C°-ANTIDERIVATIVES OF \( p \)-ADIC \( C^\infty \)-FUNCTIONS

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The purpose of this note is to prove the following theorem (for the definition of a \( C^\infty \)-function see below).

**THEOREM.** Let \( K \) be a complete non-archimedean valued field with characteristic zero. Let \( X \) be a nonempty subset of \( K \) without isolated points and let \( f : X \to K \) be a \( C^\infty \)-function. Then there is a \( C^\infty \)-function \( X \to K \) whose derivative is \( f \).

First we quote some definitions and statements from [1] which are needed for the proof. Let \( K \) and \( X \) be as above.

**Definition ([1], p. 8 175).** Let \( f : X \to K \). \( f \) is differentiable if its derivative \( a \mapsto f'(a) := \lim_{x \to a} (f(x) - f(a)) \) \((a \in X)\) exists. For \( n \in \mathbb{N} \), let \( \nabla^n X := \{(y_1, y_2, \ldots, y_n) \in X^n; \ y_i \neq y_j \text{ whenever } i \neq j \} \). The difference quotients \( \nabla_n f : \nabla^{n+1} X \to K \) \((n \in \{0, 1, 2, \ldots\})\) are given inductively by

\[ \nabla_0 f := f \]

and

\[ \nabla_n f(y_1, y_2, \ldots, y_{n+1}) := (y_1 - y_2)^{-1}(\nabla_{n-1} f(y_1, y_3, \ldots, y_{n-1}) - \nabla_{n-1} f(y_2, y_3, \ldots, y_{n+1})) \]

\((y_1, y_2, \ldots, y_{n+1}) \in \nabla^{n+1} X, \ n \in \mathbb{N} \).

\( f \) is a \( C^n \)-function \((f \in C^n(X \to K))\) if \( \nabla_n f \) can (uniquely) be extended to a continuous function \( \nabla_n f : X^{n+1} \to K \).

\( f \) is a \( C^\infty \)-function if \( f \in C^\infty(X \to K) := \bigcap_{n=0}^{\infty} C^n(X \to K) \).

**PROPOSITION ([1], p. 78, 96, 37, 116 and 123).** Let \( f : X \to K \). For each \( n \in \mathbb{N} \) the function \( \nabla_n f \) is symmetric, \( C^{n+1}(X \to K) \supset C^n(X \to K) \), if \( f \in C^0(X \to K) \) then \( f' \in C^{n+1}(X \to K) \) and \( \nabla_n f(a, a, \ldots, a) = f'(a)/n! \) for each \( a \in X \).

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If \( \lim_{x \to y} a(x - y)^n (f(x) - f(y)) = 0 \) for each \( a \in \mathbb{R} \) then \( f \in \mathcal{C}^n(X \to \mathbb{R}) \) and \( f' = 0 \). \((\text{locally})\) analytic functions are \( \mathcal{C}^\infty \)-functions.

**Definition** ([1], p. 45 and 46). - Let \( 0 < \rho < 1 \). For each \( n \in \mathbb{N} \), let \( R_n \) be a full set of representatives in \( X \) of the equivalence relation given by

\[
|x - y| < \rho^n \quad \text{for} \quad (x, y \in X)
\]

such that \( R_1 \subset R_2 \subset \ldots \). Choose \( x_0 \in R_1 \). For each \( x \in X \), \( n \in \mathbb{N} \), let \( x_n \) be determined by the conditions \( x_n \in R_n \), \( |x - x_n| < \rho^n \).

**PROPOSITION** ([1] Th. 11.2). - Let \( n \in \mathbb{N} \), \( f \in \mathcal{C}^{n-1}(X \to \mathbb{R}) \). Set

\[
P_n f(x) := \sum_{m=0}^{\infty} \frac{f^{(j)}(x)^{m}}{(j + 1)!} (x_{m+1} - x_m)^{j+1} \quad (x \in X).
\]

Then \( P_n f \) is a \( \mathcal{C}^n \)-antiderivative of \( f \).

**Proof of the theorem**. - We shall use the terminology of above.

Let \( j \in \{0, 1, 2, \ldots\} \). \( f^{(j)} \) is continuous hence locally bounded and there exists a partition of \( X \) into "closed" balls \( B_{ji} \) (relative to \( X \)) of radius \( \rho < 1 \) where \( i \) runs through some indexing set \( I_j \) such that \( f^{(j)} \) is bounded on each \( B_{ji} \). For each \( i \in I_j \), we can choose \( m_{ji} \in \mathbb{N} \) such that (recall that \( 0 < \rho < 1 \))

\[
(*) \quad \rho^{m_{ji}} \leq d(B_{ji}) < 1, \quad |f^{(j)}(x)| \rho^{m_{ji}} < |(j + 1)!| \rho^j \quad (x \in B_{ji}).
\]

Define \( F_j : X \to \mathbb{R} \) as follows. If \( x \in X \), then \( x \in B_{ji} \) for precisely one \( i \in I_j \). Set

\[
F_j(x) := \sum_{m=0}^{\infty} f^{(j)}(x_m) (x_{m+1} - x_m)^{j+1}.
\]

We shall prove that \( F := \sum_{j=0}^{\infty} F_j \) is a \( \mathcal{C}^\infty \)-antiderivative of \( f \) by means of the following steps.

(i) Each \( F_j \) is well defined.

(ii) For each \( j \in \{0, 1, 2, \ldots\} \) and for all \( i \in I_j \),

\[
|F_j(x)| \leq \rho^{m_{ji}} \quad (x \in B_{ji})
\]

so that \( F \) is well defined.

(iii) \( \sum_{j=0}^{\infty} F_j \) is a \( \mathcal{C}^{-n} \)-antiderivative of \( f \) for each \( n \in \mathbb{N} \).

(iv) For each \( n \), \( \sum_{j=n+1}^{\infty} F_j \) is a \( \mathcal{C}^n \)-function with zero derivative.

**Proof of (i).** - \( f^{(j)} \) is bounded on \( B_{ji} \), and \( \lim_{m \to \infty} (x_{m+1} - x_m) = 0 \).

**Proof of (ii).** - Let \( x \in B_{ji} \) and \( m > m_{ji} \). Then by (*) ,

\[
|x_{m+1} - x_m| \leq |x - x_m| \leq \rho^m \leq \rho^{m_{ji}} \leq d(B_{ji}).
\]
from which it follows that \( x_m \in B_{ji} \) and \( |x_{m+1} - x_m| \leq \rho_{ji}^m \). Applying the second formula of (*) with \( x \) replaced by \( x_m \), we get

\[
\frac{f(j)(x_m)}{\rho_{ji}^{j+1}} |x_{m+1} - x_m|^j \leq \rho_{ji}^m |x_{m+1} - x_m|^j = \rho_{ji}^{m+j},
\]

and (ii) is proved.

**Proof of (iii).** - The function \( F_j \) and \( x \to \sum_{m=0}^{\infty} f(j)(x_m)(x_{m+1} - x_m)^{j+1} / (j+1) \) differ (on each \( B_{ji} \), hence globally) by a locally constant function. Summation from \( j = 0 \) to \( j = n \) shows that \( \sum_{j=0}^{n} F_j - F_{n+1} \) is locally constant. By the second proposition

\[
\sum_{j=0}^{n} F_j \in C^n(X \to K) \quad \text{and} \quad (\sum_{j=0}^{n} F_j)' = f.
\]

**Proof of (iv).** - Set \( H := \sum_{j=n+1}^{\infty} F_j \). We shall prove that \( |H(x) - H(y)| \leq |x - y|^{n+1} \) for all \( x, y \in X \), which, by the first proposition implies (iv). To obtain the inequality if it suffices to prove

\[
(\ast\ast) \quad |F_j(x) - F_j(y)| \leq |x - y|^{n+1} \quad (x, y \in X) \quad \text{for each} \quad j \geq n + 1.
\]

We consider several cases.

(a) \( x \in B_{ji}, \; y \in B_{ji'}, \) where \( i \neq i' \). Then by (*),

\[
|x - y| \geq d(B_{ji}) \geq \rho_{ji}^m \quad \text{so that} \quad |x - y|^{n+1} \geq \rho_{ji}^{m(n+1)}.
\]

By (ii),

\[
|F_j(x)| \leq \rho_{ji}^{j|ji|+j}.
\]

As \( jm_{ji} + j \geq (n + 1) m_{ji} \), we have \( |F_j(x)| \leq |x - y|^{n+1} \). By symmetry, \( |F_j(y)| \leq |x - y|^{n+1} \), and \( (\ast\ast) \) follows.

(b) There is \( i \) such that \( x, y \in B_{ji} \). We may assume \( x \neq y \), there exists an \( s \in \mathbb{N} \cup \{0\} \) such that (recall that \( d(B_{ji}) < 1 \))

\[
\rho^{s+1} \leq |x - y| < \rho^s.
\]

Then \( |x - y|^{n+1} \geq \rho^{(s+1)(n+1)} \). Consider two subcases.

(b.1) \( s < m_{ji} \). Then by (ii),

\[
|F_j(x)| \leq \rho_{ji}^{j|m_{ji}|+j}
\]

and since \( jm_{ji} + j \geq (n + 1)(s + 1) + j \geq (s + 1)(n + 1) \), we have \( |F_j(x)| \leq |x - y|^{n+1} \). By symmetry \( |F_j(y)| \leq |x - y|^{n+1} \) and \( (\ast\ast) \) follows.

(b.2) \( s \geq m_{ji} \). Then since \( x_0 = y_0, \cdots, x_s = y_s \), we have, for \( m = m_{ji}, \cdots, s-1, \)
\[
\frac{f^{(j)}(x^*)}{(j+1)!} (x^*_{m+1} - x^*_m)^{j+1} = \frac{f^{(j)}(y^*)}{(j+1)!} (y^*_{m+1} - y^*_m)^{j+1}
\]

so that

\[
F_j(x) - F_j(y) = \sum_{m \geq s} \frac{f^{(j)}(x^*_m)}{(j+1)!} (x^*_{m+1} - x^*_m)^{j+1} - \sum_{m \geq s} \frac{f^{(j)}(y^*_m)}{(j+1)!} (y^*_{m+1} - y^*_m)^{j+1}.
\]

If \( m \geq s \), we have by (\(*)\) (observe that \( x^*_m \in B_{j+1} \))

\[
\left| \frac{f^{(j)}(x^*_m)}{(j+1)!} (x^*_{m+1} - x^*_m)^{j+1} \right| \leq \rho^{j-m_j + m(j+1)}
\]

and we find \( |F_j(x) - F_j(y)| \leq \rho^{j-m_j + m(j+1)} \). Using the fact that \( j \geq n + 1 \) and our assumption \( s \geq m_j \), we obtain

\[
j - m_j + s(j + 1) = (s + 1) j + s - m_j \geq (s + 1)(n + 1).
\]

By consequence

\[
|F_j(x) - F_j(y)| \leq \rho^{(s+1)(n+1)} \leq |x - y|^{n+1}
\]

which finishes the proof.

Remark. - The above construction does not give us a linear antiderivation map \( C^\infty(X \to K) \to C^\infty(X \to K) \), and it is somewhat doubtful whether there exists a linear antiderivation map \( P : C^\infty(X \to K) \to C^\infty(X \to K) \) that is continuous with respect to a natural locally convex topology ([1], p. 119) on \( C^\infty(X \to K) \).

REFERENCE