§1. Introduction and summary. (For terminology, see §2) In [2] the following separation theorem is proved. Let $K$ be spherically complete and let $E$ be a locally convex space over $K$. If $A \subseteq E$ is closed and absolutely convex and if $x \in E \setminus A$ then there is an $f \in E'$ such that

\[(*) \quad |f(A)| \leq 1, \quad f(x) = 1.\]

In order to obtain, 'real' separation one would prefer to have

\[(**) \quad |f(A)| < 1, \quad f(x) = 1\]

rather than (*). However, with the techniques used in [2], it is not clear how to arrive at such a result for densely valued fields $K$. The main obstruction is the fact that for an open absolutely convex, absorbing set $A$ its associated seminorm $q_A^*$,

\[q_A^*(x) := \inf \{ |\lambda| : \lambda \in K : x \in \lambda A \} \quad (x \in E)\]

does not determine $A$; one only has the rather 'vague' relation
\{x \in E : q_A(x) < 1\} \subset A \subset \{x \in E : q_A(x) \leq 1\}.

(Example. The open and closed convex sets \{(\xi_1, \xi_2) \in K^2 : |\xi_1| < 1, |\xi_2| \leq 1\}, \{(\xi_1, \xi_2) \in K^2 : |\xi_1| \leq 1, |\xi_2| < 1\}, \{(\xi_1, \xi_2) \in K^2 : |\xi_1| < 1, |\xi_2| < 1\} all have the associated (sem)norm \((\xi_1', \xi_2') \mapsto \max(|\xi_1'|, |\xi_2'|)\).

In this paper we shall extend the notion of a seminorm by admitting a larger range yielding the notion of a distinguishing seminorm (Definition 2). We shall prove that each absolutely convex absorbing set can be written as \(\{x \in E : p(x) < 1\}\) for some distinguishing seminorm \(p\) (Theorem 5). Next, we shall prove a Hahn Banach theorem for linear functions majorized by distinguishing seminorms (Theorem 6) and shall obtain, as a corollary, a strong separation theorem (with \((**))\) in place of \((*)\). Here the notion of distinguishing seminorm is used only in the proof, not in the formulation (Theorem 8).

Note. This separation theorem can (for Banach spaces) also be obtained as a corollary of [1], Theorem 6.21. However, one techniques differ very much from the ones used in the (clever) proof of [1]. Also, the notion of a distinguishing seminorm may very well yield new applications.

§2. Terminology

Throughout \(K\) is a non-archimedean complete non-trivially valued field with valuation \(| |, |x| := \{|\lambda| : \lambda \in \mathbb{X}\}. E\) is a vector space over \(K\).

A seminorm on \(E\) is a map \(q : E \to [0,\infty)\) satisfying (i) \(q(0) = 0\), (ii) \(q(\lambda x) = |\lambda|q(x)\) (\(x \in E, \lambda \in K\)), (iii) \(q(x+y) \leq q(x) \vee q(y)\) (\(x,y \in E\)) where \(\vee\) indicates 'maximum'.

A subset \(A\) of \(E\) is absolutely convex if it is a module over the ring \(\{\lambda \in K : |\lambda| \leq 1\}\), convex if it is an additive coset of an absolutely convex set, absorbing if \(\bigcup_{\lambda \in K} \lambda A = E\).
Each convex subset of $K$ has the form $\{ \lambda \in K : |\lambda - a| < r \}$ or $\{ \lambda \in K : |\lambda - a| \leq r \}$ for some $a \in K$, $r \in [0,\omega]$.

For a seminorm $q$ the sets $\{ x \in E : q(x) < 1 \}$ and $\{ x \in E : q(x) \leq 1 \}$ are absolutely convex and absorbing. Conversely, for an absorbing, absolutely convex subset $A$ of $E$ the associated seminorm $q_A$ defined by

$$q_A(x) := \inf\{ |\lambda| : \lambda \in K : x \in \lambda A \} \quad (x \in E)$$

is a seminorm satisfying

$$\{ x \in E : q_A(x) < 1 \} \subset A \subset \{ x \in E : q_A(x) \leq 1 \}.$$

$K$ is spherically complete if for any collection $C$ of convex subsets for which $A, B \in C$ imply $A \cap B \neq \emptyset$ we have $\cap C = \emptyset$. A locally convex space is a $K$-vector space $E$ with a topology induced by a collection of seminorms. Its dual space is denoted $E'$.

§3. **Distinguishing seminorms**

We enlarge the set $[0,\omega)$ by giving each positive real number $a$ an immediate predecessor $a^-$. Formally

$$V := [(0,\omega)] \cup (0,\omega)^-$$

where $(0,\omega)^- := \{ a^- : a \in (0,\omega) \}$ is a second copy of $(0,\omega)$. Further we define $0^- := 0$. The formulas

$$\begin{align*}
a^- < b & \iff a \leq b \\
(a, b \in (0,\omega)) \\
a^- < b^- & \iff a < b^- \iff a < b \\
(a, b \in (0,\omega)) \\
0 < a & \quad (a \in V, a \neq 0)
\end{align*}$$

define a linear ordering $<$ on $V$ extending the usual ordering on $[0,\omega)$. 
The projection map \( \pi : V \to [0,\infty) \) is defined by

\[
\begin{align*}
\pi(a) &:= a & (a \in [0,\infty)) \\
\pi(a^-) &:= a & (a \in [0,\infty)).
\end{align*}
\]

Finally we extend the multiplication on \([0,\infty)\) to a multiplication \([0,\infty) \times V \to V\) by requiring \(a \cdot b^- := (ab)^-\) for all \(a, b \in [0,\infty)\).

Proposition 1 collects some direct consequences from the definitions.

**PROPOSITION 1.**

(i) Let \(b, c \in V, b \leq c\). Then \(ab \leq ac\) for all \(a \in [0,\infty)\).

(ii) \(\pi(a \vee b) = \pi(a) \vee \pi(b)\) \((a, b \in V)\).

(iii) \(\pi(ab) = a\pi(b)\) \((a \in [0,\infty), b \in V)\).

**DEFINITION 2.** A distinguishing seminorm on \(E\) is a map \(p : E \to V\) satisfying

\[
\begin{align*}
(i) & \quad p(0) = 0 \\
(ii) & \quad p(\lambda x) = |\lambda| \ p(x) \quad (\lambda \in K, x \in E) \\
(iii) & \quad p(x+y) \leq p(x) \vee p(y) \quad (x, y \in E).
\end{align*}
\]

If, in addition, \(p(x) = 0\) implies \(x = 0\) then \(p\) is a distinguishing norm.

**Examples**

1. Any seminorm on \(E\).

2. The map \((\xi_1, \xi_2) \mapsto |\xi_1| \vee |\xi_2|^-\) \(((\xi_1, \xi_2) \in K^2)\) is a distinguishing norm on \(K^2\).

3. Let \(E\) be the space of all bounded \(K\)-valued functions on a set \(X\). Then

\[
\|f\|_\infty := \sup\{|f(x)| : x \in X\} \quad (f \in E),
\]

where the supremum is taken in \(V\), defines a distinguishing norm on \(E\).

Observe that for \(f \in E\) we have \(\|f\|_\infty \leq 1\) if and only if \(|f(x)| \leq 1\) for
all \( x \in E \) but

\[
\| f \|_\infty < 1 \iff \| f \|_\infty \leq 1 \iff |f(x)| < 1 \text{ for all } x \in E
\]

so that \( \| f \|_\infty = 1 \) if and only if \( 1 = \max\{|f(x)| : x \in X\} \).

**PROPOSITION 3.** Let \( p \) be a distinguishing seminorm on \( E \). Then \( \pi \circ p \) is a seminorm on \( E \).

**Proof.** The statement follows directly from Proposition 1 (ii), (iii).

**DEFINITION 4.** Let \( A \subseteq E \) be absorbing and absolutely convex, and let \( q_A \) be its associated seminorm. The distinguishing seminorm \( p_A \), associated to \( A \) is

\[
p_A(x) := \begin{cases} 
q_A(x) & \text{if } q_A(x) = \min\{ |\lambda| : \lambda \in K, x \in \lambda A\} \\
q_A(x) & \text{otherwise}.
\end{cases}
\]

The following theorem shows that the associated distinguishing seminorm of an absorbing absolutely convex set \( A \) determines \( A \).

**THEOREM 5.** Let \( A \subseteq E \) be absorbing and absolutely convex. Let \( p_A \) and \( q_A \) be as in Definition 4. Then \( p_A \) is a distinguishing seminorm with \( \pi \circ p_A = q_A \). Further we have

\[
A = \{ x \in E : p_A(x) < 1 \} = \{ x \in E : p_A(x) \leq 1 \}.
\]

**Proof.** Clearly \( \pi \circ p_A = q_A \). We first check the equality

\( A = \{ x \in E : p_A(x) < 1 \} \). Let \( p_A(x) < 1 \). If \( p_A(x) < 1 \) then \( q_A(x) < 1 \), so \( x \in A \). If \( p_A(x) = 1 \) then \( 1 \in \{ |\lambda| : \lambda \in K, x \in \lambda A\} \) so that \( x \in \lambda A \) for some \( \mu \in K, |\mu| = 1 \). By absolute convexity, \( x \in \mu^{-1} \mu A = A \). Conversely, let \( x \in A \). Then \( q_A(x) \leq 1 \) so that \( p_A(x) \leq q_A(x) \leq 1 \). If \( p_A(x) = 1 \) then

\[
1 = \inf\{ |\lambda| : x \in \lambda A \}
\]

is not a minimum contradicting \( x \in A \). Hence,
\[ p_A(x) < 1. \] Finally we prove the conditions (i), (ii), (iii) of Definition 2 for a distinguishing seminorm for \( p = p_A \). We have \( p_A(0) = 0^- = 0. \) To prove (ii) we may assume \( \lambda \neq 0 \) and \( p_A(x) \neq 0. \) If \( p_A(x) \in (0, \infty) \) then
\[ q_A(x) = \inf\{ |x| : x \in \tau A \} \] is not a minimum. Then neither is
\[ q_A(\lambda x) = \inf\{ |\lambda x| : \lambda x \in \tau A \} \] so that \( p_A(\lambda x) = q_A(\lambda x) = |\lambda| q_A(x) = |\lambda| p_A(x). \)
If \( p_A(x) \in (0, \infty)^- \) then \( q_A(x) = \min\{ |x| : x \in \tau A \}. \) Then also
\[ q_A(\lambda x) = \min\{ |\lambda x| : \lambda x \in \tau A \} \] so that
\[ p_A(\lambda x) = q_A(\lambda x)^- = (|\lambda| q_A(x))^- = |\lambda| q_A(x)^- = |\lambda| p_A(x). \] For the proof of the strong triangle inequality (iii) we may assume \( p_A(x+y) \neq 0. \) We distinguish two cases.

(i) \( p_A(x+y) \in (0, \infty)^- \). By Proposition 1 (ii) and \( \pi \circ p_A = q_A \), from
\[ q_A(x+y) \leq q_A(x) \lor q_A(y) \]
we obtain \( p_A(x+y) = q_A(x+y)^- \leq q_A(x)^- \lor q_A(y)^- \leq p_A(x) \lor p_A(y). \)

(ii) \( p_A(x+y) \in (0, \infty). \) Then the valuation of \( K \) is dense. Assume
\( p_A(y) \leq p_A(x). \) Suppose \( p_A(x+y) > p_A(x); \) we derive a contradiction. There is a \( \lambda \in K \) such that \( p_A(x+y) \geq |\lambda| > p_A(x). \) (In fact, if \( p_A(x) \neq p_A(x+y)^- \) then the interval \( (p_A(x), p_A(x+y)) \) contains infinitely many elements of \( |K|, \) if \( p_A(x) = p_A(x+y)^- \) then \( p_A(x+y) \in |K| \) and we may choose \( \lambda \in K \) such that \( |\lambda| = p_A(x+y). \) Then
\[ p_A(\lambda^{-1} y) \leq p_A(\lambda^{-1} x) < 1 \] so that \( \lambda^{-1} y \in A, \lambda^{-1} x \in A \) and, by absolute convexity, also \( \lambda^{-1}(x+y) \in A \) implying
\[ p_A(x+y) < |\lambda|, \] a contradiction.

§4. Hahn-Banach Theorem

For \( a \in K, a \in V \) we write \( B(a, a) = \{ \lambda \in K : |\lambda-a| \leq a \}. \)

**Theorem 6.** (Hahn-Banach Theorem) Let \( K \) be spherically complete, let \( p \)
be a distinguishing seminorm on a $K$-vector space $E$. Let $D$ be a $K$-linear subspace of $E$, let $f : D \to K$ be a $K$-linear map satisfying $|f(d)| \leq p(d)$ $(d \in D)$. Then $f$ can be extended to a $K$-linear $\tilde{f} : E \to K$ such that $|\tilde{f}(x)| \leq p(x)$ $(x \in D)$.

Proof. (It consists of checking that replacing of a seminorm by a distinguishing seminorm does not harm the well-known proof.) A simple application of Zorn's Lemma reduces the problem to the case $E = \{\lambda x + d : \lambda \in K, d \in D\}$ for some $x \in E \setminus D$. We are done if we can choose $\tilde{f}(x) = \xi \in K$ such that

$$|\lambda \xi + f(d)| \leq p(\lambda x + d) \quad (\lambda \in K, d \in D).$$

As this condition is satisfied for $\lambda = 0$ and all $d \in D$ and since for $\lambda \neq 0$

$$p(\lambda x+d) = |\lambda| \cdot p(x+\lambda^{-1}d)$$

$$|\lambda \xi + f(d)| = |\lambda| \cdot |\xi + f(\lambda^{-1}d)|$$

it suffices, by Proposition 1 (i) to produce a $\xi \in K$ such that

$$|\xi - f(d)| \leq p(x-d) \quad (d \in D),$$

i.e. we must have that

$$\bigcap_{d \in D} B(f(d), p(x-d)) \neq \emptyset.$$

By spherical completeness it suffices to show that for any $d_1, d_2 \in D$ we have $B(f(d_1), p(x-d_1)) \cap B(f(d_2), p(x-d_2)) \neq \emptyset$. But this follows easily from $|f(d_1) - f(d_2)| \leq p(d_1 - d_2) \leq p(d_1 - x) + p(x - d_2)$.

A typical application: let $\ell^\infty := \{(\xi_1, \xi_2, \ldots) : \xi_n \in K$ for all $n$, $\sup |\xi_n| < \infty\}$. Let $K$ be spherically complete. Then there is a linear function $g : \ell^\infty \to K$. 
of norm 1 such that (i) \( g((\xi_1, \xi_2, \ldots)) = \lim_{n \to \infty} \xi_n \) if \( \lim_{n \to \infty} \xi_n \) exists and

(ii) \( |g((\xi_1, \xi_2, \ldots))| < 1 \) if \( |\xi_n| < 1 \) for all \( n \in \mathbb{N} \).

(Proof: Choose in the above theorem \( D := c \) (the space of the convergent sequences), \( f((\xi_1, \xi_2, \ldots)) := \lim_{n \to \infty} ((\xi_1, \xi_2, \ldots) \in c) \), and

\[ p((\xi_1, \xi_2, \ldots)) := \sup |\xi_n|, \]

where the sup is taken in \( V \). Take \( g := \frac{f}{d} \).

§5. Separation of convex sets

Throughout §5, let \( E \) be a locally convex space over \( K \). We shall need the following observation.

**PROPOSITION 7.** An open convex subset of \( E \) is closed.

**Proof.** Any convex set is a coset of an absolutely convex set.

An open absolutely convex set is the complement of a union of cosets.

**Theorem 8.** Let \( K \) be spherically complete. Let \( A \subset E \) be closed, absolutely convex and let \( x \in E \setminus A \). Then there exists an \( f \in E' \) such that \( |f(A)| < 1 \) and \( f(x) = 1 \).

**Proof.** There is an absolutely convex open neighbourhood \( U \) of 0 such that

\((x+U) \cap A = \emptyset. \) Then \( U+A \) is absolutely convex, open, hence closed (Proposition 7). Further, \( x \notin U+A \). Thus, we may assume that \( A \) is open and closed. Then \( A \) is absorbing. Let \( p_A \) be the distinguishing seminorm of \( A \), let \( D := \{\lambda x : \lambda \in K\} \) and define \( g : D \to K \) by \( g(\lambda x) := \lambda (\lambda \in K) \). Then \( g(x) = 1 \). Since

\[ A = \{y \in E : p_A(y) < 1\} \]

(Theorem 5) and \( x \notin A \) we have \( p_A(x) \geq 1 \) so that for \( \lambda \in K \)

\[ p_A(\lambda x) = |\lambda| \cdot p_A(x) \geq |\lambda| = |g(\lambda x)|, \]

i.e. \( |g| \leq p_A \) on \( D \). By Theorem 6 \( g \) extends to a linear \( f : E \to K \) such that \( |f(y)| \leq p_A(y) \) for all \( y \in E \).
We have \( f(x) = g(x) = 1 \) and, for \( y \in A \), \( |f(y)| \leq p_A(y) < 1 \). The continuity of \( f \) follows from the continuity of \( \pi \circ p_A \) and the inequality \( |f| \leq \pi \circ p_A \).

**COROLLARY 9.** Let \( K \) be spherically complete. Each closed convex set is weakly closed.

Let \( A, B \) be convex subsets of \( E \). If \( f : E \to K \) is a linear function then \( f(A) \) and \( f(B) \) are convex in \( K \). Hence, if \( f(A) \cap f(B) = \emptyset \) then \( \text{dist}(f(A), f(B)) > 0 \). With this in mind the following definition is quite natural.

**DEFINITION 10.** Two convex subsets \( A, B \) of \( E \) are separated by an \( f \in E' \) if \( f(A) \cap f(B) = \emptyset \).

If \( A \) and \( B \) are separated by \( f \in E' \) then, since \( \text{dist}(f(A), f(B)) > 0 \) there is an open convex neighbourhood \( U \) of \( 0 \) such that \( (A + U) \cap B = \emptyset \) (if \( E \) is a normed space this is equivalent to \( \text{dist}(A, B) > 0 \)). To prove the converse we need spherical completeness.

**THEOREM 11.** Let \( K \) be spherically complete. Let \( A, B \) be convex subsets of \( E \) and suppose there is an open convex neighbourhood \( U \) of \( A \) such that \( (A + U) \cap B = \emptyset \) (observe that this condition is satisfied if \( A \) is open). Then \( A \) and \( B \) can be separated by some \( f \in E' \).

**Proof.** We may assume that \( A \) is open. Let \( C := A - B \). Then \( 0 \notin C \) and \( C = U \) (\( A - B \)) is open, convex. Choose \( c \in -C \). Then \( T := c + C \) is absolutely convex, open, hence closed and \( c \notin T \). By Theorem 8 there is an \( f \in E' \) such that \( f(c) = 1 \), \( |f(T)| < 1 \). Thus, for each \( a \in A \), \( b \in B \) we have

\[
1 > |f(c + a - b)| = |1 + f(a) - f(b)|.
\]

It follows that \( |f(a) - f(b)| = 1 \) for all \( a \in A \), \( b \in B \). In particular, \( f(A) \cap f(B) = \emptyset \).
References.
