In this note we shall prove some properties of c-compact sets that may or may not be part of the 'folklore'. The concept of c-compactness, introduced by Springer [7], takes over the role played by convex-compact sets in Functional Analysis over IR or C (or, any locally compact valued field).

Throughout, let K be a nonarchimedean nontrivially valued field with valuation | |. We assume K to be maximally (= spherically) complete. A subset A of a K-linear space E is absolutely convex if it is a submodule of E, considered as a module over the valuation ring \( \{ \lambda \in K : |\lambda| \leq 1 \} \). A set \( C \subset E \) is convex if it is either empty or an additive coset of an absolutely convex set. For a set \( X \subset E \) we denote by \( \text{co} X \) its absolutely convex hull, by \([X]\) its K-linear span.

From now on in this paper E is a locally convex space over K ([8]). We assume E to be Hausdorff.
§ 1. DEFINITION AND FIRST PROPERTIES

DEFINITION 1.1. ([7]) Let \( C \subseteq \mathbb{E} \) be a nonempty convex set. A convex filter on \( C \) is a filter of subsets of \( C \) that has a basis consisting of convex sets. \( C \) is c-compact if each convex filter on \( C \) has a cluster point in \( C \).

In other words, \( C \) is c-compact if and only if the following is true. Let \( C \) be a family of nonempty relatively closed convex subsets of \( C \) such that \( C_1, C_2 \in C \) implies \( C_1 \cap C_2 \in C \). Then \( \bigcap C \neq \emptyset \).

We quote the following properties, proved in [7].

PROPOSITION 1.2.

(i) \( K \) is c-compact.

(ii) A c-compact set is complete.

(iii) A nonempty closed convex subset of a c-compact set is c-compact.

(iv) Let \( (E_i)_{i \in I} \) be a family of Hausdorff locally convex spaces over \( K \). Suppose, for each \( i \), \( C_i \subseteq E_i \) is c-compact in \( E_i \). Then \( \bigcap_{i \in I} C_i \) is c-compact in \( \bigcap_{i \in I} E_i \).

(v) The image of a c-compact set under a continuous linear map is c-compact.

In [1] we find the following.

PROPOSITION 1.3.

(i) If \( K \) is locally compact then a bounded nonempty convex set \( C \subseteq \mathbb{E} \) is c-compact if and only if it is convex and compact.

(ii) \( K \) is c-compact if and only if \( K \) is linearly homeomorphic to a
In § 3 (Theorem 3.3) we shall characterize arbitrary c-compact sets in the spirit of Proposition 1.3 (ii). But we conclude this first section with two statements that have nothing to do with the sequel. I just want to get rid of them.

PROPOSITION 1.4.

A c-compact set is a Baire space.

Proof.

Let \( U_1, U_2, \ldots \) be (relatively) open dense subsets of a c-compact set \( C \subseteq E \). We prove that \( \cap U_n \neq \emptyset \). There exists a nonempty open convex subset \( B_1 \subseteq U_1 \). As \( U_2 \) is dense we can find a nonempty open convex set \( B_2 \subseteq B_1 \cap U_2 \). Continuing this way we find nonempty open convex sets

\[
B_1 \supseteq B_2 \supseteq \ldots \quad \text{with} \quad B_n \subseteq \bigcap_{i=1}^{n} U_i \quad \text{for each} \quad n.
\]

The open sets \( B_n \) are cosets of an additive group, hence closed. By c-compactness, \( \cap B_n \neq \emptyset \). It follows that \( \cap U_n \neq \emptyset \).

PROPOSITION 1.5.

Let \( X \subseteq E \) be closed, let \( C \subseteq E \) be c-compact. Then \( X+C \) is closed.

Proof.

Let \( z \in X+C \) (the closure of \( X+C \)), let \( \mathcal{U} \) be the collection of all absolutely convex neighbourhoods of \( 0 \). For each \( U \in \mathcal{U} \) the set \( z+U \) intersects \( X+C \) so

\[
\mathcal{C}_U := \{ c \in C : z-c \in X+U \}
\]

is not empty. \( X+U \) is a union of cosets of \( U \), so is its complement.
Therefore, $X+U$ is closed and $C_u$ is closed in $C$. Further we have
\[ C_u \cap C_v \supseteq C_{u \cap v} \quad (u, v \in U) \]

By $c$-compactness there exists a $c \in C$ such that
\[ z-c \in \bigcap_{U \in U} (X+U) = \overline{X} = X \]
i.e., $z \in X+c \subseteq X+C$.

**Remark.**

If the base field is not spherically complete there exist a complete absolutely convex compactoid $C \subseteq C_0$ and an element $a \in C_0$ such that $C+co\{a\}$ is not closed ([3], 6.25).
§ 2. LOCAL COMPACTOIDITY

DEFINITION 2.1. ([3], (6.7)) A subset X of E is a local compactoid if for each neighbourhood U of 0 in E there exists a finite dimensional K-linear subspace D of E such that \( X \subset U + D \).

PROPOSITION 2.2.

Let A be an absolutely convex subset of E. A is c-compact if and only if A is a complete local compactoid.

Proof.

For E a Banach space this is proved in [3], 6.15. Now let E be a locally convex.

(i) Assume A is c-compact. By Proposition 1.2 (ii), A is complete. To prove local compactoidity let U be an absolutely convex neighbourhood of 0 in E. There is a continuous seminorm \( p \) such that \( \{ x \in E : p(x) \leq 1 \} \subset U \).

Let \( \pi_p : E \rightarrow E_p \) be the quotient map where \( E_p \) is the canonically normed space \( E/\text{Ker} p \). Now \( \pi_p(A) \) is c-compact (Proposition 1.2 (v)) so by the above it is a local compactoid in the completion \( E_p^\sim \) of \( E_p \). By Corollary 6.15 of [3] we have \( \pi_p(A) = R + T \) where \( R \) is a compactoid and \( T \) a finite dimensional subspace of \( E_p^\sim \). Then \( T \subset E_p \). Now \( \pi_p(U) \) is open in \( E_p \) and by Katsaras' Theorem ([5], Lemma 8.1) there exist \( x_1, \ldots, x_n \in [R] \) such that \( R \subset \pi_p(U) + \text{co}(x_1, \ldots, x_n) \). Combining our knowledge on \( R \) and \( T \) we find a finite dimensional space \( F \subset [\pi_p(A)]_p \) such that \( \pi_p(A) \subset \pi_p(U) + F \). Choose a finite dimensional space \( D \subset [A]_p \) such that \( \pi_p(D) = F \). Then

\[
A \subset U + D + \text{Ker} \pi_p \subset U + D.
\]

(ii) Let A be a complete local compactoid. Let \( \Gamma \) be the collection of all continuous seminorms on E. For each \( p \in \Gamma \) we have that \( \pi_p(A) \), and also \( \pi_p(A) \), is a local compactoid in \( E_p^\sim \).
As $E^*$ is a Banach space we know that $\prod_{p \in \Gamma} \overline{p(A)}$ is $c$-compact. Then also $A_0 := \prod_{p \in \Gamma} \overline{p(A)}$ is a $c$-compact subset of $\prod_{p \in \Gamma} E^*$ (Proposition 1.1 (iv)). The canonical map $E \to \prod_{p \in \Gamma} E^*$ sends $A$ homeomorphically and linearly into $A_0$. Its image is closed in $A_0$ because $A$ is complete. Then $A$ is $c$-compact (Proposition 1.2 (iii)).

The following Proposition may look innocent.

**PROPOSITION 2.3.**

Let $A \subset E$ be absolutely convex and $c$-compact. For each neighbourhood $U$ of 0 there exists a finite dimensional absolutely convex set $F \subset A$ such that $A = U + F$.

(The crucial part is the phrase 'F $\subset A$'.) For the proof we use a lemma.

**LEMMA 2.4.**

Let $A, U$ be absolutely convex subsets of $E$, where $U$ is closed, $A$ is $c$-compact. Let $x \in E$ be such that $A \subset U + Kx$. Then there exists an $y \in E$ and an absolutely convex $C \subset K$ such that $Cy \subset A$ and $A \subset U + Cy$.

**Proof.**

Let $C := \{ c \in K : (U + cx) \cap A \neq \emptyset \}$. We have $A \subset U + CX$, $C = \{ c \in K : cx \in A + U \}$, so $C$ is absolutely convex. If $C = (0)$ then $A \subset U$ and we choose $y := 0$.

So assume $C \neq (0)$. For each $c \in C$, $c \neq 0$ define

$$H_c := c^{-1}(A \cap (cx + U)).$$

Each $H_c$ is a convex, closed, nonempty subset of $c^{-1}A$ hence $c$-compact.

Further, if $c, d \in C$, $0 < |c| \leq |d|$ then $H_d \subset H_c$. (Proof. Let $z \in H_d$. Then $dz \in A \cap (dx + U)$. By absolute convexity of $A$ and $U$,
\[ cz = \frac{c}{d} \cdot dz \in A \]

\[ cz \in \frac{c}{d} (dx + U) \subset cx + \frac{c}{d} U \subset cx + U. \]

It follows that \( cz \in A \cap (cx + U) \) i.e. \( z \in H_c \). By c-compactness there exists an \( y \in \cap H_c \). Let \( c \in C, c \neq 0 \). Then

\[ cy \in cH_c \subset A \cap (cx + U) \subset A. \]

Also, \( cy \in cx + U \) so that \( cx - cy \in U \). Let \( a \in A \). Then \( a = u + cx \) for some \( u \in U, c \in C \). We see that \( a = u + cy + cx - cy \in cy + U \). It follows that \( A \subset U + Cy \).

**Proof of Proposition 2.3.**

We may assume that \( U \) is absolutely convex. By Proposition 2.2 \( A \) is a local compactoid so there exist \( x_1, \ldots, x_n \in E \) such that

\[ A \subset U + Kx_1 + \ldots + Kx_n. \]

By the Lemma, applied to \( U + Kx_2 + \ldots + Kx_n \) in place of \( U \), there exist \( a y_1 \in E \) and an absolutely convex \( C_1 \in K \) such that

\[ C_1 y_1 \subset A \text{ and} \]

\[ A \subset U + C_1 y_1 + Kx_2 + \ldots + Kx_n \]

\[ = (U + C_1 y_1 + Kx_3 + \ldots , Kx_n) + Kx_2 \]

and we can continue. After \( n \) of these procedures we arrive at

\[ y_1, \ldots, y_n \in E, \text{ absolutely convex } C_1, \ldots, C_n \subset K \text{ such that } C_i y_i \subset A \]

for each \( i \) and \( A \subset U + C_1 y_1 + \ldots + C_n y_n \).

**Warning.**

The property of Proposition 2.3 is not shared by all absolutely convex local compactoids even when we require them to be closed! In fact we have:
EXAMPLE 2.5.

Let the valuation of $K$ be dense. Set

$$A = \{x \in c_0 : \|x\| \leq 1\}.$$  

([4], p.47).

(i) $A$ is a closed (local) compactoid for the weak topology of $c_0$.

(ii) There exists a weak neighbourhood $U$ of 0 such that for any finite dimensional set $F \subseteq A$,

$$A \not\subseteq U+F.$$  

Proof.

(i) Let $U$ be a weak neighbourhood of 0. There exists a weakly continuous seminorm $p$ such that \{x \in c_0 : p(x) \leq 1\} \subseteq U. Then $\ker p$ has finite codimension. Choose a finite dimensional space $D \subseteq c_0$ with $\pi_p(D) = \ker p$ (where as previously, $\pi_p := c_0/\ker p$ and $\pi_p : c_0 \rightarrow \ker p$ is the quotient map). We have $A \subseteq \ker p + D \subseteq U + D$ (in fact, we have shown that each subset of $c_0$ is a local compactoid for the weak topology). To prove weak closedness of $A$, let $(x_i)_{i \in I}$ be a net in $A$ converging weakly to $x \in c_0$. By [4], Lemma 4.35 (i) there exists an $f \in c_0'$, $f \neq 0$ for which $|f(x)| = \|f\| \|x\|$. We have

$$\|f\| \|x\| = |f(x)| = \lim |f(x_i)| \leq \lim \sup \|f\| \|x_i\| \leq \|f\|$$

so that $\|x\| \leq 1$.

(ii) Choose $\tau_1, \tau_2, \ldots, \in \mathbb{K}$, $0 < |\tau_1| < |\tau_2| < \ldots$, $\lim_{n \to \infty} |\tau_n| = 1$. The formula

$$f(a_1, a_2, \ldots) = \sum_{i=1}^{\infty} a_i \tau_i$$

defines an element $f \in c_0'$. Observe that $\sup_{A} |f| = 1$ but $|f(x)| < 1$ for each $x \in A$. Set $U := \{x : |f(x)| \leq \frac{1}{2}\}$, let $F$ be any finite dimensional
set in $A$. We shall arrive at $A \nsubseteq U+F$ by showing that $\sup_{U+F}|f| < 1$. To this end it suffices to prove $\sup_{F}|f| < 1$. $[F]$ is a finite dimensional subspace of $c_0$ and therefore ([4], Theorem 5.9) has an orthonormal base $x_1, \ldots, x_n$. It is easily seen that

$$ F' := \operatorname{co}\{x_1, \ldots, x_n\} \supseteq F $$

and $\sup_{F'}|f| \leq \sup_{F}|f| = \max(|f(x_1)|, \ldots, |f(x_n)|) < 1$.

Remark.

The above construction works also for the case where the base field is not spherically complete. Then $A$ is even weakly complete! ([5], Theorem 9.6 and [4], Theorem 4.17)
§ 3. A REPRESENTATION THEOREM FOR C-COMPACT SETS

LEMMA 3.1.

Let $\lambda \in K, \vert \lambda \vert > 1$. Let $G \subset E$ be closed, absolutely convex, and let $F \subset [G]$ be a finite dimensional set. If $(x_i)_{i \in I}$ is a net in $G+F$ converging to 0 then $x_i \in \lambda G$ for large $i$.

Proof.

[6], Lemma 1.3.

PROPOSITION 3.2.

(See also [2], Proposition 4, p. 93.) Let $A \subset E$ be absolutely convex, c-compact. Let $\tau'$ be a Hausdorff locally convex topology on $E$, weaker than the initial topology $\tau$. Then $\tau = \tau'$ on $A$.

Proof.

Let $(x_i)_{i \in I}$ be a net in $A$ converging to 0 for $\tau'$. Let $\lambda \in K, \vert \lambda \vert > 1$, let $U$ be an absolutely convex neighbourhood of 0 for $\tau$. Then $(\lambda^{-1} U) \cap A$ is c-compact in $(E,\tau)$ hence in $(E,\tau')$, so that $(\lambda^{-1} U) \cap A$ is $\tau'$-closed. There is (Proposition 2.3) a finite dimensional $F \subset A$ with $A \subset \lambda^{-1} U + F$. Then $A = (\lambda^{-1} U) \cap A + F$. Lemma 3.1 applies. It follows that $x_i \in \lambda (\lambda^{-1} U) \cap A \subset U$ for large $i$, so $\lim x_i = 0$ in the sense of $\tau$.

THEOREM 3.3.

Let $A \subset E$ be absolutely convex. The following are equivalent.

(a) $A$ is c-compact.

(b) $A$ is isomorphic (as a topological module over $\{\lambda \in K : \vert \lambda \vert \leq 1\}$) to a closed submodule of some power of $K$. 
Proof. 

(β) ⇒ (α). This follows from Proposition 1.2, (i), (iv), (iii). Now suppose (α). The map

\[ x \mapsto (f(x))_f \in E' \]

is a continuous linear injection \( E \to K^{E'} \) (Hahn-Banach Theorem). According to Proposition 3.2 it is a homeomorphism, if restricted to \( A \), and (β) follows.
REFERENCES


