C\(^\infty\)-ANTIDERIVATIVES OF \(p\)-ADIC \(C\(^\infty\)-FUNCTIONS

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The purpose of this note is to prove the following theorem (for the definition of a \(C\(^\infty\)-function see below).

**THEOREM.** - Let \(K\) be a complete non-archimedean valued field with characteristic zero. Let \(X\) be a nonempty subset of \(K\) without isolated points and let 
\[ f : X \to K \]
be a \(C\(^\infty\)-function. Then there is a \(C\(^\infty\)-function \(X \to K\) whose derivative is \(f\).

First we quote some definitions and statements from [1] which are needed for the proof. Let \(K\) and \(X\) be as above.

\textbf{Definition (}[1], p. 8 \& 75). - Let \(f : X \to K\). \(f\) is **differentiable** if its derivative \(a \mapsto f'(a) := \lim_{x \to a} (f(x) - f(a)) (a \in X)\) exists. For \(n \in \mathbb{N}\), let \(\nabla^n X := \{(y_1, y_2, \ldots, y_n) \in X^n ; y_i \neq y_j\ \text{whenever} \ i \neq j\}\). The **difference quotients** \(\xi_n f : \nabla^{n+1} X \to K\) (\(n \in \{0, 1, 2, \ldots\}\)) are given inductively by

\[ \xi_0 f := f \]

and

\[ \xi_n f(y_1, y_2, \ldots, y_{n+1}) := (y_1 - y_2)^{-1}(\xi_{n-1} f(y_1, y_3, \ldots, y_{n-1}) - \xi_{n-1} f(y_2, y_3, \ldots, y_{n+1})) \]

\((y_1, y_2, \ldots, y_{n+1}) \in \nabla^{n+1} X, \ n \in \mathbb{N}\).

\(f\) is a \(C^n\)-function \((f \in C^n(X \to K))\) if \(\xi_n f\) can (uniquely) be extended to a continuous function \(\xi_n f : X^{n+1} \to K\).

\(f\) is a \(C^\infty\)-function if \(f \in C^\infty(X \to K) := \bigcap_{n=0}^{\infty} C^n(X \to K)\).

\textbf{PROPOSITION ([1], p. 78, 86, 87, 116 and 123). - Let} \(f : X \to K\). For each \(n \in \mathbb{N}\) the function \(\xi n f\) is symmetric, \(C^{n-1}(X \to K) \supset C^n(X \to K)\), if \(f \in C^n(X \to K)\) then \(f' \in C^{n-1}(X \to K)\) and \(\xi_n f(a, a, \ldots, a) = f^{(n)}(a)/n!\) for each \(a \in X\).

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if \( \lim_{x,y \to a} (x - y)^n (f(x) - f(y)) = 0 \) for each \( a \in X \) then \( f \in C^r(X \to K) \) and \( f' = 0 \). (Locally) analytic functions are \( C^\infty \) functions.

Definition ([1], p. 45 and 46). Let \( 0 < \rho < 1 \). For each \( n \in N \), let \( R_n \) be a full set of representatives in \( X \) of the equivalence relation given by

\[
|x - y| < \rho^n \quad (x, y \in X)
\]

such that \( R_1 \in R_2 \subset \ldots \). Choose \( x_0 \in R_1 \). For each \( x \in X \), \( n \in N \), let \( x_n \) be determined by the conditions \( x_n \in R_n \), \( |x - x_n| < \rho^n \).

**PROPOSITION ([1] Th. 11.2).** Let \( n \in N \), \( f \in C^{n-1}(X \to K) \). Set

\[
P_n f(x) := \sum_{m=0}^{n-1} \sum_{j=0}^{\infty} \frac{f(j)(x)}{(j+1)!} (x_{m+1} - x_m)^{j+1} \quad (x \in X) .
\]

Then \( P_n f \) is a \( C^n \)-antiderivative of \( f \).

**Proof of the theorem.** We shall use the terminology of above.

Let \( j \in \{0, 1, 2, \ldots\} \). \( f(j) \) is continuous hence locally bounded and there exists a partition of \( X \) into "closed" balls \( B_{j_i} \) (relative to \( X \)) of radius \( < 1 \) where \( i \) runs through some indexing set \( I_j \) such that \( f(j) \) is bounded on each \( B_{j_i} \). For each \( i \in I_j \), we can choose \( m_{j_i} \in N \) such that (recall that \( 0 < \rho < 1 \))

\[
(*) \quad \rho m_{j_i} < d(B_{j_i}) < 1 , \quad |f(j)(x)| \rho m_{j_i} < |(j+1)| \rho^j (x \in B_{j_i}) .
\]

Define \( F_j : X \to K \) as follows. If \( x \in X \), then \( x \in B_{j_i} \) for precisely one \( i \in I_j \). Set

\[
F_j(x) := \sum_{m=0}^{m_{j_i} - 1} \frac{f(j)(x_m)}{(j+1)!} (x_{m+1} - x_m)^{j+1} .
\]

We shall prove that \( F := \sum_{j=0}^{\infty} F_j \) is a \( C^\infty \)-antiderivative of \( f \) by means of the following steps.

(i) Each \( F_j \) is well defined.

(ii) For each \( j \in \{0, 1, 2, \ldots\} \) and for all \( i \in I_j \),

\[
|F_j(x)| \leq \rho^{m_{j_i} + j} (x \in B_{j_i})
\]

so that \( F \) is well defined.

(iii) \( \sum_{j=0}^{\infty} F_j \) is a \( C^n \)-antiderivative of \( f \) for each \( n \in N \).

(iv) For each \( n \), \( \sum_{j=n+1}^{\infty} F_j \) is a \( C^n \)-function with zero derivative.

**Proof of (i).** \( f(j) \) is bounded on \( B_{j_i} \), and \( \lim_{m \to \infty} (x_{m+1} - x_m) = 0 \).

**Proof of (ii).** Let \( x \in B_{j_i} \) and \( m \geq m_{j_i} \). Then by \( (*) \),

\[
|x_{m+1} - x_m| \leq |x - x_m| \leq \rho^m \leq \rho^{m_{j_i}} \leq d(B_{j_i})
\]
from which it follows that $x_m \in B_{ji}$ and $|x_{m+1} - x_m| \leq \rho^{ji}$. Applying the second formula of $(\ast)$ with $x$ replaced by $x_m$, we set

$$f'(j)(x) \left( \sum_{m=0}^{\infty} (x_{m+1} - x_m)^{j+1} \right) \leq \rho \prod_{m} \rho^{ji} \rho^{m+1} j^{m+1} = \rho^{m+1} j,$$

and (ii) is proved.

Proof of (iii). - The function $F_j$ and $x \mapsto \sum_{m=0}^{\infty} f'(j)(x) (x_{m+1} - x_m)^{j+1} / (j+1)$ differ (on each $B_{ji}$, hence globally) by a locally constant function. Summation from $j = 0$ to $j = n$ shows that $\sum_{j=0}^{n} F_j - F_{n+1}$ is locally constant. By the second proposition

$$\sum_{j=0}^{n} F_j \in C^1(X \to \mathbb{K}) \subset C^1(X \to \mathbb{K})$$

and (ii) is proved.

Proof of (iv). - Set $H := \sum_{j=0}^{\infty} F_j$. We shall prove that $|H(x) - H(y)| \leq |x-y|^{n+1}$ for all $x, y \in X$ which, by the first proposition implies (iv). To obtain the inequality if sufficient to prove

$$(**): |F_j(x) - F_j(y)| \leq |x - y|^{n+1} (x, y \in X) \text{ for each } j \geq n+1.$$

We consider several cases.

(a) $x \in B_{j1}$, $y \in B_{j'i}$, where $i \neq i'$. Then by $(\ast)$,

$$|x - y| \geq d(B_{j1}) \geq \rho^{mji} \text{ so that } |x - y|^{n+1} \geq \rho^{mji(n+1)}.$$

By (ii),

$$|F_j(x)| \leq \rho^{mji+j}.$$

As $jm_{ji} + j \geq (n + 1) m_{ji}$, we have $|F_j(x)| \leq |x - y|^{n+1}$. By symmetry, $|F_j(y)| \leq |x - y|^{n+1}$, and (***) follows.

(b) There is $i$ such that $x, y \in B_{j1}$. We may assume $x \neq y$, there exists an $s \in N \cup \{0\}$ such that (recall that $d(B_{j1}) < 1$ )

$$\rho^{s+1} \leq |x - y| < \rho^s.$$

Then $|x - y|^{n+1} \geq \rho^{s+1}(n+1)$. Consider two subcases.

(b.1) $s < m_{ji}$. Then by (ii),

$$|F_j(x)| \leq \rho^{m_{ji}+j}$$

and since $jm_{ji} + j \geq (n + 1)(s + 1) + j \geq (s + 1)(n + 1)$, we have $|F_j(x)| \leq |x-y|^{n+1}$.

By symmetry $|F_j(y)| \leq |x - y|^{n+1}$ and (***) follows.

(b.2) $s \geq m_{ji}$. Then since $x_0 = y_0, \ldots, x_s = y_s$, we have, for $m=m_{ji}, \ldots, s-1,$
so that

\[
F_j(x) - F_j(y) = \sum_{m \geq s} \frac{f^{(j)}(x_m)}{(j + 1)!} (x_{m+1} - x_m)^{j+1} - \sum_{m \geq s} \frac{f^{(j)}(y_m)}{(j + 1)!} (y_{m+1} - y_m)^{j+1}.
\]

If \( m \geq s \), we have by (\(*\)) (observe that \( x_m \in B_{j+1} \))

\[
\left| \frac{f^{(j)}(x_m)}{(j + 1)!} (x_{m+1} - x_m)^{j+1} \right| \leq \rho^{j-m_j+1} \]

and

\[
\left| \frac{f^{(j)}(y_m)}{(j + 1)!} (y_{m+1} - y_m)^{j+1} \right| \leq \rho^{j-m_j+1}.
\]

and we find \( |F_j(x) - F_j(y)| \leq \rho^{j-m_j+s(j+1)} \). Using the fact that \( j \geq n + 1 \) and our assumption \( s \geq m_j \), we obtain

\[
j - m_j + s(j + 1) = (s + 1) j + s - m_j \geq (s + 1)(n + 1).
\]

By consequence

\[
|F_j(x) - F_j(y)| \leq \rho^{(s+1)(n+1)} \leq |x - y|^{n+1}
\]

which finishes the proof.

Remark. - The above construction does not give us a linear antiderivation map \( C^0(X \to K) \to C^0(X \to K) \), and it is somewhat doubtful whether there exists a linear antiderivation map \( P : C^0(X \to K) \to C^0(X \to K) \) that is continuous with respect to a natural locally convex topology ([1], p. 119) on \( C^0(X \to K) \).

REFERENCE