SOME PROPERTIES OF C-COMPACT SETS
IN p-ADIC SPACES

by

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In this note we shall prove some properties of c-compact sets that may or may not be part of the 'folklore'. The concept of c-compactness, introduced by Springer [7], takes over the role played by convex-compact sets in Functional Analysis over $\mathbb{R}$ or $\mathbb{C}$ (or, any locally compact valued field).

Throughout, let $K$ be a nonarchimedean nontrivially valued field with valuation $|\cdot|$. We assume $K$ to be maximally (= spherically) complete. A subset $A$ of a $K$-linear space $E$ is absolutely convex if it is a submodule of $E$, considered as a module over the valuation ring $\{\lambda \in K : |\lambda| \leq 1\}$. A set $C \subseteq E$ is convex if it is either empty or an additive coset of an absolutely convex set. For a set $X \subseteq E$ we denote by $\text{co} X$ its absolutely convex hull, by $[X]$ its $K$-linear span.

From now on in this paper $E$ is a locally convex space over $K$ ([8]). We assume $E$ to be Hausdorff.
§ 1. DEFINITION AND FIRST PROPERTIES

DEFINITION 1.1. ([7]) Let $C \subset E$ be a nonempty convex set. A **convex filter** on $C$ is a filter of subsets of $C$ that has a basis consisting of convex sets. $C$ is **c-compact** if each convex filter on $C$ has a cluster point in $C$.

In other words, $C$ is c-compact if and only if the following is true. Let $C$ be a family of nonempty relatively closed convex subsets of $C$ such that $C_1, C_2 \in C$ implies $C_1 \cap C_2 \in C$. Then $\cap C \neq \emptyset$.

We quote the following properties, proved in [7].

**PROPOSITION 1.2.**

(i) $K$ is c-compact.

(ii) A c-compact set is complete.

(iii) A nonempty closed convex subset of a c-compact set is c-compact.

(iv) Let $(E_i)_{i \in I}$ be a family of Hausdorff locally convex spaces over $K$. Suppose, for each $i$, $C_i$ is c-compact in $E_i$. Then $\bigcap_{i \in I} C_i$ is c-compact in $\bigcap_{i \in I} E_i$.

(v) The image of a c-compact set under a continuous linear map is c-compact.

In [11] we find the following.

**PROPOSITION 1.3.**

(i) If $K$ is locally compact then a bounded nonempty convex set $C \subset E$ is c-compact if and only if it is convex and compact.

(ii) $E$ is c-compact if and only if $E$ is linearly homeomorphic to a
In § 3 (Theorem 3.3) we shall characterize arbitrary $c$-compact sets in the spirit of Proposition 1.3 (ii). But we conclude this first section with two statements that have nothing to do with the sequel. I just want to get rid of them.

**Proposition 1.4.**

A $c$-compact set is a Baire space.

**Proof.**

Let $U_1, U_2, ...$ be (relatively) open dense subsets of a $c$-compact set $C \subseteq E$. We prove that $\bigcap_n U_n \neq \emptyset$. There exists a nonempty open convex subset $B_1 \subseteq U_1$. As $U_2$ is dense we can find a nonempty open convex set $B_2 \subseteq B_1 \cap U_2$. Continuing this way we find nonempty open convex sets $B_n \subseteq \bigcap_{i=1}^n U_i$ for each $n$. The open sets $B_n$ are cosets of an additive group, hence closed. By $c$-compactness, $\bigcap_n B_n \neq \emptyset$. It follows that $\bigcap_n U_n \neq \emptyset$.

**Proposition 1.5.**

Let $X \subseteq E$ be closed, let $C \subseteq E$ be $c$-compact. Then $X + C$ is closed.

**Proof.**

Let $z \in \overline{X+C}$ (the closure of $X+C$), let $\mathcal{U}$ be the collection of all absolutely convex neighbourhoods of 0. For each $U \in \mathcal{U}$ the set $z + U$ intersects $X + C$ so

$$C_U := \{ c \in C : z - c \in X + U \}$$

is not empty. $X + U$ is a union of cosets of $U$, so is its complement.
Therefore, $X+U$ is closed and $C_U$ is closed in $C$. Further we have
\[ C_U \cap C_V \supset C_U \cap V \quad (U, V \in U) \]

By $c$-compactness there exists a $c \in C$ such that
\[ z - c \in \bigcap_{U \in \mathcal{U}} (X+U) = X = X \]
i.e., $z \in X+c \subset X+C$.

Remark.

If the base field is not spherically complete there exist a complete absolutely convex compactoid $C \subset C_0$ and an element $a \in C_0$ such that $C+co(a)$ is not closed ([3], 6.25).
§ 2. LOCAL COMPACTOIDITY

DEFINITION 2.1. ([3], (6.7)) A subset X of E is a local compactoid if for each neighbourhood U of 0 in E there exists a finite dimensional K-linear subspace D of E such that X ⊂ U + D.

PROPOSITION 2.2.
Let A be an absolutely convex subset of E. A is c-compact if and only if A is a complete local compactoid.

Proof.
For E a Banach space this is proved in [3], 6.15. Now let E be a locally convex.

(i) Assume A is c-compact. By Proposition 1.2 (ii), A is complete. To prove local compactoidity let U be an absolutely convex neighbourhood of 0 in E. There is a continuous seminorm p such that \( \{ x \in E : p(x) \leq 1 \} \subset U. \)

Let \( \pi : E \to E_p \) be the quotient map where \( E_p \) is the canonically normed space \( E/\text{Ker} p. \) Now \( \pi(A) \) is c-compact (Proposition 1.2 (v)) so by the above it is a local compactoid in the completion \( E_p^e \) of \( E_p. \) By Corollary 6.15 of [3] we have \( \pi(A) = R + T \) where \( R \) is a compactoid and \( T \) a finite dimensional subspace of \( E_p^e. \) Then \( T \subset E_p. \) Now \( \pi(U) \) is open in \( E_p \) and by Katsaras' Theorem ([5], Lemma 8.1) there exist \( x_1, \ldots, x_n \in [R] \) such that \( R \subset \pi(U) + \text{co}(x_1, \ldots, x_n). \) Combining our knowledge on \( R \) and \( T \) we find a finite dimensional space \( F \subset [\pi(A)] \) such that \( \pi_p(A) \subset \pi_p(U) + F. \) Choose a finite dimensional space \( D \subset [A] \) such that \( \pi_p(D) = F. \) Then

\[
A \subset U + D + \text{Ker} \pi_p \subset U + D.
\]

(ii) Let A be a complete local compactoid. Let \( \Gamma \) be the collection of all continuous seminorms on E. For each \( p \in \Gamma \) we have that \( \pi_p(A) \), and also \( \pi_p(A) \), is a local compactoid in \( E_p^e. \)
As $E^\gamma$ is a Banach space we know that $\prod_{p \in P} \pi_p(A)$ is $c$-compact. Then also $A_0 := \prod_{p \in P} \pi_p(A)$ is a $c$-compact subset of $\prod_{p \in P} E^\gamma$ (Proposition 1.1 (iv)).

The canonical map $E \to \prod_{p \in P} E^\gamma$ sends $A$ homeomorphically and linearly into $A_0$. Its image is closed in $A_0$ because $A$ is complete. Then $A$ is $c$-compact (Proposition 1.2 (iii)).

The following Proposition may look innocent.

**Proposition 2.3.**

Let $A \subset E$ be absolutely convex and $c$-compact. For each neighbourhood $U$ of $0$ there exists a finite dimensional absolutely convex set $F \subset A$ such that $A = U + F$.

(The crucial part is the phrase 'F $\subset A$'.) For the proof we use a lemma.

**Lemma 2.4.**

Let $A, U$ be absolutely convex subsets of $E$, where $U$ is closed, $A$ is $c$-compact. Let $x \in E$ be such that $A \subset U + kx$. Then there exists an $y \in E$ and an absolutely convex $C \subset K$ such that $Cy \subset A$ and $A \subset U + Cy$.

**Proof.**

Let $C := \{c \in K : (U + cx) \cap A \neq \emptyset\}$. We have $A \subset U + cx$, $C = \{c \in K : cx \subset A + U\}$, so $C$ is absolutely convex. If $C = (0)$ then $A \subset U$ and we choose $y := 0$.

So assume $C \neq (0)$. For each $c \in C$, $c \neq 0$ define

$$H_c := \text{c}^{-1}(A \cap (cx + U)).$$

Each $H_c$ is a convex, closed, nonempty subset of $c^{-1}A$ hence $c$-compact.

Further, if $c, d \in C, 0 < |c| \leq |d|$ then $H_d \subset H_c$. (Proof. Let $z \in H_d$. Then $dz \in A \cap (dx + U)$. By absolute convexity of $A$ and $U$,}
cz = \frac{c}{d} \cdot dz \in A

cz \in \frac{c}{d} (dx+U) \subseteq cx+\frac{c}{d} U \subseteq cx+U.

It follows that \( cz \in A \cap (cx+U) \) i.e. \( z \in H_c \). By \( c \)-compactness there exists

an \( y \in \cap H_c \). Let \( c \in C \), \( c \neq 0 \). Then

\( cy \in ch_c \subseteq A \cap (cx+U) \subseteq A. \)

Also, \( cy \in cx+U \) so that \( cx-cy \in U \). Let \( a \in A \). Then \( a = u+cx \) for some

\( u \in U \), \( c \in C \). We see that \( a = u+cy+cx-cy \in cy+U \). It follows that

\( A \subseteq U+Cy \).

**Proof of Proposition 2.3.**

We may assume that \( U \) is absolutely convex. By Proposition 2.2 \( A \) is a

local compactoid so there exist \( x_1, \ldots, x_n \in E \) such that

\( A \subseteq U+Kx_1+\ldots+Kx_n \). By the Lemma, applied to \( U+Kx_2+\ldots+Kx_n \) in place of

\( U \), there exist a \( y_1 \in E \) and an absolutely convex \( C_1 \subseteq K \) such that

\( C_1y_1 \subseteq A \) and

\[
A \subseteq U + C_1y_1 + Kx_2 + \ldots + Kx_n = (U + C_1y_1 + Kx_3 + \ldots, Kx_n) + Kx_2
\]

and we can continue. After \( n \) of these procedures we arrive at

\( y_1, \ldots, y_n \in E \), absolutely convex \( C_1, \ldots, C_n \subseteq K \) such that \( C_1y_1 \subseteq A \)

for each \( i \) and \( A \subseteq U+C_1y_1+\ldots+C_ny_n \).

**Warning.**

The property of Proposition 2.3 is not shared by all absolutely convex

local compactoids even when we require them to be closed! In fact we

have:
EXAMPLE 2.5.

Let the valuation of $K$ be dense. Set

$$A = \{x \in c_0 : \|x\| \leq 1\}.$$  

([4], p.47).

(i) $A$ is a closed (local) compactoid for the weak topology of $c_0$.

(ii) There exists a weak neighbourhood $U$ of 0 such that for any finite dimensional set $F \subset A$

$$A \notin U + F.$$

Proof.

(i) Let $U$ be a weak neighbourhood of 0. There exists a weakly continuous seminorm $p$ such that $\{x \in c_0 : p(x) \leq 1\} \subset U$. Then $K_{erp}$ has finite codimension. Choose a finite dimensional space $D \subset c_0$ with $\pi_p(D) = E_p$ (where as previously, $E_p := c_0/K_{erp}$ and $\pi_p : c_0 \to E_p$ is the quotient map). We have $A \subset K_{erp} + D \subset U + D$ (in fact, we have shown that each subset of $c_0$ is a local compactoid for the weak topology). To prove weak closedness of $A$, let $(x_i)_{i \in I}$ be a net in $A$ converging weakly to $x \in c_0$. By [4], Lemma 4.35 (i) there exists an $f \in c_0'$, $f \neq 0$ for which $|f(x)| = \|f\| \|x\|$. We have

$$\|f\| \|x\| = |f(x)| = \lim |f(x_i)| \leq \lim \sup \|f\| \|x_i\| \leq \|f\|$$

so that $\|x\| \leq 1$.

(ii) Choose $\tau_1, \tau_2, \ldots \in K$, $0 < |\tau_1| < |\tau_2| < \ldots$, $\lim_{n \to \infty} |\tau_n| = 1$. The formula

$$f(a_1, a_2, \ldots) = \sum_{i=1}^{\infty} a_i \tau_i$$

defines an element $f \in c_0'$. Observe that $\sup A = 1$ but $\|f(x)\| < 1$ for each $x \in A$. Set $U := \{x : |f(x)| \leq \frac{1}{2}\}$, let $F$ be any finite dimensional
set in \(A\). We shall arrive at \(A \neq U + F\) by showing that \(\sup_{U + F} |f| < 1\). To this end it suffices to prove \(\sup_{F} |f| < 1\). \([F]\) is a finite dimensional subspace of \(c_0\) and therefore ([4], Theorem 5.9) has an orthonormal base \(x_1, \ldots, x_n\). It is easily seen that

\[
F' := \text{co} \{x_1, \ldots, x_n\} \supset F
\]

and \(\sup_{F'} |f| \leq \sup_{F'} |f| = \max(|f(x_1)|, \ldots, |f(x_n)|) < 1\).

**Remark.**

The above construction works also for the case where the base field is not spherically complete. Then \(A\) is even weakly complete! ([5], Theorem 9.6 and [4], Theorem 4.17)
§ 3. A REPRESENTATION THEOREM FOR C-COMPACT SETS

LEMMA 3.1.

Let $\lambda \in K$, $|\lambda| > 1$. Let $G \subseteq E$ be closed, absolutely convex, and let $F \subseteq [G]$ be a finite dimensional set. If $(x_i)_{i \in I}$ is a net in $G + F$ converging to 0 then $x_i \in \lambda G$ for large $i$.

Proof.

[6], Lemma 1.3.

PROPOSITION 3.2.

(See also [2], Proposition 4, p. 93.) Let $A \subseteq E$ be absolutely convex, c-compact. Let $\tau'$ be a Hausdorff locally convex topology on $E$, weaker than the initial topology $\tau$. Then $\tau = \tau'$ on $A$.

Proof.

Let $(x_i)_{i \in I}$ be a net in $A$ converging to 0 for $\tau'$. Let $\lambda \in K$, $|\lambda| > 1$, let $U$ be an absolutely convex neighbourhood of 0 for $\tau$. Then $(\lambda^{-1}U) \cap A$ is c-compact in $(E, \tau)$ hence in $(E, \tau')$, so that $(\lambda^{-1}U) \cap A$ is $\tau'$-closed. There is (Proposition 2.3) a finite dimensional $F \subseteq A$ with $A \subseteq \lambda^{-1}U + F$.

Then $A = (\lambda^{-1}U) \cap A + F$. Lemma 3.1 applies. It follows that $x_i \in \lambda(\lambda^{-1}U) \cap A \subseteq U$ for large $i$, so $\lim x_i = 0$ in the sense of $\tau$.

THEOREM 3.3.

Let $A \subseteq E$ be absolutely convex. The following are equivalent.

(a) $A$ is c-compact.

(b) $A$ is isomorphic (as a topological module over $\{\lambda \in K : |\lambda| \leq 1\}$) to a closed submodule of some power of $K$. 
Proof.

(\beta) \rightarrow (\alpha). This follows from Proposition 1.2, (i), (iv), (iii). Now suppose (\alpha). The map

\[ x \mapsto (f(x))_{f \in E'} \]

is a continuous linear injection \( E \to K^{E'} \) (Hahn-Banach Theorem). According to Proposition 3.2 it is a homeomorphism, if restricted to \( A \), and (\beta) follows.
REFERENCES


