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The purpose of this note is to prove the following theorem (for the definition of a $C^\infty$-function see below).

**THEOREM.** - Let $K$ be a complete non-archimedean valued field with characteristic zero. Let $X$ be a nonempty subset of $K$ without isolated points and let $f : X \longrightarrow K$ be a $C^\infty$-function. Then there is a $C^\infty$-function $X \longrightarrow K$ whose derivative is $f$.

First we quote some definitions and statements from [1] which are needed for the proof. Let $K$ and $X$ be as above.

**Definition ([1], p. 8 & 75).** - Let $f : X \longrightarrow K$. $f$ is differentiable if its derivative $a \mapsto f'(a) := \lim_{x \to a} (x - a)^{-1} (f(x) - f(a))$ $(a \in X)$ exists. For $n \in \mathbb{N}$, let $\nabla^n X := \{(y_1, y_2, \ldots, y_n) \in X^n ; y_i \neq y_j \text{ whenever } i \neq j \}$. The difference quotients $\nabla_n f : \nabla^{n+1} X \longrightarrow K$ $(n \in \{0, 1, 2, \ldots\})$ are given inductively by

$$\nabla_0 f := f$$

and

$$\nabla_n f(y_1, y_2, \ldots, y_{n+1}) := (y_1 - y_2)^{-1}(\nabla_{n-1} f(y_1, y_3, \ldots, y_{n-1}) - \nabla_{n-1} f(y_2, y_3, \ldots, y_{n+1}))$$

$$\nabla^n f(y_1, y_2, \ldots, y_{n+1}) \in \nabla^{n+1} X , \ n \in \mathbb{N} \} .$$

$f$ is a $C^n$-function $(f \in C^n(X \longrightarrow K))$ if $\nabla_n f$ can (uniquely) be extended to a continuous function $\nabla^n f : X^{n+1} \longrightarrow K$.

$f$ is a $C^\infty$-function if $f \in C^\infty(X \longrightarrow K) := \cap_{n=0}^\infty C^n(X \longrightarrow K)$.

**Proposition ([1], p. 78, 86, 87, 113 and 123).** - Let $f : X \longrightarrow K$. For each $n \in \mathbb{N}$ the function $\nabla_n f$ is symmetric, $C^{n+1}(X \longrightarrow K) \supset C^n(X \longrightarrow K)$, if $f \in C^n(X \longrightarrow K)$ then $f' \in C^{n+1}(X \longrightarrow K)$ and $\nabla_n f(a, a, \ldots, a) = f'(a)/n!$ for each $a \in X$.

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If \( \lim_{y \to x} (x - y)^{-n}(f(x) - f(y)) = 0 \) for each \( a \in X \) then \( f \in C^n(X \to K) \) and 
\[
f' = 0. \quad \text{(locally) analytic functions are } C^\infty \text{-functions}
\]

**Definition ([1], p. 45 and 46).** Let \( 0 < \rho < 1 \). For each \( n \in \mathbb{N} \), let \( R_n \) be a full set of representatives in \( X \) of the equivalence relation given by 
\[
|x - y| < \rho^n \quad (x, y \in X)
\]

such that \( R_1 \subseteq R_2 \subseteq \ldots \). Choose \( x_0 \in R_1 \) for each \( x \in X \), \( n \in \mathbb{N} \), let \( x_n \) be determined by the conditions \( x_n \in R_n \), \( |x - x_n| < \rho^n \).

**PROPOSITION ([1] Th. 11.2).** Let \( n \in \mathbb{N} \), \( f \in C^{n+1}(X \to K) \). Set 
\[
P_n f(x) := \sum_{m=0}^n x^{n-m} \frac{f^{(m)}(x)}{(m+1)!} (x_{m+1} - x_m)^{j+1} \quad (x \in X).
\]

Then \( P_n f \) is a \( C^n \)-antiderivative of \( f \).

**Proof of the theorem.** We shall use the terminology of above.

Let \( j \in \{0, 1, 2, \ldots\} \). \( f^{(j)} \) is continuous hence locally bounded and there exists a partition of \( X \) into "closed" balls \( B_{j,i} \) (relative to \( X \)) of radius \( < 1 \) where \( i \) runs through some indexing set \( I_j \) such that \( f^{(j)} \) is bounded on each \( B_{j,i} \). For each \( i \in I_j \), we can choose \( m_{j,i} \in \mathbb{N} \) such that (recall that \( 0 < \rho < 1 \))
\[
(*) \quad \rho m_{j,i} \leq d(B_{j,i}) < 1, \quad |f^{(j)}(x)| \rho^{m_{j,i}} < |(j + 1)^{j+1} \rho^j \quad (x \in B_{j,i}).
\]

Define \( F_j : X \to K \) as follows. If \( x \in X \), then \( x \in B_{j,i} \) for precisely one \( i \in I_j \). Set 
\[
F_j(x) := \sum_{m=0}^n f^{(m)}(x_{m+1} - x_m)^{j+1} \quad (x \in X).
\]

We shall prove that \( F := \sum_{j=0}^\infty F_j \) is a \( C^n \)-antiderivative of \( f \) by means of the following steps.

(i) Each \( F_j \) is well defined.

(ii) For each \( j \in \{0, 1, 2, \ldots\} \) and for all \( i \in I_j \), 
\[
|F_j(x)| \leq \rho^{m_{j,i}+1} \quad (x \in B_{j,i})
\]

so that \( F \) is well defined.

(iii) \( \sum_{j=0}^n F_j \) is a \( C^n \)-antiderivative of \( f \) for each \( n \in \mathbb{N} \).

(iv) For each \( n \), \( \sum_{j=n+1}^\infty F_j \) is a \( C^n \)-function with zero derivative.

**Proof of (i).** \( f^{(j)} \) is bounded on \( B_{j,i} \), and \( \lim_{m \to \infty} (x_{m+1} - x_m) = 0 \).

**Proof of (ii).** Let \( x \in B_{j,i} \) and \( m \geq m_{j,i} \). Then by (*) , 
\[
|x_{m+1} - x_m| \leq |x - x_m| \leq \rho^m \leq \rho^{m_{j,i}} \leq d(B_{j,i})
\]
from which it follows that \( x_m \in B_{ji} \) and \( |x_{m+1} - x_m| \leq \rho^{m_j} \). Applying the second formula of (*) with \( x \) replaced by \( x_m \), we get

\[
\frac{f(j)(x_m)}{f(j+1)} \cdot (x_{m+1} - x_m)^{j+1} \leq \rho^{m_j} \cdot \rho^{m_j(j+1)} = \rho^{m_jj},
\]

and (ii) is proved.

**Proof of (iii).** - The function \( F_j \) and \( x \mapsto \sum_{m=0}^{\infty} f(j)(x_m)(x_{m+1} - x_m)^{j+1}/(j+1) \) differ (on each \( B_{ji} \), hence globally) by a locally constant function. Summation from \( j = 0 \) to \( j = n \) shows that \( \sum_{j=0}^{n} F_j - F_{n+1} f \) is locally constant. By the second proof

\[
\sum_{j=0}^{n} F_j \in C^\infty(X \to \mathbb{R}) < C^\infty(X \to \mathbb{R}) \text{ and } (2n_{j=0} F_j)' = f.
\]

**Proof of (iv).** - Set \( H := \sum_{j=n}^{\infty} F_j \). We shall prove that \( |H(x)-H(y)| \leq |x-y|^{n+1} \) for all \( x, y \in X \) which, by the first proposition implies (iv). To obtain the inequality if suffices to prove

\[
(*) |F_j(x) - F_j(y)| \leq |x - y|^{n+1} \quad (x, y \in X) \text{ for each } j \geq n + 1.
\]

We consider several cases.

(a) \( x \in B_{ji}, \ y \in B_{j'i} \), where \( i \neq i' \). Then by (*),

\[
|x - y| \geq d(B_{ji}) \geq \rho^{m_j} \text{ so that } |x - y|^{n+1} \geq \rho^{m_j(n+1)}.
\]

By (ii),

\[
|F_j(x)| \leq \rho^{m_jj}.
\]

As \( jm_{ji} + j \geq (n+1)m_{ji} \), we have \( |F_j(x)| \leq |x - y|^{n+1} \). By symmetry, \( |F_j(y)| \leq |x - y|^{n+1} \), and (*) follows.

(b) There is \( i \) such that \( x, y \in B_{ji} \). We may assume \( x \neq y \), there exists an \( s \in N \cup \{0\} \) such that (recall that \( d(B_{ji}) < 1 \))

\[
\rho^s \leq |x - y| < \rho^s.
\]

Then \( |x - y|^{n+1} \geq \rho^{(s+1)(n+1)} \). Consider two subcases.

(b.1) \( s < m_{ji} \). Then by (ii),

\[
|F_j(x)| \leq \rho^{m_{ji} + j}
\]

and since \( jm_{ji} + j \geq (n+1)(s+1) + j \geq (s+1)(n+1) \), we have \( |F_j(x)| \leq |x - y|^{n+1} \). By symmetry, \( |F_j(y)| \leq |x - y|^{n+1} \) and \( (*) \) follows.

(b.2) \( s \geq m_{ji} \). Then since \( x_0 = y_0, \ldots, x_s = y_s \), we have, for \( m=m_{ji}, \ldots, s-1, \)
\[
\frac{f^{(j)}(x_m)}{(j+1)!} (x_{m+1} - x_m)^{j+1} = \frac{f^{(j)}(y_m)}{(j+1)!} (y_{m+1} - y_m)^{j+1}
\]
so that

\[
F_j(x) - F_j(y) = \sum_{m \geq s} \frac{f^{(j)}(x_m)}{(j+1)!} (x_{m+1} - x_m)^{j+1} - \sum_{m \geq s} \frac{f^{(j)}(y_m)}{(j+1)!} (y_{m+1} - y_m)^{j+1}.
\]

If \( m \geq s \), we have by (*) (observe that \( x_m \in B_{j1} \))

\[
\left| \frac{f^{(j)}(x_m)}{(j+1)!} (x_{m+1} - x_m)^{j+1} \right| \leq \rho^{j-m_{j1}+m(j+1)}
\]

\[
\left| \frac{f^{(j)}(y_m)}{(j+1)!} (y_{m+1} - y_m)^{j+1} \right| \leq \rho^{j-m_{j1}+m(j+1)}
\]

and we find \( |F_j(x) - F_j(y)| \leq \rho^{j-m_{j1}+s(j+1)} \). Using the fact that \( j \geq n + 1 \) and our assumption \( s \geq m_{j1} \), we obtain

\[
j - m_{j1} + s(j + 1) = (s + 1) j + s - m_{j1} \geq (s + 1)(n + 1).
\]

By consequence

\[
|F_j(x) - F_j(y)| \leq \rho^{(s+1)(n+1)} \leq |x - y|^{n+1}
\]

which finishes the proof.

Remark. - The above construction does not give us a linear antiderivation map \( C^0(X \rightarrow K) \rightarrow C^0(X \rightarrow K) \), and it is somewhat doubtful whether there exists a linear antiderivation map \( P : C^0(X \rightarrow K) \rightarrow C^0(X \rightarrow K) \) that is continuous with respect to a natural locally convex topology ([1], p. 119) on \( C^0(X \rightarrow K) \).

REFERENCES