Distinguishing non-archimedean seminorms

by

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§1. Introduction and summary. (For terminology, see §2) In [2] the following separation theorem is proved. Let $K$ be spherically complete and let $E$ be a locally convex space over $K$. If $A \subset E$ is closed and absolutely convex and if $x \in E \setminus A$ then there is an $f \in E'$ such that

\[
(*) \quad |f(A)| \leq 1 , \quad f(x) = 1 .
\]

In order to obtain, 'real' separation one would prefer to have

\[
(**) \quad |f(A)| < 1 , \quad f(x) = 1
\]

rather than (*). However, with the techniques used in [2], it is not clear how to arrive at such a result for densely valued fields $K$. The main obstruction is the fact that for an open absolutely convex, absorbing set $A$ its associated seminorm $q_A$,

\[
q_A(x) := \inf\{|\lambda| : \lambda \in K : x \in \lambda A\} \quad (x \in E)
\]

does not determine $A$; one only has the rather 'vague' relation
(Example. The open and closed convex sets \( \{(\xi_1, \xi_2) \in \mathbb{K}^2 : |\xi_1| \leq 1, |\xi_2| \leq 1\} \), \( \{(\xi_1, \xi_2) \in \mathbb{K}^2 : |\xi_1| < 1, |\xi_2| < 1\} \) all have the associated (semi)norm \( (\xi_1, \xi_2) \mapsto \max(|\xi_1|, |\xi_2|) \).

In this paper we shall extend the notion of a seminorm by admitting a larger range yielding the notion of a distinguishing seminorm (Definition 2). We shall prove that each absolutely convex absorbing set can be written as \( \{x \in E : p(x) < 1\} \) for some distinguishing seminorm \( p \) (Theorem 5). Next, we shall prove a Hahn Banach theorem for linear functions majorized by distinguishing seminorms (Theorem 6) and shall obtain, as a corollary, a strong separation theorem (with \( \ast \ast \)) in place of \( \ast \)). Here the notion of distinguishing seminorm is used only in the proof, not in the formulation (Theorem 8).

Note. This separation theorem can (for Banach spaces) also be obtained as a corollary of [1], Theorem 6.21. However, one techniques differ very much from the ones used in the (clever) proof of [1]. Also, the notion of a distinguishing seminorm may very well yield new applications.

§2. Terminology

Throughout \( \mathbb{K} \) is a non-archimedean complete non-trivially valued field with valuation \( |\cdot|, |\cdot|_K := \{|\lambda| : \lambda \in K\} \). \( E \) is a vector space over \( K \).

A seminorm on \( E \) is a map \( q : E \to [0,\infty) \) satisfying (i) \( q(0) = 0 \),

(ii) \( q(\lambda x) = |\lambda|q(x) \) \( (x \in E, \lambda \in K) \),

(iii) \( q(x+y) \leq q(x) \vee q(y) \) \( (x, y \in E) \)

where \( \vee \) indicates 'maximum'.

A subset \( A \) of \( E \) is absolutely convex if it is a module over the ring \( \{\lambda \in K : |\lambda| \leq 1\} \), convex if it is an additive coset of an absolutely convex set, absorbing if \( \bigcup \lambda A = E \).
Each convex subset of \( K \) has the form \( \{ \lambda \in K : |\lambda - a| < r \} \) or \( \{ \lambda \in K : |\lambda - a| \leq r \} \) for some \( a \in K, r \in [0,\infty) \).

For a seminorm \( q \) the sets \( \{ x \in E : q(x) < 1 \} \) and \( \{ x \in E : q(x) \leq 1 \} \) are absolutely convex and absorbing. Conversely, for an absorbing, absolutely convex subset \( A \) of \( E \) the associated seminorm \( q_A \) defined by

\[
q_A(x) := \inf\{ |\lambda| : \lambda \in K : x \in \lambda A \} \quad (x \in E)
\]

is a seminorm satisfying

\[
\{ x \in E : q_A(x) < 1 \} \subset A \subset \{ x \in E : q_A(x) \leq 1 \}.
\]

\( K \) is spherically complete if for any collection \( C \) of convex subsets for which \( A, B \in C \Rightarrow A \cap B \neq \emptyset \) we have \( nC = \emptyset \). A locally convex space is a \( K \)-vector space \( E \) with a topology induced by a collection of seminorms. Its dual space is denoted \( E' \).

§3. Distinguishing seminorms

We enlarge the set \([0,\infty)\) by giving each positive real number \( a \) an immediate predecessor \( a^- \). Formally

\[
V := [0,\infty)^- \cup (0,\infty)^-
\]

where \((0,\infty)^- := \{ a^- : a \in (0,\infty) \}\) is a second copy of \((0,\infty)\). Further we define \( 0^- := 0 \). The formulas

\[
a^- < b \Leftrightarrow a \leq b \quad (a, b \in (0,\infty))
a^- < b^- \Leftrightarrow a < b^- \Leftrightarrow a < b \quad (a, b \in (0,\infty))
0 < a \quad (a \in V, a \neq 0)
\]

define a linear ordering \( < \) on \( V \) extending the usual ordering on \([0,\infty)\).
The projection map \( \pi : V \to [0,\infty) \) is defined by

\[
\pi(a) := a \quad (a \in [0,\infty)) \\
\pi(a^-) := a \quad (a \in [0,\infty)).
\]

Finally we extend the multiplication on \([0,\infty)\) to a multiplication

\([0,\infty) \times V \to V\) by requiring \(a \cdot b^- := (ab)^-\) for all \(a, b \in [0,\infty)\).

Proposition 1 collects some direct consequences from the definitions.

**PROPOSITION 1.**

(i) Let \(b, c \in V, b \leq c\). Then \(ab \leq ac\) for all \(a \in [0,\infty)\).

(ii) \(\pi(a \lor b) = \pi(a) \lor \pi(b)\) \((a, b \in V)\).

(iii) \(\pi(ab) = a \pi(b)\) \((a \in [0,\infty), b \in V)\).

**DEFINITION 2.** A distinguishing seminorm on \(E\) is a map \(p : E \to V\) satisfying

(i) \(p(0) = 0\)

(ii) \(p(\lambda x) = |\lambda| p(x)\) \((\lambda \in K, x \in E)\)

(iii) \(p(x+y) \leq p(x) \lor p(y)\) \((x, y \in E)\).

If, in addition, \(p(x) = 0\) implies \(x = 0\) then \(p\) is a distinguishing norm.

**Examples**

1. Any seminorm on \(E\).

2. The map \((\xi_1, \xi_2) \mapsto |\xi_1| \lor |\xi_2|^- ((\xi_1, \xi_2) \in K^2)\) is a distinguishing norm on \(K^2\).

3. Let \(E\) be the space of all bounded \(K\)-valued functions on a set \(X\).

Then

\[
\|f\|_\infty := \sup\{|f(x)| : x \in X\} \quad (f \in E),
\]

where the supremum is taken in \(V\), defines a distinguishing norm on \(E\).

Observe that for \(f \in E\) we have \(\|f\|_\infty \leq 1\) if and only if \(|f(x)| \leq 1\) for
all $x \in E$ but

$$\| f \|_{\infty} < 1 \iff \| f \|_{\infty} \leq 1 \iff |f(x)| < 1 \text{ for all } x \in E$$

so that $\| f \|_{\infty} = 1$ if and only if $1 = \max\{|f(x)| : x \in X\}$.

**PROPOSITION 3.** Let $p$ be a distinguishing seminorm on $E$. Then $p \circ p$ is a seminorm on $E$.

**Proof.** The statement follows directly from Proposition 1 (ii), (iii).

**DEFINITION 4.** Let $A \subset E$ be absorbing and absolutely convex, and let $q_A$ be its associated seminorm. The distinguishing seminorm $p_A$, associated to $A$ is

$$p_A(x) := \begin{cases} q_A(x) & \text{if } q_A(x) = \min\{|\lambda| : \lambda \in K, x \in \lambda A\} \\ q_A(x) & \text{otherwise.} \end{cases}$$

The following theorem shows that the associated distinguishing seminorm of an absorbing absolutely convex set $A$ determines $A$.

**THEOREM 5.** Let $A \subset E$ be absorbing and absolutely convex. Let $p_A$ and $q_A$ be as in Definition 4. Then $p_A$ is a distinguishing seminorm with $p \circ p_A = q_A$. Further we have

$$A = \{x \in E : p_A(x) < 1\} = \{x \in E : p_A(x) \leq 1\}.$$ 

**Proof.** Clearly $p \circ p_A = q_A$. We first check the equality

$A = \{x \in E : p_A(x) < 1\}$. Let $p_A(x) < 1$. If $p_A(x) < 1$ then $q_A(x) < 1$, so $x \in A$. If $p_A(x) = 1$ then $1 \in \{ |\lambda| : \lambda \in K, x \in \lambda A \}$ so that $x \in \mu A$ for some $\mu \in K$, $|\mu| = 1$. By absolute convexity, $x \in \mu^{-1} \mu A = A$. Conversely, let $x \in A$. Then $q_A(x) \leq 1$ so that $p_A(x) \leq q_A(x) \leq 1$. If $p_A(x) = 1$ then

$$1 = \inf\{|\lambda| : x \in \lambda A\} \text{ is not a minimum contradicting } x \in A. \text{ Hence,}$$
Finally we prove the conditions (i), (ii), (iii) of Definition 2 for a distinguishing seminorm for \( p = p_A \). We have \( p_A(0) = 0^- = 0 \). To prove (ii) we may assume \( \lambda \neq 0 \) and \( p_A(x) \neq 0 \). If \( p_A(x) \in (0,\infty) \) then

\[
q_A(x) = \inf\{|t| : x \in \tau A\} \text{ is not a minimum. Then neither is } q_A(\lambda x) = \inf\{|t| : \lambda x \in \tau A\} \text{ so that } p_A(\lambda x) = q_A(\lambda x) = |\lambda| q_A(x) = |\lambda| p_A(x).
\]

If \( p_A(x) \in (0,\infty)^- \) then \( q_A(x) = \min\{|t| : x \in \tau A\} \). Then also

\[
q_A(\lambda x) = \min\{|t| : \lambda x \in \tau A\} \text{ so that } p_A(\lambda x) = q_A(\lambda x^-) = (|\lambda| q_A(x))^- = |\lambda| q_A(x^-) = |\lambda| p_A(x). \text{ For the proof of the strong triangle inequality (iii) we may assume } p_A(x+y) \neq 0. \text{ We distinguish two cases.}
\]

(i) \( p_A(x+y) \in (0,\infty)^- \). By Proposition 1 (ii) and \( \pi \circ p_A = q_A \), from

\[
q_A(x+y) \leq q_A(x) \lor q_A(y)
\]

we obtain \( p_A(x+y) = q_A(x+y^-) \leq q_A(x^-) \lor q_A(y^-) \leq p_A(x) \lor p_A(y) \).

(ii) \( p_A(x+y) \in (0,\infty) \). Then the valuation of \( K \) is dense. Assume \( p_A(y) \leq p_A(x) \). Suppose \( p_A(x+y) > p_A(x) \); we derive a contradiction. There is a \( \lambda \in K \) such that \( p_A(x+y) \geq |\lambda| > p_A(x) \). (In fact, if \( p_A(x) \neq p_A(x+y)^- \) then the interval \((p_A(x), p_A(x+y))\) contains infinitely many elements of \( |K| \), if \( p_A(x) = p_A(x+y)^- \) then \( p_A(x+y) \in |K| \) and we may choose \( \lambda \in K \) such that \( |\lambda| = p_A(x+y) \).) Then \( p_A(\lambda^{-1} x) \leq p_A(\lambda^{-1} x^-) \) so that \( \lambda^{-1} x \in A \) and, by absolute convexity, also \( \lambda^{-1} (x+y) \in A \) implying

\[
p_A(x+y) < |\lambda|, \text{ a contradiction.}
\]

§4. Hahn-Banach Theorem

For \( a \in K \), \( a \in V \) we write \( B(a,a) = \{\lambda \in K : |\lambda - a| \leq a\} \).

**Theorem 6.** (Hahn-Banach Theorem) Let \( K \) be spherically complete, let \( p \)
be a distinguishing seminorm on a $K$-vector space $E$. Let $D$ be a $K$-linear
subspace of $E$, let $f : D \to K$ be a $K$-linear map satisfying $|f(d)| \leq p(d)$
($d \in D$). Then $f$ can be extended to a $K$-linear \( \overline{f} : E \to K \) such that
\[ |\overline{f}(x)| \leq p(x) \quad (x \in D). \]

**Proof.** (It consists of checking that replacing of a seminorm by a
distinguishing seminorm does not harm the well-known proof.) A simple
application of Zorn's Lemma reduces the problem to the case
$E = \{ \lambda x + d : \lambda \in K, d \in D \}$ for some $x \in E \setminus D$. We are done if we can
choose $\overline{f}(x) = \xi \in K$ such that
\[ |\lambda \xi + f(d)| \leq p(\lambda x + d) \quad (\lambda \in K, d \in D). \]

As this condition is satisfied for $\lambda = 0$ and all $d \in D$ and since for
$\lambda \neq 0$
\[
\begin{align*}
p(\lambda x + d) &= |\lambda| \ p(x + \lambda^{-1}d) \\
|\lambda \xi + f(d)| &= |\lambda| \ |\xi + f(\lambda^{-1}d)|
\end{align*}
\]
it suffices, by Proposition 1 (i) to produce a $\xi \in K$ such that
\[ |\xi - f(d)| \leq p(x-d) \quad (d \in D), \]
i.e. we must have that
\[
\bigcap_{d \in D} B(\overline{f}(d), p(x-d)) \neq \emptyset.
\]
By spherical completeness it suffices to show that for any $d_1, d_2 \in D$ we
have $B(\overline{f}(d_1), p(x-d_1)) \cap B(\overline{f}(d_2), p(x-d_2)) \neq \emptyset$. But this follows easily
from $|\overline{f}(d_1) - \overline{f}(d_2)| \leq p(d_1 - d_2) \leq p(d_1 - x) \lor p(x-d_2)$.

A typical application: let $\ell^\infty := \{ (\xi_1, \xi_2, \ldots) \in K^\infty \mid \xi_n \in K \text{ for all } n, \sup_n |\xi_n| < \infty \}$.

Let $K$ be spherically complete. Then there is a linear function $g : \ell^\infty \to K$
of norm 1 such that

(i) \( g((\xi_1, \xi_2, \ldots)) = \lim_{n \to \infty} \xi_n \) if \( \lim_{n \to \infty} \xi_n \) exists and

(ii) \( |g((\xi_1, \xi_2, \ldots))| < 1 \) if \( |\xi_n| < 1 \) for all \( n \in \mathbb{N} \).

(Proof: Choose in the above theorem \( D := c \) (the space of the convergent sequences), \( f((\xi_1, \xi_2, \ldots)) := \lim_{n \to \infty} \xi_n \) \((\xi_1, \xi_2, \ldots) \in c\), and

\[ p((\xi_1, \xi_2, \ldots)) := \sup |\xi_n|, \]

where the sup is taken in \( V \). Take \( g := \frac{f}{p} \).

§5. Separation of convex sets

Throughout §5, let \( E \) be a locally convex space over \( K \). We shall need the following observation.

PROPOSITION 7. An open convex subset of \( E \) is closed.

Proof. Any convex set is a coset of an absolutely convex set.

An open absolutely convex set is the complement of a union of cosets.

Theorem 8. Let \( K \) be spherically complete. Let \( A \subset E \) be closed, absolutely convex and let \( x \in E \setminus A \). Then there exists an \( f \in E' \) such that \( |f(A)| < 1 \) and \( f(x) = 1 \).

Proof. There is an absolutely convex open neighbourhood \( U \) of 0 such that \((x+U) \cap A = \emptyset \). Then \( U+A \) is absolutely convex, open, hence closed (Proposition 7). Further, \( x \notin U+A \). Thus, we may assume that \( A \) is open and closed. Then \( A \) is absorbing. Let \( p_A \) be the distinguishing seminorm of \( A \), let \( D := \{ \lambda x : \lambda \in K \} \) and define \( g : D \to K \) by \( g(\lambda x) := \lambda \) \((\lambda \in K) \). Then

\[ g(x) = 1. \]

Since

\[ A = \{ y \in E : p_A(y) < 1 \} \]

(Theorem 5) and \( x \notin A \) we have \( p_A(x) \geq 1 \) so that for \( \lambda \in K \),

\[ p_A(\lambda x) = |\lambda| p_A(x) \geq |\lambda| = |g(\lambda x)|, \]

i.e. \( |g| \leq p_A \) on \( D \). By Theorem 6 \( g \) extends to a linear \( f : E \to K \) such that \( |f(y)| \leq p_A(y) \) for all \( y \in E \).
We have \( f(x) = g(x) = 1 \) and, for \( y \in A \), \( |f(y)| \leq p_A(y) < 1 \). The continuity of \( f \) follows from the continuity of \( \pi \circ p_A \) and the inequality \( |f| \leq \pi \circ p_A \).

**COROLLARY 9.** Let \( K \) be spherically complete. Each closed convex set is weakly closed.

Let \( A, B \) be convex subsets of \( E \). If \( f : E \to K \) is a linear function then \( f(A) \) and \( f(B) \) are convex in \( K \). Hence, if \( f(A) \cap f(B) = \emptyset \) then \( \text{dist}(f(A), f(B)) > 0 \). With this in mind the following definition is quite natural.

**DEFINITION 10.** Two convex subsets \( A, B \) of \( E \) are separated by an \( f \in E' \) if \( f(A) \cap f(B) = \emptyset \).

If \( A \) and \( B \) are separated by \( f \in E' \) then, since \( \text{dist}(f(A), f(B)) > 0 \) there is an open convex neighbourhood \( U \) of 0 such that \( (A+U) \cap B = \emptyset \) (if \( E \) is a normed space this is equivalent to \( \text{dist}(A,B) > 0 \)). To prove the converse we need spherical completeness.

**THEOREM 11.** Let \( K \) be spherically complete. Let \( A, B \) be convex subsets of \( E \) and suppose there is an open convex neighbourhood \( U \) of \( A \) such that \( (A+U) \cap B = \emptyset \) (observe that this condition is satisfied if \( A \) is open).

Then \( A \) and \( B \) can be separated by some \( f \in E' \).

**Proof.** We may assume that \( A \) is open. Let \( C := A-B \). Then \( 0 \notin C \) and \( C = U \cdot (A-b) \) is open, convex. Choose \( c \in -C \). Then \( T := c + C \) is absolutely convex, open, hence closed and \( c \notin T \). By Theorem 8 there is an \( f \in E' \) such that \( f(c) = 1, \ |f(T)| < 1 \). Thus, for each \( a \in A, b \in B \) we have

\[ 1 > |f(c + a - b)| = |1 + f(a) - f(b)|. \]

It follows that \( |f(a) - f(b)| = 1 \) for all \( a \in A, b \in B \). In particular, \( f(A) \cap f(B) = \emptyset \).
References.
