§1. Introduction and summary. (For terminology, see §2) In [2] the following separation theorem is proved. Let $K$ be spherically complete and let $E$ be a locally convex space over $K$. If $A \subset E$ is closed and absolutely convex and if $x \in E \setminus A$ then there is an $f \in E'$ such that

\begin{equation}
(*) \quad |f(A)| \leq 1, \quad f(x) = 1.
\end{equation}

In order to obtain, 'real' separation one would prefer to have

\begin{equation}
(**) \quad |f(A)| < 1, \quad f(x) = 1
\end{equation}

rather than $(*)$. However, with the techniques used in [2], it is not clear how to arrive at such a result for densely valued fields $K$. The main obstruction is the fact that for an open absolutely convex, absorbing set $A$ its associated seminorm $q_A$,

$$q_A(x) := \inf\{|\lambda| : \lambda \in K : x \in \lambda A\} \quad (x \in E)$$

does not determine $A$; one only has the rather 'vague' relation
\{x \in E : q_A(x) < 1\} \subset A \subset \{x \in E : q_A(x) \leq 1\}.

(Example. The open and closed convex sets \{(\xi_1, \xi_2) \in K^2 : |\xi_1| \leq 1, |\xi_2| \leq 1\},
\{(\xi_1, \xi_2) \in K^2 : |\xi_1| \leq 1, |\xi_2| < 1\}, \{(\xi_1, \xi_2) \in K^2 : |\xi_1| < 1, |\xi_2| < 1\} all
have the associated (sem)norm \((\xi_1, \xi_2) \mapsto \max(|\xi_1|, |\xi_2|)\).

In this paper we shall extend the notion of a seminorm by admitting
a larger range yielding the notion of a distinguishing seminorm
(Definition 2). We shall prove that each absolutely convex absorbing set
can be written as \(\{x \in E : p(x) < 1\}\) for some distinguishing seminorm \(p\)
(Theorem 5). Next, we shall prove a Hahn Banach theorem for linear
functions majorized by distinguishing seminorms (Theorem 6) and shall
obtain, as a corollary, a strong separation theorem (with \((**)) in place
of \((*)\)). Here the notion of distinguishing seminorm is used only in the
proof, not in the formulation (Theorem 8).

Note. This separation theorem can (for Banach spaces) also be obtained
as a corollary of [1], Theorem 6.21. However, one techniques differ very
much from the ones used in the (clever) proof of [1]. Also, the notion
of a distinguishing seminorm may very well yield new applications.

§2. Terminology

Throughout \(K\) is a non-archimedean complete non-trivially valued field
with valuation \(|\cdot|, |X| := \{|\lambda| : \lambda \in X\}\). \(E\) is a vector space over \(K\).

A \textbf{seminorm} on \(E\) is a map \(q : E \to [0,\infty)\) satisfying (i) \(q(0) = 0\),
(ii) \(q(\lambda x) = |\lambda|q(x)\) (\(x \in E, \lambda \in K\)), (iii) \(q(x+y) \leq q(x) \vee q(y)\) (\(x,y \in E\))
where \(\vee\) indicates 'maximum'.

A subset \(A\) of \(E\) is \textbf{absolutely convex} if it is a module over the ring
\(\{\lambda \in K : |\lambda| \leq 1\}\), \textbf{convex} if it is an additive coset of an absolutely
convex set, \textbf{absorbing} if \(\bigcup_{\lambda \in K} \lambda A = E\).
Each convex subset of $K$ has the form \( \{ \lambda \in K : |\lambda - a| < r \} \) or 
\( \{ \lambda \in K : |\lambda - a| \leq r \} \) for some \( a \in K, r \in [0,\infty] \).

For a seminorm \( q \) the sets \( \{ x \in E : q(x) < 1 \} \) and \( \{ x \in E : q(x) \leq 1 \} \) are absolutely convex and absorbing. Conversely, for an absorbing, absolutely convex subset \( A \) of \( E \) the associated seminorm \( q_A \) defined by

\[
q_A(x) := \inf\{ |\lambda| : \lambda \in K : x \in \lambda A \} \quad (x \in E)
\]

is a seminorm satisfying

\[
\{ x \in E : q_A(x) < 1 \} \subset A \subset \{ x \in E : q_A(x) \leq 1 \}.
\]

\( K \) is spherically complete if for any collection \( C \) of convex subsets for which \( A, B \in C \Rightarrow A \cap B \neq \emptyset \) we have \( nC = \emptyset \). A locally convex space is a \( K \)-vector space \( E \) with a topology induced by a collection of seminorms. Its dual space is denoted \( E' \).

§3. Distinguishing seminorms

We enlarge the set \([0,\infty)\) by giving each positive real number \( a \) an immediate predecessor \( a^- \). Formally

\[
V := [0,\infty)^- \cup (0,\infty)^-
\]

where \((0,\infty)^- := \{ a^- : a \in (0,\infty) \}\) is a second copy of \((0,\infty)\). Further we define \( 0^- := 0 \). The formulas

\[
a^- < b \leftrightarrow a \leq b \quad \quad \quad (a,b \in (0,\infty))
\]

\[
a^- < b^- \leftrightarrow a < b^- \leftrightarrow a < b \quad \quad \quad (a,b \in (0,\infty))
\]

\[
0 < a \quad \quad \quad (a \in V, a \neq 0)
\]

define a linear ordering \(<\) on \( V \) extending the usual ordering on \([0,\infty)\).
The projection map $\pi : V \to [0,\infty)$ is defined by

$$
\pi(a) := a \quad (a \in [0,\infty))
$$

Finally we extend the multiplication on $[0,\infty)$ to a multiplication $[0,\infty) \times V \to V$ by requiring $a \cdot b^\sim := (ab)^\sim$ for all $a, b \in [0,\infty)$. Proposition 1 collects some direct consequences from the definitions.

**PROPOSITION 1.**

(i) Let $b, c \in V$, $b \leq c$. Then $ab \leq ac$ for all $a \in [0,\infty)$.  

(ii) $\pi(a \lor b) = \pi(a) \lor \pi(b) \quad (a, b \in V)$.  

(iii) $\pi(ab) = a\pi(b) \quad (a \in [0,\infty), b \in V)$.

**DEFINITION 2.** A distinguishing seminorm on $E$ is a map $p : E \to V$ satisfying

(i) $p(0) = 0$

(ii) $p(\lambda x) = |\lambda| \cdot p(x) \quad (\lambda \in K, \ x \in E)$

(iii) $p(x+y) \leq p(x) \lor p(y) \quad (x, y \in E)$.

If, in addition, $p(x) = 0$ implies $x = 0$ then $p$ is a distinguishing norm.

**Examples**

1. Any seminorm on $E$.

2. The map $(\xi_1, \xi_2) \mapsto |\xi_1| \lor |\xi_2|^{-1} ((\xi_1, \xi_2) \in K^2)$ is a distinguishing norm on $K^2$.

3. Let $E$ be the space of all bounded $K$-valued functions on a set $X$. Then

$$
\|f\|_\infty := \sup\{|f(x)| : x \in X\} \quad (f \in E),
$$

where the supremum is taken in $V$, defines a distinguishing norm on $E$.

Observe that for $f \in E$ we have $\|f\|_\infty \leq 1$ if and only if $|f(x)| \leq 1$ for
all \( x \in E \) but

\[
\| f \|_{\infty} < 1 \Leftrightarrow \| f \|_{\infty} \leq 1 \Leftrightarrow |f(x)| < 1 \text{ for all } x \in E
\]

so that \( \| f \|_{\infty} = 1 \) if and only if \( 1 = \max\{|f(x)| : x \in X\} \).

**Proposition 3.** Let \( p \) be a distinguishing seminorm on \( E \). Then \( \pi \circ p \) is a seminorm on \( E \).

**Proof.** The statement follows directly from Proposition 1 (ii), (iii).

**Definition 4.** Let \( A \subseteq E \) be absorbing and absolutely convex, and let \( q_A \) be its associated seminorm. The distinguishing seminorm \( p_A \), associated to \( A \) is

\[
p_A(x) := \begin{cases} 
q_A(x) & \text{if } q_A(x) = \min\{|\lambda| : \lambda \in K, x \in \lambda A\} \\
q_A(x) & \text{otherwise.}
\end{cases}
\]

The following theorem shows that the associated distinguishing seminorm of an absorbing absolutely convex set \( A \) determines \( A \).

**Theorem 5.** Let \( A \subseteq E \) be absorbing and absolutely convex. Let \( p_A \) and \( q_A \) be as in Definition 4. Then \( p_A \) is a distinguishing seminorm with \( \pi \circ p_A = q_A \). Further we have

\[
A = \{ x \in E : p_A(x) < 1 \} = \{ x \in E : p_A(x) \leq 1 \}.
\]

**Proof.** Clearly \( \pi \circ p_A = q_A \). We first check the equality

\[
A = \{ x \in E : p_A(x) < 1 \} \subseteq \{ x \in E : p_A(x) \leq 1 \}.
\]

Let \( p_A(x) < 1 \). If \( p_A(x) < 1 \) then \( q_A(x) < 1 \), so \( x \in A \). If \( p_A(x) = 1 \) then \( 1 \in \{ |\lambda| : \lambda \in K, x \in \lambda A \} \) so that \( x \in \mu A \) for some \( \mu \in K, |\mu| = 1 \). By absolute convexity, \( x \in \mu^{-1} \mu A = A \). Conversely, let \( x \in A \). Then \( q_A(x) \leq 1 \) so that \( p_A(x) \leq q_A(x) \leq 1 \). If \( p_A(x) = 1 \) then \( 1 = \inf\{ |\lambda| : x \in \lambda A \} \) is not a minimum contradicting \( x \in A \). Hence,
$p_A(x) < 1$. Finally we prove the conditions (i), (ii), (iii) of Definition 2 for a distinguishing seminorm for $p = p_A$. We have $p_A(0) = 0 = 0$. To prove (ii) we may assume $\lambda \neq 0$ and $p_A(x) \neq 0$. If $p_A(x) \in (0,\infty)$ then

$q_A(x) = \inf\{t : x \in tA\}$ is not a minimum. Then neither is

$q_A(\lambda x) = \inf\{t : \lambda x \in tA\}$ so that $p_A(\lambda x) = q_A(\lambda x) = |\lambda| q_A(x) = |\lambda| p_A(x)$.

If $p_A(x) \in (0,\infty)^-$ then $q_A(x) = \min\{|t| : x \in tA\}$. Then also

$q_A(\lambda x) = \min\{|t| : \lambda x \in tA\}$ so that

$p_A(\lambda x) = q_A(\lambda x)^- = (|\lambda| q_A(x))^-= |\lambda| q_A(x)^-= |\lambda| p_A(x)$. For the proof of the strong triangle inequality (iii) we may assume $p_A(x+y) \neq 0$. We distinguish two cases.

(i) $p_A(x+y) \in (0,\infty)^-$. By Proposition 1 (ii) and $\pi \circ p_A = q_A$, from

$q_A(x+y) \leq q_A(x) \vee q_A(y)$

we obtain $p_A(x+y) = q_A(x+y)^- \leq q_A(x)^- \vee q_A(y)^- \leq p_A(x) \vee p_A(y)$.

(ii) $p_A(x+y) \in (0,\infty)$. Then the valuation of $K$ is dense. Assume $p_A(y) \leq p_A(x)$. Suppose $p_A(x+y) > p_A(x)$; we derive a contradiction. There is a $\lambda \in K$ such that $p_A(x+y) \geq |\lambda| > p_A(x)$. (In fact, if $p_A(x) \neq p_A(x+y)^-$ then the interval $(p_A(x), p_A(x+y))$ contains infinitely many elements of $|K|$, if $p_A(x) = p_A(x+y)^-$ then $p_A(x+y) \in |K|$ and we may choose $\lambda \in K$ such that $|\lambda| = p_A(x+y)$. Then $p_A(\lambda^{-1}y) \leq p_A(\lambda^{-1}x) < 1$ so that $\lambda^{-1}y \in A$, $\lambda^{-1}x \in A$ and, by absolute convexity, also $\lambda^{-1}(x+y) \in A$ implying $p_A(x+y) < |\lambda|$, a contradiction.

§4. Hahn-Banach Theorem

For $a \in K$, $a \in V$ we write $B(a,a) = \{\lambda \in K : |\lambda-a| \leq a\}$. 

THEOREM 6. (Hahn-Banach Theorem) Let $K$ be spherically complete, let $p$
be a distinguishing seminorm on a \(K\)-vector space \(E\). Let \(D\) be a \(K\)-linear subspace of \(E\), let \(f : D \to K\) be a \(K\)-linear map satisfying \(|f(d)| \leq p(d)\) \((d \in D)\). Then \(f\) can be extended to a \(K\)-linear \(\overline{f} : E \to K\) such that \(|\overline{f}(x)| \leq p(x)\) \((x \in D)\).

**Proof.** (It consists of checking that replacing of a seminorm by a distinguishing seminorm does not harm the well-known proof.) A simple application of Zorn's Lemma reduces the problem to the case \(E = \{\lambda x + d : \lambda \in K, d \in D\}\) for some \(x \in E\setminus D\). We are done if we can choose \(\overline{f}(x) = \xi \in K\) such that

\[
|\lambda \xi + f(d)| \leq p(\lambda x + d) \quad (\lambda \in K, d \in D).
\]

As this condition is satisfied for \(\lambda = 0\) and all \(d \in D\) and since for \(\lambda \neq 0\)

\[
p(\lambda x + d) = |\lambda| \cdot p(x + \lambda^{-1}d)
\]

\[
|\lambda \xi + f(d)| = |\lambda| \cdot |\xi + f(\lambda^{-1}d)|
\]

it suffices, by Proposition 1 (i) to produce a \(\xi \in K\) such that

\[
|\xi - f(d)| \leq p(x - d) \quad (d \in D),
\]

i.e. we must have that

\[
\bigcap_{d \in D} B(f(d), p(x - d)) \neq \emptyset.
\]

By spherical completeness it suffices to show that for any \(d_1, d_2 \in D\) we have \(B(f(d_1), p(x - d_1)) \cap B(f(d_2), p(x - d_2)) \neq \emptyset\). But this follows easily from \(|f(d_1) - f(d_2)| \leq p(d_1 - d_2) \leq p(d_1 - x) \vee p(x - d_2)\).

A typical application: Let \(\ell^\infty := \{\xi_1, \xi_2, \ldots\} : \xi_n \in K\) for all \(n\), \(\sup |\xi_n| < \infty\).

Let \(K\) be spherically complete. Then there is a linear function \(g : \ell^\infty \to K\)
of norm 1 such that (i) $g((\xi_1, \xi_2, \ldots)) = \lim_{n \to \infty} \xi_n$ if $\lim_{n \to \infty} \xi_n$ exists and
(ii) $|g((\xi_1, \xi_2, \ldots))| < 1$ if $|\xi_n| < 1$ for all $n \in \mathbb{N}$.

(Proof: Choose in the above theorem $D := c$ (the space of the convergent sequences), $f((\xi_1, \xi_2, \ldots)) := \lim_{n \to \infty} ((\xi_1, \xi_2, \ldots) \in c)$, and
$p((\xi_1, \xi_2, \ldots)) := \sup |\xi_n|$, where the sup is taken in $V$. Take $g := \frac{1}{f}$.)

§5. Separation of convex sets

Throughout §5, let $E$ be a locally convex space over $K$. We shall need the following observation.

PROPOSITION 7. An open convex subset of $E$ is closed.

Proof. Any convex set is a coset of an absolutely convex set.

An open absolutely convex set is the complement of a union of cosets.

Theorem 8. Let $K$ be spherically complete. Let $A \subset E$ be closed, absolutely convex and let $x \in E \setminus A$. Then there exists an $f \in E'$ such that $|f(A)| < 1$ and $f(x) = 1$.

Proof. There is an absolutely convex open neighbourhood $U$ of 0 such that $(x+U) \cap A = \emptyset$. Then $U+A$ is absolutely convex, open, hence closed (Proposition 7). Further, $x \notin U+A$. Thus, we may assume that $A$ is open and closed. Then $A$ is absorbing. Let $p_A$ be the distinguishing seminorm of $A$, let $D := \{\lambda x : \lambda \in K\}$ and define $g : D \to K$ by $g(\lambda x) := \lambda$ ($\lambda \in K$). Then $g(x) = 1$. Since

$$A = \{y \in E : p_A(y) < 1\}$$

(Theorem 5) and $x \notin A$ we have $p_A(x) \geq 1$ so that for $\lambda \in K$

$$p_A(\lambda x) = |\lambda| p_A(x) \geq |\lambda| = |g(\lambda x)|,$$

i.e. $|g| \leq p_A$ on $D$. By Theorem 6 $g$ extends to a linear $f : E \to K$ such that $|f(y)| \leq p_A(y)$ for all $y \in E$. 

We have $f(x) = g(x) = 1$ and, for $y \in A$, $|f(y)| \leq p_A(y) < 1$. The continuity of $f$ follows from the continuity of $\pi \circ p_A$ and the inequality $|f| \leq \pi \circ p_A$.

**COROLLARY 9.** Let $K$ be spherically complete. Each closed convex set is weakly closed.

Let $A$, $B$ be convex subsets of $E$. If $f : E \to K$ is a linear function then $f(A)$ and $f(B)$ are convex in $K$. Hence, if $f(A) \cap f(B) = \emptyset$ then $\text{dist}(f(A), f(B)) > 0$. With this in mind the following definition is quite natural.

**DEFINITION 10.** Two convex subsets $A$, $B$ of $E$ are separated by an $f \in E'$ if $f(A) \cap f(B) = \emptyset$.

If $A$ and $B$ are separated by $f \in E'$ then, since $\text{dist}(f(A), f(B)) > 0$ there is an open convex neighbourhood $U$ of $0$ such that $(A+U) \cap B = \emptyset$ (if $E$ is a normed space this is equivalent to $\text{dist}(A,B) > 0$). To prove the converse we need spherical completeness.

**THEOREM 11.** Let $K$ be spherically complete. Let $A$, $B$ be convex subsets of $E$ and suppose there is an open convex neighbourhood $U$ of $A$ such that $(A+U) \cap B = \emptyset$ (observe that this condition is satisfied if $A$ is open).

Then $A$ and $B$ can be separated by some $f \in E'$.

**Proof.** We may assume that $A$ is open. Let $C := A-B$. Then $0 \not\in C$, and $C = U \setminus (A-b)$ is open, convex. Choose $c \in -C$. Then $T := c + C$ is absolutely convex, open, hence closed and $c \not\in T$. By Theorem 8 there is an $f \in E'$ such that $f(c) = 1$, $|f(T)| < 1$. Thus, for each $a \in A$, $b \in B$ we have

$$1 > |f(c + a - b)| = |1 + f(a) - f(b)|.$$

It follows that $|f(a) - f(b)| = 1$ for all $a \in A$, $b \in B$. In particular, $f(A) \cap f(B) = \emptyset$. 

References.
