The purpose of this note is to prove the following theorem (for the definition of a \( C^\infty \)-function see below).

**Theorem.** - Let \( K \) be a complete non-archimedean valued field with characteristic zero. Let \( X \) be a nonempty subset of \( K \) without isolated points and let \( f : X \to K \) be a \( C^\infty \)-function. Then there is a \( C^\infty \)-function \( X \to K \) whose derivative is \( f \).

First we quote some definitions and statements from [1] which are needed for the proof. Let \( K \) and \( X \) be as above.

**Definition ([1], p. 87 and 75).** - Let \( f : X \to K \) be differentiable if its derivative \( a \mapsto f'(a) := \lim_{x \to a} (f(x) - f(a)) (x \in X) \) exists. For \( n \in \mathbb{N} \), let \( \nabla^n X := \{(y_1, y_2, \ldots, y_n) \in X^n ; y_i \neq y_j \) whenever \( i \neq j \} \). The difference quotients \( \xi_n f : \nabla^{n+1} X \to K \) \((n \in \{0, 1, 2, \ldots\}) \) are given inductively by

\[
\xi_0 f := f
\]

and

\[
\xi_n f(y_1, y_2, \ldots, y_{n+1}) := (y_1 - y_2)^{-1}(\xi_{n-1} f(y_1, y_3, \ldots, y_{n-1}) - \xi_{n-1} f(y_2, y_3, \ldots, y_{n+1}))
\]

((\( y_1, y_2, \ldots, y_{n+1} \)) \in \nabla^{n+1} X, n \in \mathbb{N}) .

\( f \) is a \( C^n \)-function \((f \in C^n(X \to K)) \) if \( \xi_n f \) can (uniquely) be extended to a continuous function \( \xi_n f : X^{n+1} \to K \).

\( f \) is a \( C^\infty \)-function if \( f \in C^\infty(X \to K) := \cap_{n=0}^{\infty} C^k(X \to K) \).

**Proposition ([1], p. 78, 86, 87, 116 and 123).** - Let \( f : X \to K \). For each \( n \in \mathbb{N} \) the function \( \xi_n f \) is symmetric, \( C^{n+1}(X \to K) = C^n(X \to K) \), if \( f \in C^2(X \to K) \) then \( f' \in C^{n+1}(X \to K) \) and \( \xi_n f(a, a, \ldots, a) = f'(a)/n! \) for each \( a \in X \).

(\(*) Wilhelm H. SCHIKHOF, Mathematisch Instituut, Katholieke Universiteit, Toornooiveld, NIJMEGEN (Pays-Bas).
If \( \lim_{y \to x} (x - y)^{-n}(f(x) - f(y)) = 0 \) for each \( a \in X \) then \( f \in C^p(X \to K) \) and \( f' = 0 \cdot (\text{locally}) \) analytic functions are \( C^\infty \)-functions.

Definition ([1], p. 45 and 46). Let \( 0 < \rho < 1 \). For each \( n \in N \), let \( R_n \) be a full set of representatives in \( X \) of the equivalence relation given by
\[
|x - y| < \rho^n (x, y \in X) \quad \text{such that} \quad R_1 \subset R_2 \subset \ldots \quad \text{Choose} \quad x_0 \in R_1. \quad \text{For each} \quad x \in X, \quad n \in N, \quad \text{let} \quad x_n \quad \text{be determined by the conditions} \quad x_n \in R_n, \quad |x - x_n| < \rho^n.
\]

**PROPOSITION** ([1] Th. 11.2). Let \( n \in N, \quad f \in C^{n-1}(X \to K). \) Set
\[
P_n f(x) := \sum_{m=0}^{n-1} \frac{f^{(j)}(x)}{(j + 1)!} (x_{m+1} - x_x)^{j+1} \quad (x \in X).
\]
Then \( P_n f \) is a \( C^n \)-antiderivative of \( f \).

**Proof of the theorem.** We shall use the terminology of above.

Let \( j \in \{0, 1, 2, \ldots\} \). \( f^{(j)} \) is continuous hence locally bounded and there exists a partition of \( X \) into "closed" balls \( B_{j_1} \) (relative to \( X \)) of radius \( < 1 \) where \( i \) runs through some indexing set \( I_j \) such that \( f^{(j)} \) is bounded on each \( B_{j_1} \). For each \( i \in I_j \), we can choose \( m_{j_1} \in N \) such that (recall that \( 0 < \rho < 1 \))
\[
\rho^{m_{j_1}} \leq d(B_{j_1}) < 1, \quad |f^{(j)}(x)| \rho^{m_{j_1}} < |(j + 1)| \quad \rho^j \quad (x \in B_{j_1}).
\]

Define \( F_j : X \to K \) as follows. If \( x \in X \), then \( x \in B_{j_1} \) for precisely one \( i \in I_j \). Set
\[
F_j(x) := \sum_{m=0}^{m_{j_1}} \frac{f^{(j)}(x_m)}{(j + 1)!} (x_{m+1} - x_x)^{j+1}.
\]
We shall prove that \( F := \sum_{j=0}^{\infty} F_j \) is a \( C^n \)-antiderivative of \( f \) by means of the following steps.

(i) Each \( F_j \) is well defined.

(ii) For each \( j \in \{0, 1, 2, \ldots\} \) and for all \( i \in I_j \),
\[
|F_j(x)| \leq m_{j_1}^{j+1} \quad (x \in B_{j_1})
\]
so that \( F \) is well defined.

(iii) \( \sum_{j=0}^{\infty} F_j \) is a \( C^n \)-antiderivative of \( f \) for each \( n \in N \).

(iv) For each \( n, \sum_{j=n+1}^{\infty} F_j \) is a \( C^n \)-function with zero derivative.

**Proof of (i).** \( f^{(j)} \) is bounded on \( B_{j_1} \), and \( \lim_{m \to \infty} (x_{m+1} - x_m) = 0 \).

**Proof of (ii).** Let \( x \in B_{j_1} \) and \( m \geq m_{j_1} \). Then by \( (*) \),
\[
|x_{m+1} - x_m| \leq |x - x_m| \leq \rho^m \leq \rho^{m_{j_1}} \leq d(B_{j_1})
\]

from which it follows that \( x_m \in B_{j_1} \) and \( |x_{m+1} - x_m| \leq \rho^{m_{j_1}} \). Applying the second formula of (*) with \( x \) replaced by \( x_m \), we set

\[
\frac{f(j)(x_m)}{q} (x_{m+1} - x_m)^{j+1} \leq \rho^{j_1} \rho^{m_{j_1}} \rho^{m_{j_1}(j+1)} = \rho^{m_{j_1} + j},
\]

and (ii) is proved.

**Proof of (iii).** - The function \( F_j \) and \( x \mapsto \sum_{m=0}^{\infty} f(j)(x_m)(x_{m+1} - x_m)^{j+1}/(j+1)! \) differ (on each \( B_{j_1} \), hence globally) by a locally constant function. Summation from \( j = 0 \) to \( j = n \) shows that \( \sum_{j=0}^{n} F_j - F_{n+1} \) is locally constant. By the second proposition

\[
\sum_{j=0}^{n} F_j \in C^1(\mathbb{R} \to K) \quad \text{and} \quad (\sum_{j=0}^{n} F_j)' = f.
\]

**Proof of (iv).** - Set \( H := \sum_{j=n+1}^{\infty} F_j \). We shall prove that \( |H(x) - H(y)| \leq |x - y|^{n+1} \) for all \( x, y \in \mathbb{R} \) which, by the first proposition implies (iv). To obtain the inequality if suffices to prove

\[
|F_j(x) - F_j(y)| \leq |x - y|^{n+1} \quad (x, y \in \mathbb{R}) \quad \text{for each} \quad j \geq n+1.
\]

We consider several cases.

(a) \( x \in B_{j_1}, y \in B_{j_1}' \), where \( i \neq i' \). Then by (*),

\[
|x - y| \geq d(B_{j_1}) \geq \rho^{m_{j_1}} \quad \text{so that} \quad |x - y|^{n+1} \geq \rho^{m_{j_1}(n+1)}.
\]

By (ii),

\[
|F_j(x)| \leq \rho^{m_{j_1} + j}.
\]

As \( jm_{j_1} + j \geq (n+1)m_{j_1} \), we have \( |F_j(x)| \leq |x - y|^{n+1} \). By symmetry, \( |F_j(y)| \leq |x - y|^{n+1} \), and (**) follows.

(b) There is \( i \) such that \( x, y \in B_{j_1} \). We may assume \( x \neq y \), there exists an \( s \in \mathbb{N} \cup \{0\} \) such that (recall that \( d(B_{j_1}) < 1 \))

\[
\rho^{s+1} \leq |x - y| < \rho^s.
\]

Then \( |x - y|^{n+1} \geq \rho^{s+1}(n+1) \). Consider two subcases.

(b.1) \( s < m_{j_1} \). Then by (ii),

\[
|F_j(x)| \leq \rho^{m_{j_1} + j}
\]

and since \( jm_{j_1} + j \geq (n+1)(s+1) + j \geq (s+1)(n+1) \), we have \( |F_j(x)| \leq |x - y|^{n+1} \). By symmetry \( |F_j(y)| \leq |x - y|^{n+1} \) and (**) follows.

(b.2) \( s \geq m_{j_1} \). Then since \( x_0 = y_0 , \ldots , x_s = y_s \), we have, for \( m = m_{j_1}, \ldots , s-1, \)
\[
\frac{f^{(j)}(x_m)}{(j+1)!} (x_{m+1} - x_m)^{j+1} = \frac{f^{(j)}(y_m)}{(j+1)!} (y_{m+1} - y_m)^{j+1}
\]

so that

\[
F_j(x) - F_j(y) = \sum_{m \geq s} \frac{f^{(j)}(x_m)}{(j+1)!} (x_{m+1} - x_m)^{j+1} - \sum_{m \geq s} \frac{f^{(j)}(y_m)}{(j+1)!} (y_{m+1} - y_m)^{j+1}.
\]

If \( m \geq s \), we have by (*) (observe that \( x, y \in B_{ji} \))

\[
|\frac{f^{(j)}(x_m)}{(j+1)!} (x_{m+1} - x_m)^{j+1}| \leq \rho^j m_{ji} + m(j+1)
\]

and

\[
|\frac{f^{(j)}(y_m)}{(j+1)!} (y_{m+1} - y_m)^{j+1}| \leq \rho^j m_{ji} + m(j+1)
\]

and we find \( |F_j(x) - F_j(y)| \leq \rho^j m_{ji} + s(j+1) \). Using the fact that \( j \geq n+1 \) and our assumption \( s \geq m_{ji} \), we obtain

\[
j = m_{ji} + s(j+1) = (s+1) j + s - m_{ji} \geq (s+1)(n+1).
\]

By consequence

\[
|F_j(x) - F_j(y)| \leq \rho^{(s+1)(n+1)} \leq |x - y|^{n+1}
\]

which finishes the proof.

**Remark.** - The above construction does not give us a linear antiderivation map \( C^0(X \to K) \to C^0(X \to K) \), and it is somewhat doubtful whether there exists a linear antiderivation map \( P : C^0(X \to K) \to C^0(X \to K) \) that is continuous with respect to a natural locally convex topology ([1], p. 119) on \( C^0(X \to K) \).

**Reference**