C\infty\text{-ANTIDERIVATIVES OF } p\text{-ADIC C\infty\text{-FUNCTIONS}}

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The purpose of this note is to prove the following theorem (for the definition of a C\infty\text{-function see below)}.

THEOREM. - Let K be a complete non-archimedean valued field with characteristic zero. Let X be a nonempty subset of K without isolated points and let f : X \longrightarrow K be a C\infty\text{-function. Then there is a C\infty\text{-function } X \longrightarrow K whose derivative is } f.

First we quote some definitions and statements from [1] which are needed for the proof. Let K and X be as above.

Definition ([1], p. 8 \& 75). - Let f : X \longrightarrow K. f is differentiable if its derivative a \longmapsto f'(a) := \lim_{x \to a} (f(x) - f(a)) (a \in X) exists. For n \in \mathbb{N}, let \nabla^n X := \{(y_1, y_2, \ldots, y_n) \in X^n ; y_i \neq y_j, whenever i \neq j\}. The difference quotients \nabla^n f : \nabla^{n+1} X \longrightarrow K (n \in \{0, 1, 2, \ldots\}) are given inductively by

\nabla_0 f := f

and

\nabla_n f(y_1, y_2, \ldots, y_n) := (y_1 - y_2)^{-1}(\nabla_{n-1} f(y_1, y_3, \ldots, y_{n-1}) - \nabla_{n-1} f(y_2, y_3, \ldots, y_n))

((y_1, y_2, \ldots, y_n) \in \nabla^n X, n \in \mathbb{N}).

f is a C^n\text{-function } (f \in C^n(X \longrightarrow K)) if \nabla_n f can (uniquely) be extended to a continuous function \nabla^n f : X^{n+1} \longrightarrow K.

f is a C\infty\text{-function if } f \in C\infty(X \longrightarrow K) := \cap_{n=0}^{\infty} C^n(X \longrightarrow K).

PROPOSITION ([1], p. 78, 36, 37, 116 and 123). - Let f : X \longrightarrow K. For each n \in \mathbb{N} the function \nabla_n f is symmetric, C^{n-1}(X \longrightarrow K) \supset C^n(X \longrightarrow K), if f \in C^n(X \longrightarrow K) then f' \in C^{n-1}(X \longrightarrow K) and \nabla_n f(a, a, \ldots, a) = f^{(n)}(a)/n! for each a \in X.

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\[
\lim_{x,y \to a} (x - y)^n (f(x) - f(y)) = 0 \quad \text{for each } a \in X \quad \text{then } f \in C^n(X \to K) \quad \text{and} \\
f' = 0 \quad \text{locally analytic functions are } C^\infty \text{functions}.
\]

Definition ([1], p. 45 and 46). - Let \(0 < p < 1\). For each \(n \in \mathbb{N}\), let \(R_n\) be a full set of representatives in \(X\) of the equivalence relation given by
\[
|\|x - y\| < \rho^n \quad (x, y \in X) \quad \text{such that } R_1 \subseteq R_2 \subseteq \ldots \quad \text{Choose } x_0 \in R_1. \quad \text{For each } x \in X, \\
n \in \mathbb{N}, \quad \text{let } x_n \text{ be determined by the conditions } x_n \in R_n, \quad |x - x_n| < \rho^n.
\]

PROPOSITION ([1] Th. 11.2). - Let \(n \in \mathbb{N}\), \(f \in C^{n-1}(X \to K)\). Set
\[
P_n f(x) := \sum_{m=0}^n \binom{n}{m} \frac{f^{(j)}(x)}{(j+1)!} (x_{m+1} - x_m)^{j+1} (x \in X).
\]

Then \(P_n f\) is a \(C^n\)-antiderivative of \(f\).

Proof of the theorem - We shall use the terminology of above.

Let \(j \in \{0, 1, 2, \ldots\}\). \(f^{(j)}\) is continuous hence locally bounded and there exists a partition of \(X\) into \(\text"closed"\) balls \(B_{ji}\) (relative to \(X\)) of radius \(< 1\) where \(i\) runs through some indexing set \(I_j\) such that \(f^{(j)}\) is bounded on each \(B_{ji}\). For each \(i \in I_j\), we can choose \(m_{ji} \in \mathbb{N}\) such that (recall that \(0 < p < 1\))
\[
\rho_{m_{ji}} \leq d(B_{ji}) < 1, \quad |f^{(j)}(x)| \rho_{m_{ji}} < |(j+1)^{j+1}(x \in B_{ji}).
\]

Define \(F_j : X \to K\) as follows. If \(x \in X\), then \(x \in B_{ji}\) for precisely one \(i \in I_j\). Set
\[
F_j(x) := \sum_{m=0}^n \frac{f^{(j)}(x_m)}{(j+1)!} (x_{m+1} - x_m)^{j+1}.
\]

We shall prove that \(F := \sum_{j=0}^\infty F_j\) is a \(C^n\)-antiderivative of \(f\) by means of the following steps.

(i) Each \(F_j\) is well defined.

(ii) For each \(j \in \{0, 1, 2, \ldots\}\) and for all \(i \in I_j\),
\[
|F_j(x)| \leq \rho_{m_{ji}}^{j+1} (x \in B_{ji})
\]

so that \(F\) is well defined.

(iii) \(\sum_{j=0}^n F_j\) is a \(C^n\)-antiderivative of \(f\) for each \(n \in \mathbb{N}\).

(iv) For each \(n\), \(\sum_{j=n+1}^\infty F_j\) is a \(C^n\)-function with zero derivative.

Proof of (i). - \(f^{(j)}\) is bounded on \(B_{ji}\), and \(\lim_{m \to \infty} (x_{m+1} - x_m) = 0\).

Proof of (ii). - Let \(x \in B_{ji}\) and \(m \geq m_{ji}\). Then by (\(*\)) ,
\[
|x_{m+1} - x_m| \leq |x - x_m| \leq \rho_m \leq \rho_{m_{ji}} \leq d(B_{ji})
\]
from which it follows that $x_m \in B_{ji}$ and $|x_{m+1} - x_m| \leq \rho_j^{m_{ji}}$. Applying the second formula of (*) with $x$ replaced by $x_m$, we get

$$f(j)(x_m)(x_{m+1} - x_m)^{j+1} \leq \rho_j^{j-\frac{m_{ji}}{p} - \frac{m_{ji}(j+1)}{p} = \rho_j^{j+1}$$

and (ii) is proved.

**Proof of (iii).** The function $F_j$ and $x \mapsto \sum_{m=0}^{\infty} f(j)(x_m)(x_{m+1} - x_m)^{j+1}/(j+1)$ differ (on each $B_{ji}$, hence globally) by a locally constant function. Summation from $j = 0$ to $j = n$ shows that $\sum_{j=0}^{n} F_j - F_{n+1}$ is locally constant. By the second proposition

$$\sum_{j=0}^{n} F_j \in \mathcal{C}^n(X \to \mathbb{R}) < \mathcal{C}^1(X \to \mathbb{R})$$

and (ii) is proved.

**Proof of (iv).** Set $H := \sum_{j=0}^{\infty} F_j$. We shall prove that $|H(x) - H(y)| \leq |x - y|^{n+1}$ for all $x, y \in X$ which, by the first proposition implies (iv). To obtain the inequality if suffices to prove

$$|F_j(x) - F_j(y)| \leq |x - y|^{n+1} (x, y \in X) \quad \text{for each } j \geq n + 1$$

We consider several cases.

(a) $x \in B_{ji}, y \in B_{ji}$, where $i \neq i'$. Then by (*),

$$|x - y| \geq d(B_{ji}) \geq \rho_j^{m_{ji}} \quad \text{so that } |x - y|^{n+1} \geq \rho_j^{m_{ji}(n+1)}$$

By (ii),

$$|F_j(x)| \leq \rho_j^{j m_{ji}^j}$$

As $jm_{ji}^j \geq (n + 1)m_{ji}^j$, we have $|F_j(x)| \leq |x - y|^{n+1}$. By symmetry, $|F_j(y)| \leq |x - y|^{n+1}$, and (***) follows.

(b) There is $s$ such that $x, y \in B_{ji}$. We may assume $x \neq y$, there exists an $s \in \mathbb{N} = \{0, 1, \ldots\}$ such that (recall that $d(B_{ji}) < 1$)

$$\rho^{s+1} \leq |x - y| < \rho^s$$

Then $|x - y|^{n+1} \geq \rho^{(s+1)(n+1)}$. Consider two subcases.

(b.i) $s < m_{ji}$. Then by (ii),

$$|F_j(x)| \leq \rho_j^{j m_{ji}^j}$$

and since $jm_{ji}^j \geq (n + 1)(s + 1) + j \geq (s + 1)(n + 1)$, we have $|F_j(x)| \leq |x - y|^{n+1}$. By symmetry $|F_j(y)| \leq |x - y|^{n+1}$ and (***) follows.

(b.2) $s \geq m_{ji}$. Then since $x_0 = y_0, \ldots, x_s = y_s$, we have, for $m = m_{ji}, \ldots, s-1,$
\[
\frac{f^{(j)}(x_m)}{(j+1)!} (x_{m+1} - x_m)^{j+1} = \frac{f^{(j)}(y_m)}{(j+1)!} (y_{m+1} - y_m)^{j+1}
\]

so that

\[
F_j(x) - F_j(y) = \sum_{m \geq s} \frac{f^{(j)}(x_m)}{(j+1)!} (x_{m+1} - x_m)^{j+1} - \sum_{m \geq s} \frac{f^{(j)}(y_m)}{(j+1)!} (y_{m+1} - y_m)^{j+1}.
\]

If \( m \geq s \), we have by \((*)\) (observe that \( x_m \in B_j \))

\[
|\frac{f^{(j)}(x_m)}{(j+1)!} (x_{m+1} - x_m)^{j+1}| \leq \rho^{j-m_j+1+m(j+1)}
\]

and we find

\[
|F_j(x) - F_j(y)| \leq \rho^{j-m_j+s(j+1)}.
\]

Using the fact that \( j \geq n+1 \) and our assumption \( s \geq m_j \), we obtain

\[
j - m_j + s(j + 1) = (s + 1) j + s - m_j \geq (s + 1)(n + 1).
\]

By consequence

\[
|F_j(x) - F_j(y)| \leq \rho^{(s+1)(n+1)} \leq |x - y|^{n+1}
\]

which finishes the proof.

Remark. - The above construction does not give us a linear antiderivation map

\( C^0(X \to K) \to C^0(X \to K) \), and it is somewhat doubtful whether there exists a linear antiderivation map \( P : C^0(X \to K) \to C^0(X \to K) \) that is continuous with respect to a natural locally convex topology ([1], p. 119) on \( C^0(X \to K) \).

REFERENCE