In this note we shall prove some properties of c-compact sets that may or may not be part of the 'folklore'. The concept of c-compactness, introduced by Springer [7], takes over the role played by convex-compact sets in Functional Analysis over \( \mathbb{R} \) or \( \mathbb{C} \) (or, any locally compact valued field).

Throughout, let \( K \) be a nonarchimedean nontrivially valued field with valuation \( | \cdot | \). We assume \( K \) to be maximally (= spherically) complete. A subset \( A \) of a \( K \)-linear space \( E \) is absolutely convex if it is a submodule of \( E \), considered as a module over the valuation ring \( \{ \lambda \in K : |\lambda| \leq 1 \} \). A set \( C \subset E \) is convex if it is either empty or an additive coset of an absolutely convex set. For a set \( X \subset E \) we denote by \( \operatorname{co} X \) its absolutely convex hull, by \( [X] \) its \( K \)-linear span.

From now on in this paper \( E \) is a locally convex space over \( K \) ([8]). We assume \( E \) to be Hausdorff.
§ 1. DEFINITION AND FIRST PROPERTIES

DEFINITION 1.1. ([7]) Let \( C \subseteq E \) be a nonempty convex set. A convex filter on \( C \) is a filter of subsets of \( C \) that has a basis consisting of convex sets. \( C \) is c-compact if each convex filter on \( C \) has a cluster point in \( C \).

In other words, \( C \) is c-compact if and only if the following is true. Let \( C \) be a family of nonempty relatively closed convex subsets of \( C \) such that \( C_1, C_2 \subseteq C \) implies \( C_1 \cap C_2 \subseteq C \). Then \( C \neq \emptyset \).

We quote the following properties, proved in [7].

PROPOSITION 1.2.

(i) \( K \) is c-compact.

(ii) A c-compact set is complete.

(iii) A nonempty closed convex subset of a c-compact set is c-compact.

(iv) Let \( (E_i)_{i \in I} \) be a family of Hausdorff locally convex spaces over \( K \). Suppose, for each \( i \), \( C_i \subseteq E_i \) is c-compact in \( E_i \). Then \( \bigcap_{i \in I} C_i \subseteq \bigcap_{i \in I} E_i \) is c-compact in \( \bigcap_{i \in I} E_i \).

(v) The image of a c-compact set under a continuous linear map is c-compact.

In [1] we find the following.

PROPOSITION 1.3.

(i) If \( K \) is locally compact then a bounded nonempty convex set \( C \subseteq E \) is c-compact if and only if it is convex and compact.

(ii) \( E \) is c-compact if and only if \( E \) is linearly homeomorphic to a
In § 3 (Theorem 3.3) we shall characterize arbitrary $c$-compact sets in the spirit of Proposition 1.3 (ii). But we conclude this first section with two statements that have nothing to do with the sequel. I just want to get rid of them.

**Proposition 1.4.**

A $c$-compact set is a Baire space.

**Proof.**

Let $U_1, U_2, \ldots$ be (relatively) open dense subsets of a $c$-compact set $C \subseteq E$. We prove that $\cap U_i \neq \emptyset$. There exists a nonempty open convex subset $B_1 \subseteq U_1$. As $U_2$ is dense we can find a nonempty open convex set $B_2 \subseteq B_1 \cap U_2$. Continuing this way we find nonempty open convex sets $B_1 \supseteq B_2 \supseteq \ldots$ with $B_n \subseteq \cap U_i$ for each $n$. The open sets $B_n$ are cosets of an additive group, hence closed. By $c$-compactness, $\cap B_n \neq \emptyset$. It follows that $\cap U_i \neq \emptyset$.

**Proposition 1.5.**

Let $X \subseteq E$ be closed, let $C \subseteq E$ be $c$-compact. Then $X+C$ is closed.

**Proof.**

Let $z \in \overline{X+C}$ (the closure of $X+C$), let $U$ be the collection of all absolutely convex neighbourhoods of 0. For each $U \in U$ the set $z+U$ intersects $X+C$ so

$$C_U := \{ c \in C : z-c \in X+U \}$$

is not empty. $X+U$ is a union of cosets of $U$, so is its complement.
Therefore, $X+U$ is closed and $C_U$ is closed in $C$. Further we have

$$C_U \cap C_V \supset C_U \cap V \quad (u, v \in U)$$

By $c$-compactness there exists a $c \in C$ such that

$$z-c \in \cap_{u \in U} (X+U) = \overline{X} = X$$

i.e., $z \in X+c \subseteq X+C$.

Remark.

If the base field is not spherically complete there exist a complete absolutely convex compactoid $C \subseteq C_0$ and an element $a \in C_0$ such that $C+co\{a\}$ is not closed ([3], 6.25).
§ 2. LOCAL COMPACTOIDITY

DEFINITION 2.1. ([3], (6.7)) A subset $X$ of $E$ is a local compactoid if for each neighbourhood $U$ of $0$ in $E$ there exists a finite dimensional $K$-linear subspace $D$ of $E$ such that $X \subset U + D$.

PROPOSITION 2.2.

Let $A$ be an absolutely convex subset of $E$. $A$ is $c$-compact if and only if $A$ is a complete local compactoid.

Proof.

For $E$ a Banach space this is proved in [3], 6.15. Now let $E$ be a locally convex.

(i) Assume $A$ is $c$-compact. By Proposition 1.2 (ii), $A$ is complete. To prove local compactoidity let $U$ be an absolutely convex neighbourhood of $0$ in $E$. There is a continuous seminorm $p$ such that $\{x \in E : p(x) \leq 1\} \subset U$.

Let $\pi_p : E \to E_p$ be the quotient map where $E_p$ is the canonically normed space $E/Ker p$. Now $\pi_p(A)$ is $c$-compact (Proposition 1.2 (v)) so by the above it is a local compactoid in the completion $E^\pi$ of $E_p$. By Corollary 6.15 of [3] we have $\pi_p(A) = R + T$ where $R$ is a compactoid and $T$ a finite dimensional subspace of $E^\pi_p$. Then $T \subset E_p$. Now $\pi_p(U)$ is open in $E_p$ and by Katsaras' Theorem ([5], Lemma 8.1) there exist $x_1, \ldots, x_n \in [R]$ such that $R \subset \pi_p(U) + co\{x_1, \ldots, x_n\}$. Combining our knowledge on $R$ and $T$ we find a finite dimensional space $F \subset [\pi_p(A)]$ such that $\pi_p(A) \subset \pi_p(U) + F$. Choose a finite dimensional space $D \subset [A]$ such that $\pi_p(D) = F$. Then

$$A \subset U + D + Ker \pi_p \subset U + D.$$  

(ii) Let $A$ be a complete local compactoid. Let $\Gamma$ be the collection of all continuous seminorms on $E$. For each $p \in \Gamma$ we have that $\pi_p(A)$, and also $\pi_p(A)$, is a local compactoid in $E^\pi_p$. 

As $E^\infty$ is a Banach space we know that $\prod_{p \in \Gamma} \pi_p(A)$ is $c$-compact. Then also

$A_0 := \prod_{p \in \Gamma} \pi_p(A)$ is a $c$-compact subset of $\prod_{p \in \Gamma} E^\infty$ (Proposition 1.1 (iv)).

The canonical map $E \to \prod_{p \in \Gamma} E^\infty$ sends $A$ homeomorphically and linearly into $A_0$. Its image is closed in $A_0$ because $A$ is complete. Then $A$ is $c$-compact (Proposition 1.2 (iii)).

The following Proposition may look innocent.

**Proposition 2.3.**

Let $A \subset E$ be absolutely convex and $c$-compact. For each neighbourhood $U$ of 0 there exists a finite dimensional absolutely convex set $F \subset A$ such that $A = U + F$.

(The crucial part is the phrase 'F \subset A'.) For the proof we use a lemma.

**Lemma 2.4.**

Let $A, U$ be absolutely convex subsets of $E$, where $U$ is closed, $A$ is $c$-compact. Let $x \in E$ be such that $A \subset U + Kx$. Then there exists an $y \in E$ and an absolutely convex $C \subset K$ such that $Cy \subset A$ and $A \subset U + Cy$.

**Proof.**

Let $C := \{ c \in K : (U+cx) \cap A \neq \emptyset \}$. We have $A \subset U+Cx$, $C = \{ c \in K : cx \in A+U \}$, so $C$ is absolutely convex. If $C = (0)$ then $A \subset U$ and we choose $y := 0$.

So assume $C \neq (0)$. For each $c \in C$, $c \neq 0$ define

$$H_c := c^{-1}(A \cap (cx+U)).$$

Each $H_c$ is a convex, closed, nonempty subset of $c^{-1}A$ hence $c$-compact.

Further, if $c,d \in C$, $0 < |c| \leq |d|$ then $H_d \subset H_c$. (Proof. Let $z \in H_d$. Then $dz \in A \cap (dx+U)$. By absolute convexity of $A$ and $U$,
It follows that \( cz \in A \cap (cx+U) \) i.e. \( z \in H_c \). By \( cc \)-compactness there exists an \( y \in H_c \). Let \( c \in C, c \neq 0 \). Then

\[
cy \in ch_c \subset A \cap (cx+U) \subset A.
\]

Also, \( cy \in cx+U \) so that \( cx-cy \in U \). Let \( a \in A \). Then \( a = u+cx \) for some \( u \in U, c \in C \). We see that \( a = u+cy+cx-cy \in cy+U \). It follows that \( A \subset U+Cy \).

**Proof of Proposition 2.3.**

We may assume that \( U \) is absolutely convex. By Proposition 2.2 \( A \) is a local compactoid so there exist \( x_1, \ldots, x_n \in E \) such that \( A \subset U+Kx_1+\ldots+Kx_n \). By the Lemma, applied to \( U+Kx_2+\ldots+Kx_n \) in place of \( U \), there exist \( a y_1 \in E \) and an absolutely convex \( C_1 \in K \) such that

\[
C_1 y_1 \subset A \quad \text{and}
\]

\[
A \subset U+C_1 y_1+Kx_2+\ldots+Kx_n = (U+C_1 y_1+Kx_3+\ldots+Kx_n)+Kx_2
\]

and we can continue. After \( n \) of these procedures we arrive at

\[
y_1, \ldots, y_n \in E, \text{ absolutely convex } C_1, \ldots, C_n \subset K \text{ such that } C_i y_i \subset A \quad \text{for each } i \quad \text{and } A \subset U+C_1 y_1+\ldots+C_n y_n.
\]

**Warning.**

The property of Proposition 2.3 is not shared by all absolutely convex local compactoids even when we require them to be closed! In fact we have:
EXAMPLE 2.5.

Let the valuation of \( K \) be dense. Set

\[ A = \{ x \in c_0 : \| x \| \leq 1 \}. \]

([4], p.47).

(i) \( A \) is a closed (local) compactoid for the weak topology of \( c_0 \).

(ii) There exists a weak neighbourhood \( U \) of 0 such that for any finite dimensional set \( F \subseteq A \)

\[ A \nsubseteq U + F. \]

Proof.

(i) Let \( U \) be a weak neighbourhood of 0. There exists a weakly continuous seminorm \( p \) such that \( \{ x \in c_0 : p(x) \leq 1 \} \subseteq U \). Then \( \text{Ker} p \) has finite codimension. Choose a finite dimensional space \( D \subseteq c_0 \) with \( \pi_p(D) = E_p \) (where as previously, \( E_p := c_0/\text{Ker} p \) and \( \pi_p : c_0 \rightarrow E_p \) is the quotient map). We have \( A \subseteq \text{Ker} p + D \subseteq U + D \) (in fact, we have shown that each subset of \( c_0 \) is a local compactoid for the weak topology). To prove weak closedness of \( A \), let \( (x_i) \subseteq I \) be a net in \( A \) converging weakly to \( x \in c_0 \). By [4], Lemma 4.35 (i) there exists an \( f \in c_0' \) \( f \neq 0 \) for which \( |f(x)| = \| f \| \| x \| \). We have

\[ \| f \| \| x \| = |f(x)| = \lim |f(x_i)| \leq \limsup \| f \| \| x_i \| \leq \| f \| \]

so that \( \| x \| \leq 1. \)

(ii) Choose \( \tau_1, \tau_2, \ldots, \in K, 0 < |\tau_1| < |\tau_2| < \ldots, \lim_{n \to \infty} |\tau_n| = 1 \). The formula

\[ f(a_1, a_2, \ldots) = \sum_{i=1}^{\infty} a_i \tau_i \]

defines an element \( f \in c_0' \). Observe that \( \sup A f \) = 1 but \( |f(x)| < 1 \) for each \( x \in A \). Set \( U := \{ x : |f(x)| < \frac{1}{2} \} \), let \( F \) be any finite dimensional
set in A. We shall arrive at $A \neq U+F$ by showing that $\sup_{U+F}|f| < 1$. To this end it suffices to prove $\sup_{F}|f| < 1$. [F] is a finite dimensional subspace of $c_0$ and therefore ([4], Theorem 5.9) has an orthonormal base $x_1, \ldots, x_n$. It is easily seen that

$$F' := \text{co}\{x_1, \ldots, x_n\} \supset F$$

and $\sup_{F}|f| \leq \sup_{F'}|f| = \max(|f(x_1)|, \ldots, |f(x_n)|) < 1$.

Remark.
The above construction works also for the case where the base field is not spherically complete. Then $A$ is even weakly complete! ([5], Theorem 9.6 and [4], Theorem 4.17)
§ 3. A REPRESENTATION THEOREM FOR C-COMPACT SETS

LEMMA 3.1.

Let \( \lambda \in K, \ |\lambda| > 1 \). Let \( G \subset E \) be closed, absolutely convex, and let \( F \subset [G] \) be a finite dimensional set. If \( \{x_i\} \subset I \) is a net in \( G+F \) converging to 0 then \( x_i \in \lambda \ G \) for large \( i \).

Proof.

[6], Lemma 1.3.

PROPOSITION 3.2.

(See also [2], Proposition 4, p. 93.) Let \( A \subset E \) be absolutely convex, c-compact. Let \( \tau' \) be a Hausdorff locally convex topology on \( E \), weaker than the initial topology \( \tau \). Then \( \tau = \tau' \) on \( A \).

Proof.

Let \( \{x_i\} \subset I \) be a net in \( A \) converging to 0 for \( \tau' \). Let \( \lambda \in K, \ |\lambda| > 1 \), let \( U \) be an absolutely convex neighbourhood of 0 for \( \tau \). Then \( (\lambda^{-1}U) \cap A \) is c-compact in \( (E,\tau) \) hence in \( (E,\tau') \), so that \( (\lambda^{-1}U) \cap A \) is \( \tau' \)-closed.

There is (Proposition 2.3) a finite dimensional \( F \subset A \) with \( A \subset \lambda^{-1}U+F \). Then \( A = (\lambda^{-1}U) \cap A + F \). Lemma 3.1 applies. It follows that \( x_i \in \lambda(\lambda^{-1}U) \cap A \subset U \) for large \( i \), so \( \lim x_i = 0 \) in the sense of \( \tau \).

THEOREM 3.3.

Let \( A \subset E \) be absolutely convex. The following are equivalent.

(a) \( A \) is c-compact.

(b) \( A \) is isomorphic (as a topological module over \( \{\lambda \in K : |\lambda| \leq 1\} \)) to a closed submodule of some power of \( K \).
Proof.

(β) ⇒ (α). This follows from Proposition 1.2, (i), (iv), (iii). Now suppose (α). The map

\[ x \mapsto (f(x))_f \in E' \]

is a continuous linear injection \( E \to K^{E'} \) (Hahn-Banach Theorem).

According to Proposition 3.2 it is a homeomorphism, if restricted to \( A \), and (β) follows.
REFERENCES


