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The purpose of this note is to prove the following theorem (for the definition of a $C^\infty$-function see below).

**Theorem.** - Let $K$ be a complete non-archimedean valued field with characteristic zero. Let $X$ be a nonempty subset of $K$ without isolated points and let $f : X \to K$ be a $C^\infty$-function. Then there is a $C^\infty$-function $X \to K$ whose derivative is $f$.

First we quote some definitions and statements from [1] which are needed for the proof. Let $K$ and $X$ be as above.

Definition ([1], p. 87-87). - Let $f : X \to K$. $f$ is differentiable if its derivative $a \mapsto f'(a) := \lim_{x \to a} (f(x) - f(a))$ $(a \in X)$ exists. For $n \in \mathbb{N}$, let $\nabla^1 X := \{(y_1, y_2, \ldots, y_n) \in X^n ; y_i \neq y_j \text{ whenever } i \neq j\}$. The difference quotients $\nabla_n f : \nabla^{n+1} X \to K$ $(n \in \{0, 1, 2, \ldots\})$ are given inductively by

$$\nabla_0 f := f$$

and

$$\nabla_n f(y_1, y_2, \ldots, y_{n+1}) := (y_1 - y_2)^{-1}(\nabla_{n-1} f(y_1, y_3, \ldots, y_{n-1}) - \nabla_{n-1} f(y_2, y_3, \ldots, y_{n+1}))$$

$$(y_1, y_2, \ldots, y_{n+1}) \in \nabla^{n+1} X, \ n \in \mathbb{N}.$$  

$f$ is a $C^n$-function $(f \in C^n(X \to K))$ if $\nabla_n f$ can (uniquely) be extended to a continuous function $\nabla_n^f : X^{n+1} \to K$.

$f$ is a $C^\infty$-function if $f \in C^\infty(X \to K) := \bigcap_{n=0}^{\infty} C^n(X \to K)$.

**Proposition** ([1], p. 78, 86, 87, 116 and 123). - Let $f : X \to K$. For each $n \in \mathbb{N}$ the function $\nabla_n f$ is symmetric, $C^{n+1}(X \to K) \supset C^n(X \to K)$, if $f \in C^n(X \to K)$ then $f' \in C^{n+1}(X \to K)$ and $\nabla_n f(a, a, \ldots, a) = f^{(n)}(a)/n!$ for each $a \in X$.

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\[
\lim_{x, y \to a} (x - y)^n (f(x) - f(y)) = 0 \quad \text{for each } a \in X \quad \text{then } f \in C^0(X \to K) \quad \text{and} \\
\text{and } f' = 0. \quad \text{(locally) analytic functions are } C^\infty \text{-functions}
\]

**Definition (\cite{1}, p. 45 and 46).** - Let \(0 < \rho < 1\). For each \(n \in \mathbb{N}\), let \(R_n\) be a full set of representatives in \(X\) of the equivalence relation given by
\[
|x - y| < \rho^n \quad (x, y \in X) \quad \text{such that } R_1 \subset R_2 \subset \ldots \quad \text{Choose } x_0 \in R_1. \quad \text{For each } x \in X, \quad n \in \mathbb{N}, \quad \text{let } x_n \text{ be determined by the conditions } x_n \in R_n, \quad |x - x_n| < \rho^n.
\]

**PROPOSITION (\cite{1} Th. 11.2).** - Let \(n \in \mathbb{N}\), \(f \in C^{n-1}(X \to K)\). Set
\[
P_n f(x) := \sum_{m=0}^{\infty} \sum_{j=0}^{n-1} \frac{f(j)(x)_m}{(j+1)!} (x_{m+1} - x_m)^{j+1} \quad (x \in X).
\]

Then \(P_n f\) is a \(C^n\)-antiderivative of \(f\).

**Proof of the theorem.** - We shall use the terminology of above.

Let \(j \in \{0, 1, 2, \ldots\}\), \(f(j)\) is continuous hence locally bounded and there exists a partition of \(X\) into "closed" balls \(B_{j_i}\) (relative to \(X\)) of radius \(< 1\) where \(i\) runs through some indexing set \(I_j\) such that \(f(j)\) is bounded on each \(B_{j_i}\). For each \(i \in I_j\), we can choose \(m_{j_i} \in \mathbb{N}\) such that (recall that \(0 < \rho < 1\)) \((*)\)
\[
\rho^{m_{j_i}} \leq d(B_{j_i}) < 1, \quad |f(j)(x)| \rho^{m_{j_i}} < |(j+1)\rho^j \quad (x \in B_{j_i}).
\]

Define \(F_j : X \to K\) as follows. If \(x \in X\), then \(x \in B_{j_i}\) for precisely one \(i \in I_j\). Set
\[
F_j(x) := \sum_{m=0}^{\infty} \frac{f(j)(x)_m}{(j+1)!} (x_{m+1} - x_m)^{j+1}.
\]

We shall prove that \(F := \sum_{j=0}^{\infty} F_j\) is a \(C^\infty\)-antiderivative of \(f\) by means of the following steps.

(i) Each \(F_j\) is well defined.

(ii) For each \(j \in \{0, 1, 2, \ldots\}\) and for all \(i \in I_j\),
\[
|F_j(x)| \leq m_{j_i}^j \quad (x \in B_{j_i})
\]
so that \(F\) is well defined.

(iii) \(\sum_{j=0}^{\infty} F_j\) is a \(C^n\)-antiderivative of \(f\) for each \(n \in \mathbb{N}\).

(iv) For each \(n\), \(\sum_{j=n+1}^{\infty} F_j\) is a \(C^n\)-function with zero derivative.

**Proof of (i).** - \(f(j)\) is bounded on \(B_{j_i}\), and \(\lim_{m \to \infty} (x_{m+1} - x_m) = 0\).

**Proof of (ii).** - Let \(x \in B_{j_i}\) and \(m \geq m_{j_i}\). Then by \((*)\),
\[
|x_{m+1} - x_m| \leq |x - x_m| \leq \rho^m \leq \rho^{m_{j_i}} \leq d(B_{j_i})
\]
from which it follows that $x_m \in B_{j i}$ and $|x_{m+1} - x_m| \leq \rho^{j i}$. Applying the second formula of (*) with $x$ replaced by $x_m$, we get

$$f(j)(x) \left(\frac{x_{m+1} - x_m}{j+1}\right) \leq \rho^{j i} \rho^{j i} \rho^{j i}(j+1) = \rho^{j i+j},$$

and (ii) is proved.

**Proof of (iii).** - The function $F_{j}$ and $x \mapsto \sum_{m=0}^{\infty} f(j)(x) (x_{m+1} - x_m) / (j+1)$ differ (on each $B_{j i}$, hence globally) by a locally constant function. Summation from $j=0$ to $j=n$ shows that $\sum_{j=0}^{n} F_{j} - F_{n+1} f$ is locally constant. By the second proposition

$$\sum_{j=0}^{n} F_{j} \in C^{n+1}(X \to K) \subset C^{n}(X \to K) \text{ and } (\sum_{j=0}^{n} F_{j})' = f.$$

**Proof of (iv).** - Set $H := \sum_{j=n+1}^{\infty} F_{j}$. We shall prove that $|H(x) - H(y)| \leq |x-y|^{n+1}$ for all $x, y \in X$ which, by the first proposition implies (iv). To obtain the inequality if suffices to prove

$$|F_{j}(x) - F_{j}(y)| \leq |x-y|^{n+1} \quad (x, y \in X \text{ for each } j \geq n+1).$$

We consider several cases.

(a) $x \in B_{j i}, y \in B_{j i}$, where $i \neq i'$. Then by (*),

$$|x-y| \geq d(B_{j i}) \geq \rho^{m_{j i}} \text{ so that } |x-y|^{n+1} \geq \rho^{m_{j i}(n+1)}.$$

By (ii),

$$|F_{j}(x)| \leq \rho^{j m_{j i}+j}.$$

As $j m_{j i} + j \geq (n+1) m_{j i}$, we have $|F_{j}(x)| \leq |x-y|^{n+1}$. By symmetry, $|F_{j}(y)| \leq |x-y|^{n+1}$, and (**) follows.

(b) There is $i$ such that $x, y \in B_{j i}$. We may assume $x \neq y$, there exists an $s \in \mathbb{N} \cup \{0\}$ such that (recall that $d(B_{j i}) < 1$)

$$\rho^{s+1} \leq |x-y| < \rho^{s}.$$

Then $|x-y|^{n+1} \geq \rho^{s(n+1)}$. Consider two subcases.

(b.1) $s < m_{j i}$. Then by (ii),

$$|F_{j}(x)| \leq \rho^{j m_{j i}+j}$$

and since $j m_{j i} + j \geq (n+1)(s+1) + j \geq (s+1)(n+1)$, we have $|F_{j}(x)| \leq |x-y|^{n+1}$. By symmetry $|F_{j}(y)| \leq |x-y|^{n+1}$ and (**) follows.

(b.2) $s \geq m_{j i}$. Then since $x_0 = y_0, \ldots, x_s = y_s$, we have, for $m=m_{j i}, \ldots, s-1$,
so that
\[ F_j(x) - F_j(y) = \sum_{m \geq s} \frac{f^{(j)}(x_m)}{(j+1)!} (x_{m+1} - x_m)^{j+1} - \sum_{m \geq s} \frac{f^{(j)}(y_m)}{(j+1)!} (y_{m+1} - y_m)^{j+1}. \]

If \( m \geq s \), we have by \((*)\) (observe that \( x_m \in B_{j_1} \))
\[
\left| \frac{f^{(j)}(x_m)}{(j+1)!} (x_{m+1} - x_m)^{j+1} \right| \leq \rho^{j-m_j+1+m(j+1)}
\]
\[
\left| \frac{f^{(j)}(y_m)}{(j+1)!} (y_{m+1} - y_m)^{j+1} \right| \leq \rho^{j-m_j+1+m(j+1)}
\]
and we find \( |F_j(x) - F_j(y)| \leq \rho^{j-m_j+s(j+1)} \). Using the fact that \( j \geq n+1 \) and our assumption \( s \geq m_j \), we obtain
\[
j - m_j + s(j + 1) = (s + 1) j + s - m_j \geq (s + 1)(n + 1). \]

By consequence
\[
|F_j(x) - F_j(y)| \leq \rho^{(s+1)(n+1)} \leq |x - y|^{n+1}
\]
which finishes the proof.

**Remark.** - The above construction does not give us a linear antiderivation map \( C^\alpha(X \to K) \to C^\alpha(X \to K) \), and it is somewhat doubtful whether there exists a linear antiderivation map \( P : C^\alpha(X \to K) \to C^\alpha(X \to K) \) that is continuous with respect to a natural locally convex topology ([1], p. 119) on \( C^\alpha(X \to K) \).

**REFERENCE**