SOME PROPERTIES OF C-COMPACT SETS

IN p-ADIC SPACES

by

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In this note we shall prove some properties of c-compact sets that may or may not be part of the 'folklore'. The concept of c-compactness, introduced by Springer [7], takes over the role played by convex-compact sets in Functional Analysis over \( \mathbb{R} \) or \( \mathbb{C} \) (or, any locally compact valued field).

Throughout, let \( K \) be a nonarchimedean nontrivially valued field with valuation \( |\cdot| \). We assume \( K \) to be maximally (= spherically) complete. A subset \( A \) of a \( K \)-linear space \( E \) is absolutely convex if it is a submodule of \( E \), considered as a module over the valuation ring \( \{ \lambda \in K : |\lambda| \leq 1 \} \). A set \( C \subset E \) is convex if it is either empty or an additive coset of an absolutely convex set. For a set \( X \subset E \) we denote by \( \text{co} X \) its absolutely convex hull, by \([X]\) its \( K \)-linear span.

From now on in this paper \( E \) is a locally convex space over \( K \) ([8]). We assume \( E \) to be Hausdorff.
§ 1. DEFINITION AND FIRST PROPERTIES

DEFINITION 1.1. ([7]) Let \( C \subset E \) be a nonempty convex set. A convex filter on \( C \) is a filter of subsets of \( C \) that has a basis consisting of convex sets. \( C \) is c-compact if each convex filter on \( C \) has a cluster point in \( C \).

In other words, \( C \) is c-compact if and only if the following is true. Let \( C \) be a family of nonempty relatively closed convex subsets of \( C \) such that \( C_1, C_2 \in C \) implies \( C_1 \cap C_2 \in C \). Then \( \cap C \neq \emptyset \).

We quote the following properties, proved in [7].

PROPOSITION 1.2.

1. \( K \) is c-compact.
2. A c-compact set is complete.
3. A nonempty closed convex subset of a c-compact set is c-compact.
4. Let \( (E_i)_{i \in I} \) be a family of Hausdorff locally convex spaces over \( K \). Suppose, for each \( i, C_i \in E_i \) is c-compact in \( E_i \). Then \( \cap_{i \in I} C_i \) is c-compact in \( \cap_{i \in I} E_i \).
5. The image of a c-compact set under a continuous linear map is c-compact.

In [11] we find the following.

PROPOSITION 1.3.

1. If \( K \) is locally compact then a bounded nonempty convex set \( C \subset E \) is c-compact if and only if it is convex and compact.
2. \( E \) is c-compact if and only if \( E \) is linearly homeomorphic to a
In § 3 (Theorem 3.3) we shall characterize arbitrary c-compact sets in the spirit of Proposition 1.3 (ii). But we conclude this first section with two statements that have nothing to do with the sequel. I just want to get rid of them.

**PROPOSITION 1.4.**

A c-compact set is a Baire space.

**Proof.**

Let $U_1, U_2, \ldots$ be (relatively) open dense subsets of a c-compact set $C \subseteq E$. We prove that $\cap U_n \neq \emptyset$. There exists a nonempty open convex subset $B_1 \subseteq U_1$. As $U_2$ is dense we can find a nonempty open convex set $B_2 \subseteq B_1 \cap U_2$. Continuing this way we find nonempty open convex sets $B_1 \supseteq B_2 \supseteq \ldots$ with $B_n \subseteq \cap U_i$ for each $n$. The open sets $B_n$ are cosets of an additive group, hence closed. By c-compactness, $\cap B_n \neq \emptyset$. It follows that $\cap U_n \neq \emptyset$.

**PROPOSITION 1.5.**

Let $X \subseteq E$ be closed, let $C \subseteq E$ be c-compact. Then $X+C$ is closed.

**Proof.**

Let $z \in \overline{X+C}$ (the closure of $X+C$), let $U$ be the collection of all absolutely convex neighbourhoods of 0. For each $U \in \mathcal{U}$ the set $z+U$ intersects $X+C$ so

$$C_U := \{ c \in C : z-c \in X+U \}$$

is not empty. $X+U$ is a union of cosets of $U$, so is its complement.
Therefore, \( X+U \) is closed and \( C_U \) is closed in \( C \). Further we have

\[
C_U \cap C_V \supseteq C_{U \cap V} \quad (U, V \in U)
\]

By \( c \)-compactness there exists a \( c \in C \) such that

\[
z-c \in \bigcap_{U \in U} (X+U) = \overline{X} = X
\]

i.e., \( z \in X+c \subseteq X+C \).

Remark.

If the base field is not spherically complete there exist a complete absolutely convex compactoid \( C \subset C_0 \) and an element \( a \in C_0 \) such that \( C+co\{a\} \) is not closed ([3], 6.25).
§ 2. LOCAL COMPACTOIDITY

DEFINITION 2.1. ([3], (6.7)) A subset $X$ of $E$ is a local compactoid if for each neighbourhood $U$ of $0$ in $E$ there exists a finite dimensional $K$-linear subspace $D$ of $E$ such that $X \subset U + D$.

PROPOSITION 2.2.

Let $A$ be an absolutely convex subset of $E$. $A$ is $c$-compact if and only if $A$ is a complete local compactoid.

Proof.

For $E$ a Banach space this is proved in [3], 6.15. Now let $E$ be a locally convex.

(i) Assume $A$ is $c$-compact. By Proposition 1.2 (ii), $A$ is complete. To prove local compactoidity let $U$ be an absolutely convex neighbourhood of $0$ in $E$. There is a continuous seminorm $p$ such that $\{ x \in E : p(x) \leq 1 \} \subset U$.

Let $\pi_p : E \to E_p$ be the quotient map where $E_p$ is the canonically normed space $E/Ker\, p$. Now $\pi_p(A)$ is $c$-compact (Proposition 1.2 (v)) so by the above it is a local compactoid in the completion $E_p^\sim$ of $E_p$. By Corollary 6.15 of [3] we have $\pi_p(A) = R + T$ where $R$ is a compactoid and $T$ a finite dimensional subspace of $E_p^\sim$. Then $T \subset E_p$. Now $\pi_p(U)$ is open in $E_p$ and by Katsaras' Theorem ([5], Lemma 8.1) there exist $x_1, \ldots, x_n \in [R]$ such that $R \subset \pi_p(U) + \text{co}(x_1, \ldots, x_n)$. Combining our knowledge on $R$ and $T$ we find a finite dimensional space $F \subset \pi_p(A)$ such that $\pi_p(A) \subset \pi_p(U) + F$. Choose a finite dimensional space $D \subset [A]$ such that $\pi_p(D) = F$. Then

$$A \subset U + D + \text{Ker} \, \pi_p \subset U + D.$$ 

(ii) Let $A$ be a complete local compactoid. Let $\Gamma$ be the collection of all continuous seminorms on $E$. For each $p \in \Gamma$ we have that $\pi_p(A)$, and also $\pi_p(A)$, is a local compactoid in $E_p$. 


As $E^*$ is a Banach space we know that $\prod_{p \in \Gamma} \pi_p(A)$ is c-compact. Then also $A_0 := \prod_{p \in \Gamma} \pi_p(A)$ is a c-compact subset of $\prod_{p \in \Gamma} E^*$ (Proposition 1.1 (iv)). The canonical map $E \rightarrow \prod_{p \in \Gamma} E^*$ sends $A$ homeomorphically and linearly into $A_0$. Its image is closed in $A_0$ because $A$ is complete. Then $A$ is c-compact (Proposition 1.2 (iii)).

The following Proposition may look innocent.

**Proposition 2.3.**

Let $A \subset E$ be absolutely convex and c-compact. For each neighbourhood $U$ of 0 there exists a finite dimensional absolutely convex set $F \subset A$ such that $A = U + F$.

(The crucial part is the phrase 'F $\subset A$'.) For the proof we use a lemma.

**Lemma 2.4.**

Let $A, U$ be absolutely convex subsets of $E$, where $U$ is closed, $A$ is c-compact. Let $x \in E$ be such that $A \subset U + Kx$. Then there exists an $y \in E$ and an absolutely convex $C \subset K$ such that $Cy \subset A$ and $A \subset U + Cy$.

**Proof.**

Let $C := \{c \in K : (U + cx) \cap A \neq \emptyset\}$. We have $A \subset U + Cx$, $C = \{c \in K : cx \in A + U\}$, so $C$ is absolutely convex. If $C = (0)$ then $A \subset U$ and we choose $y := 0$.

So assume $C \neq (0)$. For each $c \in C$, $c \neq 0$ define

$$H_c := c^{-1}(A \cap (cx + U)).$$

Each $H_c$ is a convex, closed, nonempty subset of $c^{-1}A$ hence c-compact.

Further, if $c, d \in C$, $0 < |c| \leq |d|$ then $H_d \subset H_c$. (Proof. Let $z \in H_d$. Then $dz \in A \cap (dx + U)$. By absolute convexity of $A$ and $U$,
cz = \frac{c}{d} \cdot dz \in A

cz \in \frac{c}{d} (dx+U) \subseteq cx+\frac{c}{d} U \subseteq cx+U.

It follows that cz \in A \cap (cx+U) \text{ i.e. } z \in H_c. \text{ By } c\text{-compactness there exists an } y \in H_c. \text{ Let } c \in C, c \neq 0. \text{ Then } c \in C, c \neq 0

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\text{Also, } cy \in cx+U \text{ so that } cx-cy \in U. \text{ Let } a \in A. \text{ Then } a = u+cx \text{ for some } u \in U, c \in C. \text{ We see that } a = u+cy+cx-cy \in cy+U. \text{ It follows that } A \subseteq U+Cy.

Proof of Proposition 2.3.

We may assume that U is absolutely convex. By Proposition 2.2 A is a local compactoid so there exist x_1,\ldots,x_n \in E \text{ such that } A \subseteq U+Kx_1+\ldots+Kx_n. \text{ By the Lemma, applied to } U+Kx_2+\ldots+Kx_n \text{ in place of } U, \text{ there exist a } y_1 \in E \text{ and an absolutely convex } C_1 \subseteq K \text{ such that } C_1y_1 \subseteq A \text{ and }

A \subseteq U + C_1y_1 + Kx_2 + \ldots + Kx_n

= (U + C_1y_1 + Kx_2 + \ldots, Kx_n) + Kx_2

and we can continue. After n of these procedures we arrive at

y_1,\ldots,y_n \in E, \text{ absolutely convex } C_1,\ldots,C_n \subseteq K \text{ such that } C_iy_i \subseteq A \text{ for each } i \text{ and } A \subseteq U+C_1y_1+\ldots+C_ny_n.

Warning.

The property of Proposition 2.3 is not shared by all absolutely convex local compactoids even when we require them to be closed! In fact we have:
EXAMPLE 2.5.

Let the valuation of $K$ be dense. Set

$$A = \{x \in c_0 : \|x\| \leq 1\}.$$  

([4], p.47).

(i) $A$ is a closed (local) compactoid for the weak topology of $c_0$.

(ii) There exists a weak neighbourhood $U$ of 0 such that for any finite dimensional set $F \subset A$

$$A \cap U + F.$$  

Proof.

(i) Let $U$ be a weak neighbourhood of 0. There exists a weakly continuous seminorm $p$ such that $\{x \in c_0 : p(x) \leq 1\} \subset U$. Then $\text{Kerp}$ has finite codimension. Choose a finite dimensional space $D \subset c_0$ with $\pi_p(D) = E_p$ (where as previously, $E_p := c_0/\text{Kerp}$ and $\pi_p : c_0 \to E_p$ is the quotient map). We have $A \subset \text{Kerp} + D \subset U + D$ (in fact, we have shown that each subset of $c_0$ is a local compactoid for the weak topology). To prove weak closedness of $A$, let $(x_i)_{i \in I}$ be a net in $A$ converging weakly to $x \in c_0$. By [4], Lemma 4.35 (i) there exists an $f \in c_0'$ $f \neq 0$ for which $|f(x)| = \|f\| \|x\|$. We have

$$\|f\| \|x\| = |f(x)| = \lim |f(x_i)| \leq \lim \sup \|f\| \|x_i\| \leq \|f\|$$  

so that $\|x\| \leq 1$.

(ii) Choose $r_1, r_2, \ldots, \in K$, $0 < |r_1| < |r_2| < \ldots$, $\lim |r_n| = 1$. The formula

$$f(a_1, a_2, \ldots) = \sum_{i=1}^{\infty} a_i r_i$$  

defines an element $f \in c_0'$. Observe that $\sup_{A} |f| = 1$ but $|f(x)| < 1$ for each $x \in A$. Set $U := \{x : |f(x)| \leq \frac{1}{2}\}$, let $F$ be any finite dimensional
set in \( A \). We shall arrive at \( A \subseteq U + F \) by showing that \( \sup_{U + F} |f| < 1 \). To this end it suffices to prove \( \sup_F |f| < 1 \). \( F \) is a finite dimensional subspace of \( c_0 \) and therefore ([4], Theorem 5.9) has an orthonormal base \( x_1, \ldots, x_n \).

It is easily seen that

\[
F' := \text{co} \{x_1, \ldots, x_n\} \supseteq F
\]

and \( \sup_F |f| \leq \sup_{F'} |f| = \max(|f(x_1)|, \ldots, |f(x_n)|) < 1 \).

**Remark.**

The above construction works also for the case where the base field is not spherically complete. Then \( A \) is even weakly complete! ([5], Theorem 9.6 and [4], Theorem 4.17).
§ 3. A REPRESENTATION THEOREM FOR C-COMPACT SETS

LEMMA 3.1.

Let \( \lambda \in K, |\lambda| > 1 \). Let \( G \subseteq E \) be closed, absolutely convex, and let \( F \subseteq [G] \) be a finite dimensional set. If \( (x_i)_{i \in I} \) is a net in \( G + F \) converging to \( 0 \) then \( x_i \in \lambda G \) for large \( i \).

Proof.

[6], Lemma 1.3.

PROPOSITION 3.2.

(See also [2], Proposition 4, p. 93.) Let \( A \subseteq E \) be absolutely convex, c-compact. Let \( \tau' \) be a Hausdorff locally convex topology on \( E \), weaker than the initial topology \( \tau \). Then \( \tau = \tau' \) on \( A \).

Proof.

Let \( (x_i)_{i \in I} \) be a net in \( A \) converging to \( 0 \) for \( \tau' \). Let \( \lambda \in K, |\lambda| > 1 \), let \( U \) be an absolutely convex neighbourhood of \( 0 \) for \( \tau \). Then \( (\lambda^{-1}U) \cap A \) is c-compact in \( (E, \tau) \) hence in \( (E, \tau') \), so that \( (\lambda^{-1}U) \cap A \) is \( \tau' \)-closed. There is (Proposition 2.3) a finite dimensional \( F \subseteq A \) with \( A \subseteq \lambda^{-1}U + F \). Then \( A = (\lambda^{-1}U) \cap A + F \). Lemma 3.1 applies. It follows that \( x_i \in \lambda(\lambda^{-1}U) \cap A + F \) for large \( i \), so \( \lim x_i = 0 \) in the sense of \( \tau \).

THEOREM 3.3.

Let \( A \subseteq E \) be absolutely convex. The following are equivalent.

(a) \( A \) is c-compact.

(b) \( A \) is isomorphic (as a topological module over \( \{\lambda \in K : |\lambda| \leq 1\} \)) to a closed submodule of some power of \( K \).
Proof.

$\beta \Rightarrow (a)$. This follows from Proposition 1.2, (i), (iv), (iii). Now suppose $(a)$. The map

$$x \mapsto (f(x))_f \in E'$$

is a continuous linear injection $E \to K^E$ (Hahn-Banach Theorem). According to Proposition 3.2 it is a homeomorphism, if restricted to $A$, and $(\beta)$ follows.
REFERENCES


