Distinguishing non-archimedean seminorms
by
W.H. Schikhof

§1. Introduction and summary. (For terminology, see §2) In [2] the following separation theorem is proved. Let \( K \) be spherically complete and let \( E \) be a locally convex space over \( K \). If \( A \subset E \) is closed and absolutely convex and if \( x \in E \setminus A \) then there is an \( f \in E' \) such that

\[ |f(A)| < 1 \quad , \quad f(x) = 1. \tag{*} \]

In order to obtain, 'real' separation one would prefer to have

\[ |f(A)| < 1 \quad , \quad f(x) = 1 \tag{**} \]

rather than \((*)\). However, with the techniques used in [2], it is not clear how to arrive at such a result for densely valued fields \( K \). The main obstruction is the fact that for an open absolutely convex, absorbing set \( A \) its associated seminorm \( q_A \),

\[ q_A(x) := \inf\{|\lambda| : \lambda \in K : x \in \lambda A\} \quad (x \in E) \]

does not determine \( A \); one only has the rather 'vague' relation
\{x \in E : q^*_A(x) < 1\} \subset A \subset \{x \in E : q^*_A(x) \leq 1\}.

(Example. The open and closed convex sets \((\xi_1, \xi_2) \in K^2 : |\xi_1| \leq 1, |\xi_2| \leq 1\), \((\xi_1, \xi_2) \in K^2 : |\xi_1| \leq 1, |\xi_2| < 1\), \((\xi_1, \xi_2) \in K^2 : |\xi_1| < 1, |\xi_2| < 1\) all have the associated (semi)norm \((\xi_1, \xi_2) \mapsto \max(|\xi_1|, |\xi_2|)\).

In this paper we shall extend the notion of a seminorm by admitting a larger range yielding the notion of a distinguishing seminorm (Definition 2). We shall prove that each absolutely convex absorbing set can be written as \(\{x \in E : p(x) < 1\}\) for some distinguishing seminorm \(p\) (Theorem 5). Next, we shall prove a Hahn Banach theorem for linear functions majorized by distinguishing seminorms (Theorem 6) and shall obtain, as a corollary, a strong separation theorem (with \((**\)) in place of (**) ). Here the notion of distinguishing seminorm is used only in the proof, not in the formulation (Theorem 8).

Note. This separation theorem can (for Banach spaces) also be obtained as a corollary of [1], Theorem 6.21. However, one techniques differ very much from the ones used in the (clever) proof of [1]. Also, the notion of a distinguishing seminorm may very well yield new applications.

§2. Terminology

Throughout \(K\) is a non-archimedean complete non-trivially valued field with valuation \(|\cdot|, |x| := \{|\lambda| : \lambda \in K\}\). \(E\) is a vector space over \(K\).

A seminorm on \(E\) is a map \(q : E \to [0, \infty)\) satisfying (i) \(q(0) = 0\), (ii) \(q(\lambda x) = |\lambda|q(x) (x \in E, \lambda \in K)\), (iii) \(q(x+y) \leq q(x) \vee q(y) (x,y \in E)\)

where \(\vee\) indicates 'maximum'.

A subset \(A\) of \(E\) is absolutely convex if it is a module over the ring \(\{\lambda \in K : |\lambda| \leq 1\}\), convex if it is an additive coset of an absolutely convex set, absorbing if \(\cup_{\lambda \in K} \lambda A = E\).
Each convex subset of $K$ has the form \( \{ \lambda \in K : |\lambda - a| < r \} \) or \( \{ \lambda \in K : |\lambda - a| \leq r \} \) for some $a \in K$, $r \in [0,\infty]$. 

For a seminorm $q$ the sets \( \{ x \in E : q(x) < 1 \} \) and \( \{ x \in E : q(x) \leq 1 \} \) are absolutely convex and absorbing. Conversely, for an absorbing, absolutely convex subset $A$ of $E$ the associated seminorm $q_A$ defined by 

$$q_A(x) := \inf\{ |\lambda| : \lambda \in K : x \in \lambda A \} \quad (x \in E)$$

is a seminorm satisfying 

$$\{ x \in E : q_A(x) < 1 \} \subset A \subset \{ x \in E : q_A(x) \leq 1 \}.$$ 

$K$ is spherically complete if for any collection $C$ of convex subsets for which $A, B \in C \implies A \cap B \neq \emptyset$ we have $\cap C = \emptyset$. A locally convex space is a $K$-vector space $E$ with a topology induced by a collection of seminorms. Its dual space is denoted $E'$. 

§3. **Distinguishing seminorms**

We enlarge the set $[0,\infty)$ by giving each positive real number $a$ an immediate predecessor $a^-$. Formally 

$$V := [0,\infty) \cup (0,\infty)^-$$ 

where $(0,\infty)^- := \{ a^- : a \in (0,\infty) \}$ is a second copy of $(0,\infty)$. Further we define $0^- := 0$. The formulas 

\[
\begin{align*}
    a^- < b & \iff a \leq b \quad (a, b \in (0,\infty)) \\
    a^- < b^- & \iff a < b^- \iff a < b \quad (a, b \in (0,\infty)) \\
    0 < a & \quad (a \in V, a \neq 0)
\end{align*}
\]

define a linear ordering $<$ on $V$ extending the usual ordering on $[0,\infty)$. 

The projection map \( \pi : V \to [0,\infty) \) is defined by

\[
\pi(a) := a \quad (a \in [0,\infty))
\]

\[
\pi(a^-) := a \quad (a \in [0,\infty)).
\]

Finally we extend the multiplication on \([0,\infty)\) to a multiplication \([0,\infty) \times V \to V\) by requiring \(a \cdot b^- := (ab)^-\) for all \(a, b \in [0,\infty)\).

Proposition 1 collects some direct consequences from the definitions.

**Proposition 1.**

(i) Let \(b, c \in V, b \leq c\). Then \(ab \leq ac\) for all \(a \in [0,\infty)\).

(ii) \(\pi(a \vee b) = \pi(a) \vee \pi(b)\) \((a, b \in V)\).

(iii) \(\pi(ab) = a\pi(b)\) \((a \in [0,\infty), b \in V)\).

**Definition 2.** A distinguishing seminorm on \(E\) is a map \(p : E \to V\) satisfying

(i) \(p(0) = 0\)

(ii) \(p(\lambda x) = |\lambda| p(x)\) \((\lambda \in K, x \in E)\)

(iii) \(p(x+y) \leq p(x) \vee p(y)\) \((x, y \in E)\).

If, in addition, \(p(x) = 0\) implies \(x = 0\) then \(p\) is a distinguishing norm.

**Examples**

1. Any seminorm on \(E\).

2. The map \((\xi_1, \xi_2) \mapsto |\xi_1| \vee |\xi_2|^-\) \(((\xi_1, \xi_2) \in K^2)\) is a distinguishing norm on \(K^2\).

3. Let \(E\) be the space of all bounded \(K\)-valued functions on a set \(X\).

Then

\[
\|f\|_\infty := \sup\{|f(x)| : x \in X\} \quad (f \in E),
\]

where the supremum is taken in \(V\), defines a distinguishing norm on \(E\).

Observe that for \(f \in E\) we have \(\|f\|_\infty \leq 1\) if and only if \(|f(x)| \leq 1\) for
all \( x \in E \) but

\[ \| f \|_\infty < 1 \iff \| f \|_\infty \leq 1^- \iff |f(x)| < 1 \text{ for all } x \in E \]

so that \( \| f \|_\infty = 1 \) if and only if \( 1 = \max\{|f(x)| : x \in X\} \).

**PROPOSITION 3.** Let \( p \) be a distinguishing seminorm on \( E \). Then \( \pi \circ p \) is a seminorm on \( E \).

**Proof.** The statement follows directly from Proposition 1 (ii), (iii).

**DEFINITION 4.** Let \( A \subset E \) be absorbing and absolutely convex, and let \( q_A \) be its associated seminorm. The distinguishing seminorm \( p_A \), associated to \( A \) is

\[ p_A(x) := \begin{cases} 
q_A(x) & \text{if } q_A(x) = \min\{|\lambda| : \lambda \in K, x \in \lambda A\} \\
q_A(x) & \text{otherwise.}
\end{cases} \]

The following theorem shows that the associated distinguishing seminorm of an absorbing absolutely convex set \( A \) determines \( A \).

**THEOREM 5.** Let \( A \subset E \) be absorbing and absolutely convex. Let \( p_A \) and \( q_A \) be as in Definition 4. Then \( p_A \) is a distinguishing seminorm with \( \pi \circ p_A = q_A \). Further we have

\[ A = \{x \in E : p_A(x) < 1\} = \{x \in E : p_A(x) \leq 1^-\}. \]

**Proof.** Clearly \( \pi \circ p_A = q_A \). We first check the equality

\( A = \{x \in E : p_A(x) < 1\} \). Let \( p_A(x) < 1 \). If \( p_A(x) < 1^- \) then \( q_A(x) < 1 \), so \( x \in A \). If \( p_A(x) = 1^- \) then \( 1 \in \{ |\lambda| : \lambda \in K, x \in \lambda A \} \) so that \( x \in \mu A \) for some \( \mu \in K, |\mu| = 1 \). By absolute convexity, \( x \in \mu^{-1}\mu A = A \). Conversely, let \( x \in A \). Then \( q_A(x) \leq 1 \) so that \( p_A(x) \leq q_A(x) \leq 1 \). If \( p_A(x) = 1 \) then \( 1 = \inf\{|\lambda| : x \in \lambda A\} \) is not a minimum contradicting \( x \in A \). Hence,
Finally we prove the conditions (i), (ii), (iii) of Definition 2 for a distinguishing seminorm for $p = p_A$. We have $p_A(0) = 0^- = 0$. To prove (ii) we may assume $\lambda \neq 0$ and $p_A(x) \neq 0$. If $p_A(x) \in (0,\omega)$ then

$q_A(x) = \inf\{|t| : x \in \tau A\}$ is not a minimum. Then neither is

$q_A(\lambda x) = \inf\{|t| : \lambda x \in \tau A\}$ so that $p_A(\lambda x) = q_A(\lambda x) = |\lambda| q_A(x) = |\lambda| p_A(x)$.

If $p_A(x) \in (0,\omega)^-$ then $q_A(x) = \min\{|t| : x \in \tau A\}$. Then also

$q_A(\lambda x) = \min\{|t| : \lambda x \in \tau A\}$ so that

$p_A(\lambda x) = q_A(\lambda x)^- = (|\lambda| q_A(x))^- = |\lambda| q_A(x)^- = |\lambda| p_A(x)$. For the proof of the strong triangle inequality (iii) we may assume $p_A(x+y) \neq 0$. We distinguish two cases.

(i) $p_A(x+y) \in (0,\omega)^-$. By Proposition 1 (ii) and $\pi \circ p_A = q_A$, from

$q_A(x+y) \leq q_A(x) \vee q_A(y)$

we obtain $p_A(x+y) = q_A(x+y)^- \leq q_A(x)^- \vee q_A(y)^- \leq p_A(x) \vee p_A(y)$.

(ii) $p_A(x+y) \in (0,\omega)$. Then the valuation of $K$ is dense. Assume

$p_A(y) \leq p_A(x)$. Suppose $p_A(x+y) > p_A(x)$; we derive a contradiction. There is a $\lambda \in K$ such that $p_A(x+y) \geq |\lambda| > p_A(x)$. (In fact, if $p_A(x) \neq p_A(x+y)^-$ then the interval $(p_A(x), p_A(x+y))$ contains infinitely many elements of $|K|$, if $p_A(x) = p_A(x+y)^-$ then $p_A(x+y) \in |K|$ and we may choose $\lambda \in K$ such that $|\lambda| = p_A(x+y)$.) Then $p_A(\lambda^{-1} y) \leq p_A(\lambda^{-1} x) < 1$ so that $\lambda^{-1} y \in A$, $\lambda^{-1} x \in A$ and, by absolute convexity, also $\lambda^{-1}(x+y) \in A$ implying

$p_A(x+y) < |\lambda|$, a contradiction.

§4. Hahn-Banach Theorem

For $\alpha \in K$, $a \in V$ we write $B(\alpha, a) = \{ \lambda \in K : |\lambda - \alpha| \leq a \}$.

**Theorem 6.** (Hahn-Banach Theorem) Let $K$ be spherically complete, let $p$
be a distinguishing seminorm on a K-vector space E. Let D be a K-linear subspace of E, let f : D → K be a K-linear map satisfying |f(d)| ≤ p(d) (d ∈ D). Then f can be extended to a K-linear \( \overline{f} : E \rightarrow K \) such that

\[ |\overline{f}(x)| \leq p(x) \quad (x \in D). \]

Proof. (It consists of checking that replacing of a seminorm by a distinguishing seminorm does not harm the well-known proof.) A simple application of Zorn's Lemma reduces the problem to the case 

\[ E = \{\lambda x + d : \lambda \in K, d \in D\} \]

for some \( x \in E \setminus D \). We are done if we can choose \( \overline{f}(x) = \xi \in K \) such that

\[ |\lambda \xi + f(d)| \leq p(\lambda x + d) \quad (\lambda \in K, d \in D). \]

As this condition is satisfied for \( \lambda = 0 \) and all \( d \in D \) and since for \( \lambda \neq 0 \)

\[ p(\lambda x+d) = |\lambda| \ p(x+\lambda^{-1}d) \]

\[ |\lambda \xi + f(d)| = |\lambda| \ |\xi + f(\lambda^{-1}d)| \]

it suffices, by Proposition 1 (i) to produce a \( \xi \in K \) such that

\[ |\xi - f(d)| \leq p(x-d) \quad (d \in D), \]

i.e. we must have that

\[ \cap_{d \in D} B(f(d), p(x-d)) \neq \emptyset. \]

By spherical completeness it suffices to show that for any \( d_1, d_2 \in D \) we have \( B(f(d_1), p(x-d_1)) \cap B(f(d_2), p(x-d_2)) \neq \emptyset \). But this follows easily from \( |f(d_1) - f(d_2)| \leq p(d_1 - d_2) \leq p(d_1 - x) \lor p(x - d_2) \).

A typical application: Let \( \ell^\infty := \{(\xi_1, \xi_2, \ldots) : \xi_n \in K \text{ for all } n, \sup|\xi_n| < \infty\} \).

Let K be spherically complete. Then there is a linear function \( g : \ell^\infty \rightarrow K \)
of norm 1 such that (i) \( g((\xi_1, \xi_2, \ldots)) = \lim_{n \to \infty} \xi_n \) if \( \lim_{n \to \infty} \xi_n \) exists and
(ii) \( |g((\xi_1, \xi_2, \ldots))| < 1 \) if \( |\xi_n| < 1 \) for all \( n \in \mathbb{N} \).

(Proof: Choose in the above theorem \( D := c \) (the space of the convergent sequences), \( f((\xi_1, \xi_2, \ldots)) := \lim_{n \to \infty} ((\xi_1, \xi_2, \ldots) \in c) \), and \( p((\xi_1, \xi_2, \ldots)) := \sup |\xi_n| \), where the sup is taken in \( V \). Take \( g := \frac{1}{f} \).)

§5. Separation of convex sets

Throughout §5, let \( E \) be a locally convex space over \( K \). We shall need the following observation.

PROPOSITION 7. An open convex subset of \( E \) is closed.

Proof. Any convex set is a coset of an absolutely convex set.

An open absolutely convex set is the complement of a union of cosets.

Theorem 8. Let \( K \) be spherically complete. Let \( A \subset E \) be closed, absolutely convex and let \( x \in E \setminus A \). Then there exists an \( f \in E' \) such that \( |f(A)| < 1 \) and \( f(x) = 1 \).

Proof. There is an absolutely convex open neighbourhood \( U \) of 0 such that \( (x+U) \cap A = \emptyset \). Then \( U+A \) is absolutely convex, open, hence closed (Proposition 7). Further, \( x \notin U+A \). Thus, we may assume that \( A \) is open and closed. Then \( A \) is absorbing. Let \( p_A \) be the distinguishing seminorm of \( A \), let \( D := \{ \lambda x : \lambda \in K \} \) and define \( g : D \to K \) by \( g(\lambda x) := \lambda \) (\( \lambda \in K \)). Then \( g(x) = 1 \). Since

\[ A = \{ y \in E : p_A(y) < 1 \} \]

(Theorem 5) and \( x \notin A \) we have \( p_A(x) \geq 1 \) so that for \( \lambda \in K \)

\[ p_A(\lambda x) = |\lambda| p_A(x) \geq |\lambda| = |g(\lambda x)|, \text{ i.e. } |g| \leq p_A \text{ on } D. \]

By Theorem 6 \( g \) extends to a linear \( f : E \to K \) such that \( |f(y)| \leq p_A(y) \) for all \( y \in E \).
We have $f(x) = g(x) = 1$ and, for $y \in A$, $|f(y)| \leq p_A(y) < 1$. The continuity of $f$ follows from the continuity of $\pi \circ p_A$ and the inequality $|f| \leq \pi \circ p_A$.

**COROLLARY 9.** Let $K$ be spherically complete. Each closed convex set is weakly closed.

Let $A, B$ be convex subsets of $E$. If $f : E \to K$ is a linear function then $f(A)$ and $f(B)$ are convex in $K$. Hence, if $f(A) \cap f(B) = \emptyset$ then $\text{dist}(f(A), f(B)) > 0$. With this in mind the following definition is quite natural.

**DEFINITION 10.** Two convex subsets $A, B$ of $E$ are separated by an $f \in E'$ if $f(A) \cap f(B) = \emptyset$.

If $A$ and $B$ are separated by $f \in E'$ then, since $\text{dist}(f(A), f(B)) > 0$ there is an open convex neighbourhood $U$ of $0$ such that $(A+U) \cap B = \emptyset$ (if $E$ is a normed space this is equivalent to $\text{dist}(A,B) > 0$). To prove the converse we need spherical completeness.

**THEOREM 11.** Let $K$ be spherically complete. Let $A, B$ be convex subsets of $E$ and suppose there is an open convex neighbourhood $U$ of $A$ such that $(A+U) \cap B = \emptyset$ (observe that this condition is satisfied if $A$ is open).

Then $A$ and $B$ can be separated by some $f \in E'$.

**Proof.** We may assume that $A$ is open. Let $C := A - B$. Then $0 \not\in C$, and $C = U$ is open, convex. Choose $c \in -C$. Then $T := c + C$ is absolutely convex, open, hence closed and $c \not\in T$. By Theorem 8 there is an $f \in E'$ such that $f(c) = 1$, $|f(T)| < 1$. Thus, for each $a \in A$, $b \in B$ we have

$$1 > |f(c + a - b)| = |1 + f(a) - f(b)|.$$

It follows that $|f(a) - f(b)| = 1$ for all $a \in A$, $b \in B$. In particular, $f(A) \cap f(B) = \emptyset$. 
References.
