The purpose of this note is to prove the following theorem (for the definition of a $C^\infty$-function see below).

**Theorem**. Let $K$ be a complete non-Archimedean valued field with characteristic zero. Let $X$ be a nonempty subset of $K$ without isolated points and let $f : X \rightarrow K$ be a $C^\infty$-function. Then there is a $C^\infty$-function $X \rightarrow K$ whose derivative is $f$.

First we quote some definitions and statements from [1] which are needed for the proof. Let $K$ and $X$ be as above.

**Definition** ([1], p. 8). Let $f : X \rightarrow K$. $f$ is differentiable if its derivative $a \mapsto f'(a) := \lim_{x \rightarrow a} (f(x) - f(a))$ exists. For $n \in \mathbb{N}$, let $\nabla^n X := \{(y_1, y_2, \ldots, y_n) \in X^n : y_i \neq y_j \text{ whenever } i \neq j\}$. The difference quotients $\nabla_n f : \nabla^{n+1} X \rightarrow K$ are given inductively by

$$
\nabla^0 f := f
$$

and

$$
\nabla_n f(y_1, y_2, \ldots, y_{n+1}) := (y_1 - y_2)^{-1}(\nabla_{n-1} f(y_1, y_3, \ldots, y_{n-1}) - \nabla_{n-1} f(y_2, y_3, \ldots, y_{n+1}))
$$

for $((y_1, y_2, \ldots, y_{n+1}) \in \nabla^{n+1} X, n \in \mathbb{N})$.

$f$ is a $C^n$-function ($f \in C^n(X \rightarrow K)$) if $\nabla_n f$ can (uniquely) be extended to a continuous function $\nabla^n f : X^{n+1} \rightarrow K$.

$f$ is a $C^\infty$-function if $f \in C^\infty(X \rightarrow K) := \cap_{n=0}^{\infty} C^n(X \rightarrow K)$.

**Proposition** ([1], p. 78, 36, 37, 116 and 123). Let $f : X \rightarrow K$. For each $n \in \mathbb{N}$ the function $\nabla_n f$ is symmetric, $C^{n+1}(X \rightarrow K) \supset C^n(X \rightarrow K)$, if $f \in C^n(X \rightarrow K)$ then $f' \in C^{n+1}(X \rightarrow K)$ and $\nabla_n f(a, a, \ldots, a) = f^{(n)}(a)/n!$ for each $a \in X$.

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If \( \lim_{x \to a} (x - y)^n (f(x) - f(y)) = 0 \) for each \( a \in X \) then \( f \in C^p(X \to K) \) and \( f' = 0 \). (Locally) analytic functions are \( C^\infty \)-functions.

**Definition ([1], p. 45 and 46).** Let \( 0 < \rho < 1 \). For each \( n \in \mathbb{N} \), let \( R_n \) be a full set of representatives in \( X \) of the equivalence relation given by \( |x - y| < \rho^n \) \( (x, y \in X) \) such that \( R_1 \subset R_2 \subset \cdots \). Choose \( x_0 \in R_1 \). For each \( y \in X \), \( n \in \mathbb{N} \), let \( x_n \) be determined by the conditions \( x_n \in R_n \), \( |x - x_n| < \rho^n \).

**PROPOSITION ([1] Th. 11.2).** Let \( n \in \mathbb{N} \), \( f \in C^{n-1}(X \to K) \). Set

\[
P_n f(x) := \sum_{m=0}^{\infty} \frac{f^{(j)}(x)}{(j + 1)!} (x_{m+1} - x_m)^{j+1} \quad (x \in X).
\]

Then \( P_n f \) is a \( C^n \)-antiderivative of \( f \).

**Proof of the theorem.** We shall use the terminology of above.

Let \( j \in \{0, 1, 2, \ldots\} \). \( f^{(j)} \) is continuous hence locally bounded and there exists a partition of \( X \) into "closed" balls \( B_{i_j} \) (relative to \( X \) ) of radius \( < 1 \) where \( i \) runs through some indexing set \( I_j \) such that \( f^{(j)} \) is bounded on each \( B_{i_j} \). For each \( i \in I_j \), we can choose \( m_{i_j} \in \mathbb{N} \) such that (recall that \( 0 < \rho < 1 \))

\[
\rho^{m_{i_j}} \leq d(B_{i_j}) < 1, \quad |f^{(j)}(x)| \rho^{m_{i_j}} < |(j + 1)!| \rho^j \quad (x \in B_{i_j}).
\]

Define \( F_j : X \to K \) as follows. If \( x \in X \), then \( x \in B_{i_j} \) for precisely one \( i \in I_j \). Set

\[
F_j(x) := \sum_{m=0}^{m_{i_j}} f^{(j)}(x) \frac{(x_{m+1} - x_m)^{j+1}}{(j + 1)!}.
\]

We shall prove that \( F := \sum_{j=0}^{\infty} F_j \) is a \( C^n \)-antiderivative of \( f \) by means of the following steps.

(i) Each \( F_j \) is well defined.

(ii) For each \( j \in \{0, 1, 2, \ldots\} \) and for all \( i \in I_j \),

\[
|F_j(x)| < \rho^{m_{i_j}+1} \quad (x \in B_{i_j})
\]

so that \( F \) is well defined.

(iii) \( \sum_{j=0}^{\infty} F_j \) is a \( C^n \)-antiderivative of \( f \) for each \( n \in \mathbb{N} \).

(iv) For each \( n \), \( \sum_{j=n+1}^{\infty} F_j \) is a \( C^n \)-function with zero derivative.

**Proof of (i).** \( f^{(j)} \) is bounded on \( B_{i_j} \), and \( \lim_{m \to \infty} (x_{m+1} - x_m) = 0 \).

**Proof of (ii).** Let \( x \in B_{i_j} \) and \( m > m_{i_j} \). Then by (*)

\[
|x_{m+1} - x_m| \leq |x - x_m| \leq \rho^{m} \leq \rho^{m_{i_j}} \leq d(B_{i_j})
\]

so that \( F_j \) is well defined.
from which it follows that \( x_m \in B_{j_1} \) and \( |x_{m+1} - x_m| \leq \rho^{j_1} \). Applying the second formula of (*) with \( x \) replaced by \( x_m \), we get

\[
f(j)(x)_m \left( x_{m+1} - x_m \right)^{j_1} \leq \rho^{j_1} \rho^{m_j_1} (j+1) = \rho^{m_j_1+j},
\]

and (ii) is proved.

Proof of (iii). - The function \( F_j \) and \( x \mapsto \sum_{m=0}^\infty f(j)(x)_m \left( x_{m+1} - x_m \right)^{j_1+1}/(j+1) \) differ (on each \( B_{j_1} \), hence globally) by a locally constant function. Summation from \( j = 0 \) to \( j = n \) shows that \( \sum_{j=0}^n F_j - F_{n+1} \) is locally constant. By the second proposition

\[
\sum_{j=0}^n F_j \in C^N(X \to K) \subseteq C^N(X \to K) \quad \text{and} \quad \left( \sum_{j=0}^n F_j \right)' = f.
\]

Proof of (iv). - Set \( H := \sum_{j=n+1}^\infty F_j \). We shall prove that \( |H(x) - H(y)| \leq |x-y|^{n+1} \) for all \( x, y \in X \) which, by the first proposition implies (iv). To obtain the inequality if sufficient to prove

\[
(\star \star) \quad |F_j(x) - F_j(y)| \leq |x-y|^{n+1} \quad (x, y \in X) \quad \text{for each } j \geq n + 1.
\]

We consider several cases.

(a) \( x \in B_{j_1}, \ y \in B_{j_1'}, \) where \( i \neq i' \). Then by (\star),

\[
|x - y| \geq d(B_{j_1}) \geq \rho^{j_1} \quad \text{so that} \quad |x - y|^{n+1} \geq \rho^{m_j_1(n+1)}.
\]

By (ii),

\[
|F_j(x)| \leq \rho^{j_1 m_j_1+j}.
\]

As \( jm_{j_1} + j \geq (n+1)m_j_1 \), we have \( |F_j(x)| \leq |x-y|^{n+1} \). By symmetry, \( |F_j(y)| \leq |x-y|^{n+1} \), and \((\star \star)\) follows.

(b) There is \( i \) such that \( x, y \in B_{j_1} \). We may assume \( x \neq y \), there exists an \( s \in \mathbb{N} \cup \{0\} \) such that (recall that \( d(B_{j_1}) < 1 \))

\[
\rho^{s+1} \leq |x-y| < \rho^{s}.
\]

Then \( |x-y|^{n+1} \geq \rho^{(s+1)(n+1)} \). Consider two subcases.

(b.1) \( s < m_{j_1} \). Then by (ii),

\[
|F_j(x)| \leq \rho^{j_1 m_j_1+j}
\]

and since \( jm_{j_1} + j \geq (n+1)(s+1) + j \geq (s+1)(n+1) \), we have \( |F_j(x)| \leq |x-y|^{n+1} \). By symmetry, \( |F_j(y)| \leq |x-y|^{n+1} \) and \((\star \star)\) follows.

(b.2) \( s \geq m_{j_1} \). Then since \( x_0 = y_0, \ldots, x_s = y_s \), we have, for \( m = m_{j_1}, \ldots, s-1, \ldots, n+1 \),
\[
\frac{f^{(j)}(x_m)}{(j+1)!} (x_{m+1} - x_m)^{j+1} = \frac{f^{(j)}(y_m)}{(j+1)!} (y_{m+1} - y_m)^{j+1}
\]
so that
\[
F_j(x) - F_j(y) = \sum_{m \geq s} \frac{f^{(j)}(x_m)}{(j+1)!} (x_{m+1} - x_m)^{j+1} - \sum_{m \geq s} \frac{f^{(j)}(y_m)}{(j+1)!} (y_{m+1} - y_m)^{j+1}.
\]

If \( m \geq s \), we have by (\( \ast \)) (observe that \( x_m \in B_{j_1} \))
\[
\left| \frac{f^{(j)}(x_m)}{(j+1)!} (x_{m+1} - x_m)^{j+1} \right| \leq \rho^{j-m_{j_1}^1+m(j+1)}
\]
\[
\left| \frac{f^{(j)}(y_m)}{(j+1)!} (y_{m+1} - y_m)^{j+1} \right| \leq \rho^{j-m_{j_1}^1+m(j+1)}
\]
and we find \( |F_j(x) - F_j(y)| \leq \rho^{-j-m_{j_1}^1+s(j+1)}. \) Using the fact that \( j \geq n+1 \) and our assumption \( s \geq m_{j_1} \), we obtain
\[
j - m_{j_1}^1 + s(j + 1) = (s + 1) j + s - m_{j_1}^1 \geq (s + 1)(n + 1).
\]
By consequence
\[
|F_j(x) - F_j(y)| \leq \rho^{(s+1)(n+1)} \leq |x - y|^{n+1}
\]
which finishes the proof.

Remark. - The above construction does not give us a linear antiderivation map
\( C^\infty(X \rightarrow K) \rightarrow C^\infty(X \rightarrow K) \), and it is somewhat doubtful whether there exists a linear antiderivation map \( P : C^\infty(X \rightarrow K) \rightarrow C^\infty(X \rightarrow K) \) that is continuous with respect to a natural locally convex topology ([1], p. 119) on \( C^\infty(X \rightarrow K) \).

REFERENCE