Distinguishing non-archimedean seminorms

by

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§1. Introduction and summary. (For terminology, see §2) In [2] the following separation theorem is proved. Let $K$ be spherically complete and let $E$ be a locally convex space over $K$. If $A \subseteq E$ is closed and absolutely convex and if $x \in E \setminus A$ then there is an $f \in E'$ such that

\[ (*) \quad |f(A)| < 1 \quad , \quad f(x) = 1. \]

In order to obtain, 'real' separation one would prefer to have

\[ (**) \quad |f(A)| < 1 \quad , \quad f(x) = 1 \]

rather than (*). However, with the techniques used in [2], it is not clear how to arrive at such a result for densely valued fields $K$. The main obstruction is the fact that for an open absolutely convex, absorbing set $A$ its associated seminorm $q_A'$

\[ q_A'(x) := \inf \{|\lambda| : \lambda \in K : x \in \lambda A\} \quad (x \in E) \]

does not determine $A$; one only has the rather 'vague' relation
\{x \in E : q_A(x) < 1\} \subset A \subset \{x \in E : q_A(x) \leq 1\}.

(Example. The open and closed convex sets \{(\xi_1, \xi_2) \in K^2 : |\xi_1| \leq 1, |\xi_2| \leq 1\}, \{(\xi_1, \xi_2) \in K^2 : |\xi_1| \leq 1, |\xi_2| < 1\}, \{(\xi_1, \xi_2) \in K^2 : |\xi_1| < 1, |\xi_2| < 1\} all have the associated (semi)norm \( (\xi_1, \xi_2) \mapsto \max(|\xi_1|, |\xi_2|) \).

In this paper we shall extend the notion of a seminorm by admitting a larger range yielding the notion of a distinguishing seminorm (Definition 2). We shall prove that each absolutely convex absorbing set can be written as \( \{x \in E : p(x) < 1\} \) for some distinguishing seminorm \( p \) (Theorem 5). Next, we shall prove a Hahn Banach theorem for linear functions majorized by distinguishing seminorms (Theorem 6) and shall obtain, as a corollary, a strong separation theorem (with \((**)\) in place of \((*)\)). Here the notion of distinguishing seminorm is used only in the proof, not in the formulation (Theorem 8).

Note. This separation theorem can (for Banach spaces) also be obtained as a corollary of [1], Theorem 6.21. However, one techniques differ very much from the ones used in the (clever) proof of [1]. Also, the notion of a distinguishing seminorm may very well yield new applications.

§2. Terminology

Throughout \( K \) is a non-archimedean complete non-trivially valued field with valuation \(| |, |X| := \{ |\lambda| : \lambda \in K\} \). \( E \) is a vector space over \( K \).

A seminorm on \( E \) is a map \( q : E \to [0, \infty) \) satisfying \( i \) \( q(0) = 0 \), \( ii \) \( q(\lambda x) = |\lambda|q(x) \) (\( x \in E, \lambda \in K \)), \( iii \) \( q(x+y) \leq q(x) \vee q(y) \) (\( x,y \in E \)) where \( \vee \) indicates 'maximum'.

A subset \( A \) of \( E \) is absolutely convex if it is a module over the ring \( \{ \lambda \in K : |\lambda| \leq 1\} \), convex if it is an additive coset of an absolutely convex set, absorbing if \( \bigcup_{\lambda \in K} \lambda A = E \).
Each convex subset of $K$ has the form \( \{ \lambda \in K : |\lambda - a| < r \} \) or \( \{ \lambda \in K : |\lambda - a| \leq r \} \) for some $a \in K$, $r \in [0,\omega]$. 

For a seminorm $q$ the sets \( \{ x \in E : q(x) < 1 \} \) and \( \{ x \in E : q(x) \leq 1 \} \) are absolutely convex and absorbing. Conversely, for an absorbing, absolutely convex subset $A$ of $E$ the associated seminorm $q_A$ defined by 

\[
q_A(x) := \inf(\{ |\lambda| : \lambda \in K : x \in \lambda A \}) \quad (x \in E)
\]

is a seminorm satisfying 

\[
\{ x \in E : q_A(x) < 1 \} \subset A \subset \{ x \in E : q_A(x) \leq 1 \}.
\]

$K$ is spherically complete if for any collection $C$ of convex subsets for which $A, B \in C \Rightarrow A \cap B \neq \emptyset$ we have $nC = \emptyset$. A locally convex space is a $K$-vector space $E$ with a topology induced by a collection of seminorms. Its dual space is denoted $E'$. 

§3. Distinguishing seminorms

We enlarge the set $[0,\omega]$ by giving each positive real number $a$ an immediate predecessor $a^-$. Formally 

\[
V := [0,\omega) \cup (0,\omega)^-
\]

where $(0,\omega)^- := \{ a^- : a \in (0,\omega) \}$ is a second copy of $(0,\omega)$. Further we define $0^- := 0$. The formulas 

\[
\begin{align*}
a^- < b & \iff a \leq b \quad (a, b \in (0,\omega)) \\
a^- < b^- & \iff a < b^- \iff a < b \quad (a, b \in (0,\omega)) \\
0 < a & \quad (a \in V, a \neq 0)
\end{align*}
\]

define a linear ordering $<$ on $V$ extending the usual ordering on $[0,\omega)$. 

The projection map \( \pi : V \to [0,\infty) \) is defined by

\[
\pi(a) := a \quad (a \in [0,\infty))
\]

\[
\pi(a^-) := a \quad (a \in [0,\infty)).
\]

Finally we extend the multiplication on \([0,\infty)\) to a multiplication

\([0,\infty) \times V \to V\) by requiring \(a \cdot b^- := (ab)^-\) for all \(a, b \in [0,\infty)\).

Proposition 1 collects some direct consequences from the definitions.

**Proposition 1.**

(i) Let \(b, c \in V, b \leq c\). Then \(ab \leq ac\) for all \(a \in [0,\infty)\).

(ii) \(\pi(a \vee b) = \pi(a) \vee \pi(b)\) \((a, b \in V)\).

(iii) \(\pi(ab) = a\pi(b)\) \((a \in [0,\infty), b \in V)\).

**Definition 2.** A distinguishing seminorm on \(E\) is a map \(p : E \to V\) satisfying

(i) \(p(0) = 0\)

(ii) \(p(\lambda x) = |\lambda| p(x)\) \((\lambda \in K, x \in E)\)

(iii) \(p(x+y) \leq p(x) \vee p(y)\) \((x, y \in E)\).

If, in addition, \(p(x) = 0\) implies \(x = 0\) then \(p\) is a distinguishing norm.

**Examples**

1. Any seminorm on \(E\).

2. The map \((\xi_1, \xi_2) \mapsto |\xi_1| \vee |\xi_2|^- ((\xi_1, \xi_2) \in K^2)\) is a distinguishing norm on \(K^2\).

3. Let \(E\) be the space of all bounded \(K\)-valued functions on a set \(X\).

Then

\[
\|f\|_\infty := \sup\{|f(x)| : x \in X\} \quad (f \in E),
\]

where the supremum is taken in \(V\), defines a distinguishing norm on \(E\).

Observe that for \(f \in E\) we have \(\|f\|_\infty \leq 1\) if and only if \(|f(x)| \leq 1\) for
all \( x \in E \) but

\[
\| f \|_\infty < 1 \iff \| f \|_\infty \leq 1 \iff |f(x)| < 1 \text{ for all } x \in E
\]

so that \( \| f \|_\infty = 1 \) if and only if \( 1 = \max\{|f(x)| : x \in X\} \).

**PROPOSITION 3.** Let \( p \) be a distinguishing seminorm on \( E \). Then \( \pi \circ p \) is a seminorm on \( E \).

**Proof.** The statement follows directly from Proposition 1 (ii), (iii).

**DEFINITION 4.** Let \( A \subseteq E \) be absorbing and absolutely convex, and let \( q_A \) be its associated seminorm. The **distinguishing seminorm** \( p_A \), associated to \( A \) is

\[
p_A(x) := \begin{cases} 
q_A(x) & \text{if } q_A(x) = \min\{|\lambda| : \lambda \in K, x \in \lambda A\} \\
q_A(x) & \text{otherwise.}
\end{cases}
\]

The following theorem shows that the associated distinguishing seminorm of an absorbing absolutely convex set \( A \) determines \( A \).

**THEOREM 5.** Let \( A \subseteq E \) be absorbing and absolutely convex. Let \( p_A \) and \( q_A \) be as in Definition 4. Then \( p_A \) is a distinguishing seminorm with \( \pi \circ p_A = q_A \). Further we have

\[
A = \{ x \in E : p_A(x) < 1 \} = \{ x \in E : p_A(x) \leq 1 \}.
\]

**Proof.** Clearly \( \pi \circ p_A = q_A \). We first check the equality

\[
A = \{ x \in E : p_A(x) < 1 \}. \text{ Let } p_A(x) < 1. \text{ If } p_A(x) < 1 \text{ then } q_A(x) < 1, \text{ so } x \in A. \text{ If } p_A(x) = 1 \text{ then } 1 \in \{ |\lambda| : \lambda \in K, x \in \lambda A \} \text{ so that } x \in \mu A \text{ for some } \mu \in K, |\mu| = 1. \text{ By absolute convexity, } x \in \mu^{-1} \mu A = A. \text{ Conversely, let } x \in A. \text{ Then } q_A(x) \leq 1 \text{ so that } p_A(x) \leq q_A(x) \leq 1. \text{ If } p_A(x) = 1 \text{ then } 1 = \inf\{|\lambda| : x \in \lambda A\} \text{ is not a minimum contradicting } x \in A. \text{ Hence,}\
\]
Finally we prove the conditions (i), (ii), (iii) of Definition 2 for a distinguishing seminorm for $p = p_A$. We have $p_A(0) = 0^- = 0$. To prove (ii) we may assume $\lambda \neq 0$ and $p_A(x) \neq 0$. If $p_A(x) \in (0,\infty)$ then

$q_A(x) = \inf\{|\tau| : x \in \tau A\}$ is not a minimum. Then neither is

$q_A(\lambda x) = \inf\{|\tau| : \lambda x \in \tau A\}$ so that $p_A(\lambda x) = q_A(\lambda x) = |\lambda|q_A(x) = |\lambda|p_A(x)$.

If $p_A(x) \in (0,\infty)$ then $q_A(x) = \min\{|\tau| : x \in \tau A\}$. Then also

$q_A(\lambda x) = \min\{|\tau| : \lambda x \in \tau A\}$ so that

$p_A(\lambda x) = q_A(\lambda x)^- = (|\lambda|q_A(x))^-- = |\lambda|q_A(x)^-- = |\lambda|p_A(x)$. For the proof of the strong triangle inequality (iii) we may assume $p_A(x+y) \neq 0$. We distinguish two cases.

(i) $p_A(x+y) \in (0,\infty)^-$. By Proposition 1 (ii) and $\pi \circ p_A = q_A$, from

$q_A(x+y) \leq q_A(x) \lor q_A(y)$

we obtain $p_A(x+y) = q_A(x+y)^- \leq q_A(x)^- \lor q_A(y)^- \leq p_A(x) \lor p_A(y)$.

(ii) $p_A(x+y) \in (0,\infty)$. Then the valuation of $K$ is dense. Assume

$p_A(y) \leq p_A(x)$. Suppose $p_A(x+y) > p_A(x)$; we derive a contradiction. There is a $\lambda \in K$ such that $p_A(x+y) \geq |\lambda| > p_A(x)$. (In fact, if $p_A(x) \neq p_A(x+y)^-$ then the interval $(p_A(x), p_A(x+y))$ contains infinitely many elements of $|K|$, if $p_A(x) = p_A(x+y)^-$ then $p_A(x+y) \in |K|$ and we may choose $\lambda \in K$ such that $|\lambda| = p_A(x+y)$. Then $p_A(\lambda^{-1}y) \leq p_A(\lambda^{-1}x) < 1$ so that $\lambda^{-1}y \in A$, $\lambda^{-1}x \in A$ and, by absolute convexity, also $\lambda^{-1}(x+y) \in A$ implying $p_A(x+y) < |\lambda|$, a contradiction.

§4. Hahn-Banach Theorem

For $a \in K$, $a \in V$ we write $B(a,a) = \{\lambda \in K : |\lambda-a| \leq a\}$.

THEOREM 6. (Hahn-Banach Theorem) Let $K$ be spherically complete, let $p$
be a distinguishing seminorm on a $K$-vector space $E$. Let $D$ be a $K$-linear 
subspace of $E$, let $f : D \to K$ be a $K$-linear map satisfying $|f(d)| \leq p(d)$ 
(d $\in$ D). Then $f$ can be extended to a $K$-linear $\bar{f} : E \to K$ such that 
$|\bar{f}(x)| \leq p(x)$ (x $\in$ D).

**Proof.** (It consists of checking that replacing of a seminorm by a 
distinguishing seminorm does not harm the well-known proof.) A simple 
application of Zorn's Lemma reduces the problem to the case 
$E = \{\lambda x + d : \lambda \in K, d \in D\}$ for some $x \in E \setminus D$. We are done if we can 
choose $\bar{f}(x) = \xi \in K$ such that 

$$|\lambda \xi + f(d)| \leq p(\lambda x + d) \quad (\lambda \in K, d \in D).$$

As this condition is satisfied for $\lambda = 0$ and all $d \in D$ and since for 
$\lambda \neq 0$

$$p(\lambda x + d) = |\lambda| p(x + \lambda^{-1}d)$$

$$|\lambda \xi + f(d)| = |\lambda| |\xi + f(\lambda^{-1}d)|$$

it suffices, by Proposition 1 (i) to produce a $\xi \in K$ such that 

$$|\xi - f(d)| \leq p(x - d) \quad (d \in D),$$

i.e. we must have that

$$\bigcap_{d \in D} B(f(d), p(x - d)) \neq \emptyset.$$

By spherical completeness it suffices to show that for any $d_1, d_2 \in D$ we 
have $B(f(d_1), p(x - d_1)) \cap B(f(d_2), p(x - d_2)) \neq \emptyset$. But this follows easily 
from $|f(d_1) - f(d_2)| \leq p(d_1 - d_2) \leq p(d_1 - x) + p(x - d_2)$.

A typical application: let $l^\infty := \{(\xi_1, \xi_2, \ldots) : \xi_n \in K$ for all $n$, 
$
sup_{n} |\xi_n| < \infty\}$. 

Let $K$ be spherically complete. Then there is a linear function $g : l^\infty \to K$
of norm 1 such that (i) \( g((\xi_1, \xi_2, \ldots)) = \lim_{n \to \infty} \xi_n \) if \( \lim_{n \to \infty} \xi_n \) exists and (ii) \( |g((\xi_1, \xi_2, \ldots))| < 1 \) if \( |\xi_n| < 1 \) for all \( n \in \mathbb{N} \).

(Proof: Choose in the above theorem \( D := c \) (the space of the convergent sequences), \( f((\xi_1, \xi_2, \ldots)) := \lim_{n \to \infty} \xi_n \ ( (\xi_1, \xi_2, \ldots) \in c) \), and \( p((\xi_1, \xi_2, \ldots)) := \sup |\xi_n| \), where the sup is taken in \( V \). Take \( g := \frac{1}{f} \).)

§5. Separation of convex sets
Throughout §5, let \( E \) be a locally convex space over \( K \). We shall need the following observation.

PROPOSITION 7. An open convex subset of \( E \) is closed.
Proof. Any convex set is a coset of an absolutely convex set.
An open absolutely convex set is the complement of a union of cosets.

Theorem 8. Let \( K \) be spherically complete. Let \( A \subset E \) be closed, absolutely convex and let \( x \in E \backslash A \). Then there exists an \( f \in E' \) such that \( |f(A)| < 1 \) and \( f(x) = 1 \).

Proof. There is an absolutely convex open neighbourhood \( U \) of 0 such that \((x+U) \cap A = \emptyset \). Then \( U+A \) is absolutely convex, open, hence closed (Proposition 7). Further, \( x \notin U+A \). Thus, we may assume that \( A \) is open and closed. Then \( A \) is absorbing. Let \( p_A \) be the distinguishing seminorm of \( A \), let \( D := \{\lambda x : \lambda \in K\} \) and define \( g : D \to K \) by \( g(\lambda x) := \lambda \ (\lambda \in K) \). Then \( g(x) = 1 \). Since

\[ A = \{y \in E : p_A(y) < 1\} \]

(Theorem 5) and \( x \notin A \) we have \( p_A(x) \geq 1 \) so that for \( \lambda \in K \)

\[ p_A(\lambda x) = |\lambda| \ p_A(x) \geq |\lambda| = |g(\lambda x)|, \ i.e. \ |g| \leq p_A \text{ on } D. \] By Theorem 6 \( g \) extends to a linear \( f : E \to K \) such that \( |f(y)| \leq p_A(y) \) for all \( y \in E \).
We have \( f(x) = g(x) = 1 \) and, for \( y \in A \), \( |f(y)| \leq p_A(y) < 1 \). The continuity of \( f \) follows from the continuity of \( \pi \circ p_A \) and the inequality \( |f| \leq \pi \circ p_A \).

**COROLLARY 9.** Let \( K \) be spherically complete. Each closed convex set is weakly closed.

Let \( A, B \) be convex subsets of \( E \). If \( f : E \to K \) is a linear function then \( f(A) \) and \( f(B) \) are convex in \( K \). Hence, if \( f(A) \cap f(B) = \emptyset \) then \( \text{dist}(f(A), f(B)) > 0 \). With this in mind the following definition is quite natural.

**DEFINITION 10.** Two convex subsets \( A, B \) of \( E \) are separated by an \( f \in E' \) if \( f(A) \cap f(B) = \emptyset \).

If \( A \) and \( B \) are separated by \( f \in E' \) then, since \( \text{dist}(f(A), f(B)) > 0 \) there is an open convex neighbourhood \( U \) of \( 0 \) such that \( (A+U) \cap B = \emptyset \) (if \( E \) is a normed space this is equivalent to \( \text{dist}(A,B) > 0 \)). To prove the converse we need spherical completeness.

**THEOREM 11.** Let \( K \) be spherically complete. Let \( A, B \) be convex subsets of \( E \) and suppose there is an open convex neighbourhood \( U \) of \( A \) such that \( (A+U) \cap B = \emptyset \) (observe that this condition is satisfied if \( A \) is open).

Then \( A \) and \( B \) can be separated by some \( f \in E' \).

**Proof.** We may assume that \( A \) is open. Let \( C := A - B \). Then \( 0 \notin C \), and \( C = U \) (\( A-b \)) is open, convex. Choose \( c \in -C \). Then \( T := c + C \) is absolutely convex, open, hence closed and \( c \notin T \). By Theorem 8 there is an \( f \in E' \) such that \( f(c) = 1 \), \( |f(T)| < 1 \). Thus, for each \( a \in A, b \in B \) we have

\[ 1 > |f(c + a - b)| = |1 + f(a) - f(b)|. \]

It follows that \( |f(a) - f(b)| = 1 \) for all \( a \in A, b \in B \). In particular, \( f(A) \cap f(B) = \emptyset \).
References.
