The purpose of this note is to prove the following theorem (for the definition of a C°-function see below).

**THEOREM.** - Let $K$ be a complete non-archimedean valued field with characteristic zero. Let $X$ be a nonempty subset of $K$ without isolated points and let $f : X \to K$ be a $C^n$-function. Then there is a $C^n$-function $X \to K$ whose derivative is $f$.

First we quote some definitions and statements from [1] which are needed for the proof. Let $K$ and $X$ be as above.

**Definition ([1], p. 8-75).** - Let $f : X \to K$. $f$ is **differentiable** if its derivative $a \mapsto f'(a) := \lim_{x \to a} (x - a)^{-1} (f(x) - f(a))$ (a $\in X$) exists. For $n \in \mathbb{N}$, let $\nu^n X := \{(y_1, y_2, \ldots, y_n) \in X^n; y_i \neq y_j$ whenever $i \neq j\}$. The difference quotients $\xi_n f : \nu^{n+1} X \to K$ ($n \in \{0, 1, 2, \ldots\}$) are given inductively by

$$\xi_0 f := f$$

and

$$\xi_n f(y_1, y_2, \ldots, y_{n+1}) := (y_1 - y_2)^{-1} (\xi_{n-1} f(y_1, y_3, \ldots, y_{n-1}) - \xi_{n-1} f(y_2, y_3, \ldots, y_{n+1}))$$

$$((y_1, y_2, \ldots, y_{n+1}) \in \nu^{n+1} X, n \in \mathbb{N}).$$

$f$ is a $\mathcal{C}^n$-function ($f \in \mathcal{C}^n(X \to K)$) if $\xi_n f$ can (uniquely) be extended to a **continuous function** $\xi_n f : X^{n+1} \to K$.

$f$ is a $\mathcal{C}^\infty$-function if $f \in \mathcal{C}^\infty(X \to K) := \bigcap_{n=0}^{\infty} \mathcal{C}^n(X \to K)$.

**Proposition ([1], p. 78, 86, 87, 116 and 123).** - Let $f : X \to K$. For each $n \in \mathbb{N}$ the function $\xi_n f$ is symmetric, $\mathcal{C}^{n-1}(X \to K) \supset \mathcal{C}^n(X \to K)$, if $f \in \mathcal{C}^n(X \to K)$ then $f' \in \mathcal{C}^{n-1}(X \to K)$ and $\xi_n f(a, a, \ldots, a) = f^{(n)}(a)/n!$ for each $a \in X$,

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If \( \lim_{(x, y) \to (a, b)} (f(x) - f(y)) = 0 \) for each \( a \in X \) then \( f \in C^1(X \to K) \) and \( f' = 0 \). (Locally) analytic functions are \( C^\infty \)-functions.

**Definition** ([1], p. 45 and 46). - Let \( 0 < \rho < 1 \). For each \( n \in \mathbb{N} \), let \( R_n \) be a full set of representatives in \( X \) of the equivalence relation given by \( |x - y| < \rho^n \) (\( x, y \in X \)) such that \( R_1 \subset R_2 \subset \ldots \). Choose \( x_0 \in R_1 \). For each \( x \in X \), \( n \in \mathbb{N} \), let \( x_n \) be determined by the conditions \( x_n \in R_n \), \( |x - x_n| < \rho^n \).

**PROPOSITION** ([1] Th. 11.2). - Let \( n \in \mathbb{N} \), \( f \in C^{n-1}(X \to K) \). Set
\[
P_n f(x) := \sum_{m=0}^{n} \left( \frac{f^{(j)}(x)}{(j + 1)!} (x_{m+1} - x_m)^{j+1} \right) (x \in X).
\]

Then \( P_n f \) is a \( C^n \)-antiderivative of \( f \).

**Proof of the theorem.** - We shall use the terminology of above.

Let \( j \in \{0, 1, 2, \ldots\ \} \). \( f^{(j)} \) is continuous hence locally bounded and there exists a partition of \( X \) into "closed" balls \( B_{j1} \) (relative to \( X \)) of radius \( < 1 \) where \( i \) runs through some indexing set \( I_j \) such that \( f^{(j)} \) is bounded on each \( B_{j1} \). For each \( i \in I_j \), we can choose \( m_{j1} \in \mathbb{N} \) such that (recall that \( 0 < \rho < 1 \))
\[
(*) \quad \rho^{m_{j1}} \leq d(B_{j1}) < 1, \quad |f^{(j)}(x)| \rho^{m_{j1}} < |(j + 1)i| \rho^j \quad (x \in B_{j1}).
\]

Define \( F_j : X \to K \) as follows. If \( x \in X \), then \( x \in B_{j1} \) for precisely one \( i \in I_j \). Set
\[
F_j(x) := \sum_{m_{j1}}^{m} \frac{f^{(j)}(x)}{(j + 1)!} (x_{m+1} - x_m)^{j+1} \quad (x \in X).
\]

We shall prove that \( F := \sum_{j=0}^{\infty} F_j \) is a \( C^\infty \)-antiderivative of \( f \) by means of the following steps.

(i) Each \( F_j \) is well defined.

(ii) For each \( j \in \{0, 1, 2, \ldots\ \} \) and for all \( i \in I_j \),
\[
|F_j(x)| \leq \rho^{m_{j1}+j} \quad (x \in B_{j1})
\]
so that \( F \) is well defined.

(iii) \( \sum_{j=0}^{n} F_j \) is a \( C^n \)-antiderivative of \( f \) for each \( n \in \mathbb{N} \).

(iv) For each \( n \), \( \sum_{j=n+1}^{\infty} F_j \) is a \( C^n \)-function with zero derivative.

**Proof of (i).** - \( f^{(j)} \) is bounded on \( B_{j1} \), and \( \lim_{m \to \infty} (x_{m+1} - x_m) = 0 \).

**Proof of (ii).** - Let \( x \in B_{j1} \) and \( m > m_{j1} \). Then by (*) ,
\[
|x_{m+1} - x_m| \leq |x - x_m| \leq \rho^m \leq \rho^{m_{j1}} \leq d(B_{j1})
\]
from which it follows that \( x_m \in B_{ji} \) and \( |x_{m+1} - x_m| \leq \rho^{m+1} \). Applying the second formula of \((\ast)\) with \( x \) replaced by \( x_m \), we get

\[
\left| \frac{f^*(x)}{(j+1)!} (x_{m+1} - x_m)^{j+1} \right| \leq \rho^j \rho^{m+1} \rho^{m+1} (j+1) = \rho^{m+1} \]

and \((\ast\ast)\) is proved.

Proof of (iii). - The function \( F_j \) and \( x \mapsto \sum_{m=0}^{\infty} \frac{f^*(x)(x_{m+1} - x_m)^{j+1}}{(j+1)!} \) differ (on each \( B_{ji} \), hence globally) by a locally constant function. Summation from \( j = 0 \) to \( j = n \) shows that \( \sum_{j=0}^{n} F_j - F_{n+1} \) is locally constant. By the second proposition

\[
\sum_{j=0}^{n} F_j \in C^0(X \to \mathbb{K}) \quad \text{and} \quad \left( \sum_{j=0}^{n} F_j \right)' = f.
\]

Proof of (iv). - Set \( H := \sum_{j=n+1}^{\infty} F_j \). We shall prove that \( |H(x) - H(y)| \leq |x - y|^{n+1} \) for all \( x, y \in X \) which, by the first proposition implies (iv). To obtain the inequality if suffices to prove

\[
(*) \quad |F_j(x) - F_j(y)| \leq |x - y|^{n+1} \quad (x, y \in X) \quad \text{for each} \quad j \geq n + 1.
\]

We consider several cases.

(a) \( x \in B_{ji} \), \( y \in B_{j'i} \), where \( i \neq i' \). Then by \((\ast)\),

\[
|x - y| \geq d(B_{ji}) \geq \rho^{m_{ji}} \quad \text{so that} \quad |x - y|^{n+1} \geq \rho^{m_{ji}}(n+1).
\]

By (ii),

\[
|F_j(x)| \leq \rho^{m_{ji}+j}.
\]

As \( jm_{ji} + j \geq (n+1)m_{ji} \), we have \( |F_j(x)| \leq |x - y|^{n+1} \). By symmetry,

\[
|F_j(y)| \leq |x - y|^{n+1}, \quad \text{and} \quad (*) \quad \text{follows}.
\]

(b) There is \( i \) such that \( x \), \( y \in B_{ji} \). We may assume \( x \neq y \), there exists an \( s \in \mathbb{N} \cup \{0\} \) such that (recall that \( d(B_{ji}) < \rho \))

\[
\rho^{s+1} \leq |x - y| < \rho^s.
\]

Then \( |x - y|^{n+1} \geq \rho^{(s+1)(n+1)} \). Consider two subcases.

(b.1) \( s < m_{ji} \). Then by (ii),

\[
|F_j(x)| \leq \rho^{m_{ji}+j}
\]

and since \( jm_{ji} + j \geq (n+1)(s+1) + j \geq (s+1)(n+1) \), we have \( |F_j(x)| \leq |x - y|^{n+1} \). By symmetry \( |F_j(y)| \leq |x - y|^{n+1} \) and \((**)\) follows.

(b.2) \( s \geq m_{ji} \). Then since \( x_0 = y_0, \ldots, x_s = y_s \), we have, for \( m = m_{ji}, \ldots, s-1, \ldots, n+1 \),
\[ \frac{f^{(j)}}{(j+1)!} (x^{m+1}_m - x^m_m)^{j+1} = \frac{f^{(j)}}{(j+1)!} (y^{m+1}_m - y^m_m)^{j+1} \]

so that

\[ F_j(x) - F_j(y) = \sum_{m \geq s} \frac{f^{(j)}}{(j+1)!} (x^{m+1}_m - x^m_m)^{j+1} - \sum_{m \geq s} \frac{f^{(j)}}{(j+1)!} (y^{m+1}_m - y^m_m)^{j+1} . \]

If \( m \geq s \), we have by \((*)\) (observe that \( x_m \in B_j \))

\[ \left| \frac{f^{(j)}}{(j+1)!} (x^{m+1}_m - x^m_m)^{j+1} \right| \leq \rho^{j-m_j+1} \]

\[ \left| \frac{f^{(j)}}{(j+1)!} (y^{m+1}_m - y^m_m)^{j+1} \right| \leq \rho^{j-m_j+1} \]

and we find \( |F_j(x) - F_j(y)| \leq \rho^{j-m_j+s(j+1)} \) . Using the fact that \( j \geq n+1 \) and our assumption \( s \geq m_j \), we obtain

\[ j - m_j + s(j+1) = (s+1) \ j + s - m_j \geq (s+1)(n+1) . \]

By consequence

\[ |F_j(x) - F_j(y)| \leq \rho^{(s+1)(n+1)} \leq |x - y|^{n+1} \]

which finishes the proof.

Remark. - The above construction does not give us a linear antiderivation map \( C^0(X \rightarrow K) \rightarrow C^0(X \rightarrow K) \), and it is somewhat doubtful whether there exists a linear antiderivation map \( P : C^0(X \rightarrow K) \rightarrow C^0(X \rightarrow K) \) that is continuous with respect to a natural locally convex topology ([1], p. 119) on \( C^0(X \rightarrow K) \).

REFERENCE