (0)$. For each $c \in C$, $c \neq 0$ define

$$H_c := c^{-1}(A \cap (cx + U)).$$

Each $H_c$ is a convex, closed, nonempty subset of $c^{-1}A$ hence $c$-compact.

Further, if $c, d \in C$, $0 < |c| \leq |d|$ then $H_d \subset H_c$. (Proof. Let $z \in H_d$. Then $dz \in A \cap (dx + U)$. By absolute convexity of $A$ and $U$,
\[ cz = \frac{c}{d} \cdot dz \in A \]

\[ cz \in \frac{c}{d} \cdot (dx + U) \subseteq cx + \frac{c}{d} U \subseteq cx + U. \]

It follows that \( cz \in A \cap (cx + U) \) i.e. \( z \in H_c \). By \( c \)-compactness there exists an \( y \in H_c \). Let \( c \in C, \ c \neq 0 \). Then

\[ cy \in cH_c \subseteq A \cap (cx + U) \subseteq A. \]

Also, \( cy \in cx + U \) so that \( cx - cy \in U \). Let \( a \in A \). Then \( a = u + cx \) for some \( u \in U, c \in C \). We see that \( a = u + cy + cx - cy \in cy + U \). It follows that \( A \subseteq U + cy \).

**Proof of Proposition 2.3.**

We may assume that \( U \) is absolutely convex. By Proposition 2.2 \( A \) is a local compactoid so there exist \( x_1, \ldots, x_n \in E \) such that

\[ A \subseteq U + Kx_1 + \ldots + Kx_n. \]

By the Lemma, applied to \( U + Kx_1 + \ldots + Kx_n \) in place of \( U \), there exist a \( y_1 \in E \) and an absolutely convex \( C_1 \subseteq K \) such that

\[ C_1 y_1 \subseteq A \quad \text{and} \quad A \subseteq U + C_1 y_1 + Kx_2 + \ldots + Kx_n \]

and we can continue. After \( n \) of these procedures we arrive at

\[ y_1, \ldots, y_n \in E, \text{ absolutely convex } C_1, \ldots, C_n \subseteq K \text{ such that } C_i y_i \subseteq A \]

for each \( i \) and \( A \subseteq U + C_1 y_1 + \ldots + C_n y_n \).

**Warning.**

The property of Proposition 2.3 is not shared by **all** absolutely convex local compactoids even when we require them to be closed! In fact we have:
EXAMPLE 2.5.

Let the valuation of $K$ be dense. Set

$$A = \{x \in c_0 : \|x\| \leq 1\}.$$  

([4], p.47).

(i) $A$ is a closed (local) compactoid for the weak topology of $c_0$.

(ii) There exists a weak neighbourhood $U$ of 0 such that for any finite dimensional set $F \subseteq A$

$$A \not\supseteq U + F.$$  

Proof.

(i) Let $U$ be a weak neighbourhood of 0. There exists a weakly continuous seminorm $p$ such that $\{x \in c_0 : p(x) \leq 1\} \subseteq U$. Then $\ker p$ has finite codimension. Choose a finite dimensional space $D \subseteq c_0$ with $\pi_p(D) = E_p$ (where as previously, $E_p := c_0/\ker p$ and $\pi_p : c_0 \to E_p$ is the quotient map). We have $A \subseteq \ker p + D \subseteq U + D$ (in fact, we have shown that each subset of $c_0$ is a local compactoid for the weak topology). To prove weak closedness of $A$, let $(x_i)_{i \in I}$ be a net in $A$ converging weakly to $x \in c_0$. By [4, Lemma 4.35 (i)] there exists an $f \in c_0'$, $f \neq 0$ for which $|f(x_i)| = \|f\| \|x_i\|$. We have

$$\|f\| \|x_i\| = |f(x_i)| \leq \limsup_{n \to \infty} \|f\| \|x_i\| \leq \|f\|$$

so that $\|x_i\| \leq 1$.

(ii) Choose $\tau_1, \tau_2, \ldots, \in \mathbb{R}$, $0 < |\tau_1| < |\tau_2| < \ldots$, $\lim_{n \to \infty} |\tau_n| = 1$. The formula

$$f(a_1, a_2, \ldots) = \sum_{i=1}^{\infty} a_i \tau_i$$

defines an element $f \in c_0'$. Observe that $\sup A = 1$ but $|f(x)| < 1$ for each $x \in A$. Set $U := \{x : |f(x)| \leq \frac{1}{2}\}$, let $F$ be any finite dimensional
set in A. We shall arrive at $A \not\subset U+F$ by showing that $\sup_{U+F} |f| < 1$. To this end it suffices to prove $\sup_{F} |f| < 1$. If $F$ is a finite dimensional subspace of $c_0$ and therefore ([4], Theorem 5.9) has an orthonormal base $x_1, \ldots, x_n$. It is easily seen that

$$F' := \text{co} \{x_1, \ldots, x_n\} \supset F$$

and $\sup_{F} |f| \leq \sup_{F'} |f| = \max(|f(x_1)|, \ldots, |f(x_n)|) < 1$.

Remark.

The above construction works also for the case where the base field is not spherically complete. Then $A$ is even weakly complete! ([5], Theorem 9.6 and [4], Theorem 4.17)
§ 3. A REPRESENTATION THEOREM FOR C-COMPACT SETS

LEMMA 3.1.
Let $\lambda \in K$, $|\lambda| > 1$. Let $G \subset E$ be closed, absolutely convex, and let $F \subset [G]$ be a finite dimensional set. If $(x_i)_{i \in I}$ is a net in $G + F$ converging to 0 then $x_i \in \lambda G$ for large $i$.

Proof.
[6], Lemma 1.3.

PROPOSITION 3.2.
(See also [2], Proposition 4, p. 93.) Let $A \subset E$ be absolutely convex, c-compact. Let $\tau'$ be a Hausdorff locally convex topology on $E$, weaker than the initial topology $\tau$. Then $\tau = \tau'$ on $A$.

Proof.
Let $(x_i)_{i \in I}$ be a net in $A$ converging to 0 for $\tau'$. Let $\lambda \in K$, $|\lambda| > 1$, let $U$ be an absolutely convex neighbourhood of 0 for $\tau$. Then $(\lambda^{-1}U) \cap A$ is c-compact in $(E, \tau)$ hence in $(E, \tau')$, so that $(\lambda^{-1}U) \cap A$ is $\tau'$-closed.
There is (Proposition 2.3) a finite dimensional $F \subset A$ with $A \subset \lambda^{-1}U + F$.
Then $A = (\lambda^{-1}U) \cap A + F$. Lemma 3.1 applies. It follows that $x_i \in \lambda(\lambda^{-1}U) \cap A \subset U$ for large $i$, so $\lim x_i = 0$ in the sense of $\tau$.

THEOREM 3.3.
Let $A \subset E$ be absolutely convex. The following are equivalent.
(a) $A$ is c-compact.
(b) $A$ is isomorphic (as a topological module over $\{\lambda \in K : |\lambda| \leq 1\}$) to a closed submodule of some power of $K$. 
Proof.

$(\beta) \Rightarrow (\alpha)$. This follows from Proposition 1.2, (i), (iv), (iii). Now suppose $(\alpha)$. The map

$$x \mapsto (f(x))_{f \in E'}$$

is a continuous linear injection $E \to K^{E'}$ (Hahn-Banach Theorem). According to Proposition 3.2 it is a homeomorphism, if restricted to $A$, and $(\beta)$ follows.
REFERENCES


