SOME PROPERTIES OF C-COMPACT SETS

IN p-ADIC SPACES

by

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In this note we shall prove some properties of c-compact sets that may or may not be part of the 'folklore'. The concept of c-compactness, introduced by Springer [7], takes over the role played by convex-compact sets in Functional Analysis over \( \mathbb{R} \) or \( \mathbb{C} \) (or, any locally compact valued field).

Throughout, let \( K \) be a nonarchimedean nontrivially valued field with valuation \(| |\). We assume \( K \) to be maximally (= spherically) complete. A subset \( A \) of a \( K \)-linear space \( E \) is absolutely convex if it is a submodule of \( E \), considered as a module over the valuation ring \( \{ \lambda \in K : |\lambda| \leq 1 \} \). A set \( C \subset E \) is convex if it is either empty or an additive coset of an absolutely convex set. For a set \( X \subset E \) we denote by \( \text{co} \ X \) its absolutely convex hull, by \( [X] \) its \( K \)-linear span.

From now on in this paper \( E \) is a locally convex space over \( K \) ([8]). We assume \( E \) to be Hausdorff.
§ 1. DEFINITION AND FIRST PROPERTIES

DEFINITION 1.1. ([7]) Let \( C \subseteq E \) be a nonempty convex set. A convex filter on \( C \) is a filter of subsets of \( C \) that has a basis consisting of convex sets. \( C \) is \( c \)-compact if each convex filter on \( C \) has a cluster point in \( C \).

In other words, \( C \) is \( c \)-compact if and only if the following is true. Let \( C \) be a family of nonempty relatively closed convex subsets of \( C \) such that \( C_1, C_2 \in C \) implies \( C_1 \cap C_2 \in C \). Then \( \cap C \neq \emptyset \).

We quote the following properties, proved in [7].

PROPOSITION 1.2.

(i) \( K \) is \( c \)-compact.

(ii) A \( c \)-compact set is complete.

(iii) A nonempty closed convex subset of a \( c \)-compact set is \( c \)-compact.

(iv) Let \( (E_i)_{i \in I} \) be a family of Hausdorff locally convex spaces over \( K \). Suppose, for each \( i, \) \( C_i \subseteq E_i \) is \( c \)-compact in \( E_i \). Then \( \bigcap_{i \in I} C_i \) is \( c \)-compact in \( \bigcap_{i \in I} E_i \).

(v) The image of a \( c \)-compact set under a continuous linear map is \( c \)-compact.

In [1] we find the following.

PROPOSITION 1.3.

(i) If \( K \) is locally compact then a bounded nonempty convex set \( C \subseteq E \) is \( c \)-compact if and only if it is convex and compact.

(ii) \( E \) is \( c \)-compact if and only if \( E \) is linearly homeomorphic to
In § 3 (Theorem 3.3) we shall characterize arbitrary c-compact sets in the spirit of Proposition 1.3 (ii). But we conclude this first section with two statements that have nothing to do with the sequel. I just want to get rid of them.

PROPOSITION 1.4.
A c-compact set is a Baire space.

Proof.
Let $U_1, U_2, \ldots$ be (relatively) open dense subsets of a c-compact set $C \subseteq E$. We prove that $\bigcap_n U_n \neq \emptyset$. There exists a nonempty open convex subset $B_1 \subseteq U_1$. As $U_2$ is dense we can find a nonempty open convex set $B_2 \subseteq B_1 \cap U_2$. Continuing this way we find nonempty open convex sets $B_n$ with $B_1 \supseteq B_2 \supseteq \ldots$ and $B_n \subseteq \bigcap_{i=1}^n U_i$ for each $n$. The open sets $B_n$ are cosets of an additive group, hence closed. By c-compactness, $\bigcap_n B_n \neq \emptyset$. It follows that $\bigcap_n U_n \neq \emptyset$.

PROPOSITION 1.5.
Let $X \subseteq E$ be closed, let $C \subseteq E$ be c-compact. Then $X + C$ is closed.

Proof.
Let $z \in \overline{X+C}$ (the closure of $X+C$), let $U$ be the collection of all absolutely convex neighbourhoods of 0. For each $U \in \mathcal{U}$ the set $z+U$ intersects $X+C$ so

$$C_U := \{ c \in C : z-c \in X+U \}$$

is not empty. $X+U$ is a union of cosets of $U$, so is its complement.
Therefore, $X+U$ is closed and $C_U$ is closed in $C$. Further we have

$$C_U \cap C_V \supseteq C_{U \cap V} \quad (U, V \in U)$$

By $c$-compactness there exists a $c \in C$ such that

$$z-c \in \bigcap_{U \in U} (X+U) = X = X$$

i.e., $z \in X+c \subseteq X+C$.

**Remark.**

If the base field is not spherically complete there exist a complete absolutely convex compactoid $C \subseteq C_0$ and an element $a \in C_0$ such that $C+a$ is not closed ([3], 6.25).
§ 2. LOCAL COMPACTOIDITY

DEFINITION 2.1. ([3], (6.7)) A subset $X$ of $E$ is a local compactoid if for each neighbourhood $U$ of 0 in $E$ there exists a finite dimensional $K$-linear subspace $D$ of $E$ such that $X \subseteq U + D$.

PROPOSITION 2.2.

Let $A$ be an absolutely convex subset of $E$. $A$ is $c$-compact if and only if $A$ is a complete local compactoid.

Proof.

For $E$ a Banach space this is proved in [3], 6.15. Now let $E$ be a locally convex.

(i) Assume $A$ is $c$-compact. By Proposition 1.2 (ii), $A$ is complete. To prove local compactoidity let $U$ be an absolutely convex neighbourhood of 0 in $E$. There is a continuous seminorm $p$ such that $\{x \in E : p(x) \leq 1\} \subseteq U$.

Let $\pi_p : E \rightarrow E_p$ be the quotient map where $E_p$ is the canonically normed space $E/\ker p$. Now $\pi_p(A)$ is $c$-compact (Proposition 1.2 (v)) so by the above it is a local compactoid in the completion $E_p^\sim$ of $E_p$. By Corollary 6.15 of [3] we have $\pi_p(A) = R + T$ where $R$ is a compactoid and $T$ a finite dimensional subspace of $E_p^\sim$. Then $T \subseteq E_p$. Now $\pi_p(U)$ is open in $E_p$ and by Katsaras' Theorem ([5], Lemma 8.1) there exist $x_1, \ldots, x_n \in [R]$ such that $R \subseteq \pi_p(U) + \text{co}(x_1, \ldots, x_n)$. Combining our knowledge on $R$ and $T$ we find a finite dimensional space $F \subseteq [\pi_p(A)]$ such that $\pi_p(A) \subseteq \pi_p(U) + F$. Choose a finite dimensional space $D \subseteq [A]$ such that $\pi_p(D) = F$. Then

$$A \subseteq U + D + \ker \pi_p \subseteq U + D.$$ 

(ii) Let $A$ be a complete local compactoid. Let $\Gamma$ be the collection of all continuous seminorms on $E$. For each $p \in \Gamma$ we have that $\pi_p(A)$, and also

$\pi_p(A)$, is a local compactoid in $E_p^\sim$. 

As $E^\sim$ is a Banach space we know that $\prod_{\mathcal{P}} \overset{\sim}{\pi}(A)$ is $c$-compact. Then also

$$A_0 := \prod_{\mathcal{P}} \pi(\overset{\sim}{\pi}(A))$$

is $c$-compact subset of $\prod_{\mathcal{P}} E^\sim$ (Proposition 1.1 (iv)).

The canonical map $E \to \prod_{\mathcal{P}} E^\sim$ sends $A$ homeomorphically and linearly into $A_0$. Its image is closed in $A_0$ because $A$ is complete. Then $A$ is $c$-compact (Proposition 1.2 (iii)).

The following Proposition may look innocent.

**PROPOSITION 2.3.**

Let $A \subseteq E$ be absolutely convex and $c$-compact. For each neighbourhood $U$ of 0 there exists a finite dimensional absolutely convex set $F \subseteq A$ such that $A = U + F$.

(The crucial part is the phrase 'F $\subseteq A$'.) For the proof we use a lemma.

**LEMMA 2.4.**

Let $A, U$ be absolutely convex subsets of $E$, where $U$ is closed, $A$ is $c$-compact. Let $x \in E$ be such that $A \subseteq U + Kx$. Then there exists an $y \in E$ and an absolutely convex $C \subseteq K$ such that $Cy \subseteq A$ and $A \subseteq U + Cy$.

**Proof.**

Let $C := \{ c \in K : (U + cx) \cap A \neq \emptyset \}$. We have $A \subseteq U + cx$, $C = \{ c \in K : cx \in A + U \}$, so $C$ is absolutely convex. If $C = (0)$ then $A \subseteq U$ and we choose $y := 0$.

So assume $C \neq (0)$. For each $c \in C$, $c \neq 0$ define

$$H_c := c^{-1}(A \cap (cx + U))$$

Each $H_c$ is a convex, closed, nonempty subset of $c^{-1}A$ hence $c$-compact.

Further, if $c, d \in C$, $0 < |c| \leq |d|$ then $H_d \subseteq H_c$. (Proof. Let $z \in H_d$. Then $dz \in A \cap (dx + U)$. By absolute convexity of $A$ and $U$,
It follows that \( cz \in A \cap (cx+U) \) i.e. \( z \in H_c \). By \( c \)-compactness there exists an \( y \in H_c \). Let \( c \in C \), \( c \neq 0 \). Then

\[
\begin{align*}
\forall y \in H_c, \quad cy \in ch_c \subseteq A \cap (cx+U) \subseteq A.
\end{align*}
\]

Also, \( cy \in cx+U \) so that \( cx-cy \in U \). Let \( a \in A \). Then \( a = u+cx \) for some \( u \in U \), \( c \in C \). We see that \( a = u+cy+cx-cy \in cy+U \). It follows that \( A \subseteq U+Cy \).

**Proof of Proposition 2.3.**

We may assume that \( U \) is absolutely convex. By Proposition 2.2 \( A \) is a local compactoid so there exist \( x_1, \ldots, x_n \in E \) such that

\[
A \subseteq U + Kx_1 + \ldots + Kx_n.
\]

By the Lemma, applied to \( U + Kx_1 + \ldots + Kx_n \) in place of \( U \), there exist \( a y_1 \in E \) and an absolutely convex \( C_1 \in K \) such that

\[
C_1 y_1 \subseteq A
\]

and

\[
A \subseteq U + C_1 y_1 + Kx_2 + \ldots + Kx_n
\]

and we can continue. After \( n \) of these procedures we arrive at

\[
y_1, \ldots, y_n \in E, \text{ absolutely convex } C_1, \ldots, C_n \subseteq K \text{ such that } C_i y_i \subseteq A
\]

for each \( i \) and \( A \subseteq U + C_1 y_1 + \ldots + C_n y_n \).

**Warning.**

The property of Proposition 2.3 is not shared by all absolutely convex local compactoids even when we require them to be closed. In fact we have:
EXAMPLE 2.5.

Let the valuation of $K$ be dense. Set

$$A = \{x \in c_0 : \|x\| \leq 1\}.$$ 

([4], p.47).

(i) $A$ is a closed (local) compactoid for the weak topology of $c_0$.

(ii) There exists a weak neighbourhood $U$ of 0 such that for any finite dimensional set $F \subset A$

$$A \not\subset U+F.$$ 

Proof.

(i) Let $U$ be a weak neighbourhood of 0. There exists a weakly continuous seminorm $p$ such that $\{x \in c_0 : p(x) \leq 1\} \subset U$. Then $\text{Ker} p$ has finite codimension. Choose a finite dimensional space $D \subset c_0$ with $\pi_p(D) = E_p$ (where as previously, $E_p := c_0/\text{Ker} p$ and $\pi_p : c_0 \to E_p$ is the quotient map). We have $A \subset \text{Ker} p + D \subset U + D$ (in fact, we have shown that each subset of $c_0$ is a local compactoid for the weak topology). To prove weak closedness of $A$, let $(x_i)_i \in I$ be a net in $A$ converging weakly to $x \in c_0$. By [4], Lemma 4.35 (i) there exists an $f \in c_0^*$, $f \not= 0$ for which $|f(x)| = \|f\| \|x\|$. We have

$$\|f\| \|x\| = |f(x)| = \lim |f(x_i)| \leq \lim \sup \|f\| \|x_i\| \leq \|f\|$$

so that $\|x\| \leq 1$.

(ii) Choose $\tau_1, \tau_2, \ldots, \in \mathbb{K}$, $0 < |\tau_1| < |\tau_2| < \ldots$, $\lim_{n \to \infty} |\tau_n| = 1$. The formula

$$f(a_1, a_2, \ldots) = \sum_{i=1}^{\infty} a_i \tau_i$$

defines an element $f \in c_0^*$. Observe that $\sup_A |f| = 1$ but $|f(x)| < 1$ for each $x \in A$. Set $U := \{x : |f(x)| \leq \frac{1}{2}\}$, let $F$ be any finite dimensional
set in $A$. We shall arrive at $A \not\in U + F$ by showing that $\sup_{U+F}|f| < 1$. To this end it suffices to prove $\sup_{F}|f| < 1$. $[F]$ is a finite dimensional subspace of $c_0$ and therefore ([4], Theorem 5.9) has an orthonormal base $x_1, \ldots, x_n$. It is easily seen that

$$F' := \text{co}\{x_1, \ldots, x_n\} \supset F$$

and $\sup_{F}|f| \leq \sup_{F'}|f| = \max(|f(x_1)|, \ldots, |f(x_n)|) < 1.$

Remark.

The above construction works also for the case where the base field is not spherically complete. Then $A$ is even weakly complete! ([5], Theorem 9.6 and [4], Theorem 4.17)
§ 3. A REPRESENTATION THEOREM FOR C-COMPACT SETS

LEMMA 3.1.

Let \( \lambda \in K, |\lambda| > 1 \). Let \( G \subseteq E \) be closed, absolutely convex, and let \( F \subseteq \{G\} \) be a finite dimensional set. If \((x_i)_{i \in I} \) is a net in \( G + F \) converging to \( 0 \) then \( x_i \in \lambda G \) for large \( i \).

Proof.

[6], Lemma 1.3.

PROPOSITION 3.2.

(See also [2], Proposition 4, p. 93.) Let \( A \subseteq E \) be absolutely convex, c-compact. Let \( \tau' \) be a Hausdorff locally convex topology on \( E \), weaker than the initial topology \( \tau \). Then \( \tau = \tau' \) on \( A \).

Proof.

Let \((x_i)_{i \in I} \) be a net in \( A \) converging to \( 0 \) for \( \tau' \). Let \( \lambda \in K, |\lambda| > 1 \), let \( U \) be an absolutely convex neighbourhood of \( 0 \) for \( \tau \). Then \( (\lambda^{-1}U) \cap A \) is c-compact in \((E,\tau)\) hence in \((E,\tau')\), so that \( (\lambda^{-1}U) \cap A \) is \( \tau' \)-closed.

There is (Proposition 2.3) a finite dimensional \( F \subseteq A \) with \( A \subseteq \lambda^{-1}U + F \).

Then \( A = (\lambda^{-1}U) \cap A + F \). Lemma 3.1 applies. It follows that \( x_i \in \lambda(\lambda^{-1}U) \cap A \subseteq U \) for large \( i \), so \( \lim x_i = 0 \) in the sense of \( \tau \).

THEOREM 3.3.

Let \( A \subseteq E \) be absolutely convex. The following are equivalent.

(a) \( A \) is c-compact.

(b) \( A \) is isomorphic (as a topological module over \( \{\lambda \in K : |\lambda| \leq 1\} \)) to a closed submodule of some power of \( K \).
Proof.

(β) ⇒ (α). This follows from Proposition 1.2, (i), (iv), (iii). Now suppose (α). The map

\[ x \mapsto (f(x))_{f \in E'} \]

is a continuous linear injection \( E \to K^E' \) (Hahn-Banach Theorem).
According to Proposition 3.2 it is a homeomorphism, if restricted
to \( A \), and (β) follows.
REFERENCES


