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The purpose of this note is to prove the following theorem (for the definition of a $C^\infty$-function see below).

**Theorem.** - Let $K$ be a complete non-archimedean valued field with characteristic zero. Let $X$ be a nonempty subset of $K$ without isolated points and let $f : X \to K$ be a $C^\infty$-function. Then there is a $C^\infty$-function $X \to K$ whose derivative is $f$.

First we quote some definitions and statements from [1] which are needed for the proof. Let $K$ and $X$ be as above.

Definition ([1], p. 8 ff, 75). - Let $f : X \to K$. $f$ is differentiable if its derivative $a \mapsto f'(a) := \lim_{x \to a} (x - a)^{-1} (f(x) - f(a))$ $(a \in X)$ exists. For $n \in \mathbb{N}$, let $\mathcal{V}^n X := \{(y_1, y_2, \ldots, y_n) \in X^n ; y_i \neq y_j$ whenever $i \neq j\}$. The difference quotients $\partial_n f : \mathcal{V}^{n+1} X \to K$ $(n \in \{0, 1, 2, \ldots\})$ are given inductively by

$$\partial_0 f := f$$

and

$$\partial_n f(y_1, y_2, \ldots, y_{n+1}) := (y_1 - y_2)^{-1}(\partial_{n-1} f(y_1, y_3, \ldots, y_{n-1}) - \partial_{n-1} f(y_2, y_3, \ldots, y_{n+1}))$$

$((y_1, y_2, \ldots, y_{n+1}) \in \mathcal{V}^{n+1} X, n \in \mathbb{N})$.

$f$ is a $C^n$-function ($f \in C^n(X \to K)$) if $\partial_n f$ can (uniquely) be extended to a continuous function $\partial_n f : X^{n+1} \to K$.

$f$ is a $C^\infty$-function if $f \in C^\infty(X \to K) := \cap_{n=0}^\infty C^n(X \to K)$.

Proposition ([1], p. 78, 86, 87, 116 and 123). - Let $f : X \to K$. For each $n \in \mathbb{N}$ the function $\partial_n f$ is symmetric, $C^{n+1}(X \to K) \supset C^n(X \to K)$, if $f \in C^n(X \to K)$ then $f' \in C^{n+1}(X \to K)$ and $\partial_n f(a, a, \ldots, a) = f^{(n)}(a)/n!$ for each $a \in X$.

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if \( \lim_{y \to x} (x - y)^{\alpha} (f(x) - f(y)) = 0 \) for each \( \alpha \in \mathbb{X} \) then \( f \in C^{\alpha}(X \to K) \) and 
\( f' = 0 \). (locally) analytic functions are \( C^\infty \) functions.

**Definition ([1], p. 45 and 46).** - Let \( 0 < \rho < 1 \). For each \( n \in \mathbb{N} \), let \( R_n \) be a full set of representatives in \( X \) of the equivalence relation given by 
\( |x - y| < \rho^n \) \( (x, y \in X) \) such that \( R_1 \subseteq R_2 \subseteq \ldots \). Choose \( x_0 \in R_1 \). For each \( x \in \mathbb{X} \), \( n \in \mathbb{N} \), let \( x_n \) be determined by the conditions \( x_n \in R_n \), \( |x - x_n| < \rho^n \).

**Proposition ([1] Th. 11.2).** - Let \( n \in \mathbb{N} \), \( f \in C^{n-1}(X \to K) \). Set
\[
P_n f(x) := \sum_{m=0}^{\infty} \frac{f^{(j)}(x)_m^{m-1}}{(j + 1)!} (x_m + 1 - x_m)^{j+1} (x \in X).
\]

Then \( P_n f \) is a \( C^n \)-antiderivative of \( f \).

**Proof of the theorem.** - We shall use the terminology of above.

Let \( j \in \{0, 1, 2, \ldots\} \). \( f^{(j)} \) is continuous hence locally bounded and there exists a partition of \( X \) into "closed" balls \( B_{j+1} \) (relative to \( X \)) of radius \( < 1 \) where \( i \) runs through some indexing set \( I_j \) such that \( f^{(j)} \) is bounded on each \( B_{j+1} \). For each \( i \in I_j \), we can choose \( m_{j+1} \in \mathbb{N} \) such that (recall that \( 0 < \rho < 1 \))
\[
(*) \rho m_{j+1} < d(B_{j+1}) < 1, \ |f^{(j)}(x)| \rho m_{j+1} < |(j + 1)i| \rho^j (x \in B_{j+1}).
\]

Define \( F_j : X \to K \) as follows. If \( x \in X \), then \( x \in B_{j+1} \) for precisely one \( i \in I_j \). Set
\[
F_j(x) := \sum_{m=0}^{\infty} \frac{f^{(j)}(x_m)}{(j + 1)!} (x_m + 1 - x_m)^{j+1}.
\]

We shall prove that \( F := \sum_{j=0}^{\infty} F_j \) is a \( C^\infty \)-antiderivative of \( f \) by means of the following steps.

(i) Each \( F_j \) is well defined.

(ii) For each \( j \in \{0, 1, 2, \ldots\} \) and for all \( i \in I_j \),
\[
|F_j(x)| \leq \rho m_{j+1} \rho^j (x \in B_{j+1})
\]
so that \( F \) is well defined.

(iii) \( \sum_{j=0}^{\infty} F_j \) is a \( C^n \)-antiderivative of \( f \) for each \( n \in \mathbb{N} \).

(iv) For each \( n \), \( \sum_{j=n+1}^{\infty} F_j \) is a \( C^n \)-function with zero derivative.

**Proof of (i).** - \( f^{(j)} \) is bounded on \( B_{j+1} \), and \( \lim_{m \to \infty} (x_{m+1} - x_m) = 0 \).

**Proof of (ii).** - Let \( x \in B_{j+1} \) and \( m \geq m_{j+1} \). Then by (*)
\[
|x_{m+1} - x_m| \leq |x - x_m| \leq \rho^m \leq \rho m_{j+1} \leq d(B_{j+1}).
\]
from which it follows that $x_m \in B_{ji}$ and $|x_{m+1} - x_m| \leq \rho_{ji}^{m+1}$. Applying the second formula of ($\star$) with $x$ replaced by $x_m$, we get

$$f(j)(x_m) \left| \frac{x_{m+1} - x_m}{(j+1)!} \right|^{j+1} \leq \rho_{ji}^{p+1} \rho_{ji}(j+1) = \rho_{ji}^{m+1},$$

and (ii) is proved.

**Proof of (iii).** - The function $P_j$ and $x \mapsto \sum_{m=0}^{\infty} f(j)(x_m)(x_{m+1} - x_m)^{j+1}/(j+1)!$ differ (on each $B_{ji}$, hence globally) by a locally constant function. Summation from $j=0$ to $j=n$ shows that $\sum_{j=0}^{n} P_j - P_{n+1}$ is locally constant. By the second proposition

$$\sum_{j=0}^{n} P_j \in C^n(X \to \mathbb{K}) \subset C(X \to \mathbb{K})$$

and (iii) is proved.

**Proof of (iv).** - Set $H := \sum_{j=n+1}^{\infty} P_j$. We shall prove that $|H(x) - H(y)| \leq |x - y|^{n+1}$ for all $x, y \in X$ which, by the first proposition implies (iv). To obtain the inequality if suffices to prove

$$|F_j(x) - F_j(y)| \leq |x - y|^{n+1} \quad (x, y \in X) \text{ for each } j \geq n + 1.$$

We consider several cases.

(a) $x \in B_{ji}$, $y \in B_{i'}$, where $i \neq i'$. Then by ($\star$),

$$|x - y| \geq d(B_{ji}) \geq \rho_{ji}^{m+1} \text{ so that } |x - y|^{n+1} \geq \rho_{ji}^{m+1}.$$

By (ii),

$$|F_j(x)| \leq \rho_{ji}^{m+1}.$$

As $jm_{ji} + j \geq (n+1)m_{ji}$, we have $|F_j(x)| \leq |x - y|^{n+1}$. By symmetry, $|F_j(y)| \leq |x - y|^{n+1}$, and ($\star$) follows.

(b) There is $i$ such that $x, y \in B_{ji}$. We may assume $x \neq y$, there exists an $s \in \mathbb{N} \cup \{0\}$ such that (recall that $d(B_{ji}) < 1$)

$$\rho^{s+1} \leq |x - y| < \rho^s.$$

Then $|x - y|^{n+1} \geq \rho^{(s+1)(n+1)}$. Consider two subcases.

(b.1) $s < m_{ji}$. Then by (ii),

$$|F_j(x)| \leq \rho_{ji}^{m_{ji}+1}$$

and since $jm_{ji} + j \geq (n+1)(s+1) + j \geq (s+1)(n+1)$, we have $|F_j(x)| \leq |x - y|^{n+1}$. By symmetry $|F_j(y)| \leq |x - y|^{n+1}$ and ($\star$) follows.

(b.2) $s \geq m_{ji}$. Then since $x_0 = y_0, \ldots, x_s = y_s$, we have, for $m=m_{ji}, \ldots, s-1$,
\[
\frac{f^{(j)}(x_m)}{(j+1)!} (x_{m+1} - x_m)^{j+1} = \frac{f^{(j)}(y_m)}{(j+1)!} (y_{m+1} - y_m)^{j+1}
\]
so that
\[
F_j(x) - F_j(y) = \sum_{m \geq s} \frac{f^{(j)}(x_m)}{(j+1)!} (x_{m+1} - x_m)^{j+1} - \sum_{m \geq s} \frac{f^{(j)}(y_m)}{(j+1)!} (y_{m+1} - y_m)^{j+1}.
\]

If \( m \geq s \), we have by (*) (observe that \( x_m \in B_{ji} \))
\[
\frac{f^{(j)}(x_m)}{(j+1)!} (x_{m+1} - x_m)^{j+1} \leq \rho^{j-m_{ji}+m(j+1)}
\]
\[
\frac{f^{(j)}(y_m)}{(j+1)!} (y_{m+1} - y_m)^{j+1} \leq \rho^{j-m_{ji}+m(j+1)}
\]
and we find \( |F_j(x) - F_j(y)| \leq \rho^{j-m_{ji}+s(j+1)} \). Using the fact that \( j \geq n+1 \) and our assumption \( s \geq m_{ji} \), we obtain
\[
j - m_{ji} + s(j+1) = (s+1)j + s - m_{ji} \geq (s+1)(n+1).
\]

By consequence
\[
|F_j(x) - F_j(y)| \leq \rho^{(s+1)(n+1)} \leq |x - y|^{n+1}
\]
which finishes the proof.

**Remark.** - The above construction does not give us a linear antiderivation map \( C^\infty(X \to K) \to C^\infty(X \to K) \), and it is somewhat doubtful whether there exists a linear antiderivation map \( P : C^\infty(X \to K) \to C^\infty(X \to K) \) that is continuous with respect to a natural locally convex topology ([1], p. 119) on \( C^\infty(X \to K) \).

**Reference**