In this note we shall prove some properties of c-compact sets that may or may not be part of the 'folklore'. The concept of c-compactness, introduced by Springer [7], takes over the role played by convex-compact sets in Functional Analysis over \( \mathbb{R} \) or \( \mathbb{C} \) (or, any locally compact valued field).

Throughout, let \( K \) be a nonarchimedean nontrivially valued field with valuation \( | | \). We assume \( K \) to be maximally (= spherically) complete. A subset \( A \) of a \( K \)-linear space \( E \) is absolutely convex if it is a submodule of \( E \), considered as a module over the valuation ring \( \{ \lambda \in K : |\lambda| \leq 1 \} \). A set \( C \subset E \) is convex if it is either empty or an additive coset of an absolutely convex set. For a set \( X \subset E \) we denote by \( \text{co} \ X \) its absolutely convex hull, by \( [X] \) its \( K \)-linear span.

From now on in this paper \( E \) is a locally convex space over \( K \) ([8]). We assume \( E \) to be Hausdorff.
§ 1. DEFINITION AND FIRST PROPERTIES

DEFINITION 1.1. ([7]) Let $C \subseteq \mathbb{E}$ be a nonempty convex set. A convex filter on $C$ is a filter of subsets of $C$ that has a basis consisting of convex sets. $C$ is $\sigma$-compact if each convex filter on $C$ has a cluster point in $C$.

In other words, $C$ is $\sigma$-compact if and only if the following is true. Let $C$ be a family of nonempty relatively closed convex subsets of $C$ such that $C_1, C_2 \in C$ implies $C_1 \cap C_2 \in C$. Then $\cap C \neq \emptyset$.

We quote the following properties, proved in [7].

PROPOSITION 1.2.
(i) $K$ is $\sigma$-compact.
(ii) A $\sigma$-compact set is complete.
(iii) A nonempty closed convex subset of a $\sigma$-compact set is $\sigma$-compact.
(iv) Let $(E_i)_{i \in I}$ be a family of Hausdorff locally convex spaces over $K$. Suppose, for each $i$, $C_i$ is $\sigma$-compact in $E_i$. Then $\bigcap_{i \in I} C_i$ is $\sigma$-compact in $\bigcap_{i \in I} E_i$.
(v) The image of a $\sigma$-compact set under a continuous linear map is $\sigma$-compact.

In [1] we find the following.

PROPOSITION 1.3.
(i) If $K$ is locally compact then a bounded nonempty convex set $C \subseteq \mathbb{E}$ is $\sigma$-compact if and only if it is convex and compact.
(ii) $K$ is $\sigma$-compact if and only if $\mathbb{E}$ is linearly homeomorphic to a
In § 3 (Theorem 3.3) we shall characterize arbitrary c-compact sets in the spirit of Proposition 1.3 (ii). But we conclude this first section with two statements that have nothing to do with the sequel. I just want to get rid of them.

**Proposition 1.4.**

A c-compact set is a Baire space.

**Proof.**

Let \( U_1, U_2, \ldots \) be (relatively) open dense subsets of a c-compact set \( C \subseteq E \). We prove that \( \cap U_n \neq \emptyset \). There exists a nonempty open convex subset \( B_1 \subseteq U_1 \). As \( U_2 \) is dense we can find a nonempty open convex set \( B_2 \subseteq B_1 \cap U_2 \). Continuing this way we find nonempty open convex sets \( B_n \subseteq \cap U_i \) for each \( n \). The open sets \( B_n \) are cosets of an additive group, hence closed. By c-compactness, \( \cap B_n \neq \emptyset \). It follows that \( \cap U_n \neq \emptyset \).

**Proposition 1.5.**

Let \( X \subseteq E \) be closed, let \( C \subseteq E \) be c-compact. Then \( X+C \) is closed.

**Proof.**

Let \( z \in \overline{X+C} \) (the closure of \( X+C \)), let \( \mathcal{U} \) be the collection of all absolutely convex neighbourhoods of 0. For each \( U \in \mathcal{U} \) the set \( z+U \) intersects \( X+C \) so

\[
C_U := \{ c \in C : z-c \in X+U \}
\]

is not empty. \( X+U \) is a union of cosets of \( U \), so is its complement.
Therefore, $X+U$ is closed and $C_U$ is closed in $C$. Further we have

$$C_U \cap C_V \supseteq C_U \cap V \quad (U, V \in U)$$

By $c$-compactness there exists a $c \in C$ such that

$$z-c \in \cap \ (X+U) = \overline{X} = X$$

$$U \in U$$

i.e., $z \in X+c \subseteq X+C$.

Remark.

If the base field is not spherically complete there exist a complete absolutely convex compactoid $C \subset c_0$ and an element $a \in c_0$ such that $C+co\{a\}$ is not closed ([3], 6.25).
§ 2. LOCAL COMPACTOIDITY

DEFINITION 2.1. ([3], (6.7)) A subset $X$ of $E$ is a local compactoid if for each neighbourhood $U$ of 0 in $E$ there exists a finite dimensional $K$-linear subspace $D$ of $E$ such that $X \subset U + D$.

PROPOSITION 2.2.

Let $A$ be an absolutely convex subset of $E$. $A$ is $c$-compact if and only if $A$ is a complete local compactoid.

Proof.

For $E$ a Banach space this is proved in [3], 6.15. Now let $E$ be a locally convex.

(i) Assume $A$ is $c$-compact. By Proposition 1.2 (ii), $A$ is complete. To prove local compactoidity let $U$ be an absolutely convex neighbourhood of 0 in $E$. There is a continuous seminorm $p$ such that $\{x \in E : p(x) \leq 1\} \subset U$.

Let $\pi_p : E \rightarrow E_p$ be the quotient map where $E_p$ is the canonically normed space $E/\text{Ker}p$. Now $\pi_p(A)$ is $c$-compact (Proposition 1.2 (v)) so by the above it is a local compactoid in the completion $E_p^\sim$ of $E_p$. By Corollary 6.15 of [3] we have $\pi_p(A) = R + T$ where $R$ is a compactoid and $T$ a finite dimensional subspace of $E_p^\sim$. Then $T \subset E_p$. Now $\pi_p(U)$ is open in $E_p$ and by Katsaras' Theorem ([5], Lemma 8.1) there exist $x_1, \ldots, x_n \in \mathbb{R}$ such that $R \subset \pi_p(U) + \text{co}(x_1, \ldots, x_n)$. Combining our knowledge on $R$ and $T$ we find a finite dimensional space $F \subset [\pi_p(A)]$ such that $\pi_p(A) \subset \pi_p(U) + F$. Choose a finite dimensional space $D \subset [A]$ such that $\pi_p(D) = F$. Then

$$A \subset U + D + \text{Ker} \pi_p \subset U + D.$$ 

(ii) Let $A$ be a complete local compactoid. Let $\Gamma$ be the collection of all continuous seminorms on $E$. For each $p \in \Gamma$ we have that $\pi_p(A)$, and also $\pi_p(A)$, is a local compactoid in $E_p^\sim$. 


As $E^\ast$ is a Banach space we know that $\pi_p(A)$ is $c$-compact. Then also

$A_0 := \prod_{p \in \Gamma} \pi_p(A)$ is a $c$-compact subset of $\prod_{p \in \Gamma} E^\ast$ (Proposition 1.1 (iv)).

The canonical map $E \rightarrow \prod_{p \in \Gamma} E^\ast$ sends $A$ homeomorphically and linearly into $A_0$. Its image is closed in $A_0$ because $A$ is complete. Then $A$ is $c$-compact (Proposition 1.2 (iii)).

The following Proposition may look innocent.

**PROPOSITION 2.3.**

Let $A \subset E$ be absolutely convex and $c$-compact. For each neighbourhood $U$ of 0 there exists a finite dimensional absolutely convex set $F \subset A$ such that $A \subset U+F$.

(The crucial part is the phrase 'F $\subset A$'.) For the proof we use a lemma.

**LEMMA 2.4.**

Let $A, U$ be absolutely convex subsets of $E$, where $U$ is closed, $A$ is $c$-compact. Let $x \in E$ be such that $A \subset U+Kx$. Then there exists an $y \in E$ and an absolutely convex $C \subset K$ such that $Cy \subset A$ and $A \subset U+Cy$.

**Proof.**

Let $C := \{ c \in K : (U+cx) \cap A \neq \emptyset \}$. We have $A \subset U+cx$, $C = \{ c \in K : cx \in A+U \}$, so $C$ is absolutely convex. If $C = \{0\}$ then $A \subset U$ and we choose $y := 0$.

So assume $C \neq \{0\}$. For each $c \in C$, $c \neq 0$ define

$H_c := c^{-1}(A \cap (cx+U))$.

Each $H_c$ is a convex, closed, nonempty subset of $c^{-1}A$ hence $c$-compact.

Further, if $c, d \in C$, $0 < |c| \leq |d|$ then $H_d \subset H_c$. (Proof. Let $z \in H_d$. Then $dz \in A \cap (dx+U)$. By absolute convexity of $A$ and $U$,
\[ \begin{align*}
  c z &= \frac{c}{d} \cdot dz \in A \\
  c z \in \frac{c}{d} (dx+U) \subset cx+\frac{c}{d} U \subset cx+U.
\end{align*} \]

It follows that \( c z \in A \cap (cx+U) \), i.e., \( z \in H_c \).
By \( c \)-compactness there exists an \( y \in H_c \). Let \( c \in C \), \( c \neq 0 \). Then \( c \neq 0 \)

\[ c y \in ch_c \subset A \cap (cx+U) \subset A. \]

Also, \( c y \in cx+U \) so that \( cx-cy \in U \). Let \( a \in A \). Then \( a = u+cx \) for some \( u \in U \), \( c \in C \). We see that \( a = u+cy+cx-cy \in cy+U \). It follows that \( A \subset U+Cy \).

**Proof of Proposition 2.3.**

We may assume that \( U \) is absolutely convex. By Proposition 2.2 \( A \) is a local compactoid so there exist \( x_1, \ldots, x_n \in E \) such that \( A \subset U+Kx_1+\ldots+Kx_n \). By the Lemma, applied to \( U+Kx_2+\ldots+Kx_n \) in place of \( U \), there exist a \( y_1 \in E \) and an absolutely convex \( C_1 \in K \) such that \( C_1 y_1 \subset A \) and

\[ A \subset U + C_1 y_1 + Kx_2 + \ldots + Kx_n = (U + C_1 y_1 + Kx_3 + \ldots, Kx_n) + Kx_2 \]

and we can continue. After \( n \) of these procedures we arrive at \( y_1, \ldots, y_n \in E \), absolutely convex \( C_1, \ldots, C_n \subset K \) such that \( C_i y_i \subset A \) for each \( i \) and \( A \subset U+C_1 y_1+\ldots+C_n y_n \).

**Warning.**

The property of Proposition 2.3 is not shared by all absolutely convex local compactoids even when we require them to be closed! In fact we have:
EXAMPLE 2.5.

Let the valuation of $K$ be dense. Set

$$A = \{x \in c_0 : \|x\| \leq 1\}.$$

([4], p.47).

(i) $A$ is a closed (local) compactoid for the weak topology of $c_0$.

(ii) There exists a weak neighbourhood $U$ of $0$ such that for any finite dimensional set $F \subseteq A$

$$A \not\subseteq U+F.$$

Proof.

(i) Let $U$ be a weak neighbourhood of $0$. There exists a weakly continuous seminorm $p$ such that $\{x \in c_0 : p(x) \leq 1\} \subseteq U$. Then $Ker p$ has finite codimension. Choose a finite dimensional space $D \subseteq c_0$ with $\pi_p(D) = E_p$ (where as previously, $E_p := c_0/Ker p$ and $\pi_p : c_0 \to E_p$ is the quotient map). We have $A \subseteq Ker p+D \subseteq U+D$ (in fact, we have shown that each subset of $c_0$ is a local compactoid for the weak topology). To prove weak closedness of $A$, let $(x_i)_{i \in I}$ be a net in $A$ converging weakly to $x \in c_0$. By [4], Lemma 4.35 (i) there exists an $f \in c_0'$ $f \neq 0$ for which $|f(x)| = \|f\| \|x\|$. We have

$$\|f\| \|x\| = |f(x)| = \lim_{i \to \infty} |f(x_i)| \leq \lim \sup \|f\| \|x_i\| \leq \|f\|$$

so that $\|x\| \leq 1$.

(ii) Choose $\tau_1, \tau_2, \ldots, \epsilon K$, $0 < |\tau_1| < |\tau_2| < \ldots$, $\lim_{n \to \infty} |\tau_n| = 1$. The formula

$$f(a_1, a_2, \ldots) = \sum_{i=1}^{\infty} a_i \tau_i$$

defines an element $f \in c_0'$. Observe that $\sup_{A} f = 1$ but $|f(x)| < 1$ for each $x \in A$. Set $U := \{x : \|f(x)\| < \frac{1}{2}\}$, let $F$ be any finite dimensional
set in $A$. We shall arrive at $A \neq U + F$ by showing that $\sup_{U + F} |f| < 1$. To this end it suffices to prove $\sup_{F} |f| < 1$. $[F]$ is a finite dimensional subspace of $c_0$ and therefore ([4], Theorem 5.9) has an orthonormal base $x_1, \ldots, x_n$. It is easily seen that

$$F' := \text{co} \{x_1, \ldots, x_n\} \supset F$$

and $\sup_{F'} |f| \leq \sup_{F} |f| = \max(|f(x_1)|, \ldots, |f(x_n)|) < 1$.

**Remark.**

The above construction works also for the case where the base field is not spherically complete. Then $A$ is even weakly complete! ([5], Theorem 9.6 and [4], Theorem 4.17)
§ 3. A REPRESENTATION THEOREM FOR C-COMPACT SETS

**LEMMA 3.1.**

Let $\lambda \in K$, $|\lambda| > 1$. Let $G \subseteq E$ be closed, absolutely convex, and let $F \subseteq [G]$ be a finite dimensional set. If $(x_i)_{i \in I}$ is a net in $G+F$ converging to 0 then $x_i \in \lambda G$ for large $i$.

**Proof.**

[6], Lemma 1.3.

**PROPOSITION 3.2.**

(See also [2], Proposition 4, p. 93.) Let $A \subseteq E$ be absolutely convex, c-compact. Let $\tau'$ be a Hausdorff locally convex topology on $E$, weaker than the initial topology $\tau$. Then $\tau = \tau'$ on $A$.

**Proof.**

Let $(x_i)_{i \in I}$ be a net in $A$ converging to 0 for $\tau'$. Let $\lambda \in K$, $|\lambda| > 1$, let $U$ be an absolutely convex neighbourhood of 0 for $\tau$. Then $(\lambda^{-1}U) \cap A$ is c-compact in $(E,\tau)$ hence in $(E,\tau')$, so that $(\lambda^{-1}U) \cap A$ is $\tau'$-closed. There is (Proposition 2.3) a finite dimensional $F \subseteq A$ with $A \subseteq \lambda^{-1}U+F$. Then $A = (\lambda^{-1}U) \cap A + F$. Lemma 3.1 applies. It follows that $x_i \in \lambda(\lambda^{-1}U) \cap A \subseteq U$ for large $i$, so $\lim x_i = 0$ in the sense of $\tau$.

**THEOREM 3.3.**

Let $A \subseteq E$ be absolutely convex. The following are equivalent.

(a) $A$ is c-compact.

(b) $A$ is isomorphic (as a topological module over $\{\lambda \in K : |\lambda| \leq 1\}$) to a closed submodule of some power of $K$. 
Proof.

$(\beta) \Rightarrow (\alpha)$. This follows from Proposition 1.2, (i), (iv), (iii). Now suppose $(\alpha)$. The map

$$x \mapsto (f(x))_f \in E'$$

is a continuous linear injection $E \to K^{E'}$ (Hahn-Banach Theorem).

According to Proposition 3.2 it is a homeomorphism, if restricted to $A$, and $(\beta)$ follows.
REFERENCES


