The following full text is a publisher's version.

For additional information about this publication click this link.
http://hdl.handle.net/2066/57037

Please be advised that this information was generated on 2017-12-11 and may be subject to change.
UNIQUENESS OF THE BANACH ALGEBRA TOPOLOGY

FOR NON-ARCHIMEDEAN ALGEBRAS

W.H. Schikhof

1. Introduction.

An algebra $A$ over $\mathbb{R}$ or $\mathbb{C}$ is said to have a unique Banach algebra topology if any two Banach algebra norms on $A$ are equivalent. Johnson's theorem \cite{2} is very satisfactory; it states that a semi-simple algebra over $\mathbb{R}$ or $\mathbb{C}$ has this property.

In this note we are concerned with the non-archimedean analogue. Thus, let $A$ be an algebra over a complete non-archimedean valued field $K$. We say that $A$ has UBAT (a unique Banach algebra topology) if each two (non-archimedean) Banach algebra norms on $A$ are equivalent. Our problem is to find reasonable conditions on $A$ implying the UBAT property.

It is known \cite{3} that in the non-archimedean case semi-simple algebras (even commutative fields) may fail to have UBAT. In fact, we have

1.1 EXAMPLE. Let $p$ be a prime. Let $\mathbb{C}_p$ be the completion (with respect to the natural valuation $|\cdot|$) of the algebraic closure of the field $\mathbb{Q}_p$ of the $p$-adic numbers. Then $(\mathbb{C}_p, |\cdot|)$ is a valued field and a $\mathbb{Q}_p$-Banach algebra. There exists a valuation $|\cdot|$ on $\mathbb{C}_p$, not equivalent to $|\cdot|$, for which $(\mathbb{C}_p, |\cdot|)$ is also a $\mathbb{Q}_p$-Banach algebra.
PROOF. It is well known that $\mathbb{C}$ is algebraically closed. Let $I$ be a maximal set of algebraically independent elements over $\mathbb{Q}$. Then $\bigcup_p \mathbb{Q}_p(I) \subseteq \mathbb{C}_p$, $\mathbb{C}_p$ is the algebraic closure of $\bigcup_p \mathbb{Q}_p(I)$, $I \neq \emptyset$. Fix $x \in I$, and define $\sigma : \bigcup_p \mathbb{Q}_p(I) \to \bigcup_p \mathbb{Q}_p(I)$ by $\sigma(X) = px$ and $\sigma(Y) = Y$ for $Y \in I, Y \neq X$. Then $\sigma$ is an endomorphism $\bigcup_p \mathbb{Q}_p(I) \to \bigcup_p \mathbb{Q}_p(I)$ that can be extended to an endomorphism $\tilde{\sigma} : \bigcup_p \mathbb{Q}_p(I) \to \bigcup_p \mathbb{Q}_p(I)$ that can be extended to an endomorphism $\tilde{\sigma} : \mathbb{C}_p \to \mathbb{C}_p$. It is easy to see that $\tilde{\sigma}$ is also a $\mathbb{Q}_p$-algebra homomorphism. Define $|\cdot|$ via

$$|x|' := |\tilde{\sigma}(x)| \quad (x \in \mathbb{C}_p).$$

Then $|\cdot|$ is not equivalent to $|\cdot|$ since $|x^n|' = |p^n|x^n|$, so there is no $c > 0$ for which $|\cdot| \geq c|\cdot|$. The rest is obvious.

With 1.1 in mind it is rather surprising that we can prove that a $K$-Banach algebra whose norm is multiplicative and that is not a field has UBAT (see 4.7). Further results are:

Tate algebras without nilpotents $\neq 0$ have UBAT. (4.4)

$L(E)$ has UBAT if $E$ is a well-behaved Banach space. (5.4).

For background information on non-archimedean fields, Banach spaces and algebras we refer to [5].

In the sequel $K$ is a non-archimedean non-trivially valued complete field.

Instead of "$A$ is a $K$-Banach algebra with respect to the norms $||\cdot||_1'$ and $||\cdot||_2$" we will sometimes use the expression "$(A, ||\cdot||_1', ||\cdot||_2)$ is bicomplete".

2. Algebras of functions.

Theorem 2.1 is more or less contained in [3].

Let $X$ be a nonempty set. For $f \in K^X$ set
23-03

\[ \| f \|_\infty := \sup \{ |f(x)| : x \in X \} \text{ (possibly } \infty) \]. A function algebra is a K-algebra that is, for some X, (algebraically isomorphic to) a sub-algebra of \( X^X \). Without much effort we can prove

2.1 THEOREM. Let \( F \) be a function algebra. Then

(i) \( F \) has UBAT

(ii) If \( \| \| \) is a Banach algebra norm on \( F \) then

\[ \| \| \geq \| \|_\infty \].

PROOF. Let \( \| \| \) be a Banach algebra norm on \( F \). Let \( a \in X \). The map

\[ f \mapsto f(a) \in F \] (\( a \in X \)) is a homomorphism: \( F \to K \), so by [5] it has norm \( \leq 1: |f(a)| \leq \| f \|. \) It follows that \( \| \| \|_\\infty \leq \| \|. \) Now let \( \| \|_1 \) and \( \| \|_2 \) be two Banach algebra norms on \( F \). We prove that the identity:

\[ (F, \| \|_1) \to (F, \| \|_2) \] is continuous. Let \( f, f_1, f_2, \ldots \in F \) such that

\[ \| f_n \|_1 \to 0, \| f_n - f \|_2 \to 0. \] By the foregoing, \( \| f_n \|_\infty \to 0, \| f_n - f \|_\infty \to 0, \) so \( f = 0. \) Continuity follows after applying the closed graph theorem. 

3. The separating seminorm.

(This is a non-archimedean version of [4], (2.5.1))

3.1 DEFINITION. Let \( \| \|_1 \) and \( \| \|_2 \) be norms on a K-vector space \( E \).

The function \( \Delta : E \to \mathbb{R} \) defined by

\[ \Delta(s) := \inf \{ \max(\| x \|_1, \| y \|_2) : x + y = s \} \quad (s \in E) \]

is called the separating seminorm of \( \| \|_1 \) and \( \| \|_2 \).

One easily checks that \( \Delta \) is the largest among the (non-archimedean) seminorms that are \( \leq \| \|_1 \) and \( \leq \| \|_2 \). As in [4] we have

3.2 LEMMA In case \( \| \|_1 \) and \( \| \|_2 \) are complete norms on \( E \) then:

\( \Delta \) is a norm \( \leftrightarrow \| \|_1 \sim \| \|_2 \).
3.3 **LEMMA.** Let $A$ be a normed $K$-algebra with respect to $\| \cdot \|_1$ and $\| \cdot \|_2$, and let $\Delta$ be its separating seminorm. Then Ker $\Delta$ is a two-sided ideal that is closed with respect to both norms. In fact, we have for $s,t \in A$:
\[
\Delta(st) \leq \Delta(s) \max(\| t \|_1, \| t \|_2) \\
\Delta(st) \leq \Delta(t) \max(\| s \|_1, \| s \|_2).
\]

The proofs of 3.2 and 3.3 are elementary and similar to the ones in [4], (2.5) and are omitted.

3.4 **DEFINITION.** For a linear subspace $D$ of an algebra $A$ that is normed by $\| \cdot \|_1, \| \cdot \|_2$ we set for $d \in D$:
\[
\Delta_D(d) := \inf\{\max(\| x \|_1, \| y \|_2) : x, y \in D, x+y = d\}
\]
($\Delta_D$ is the separating seminorm of the restriction of $\| \cdot \|_1$ and $\| \cdot \|_2$ to $D$).

We have the following elementary facts concerning the behaviour of $\Delta$ with respect to subalgebras and quotients:

3.5 **LEMMA.** With the notations as above we have

(i) $\Delta_D \geq \Delta|_D$, so Ker $\Delta_D \subset$ Ker $\Delta \cap D$.

(ii) Let $D$ be a left ideal, then for $x \in A, t \in D$:
\[
\Delta_D(xt) \leq \Delta(x) \max(\| t \|_1, \| t \|_2) \\
\Delta_D(xt) \leq \max(\| x \|_1, \| x \|_2) \Delta_D(t), \text{ so } \text{Ker } \Delta_D \text{ is a left ideal in } A, \text{ satisfying}
\]

(Ker $\Delta$ $\cdot D \subset$ Ker $\Delta_D \subset$ Ker $\Delta \cap D$).

**PROOF.** (i) For $t \in D$ we have $\Delta(t) = \inf\{\max(\| x \|_1, \| y \|_2) : x+y = t, x, y \in A\} \leq \inf\{\max(\| x \|_1, \| y \|_2) : x+y = t, x, y \in D\} = \Delta_D(t)$.
(ii) \( \Delta_D(xt) = \inf_{z \in D} \max_{y \in A} (\| z \|_1, \| xt - z \|_2) \leq \inf_{y \in A} \max_{t \in D} (\| yt \|_1, \| xt - yt \|_2) \leq \inf_{y \in A} \max_{t \in D} (\| y \|_1, \| t \|_2, \| x - y \|_2). \)

Also, \( \Delta_D(xt) = \inf_{z \in D} \max_{y \in D} (\| z \|_1, \| xt - z \|_2) \leq \inf_{d \in D} \max_{z \in D} (\| zd \|_1, \| xt - zd \|_2) \leq \max(\| x \|_1, \| x \|_2) \cdot \Delta_D(t). \)

3.6 LEMMA. Let \((A, \| \cdot \|_1, \| \cdot \|_2)\) be bicomplete. Suppose \( D \) is a closed linear subspace with respect to both \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \), and suppose that the quotient norms on \( A/D \) are equivalent. Then \( \text{Ker} \Delta \subseteq D \).

PROOF. Let \( \Delta(x) = 0 \) for some \( x \in A \). Then there are \( x_1, x_2, \ldots \) in \( A \) such that \( \| x - x_n \|_1 \to 0 \), \( \| x_n \|_2 \to 0 \). Let \( \pi : A \to A/D \) be the quotient map. Then \( \lim \pi(x_n) = \pi(x) \) for the first quotient norm and \( \lim \pi(x_n) = 0 \) for the second one. Hence, \( \pi(x) = 0 \), i.e., \( x \in D \).

A subset of a K-algebra \( A \) is called universally closed if it is closed with respect to each Banach algebra topology on \( A \). (In case \( A \) has no Banach algebra topology then, by definition, each subset of \( A \) is universally closed). Examples of universally closed sets are

(i) \( \emptyset, A, \) singletons, finite dimensional linear subspaces.

(ii) For each set \( X \subseteq A \) its commutant \( X' := \{ y \in A : yx = xy \text{ for all } x \in X \} \), in particular, the center of \( A \).

(iii) For each \( X \subseteq A \) the left and right annihilator of \( X \):

\[
\begin{align*}
\downarrow X & := \{ y \in A : yx = 0 \text{ for all } x \in X \} \\
X^\perp & := \{ y \in A : xy = 0 \text{ for all } x \in X \}.
\end{align*}
\]

(iv) For each idempotent \( e \) of \( A \) the left ideal \( Ae \), the right ideal \( eA \), the subalgebra \( eAe \).
(v) Maximal modular left, right, two-sided ideals.

(vi) If \( A \) is unitary, the set of the non-invertible elements of \( A \).

We proceed by stating some corollaries of the lemmas 3.5, and 3.6.

3.7 LEMMA. Let \( (A, \| \|_1, \| \|_2) \) be bicomplete and let \( e \) be an idempotent in \( A \). Then \( \Delta | eA \) is equivalent to \( \Delta_{eA} \)
\( \Delta | Ae \) is equivalent to \( \Delta_{eA} \)
\( \Delta | eAe \) is equivalent to \( \Delta_{eAe} \).

PROOF. For \( s \in eA \) we have \( \Delta_{eA}(s) = \Delta_{eA}(es) \leq \text{bij 3.5} \leq \max(\| e \|_1, \| e \|_2) \Delta(s) \leq \max(\| e \|_1, \| e \|_2) \Delta_{eA}(s) \). The other proofs are similar.

3.8 LEMMA. Let \( (A, \| \|_1, \| \|_2) \) be bicomplete, and let \( I \) be a universally closed left ideal, that, as a \( K \)-algebra, has UBAT. Then \( \text{Ker } \Delta \subset \frac{1}{I} \).

PROOF. \( (I, \| \|_1, \| \|_2) \) is bicomplete, \( I \) has UBAT, so \( \| \|_1 \sim \| \|_2 \) on \( I \). Thus \( \Delta_{\frac{1}{I}} \) is a norm: \( \text{Ker } \Delta_{\frac{1}{I}} = \{ 0 \} \). By 3.5, \( (\text{Ker } \Delta) \cdot I = \{ 0 \} \) so \( \text{Ker } \Delta \subset \frac{1}{I} \).

3.9 THEOREM. Let \( I \) be a universally closed two-sided ideal in a \( K \)-algebra \( A \). Suppose that \( I \cap \frac{1}{I} \cap I = \{ 0 \} \) (this is true, for example, if for any two-sided ideal \( J \) in \( A \), \( J^2 = \{ 0 \} \) implies \( J = \{ 0 \} \)). Then, if \( I, A/I \) have UBAT then so has \( A \).

PROOF. By 3.6 and 3.5, if \( (A, \| \|_1, \| \|_2) \) is bicomplete then \( \text{Ker } \Delta \subset I \cap \frac{1}{I} \cap I = \{ 0 \} : \Delta \) is a norm, so \( \| \|_1 \sim \| \|_2 \).

3.10 THEOREM. ([3], (1.1)) Let \( A \) be a \( K \)-algebra. Suppose the intersection of the maximal modular left (right, two-sided) ideals
with finite codimension is zero. Then A has UBAT.

PROOF. Maximal modular left (right, two-sided) ideals are universally closed. Now apply 3.6.

4. Topological zero divisors.

In this section we will show that in many cases, for a bicomplete algebra, the ideal Ker A consists only of topological zero divisors. (Compare [4] (2.5.6)). We first consider unitary algebras.

Let A be a K-Banach algebra with identity 1. Set

\[ A^1 := \{ x \in A : x^{-1} \text{ exists} \} \]

Then \( A^1 \) is open.

Let us call \( T(A) := A^1 \).

An element \( x \in A \) is called a strong two-sided topological zero divisor iff there exist \( s_1, s_2, \ldots \in A \) such that \( \inf ||s_n|| > 0 \) and

\[ \lim_{n \to \infty} s_n x = \lim_{n \to \infty} x s_n = 0. \]

4.1 LEMMA. Let A be a K-Banach algebra with unit. Then

\[ x \in T(A) \setminus A^1 \Rightarrow x \text{ is a strong two-sided topological zero divisor.} \]

Proof: Let \( x \in T(A) \setminus A^1 \). Then there are \( x_n \in A^1 \) such that \( \lim_{n \to \infty} x_n = x \).

Then we claim that \( ||x_n^{-1}|| \) is unbounded. Suppose namely that

\[ \sup_n ||x_n^{-1}|| = M < \infty \] then for \( n,m \in \mathbb{N} \).

\[ ||x_n^{-1}x_m^{-1}|| = ||x_n^{-1}(x_n^{-1}-x_n^{-1})x_m^{-1}|| \leq M^2 ||x_n^{-1}-x_m^{-1}||, \text{ so } y := \lim_{n \to \infty} x_n^{-1} \text{ exists.} \]

But then \( xy = yx = 1 : x \) would be invertible, a contradiction.

By taking a suitable subsequence, assume \( \lim_{n \to \infty} ||x_n^{-1}|| = \infty \).

There are \( \lambda_n \in \mathbb{K}, c_1, c_2 \in \mathbb{R}^+ \) such that
23-08

\[ \left\| x_n^{-1} \right\| \leq \frac{c_1}{\lambda_n} \leq c_2 \quad (n \in \mathbb{N}). \]

Then \( \lim_{n \to \infty} \lambda_n = \infty \) and

\[ \frac{x_n^{-1}}{\lambda_n} = \frac{(x-x_n)x_n^{-1}}{\lambda_n} + \frac{1}{\lambda_n} \to 0 \quad (\text{if } n \to \infty) \]

hence \( x_n \to 0 \), where \( s_n := \lambda_n^{-1} x_n^{-1} \).

Analogously,

\[ \frac{s_n x}{\lambda_n} = \frac{x_n^{-1} x - x_n^{-1} (x-x_n)}{\lambda_n} + \frac{1}{\lambda_n} \to 0 \quad (\text{if } n \to \infty). \]

Thus indeed, \( x \) is a strong two-sided topological zero divisor in the above sense.

For a bicomplete algebra with unit \((A, \| \cdot \|_1, \| \cdot \|_2)\) let us define \( T^1(A) \) (resp. \( T^2(A) \)) to be the closure of \( A^i \) with respect to \( \| \cdot \|_1 \) (resp. \( \| \cdot \|_2 \)). We have

4.2 LEMMA. Let \((A, \| \cdot \|_1, \| \cdot \|_2)\) be a bicomplete algebra with unit.

Then \( \text{Ker} \Delta \subseteq T^1(A) \cap T^2(A) \).

PROOF. Choose \( \lambda_1, \lambda_2, \ldots \in \mathbb{K} \) such that \( \| \lambda_n \| \geq n \) \((n \in \mathbb{N})\). Let \( \Delta(x) = 0 \) for some \( x \in A \). Let \( n \in \mathbb{N} \). Then \( \Delta(\lambda_n x) = 0 \), so there is a sequence \( x_1, x_2, \ldots \) in \( A \) such that \( \lim_{k \to \infty} \| x_k \|_1 = 0 \), \( \lim_{k \to \infty} \| \lambda_n x - x_k \|_2 = 0 \). So \( 1 - x_k \) is invertible for large \( k \). It follows that \( 1 - \lambda_n x \in T^2(A) \), hence so is \( x^{-\lambda_n^{-1}} \). Now \( x = \lim_{n \to \infty} (x - \lambda_n^{-1}) \) (with respect to \( \| \cdot \|_2 \)), so \( x \in T^2(A) \).

Similarly, \( x \in T^1(A) \).

Thus we have the following alternative.

4.3 THEOREM. Let \((A, \| \cdot \|_1, \| \cdot \|_2)\) be a bicomplete algebra with unit.

and with separating seminorm \( \Delta \). Then we have either (i) or (ii):

\[ \text{(i) } \| \Delta(x) \|_1 < \infty \]
\[ \text{(ii) } \| \Delta(x) \|_2 < \infty. \]
(i) Ker $\Delta = A$, $A = T_1(A) = T_2(A)$. If an element of $A$ is not invertible then it is a strong two-sided topological zero divisor with respect to both norms.

(ii) Ker $\Delta$ is a proper ideal. Ker $\Delta$ consists only of strong two-sided topological zero divisors with respect to both norms.

NOTE. In contrast to the classical theory, case (i) can occur. In fact the separating seminorm of $| \cdot |$ and $| \cdot |'$ in Example (1.1) must be zero.

An example of case (i) in which $A$ is not a field can easily be made. Let $A := \mathbb{C}_p \times \mathbb{C}_p$ with pointwise operations. Let

$$
\| (a_1, a_2) \| := \max(|a_1|, |a_2|), \quad \text{if } (a_1, a_2) \in A
$$

$$
\| (a_1, a_2) \|' := \max(|a_1'|, |a_2'|), \quad \text{if } (a_1, a_2) \in A
$$

Then $(A, \| \cdot \|, \| \cdot \|')$ is a bicomplete $\mathbb{Q}_p$-algebra, is not a field.

The separating seminorm is zero. (Bij 3.5 (i), with $D := (0) \times \mathbb{C}_p$, we have $\Delta(0,1) = 0$. Similarly, $\Delta(1,0) = 0$ so $\Delta = 0$).

A Tate algebra is a quotient of $K[X_1, \ldots, X_n]$, where the latter is the algebra of formal power series in $X_1, \ldots, X_n$ of which the coefficients tend to zero. (see [1] and [6]). We have the following application of 4.3.

4.4 THEOREM. Let $(A, \| \cdot \|, \| \cdot \|')$ be a bicomplete Tate algebra with separating seminorm $\Delta$. Then Ker $\Delta$ consists of only nilpotent elements. In particular, a Tate algebra without nilpotents $\neq 0$ has UBAT.

PROOF. Since $A$ is noetherian ([3] 1.5) each ideal in $A$ is universally closed. Let $P$ be a prime ideal of $A$. Then $A/P$ is a noetherian Banach
algebra with respect to both quotient norms, (again denoted by \( \| \|_1 \) and \( \| \|_2 \)). Now \( A/P \) has maximal ideals of finite codimension ([6] (4.5)), so the separating seminorm of the norms on \( A/P \) is nonzero by 3.6. Theorem 4.2 (ii) tells us that its kernel consists only of topological zero divisors with respect to both norms.

On the other hand for any \( x \in A/P, x \neq 0 \) the map \( t \mapsto tx (t \in A/P) \) is a bijection of \( A/P \) onto the principal ideal \( I \) generated by \( x \) (\( A/P \) has no zero divisors). \( I \) is universally closed in \( A/P \), the norms \( tx \mapsto \| t \|_1 \) and \( tx \mapsto \| tx \|_1 (t \in A/P) \) on \( I \) are complete, the latter is majorized by the first. By the open mapping theorem they are equivalent: there is a \( c > 0 \) such that \( \| tx \|_1 \geq c \| t \|_1 (t \in A/P) \). It follows that \( x \) is not a topological zero with respect to \( \| \|_1 \).

Combining the results of the two previous paragraphs we conclude that \( \| \|_1 \) and \( \| \|_2 \) induce equivalent quotient norms on \( A/P \). By 3.6, \( \ker A \) is contained in the intersection of all prime ideals of \( A \), hence consists only of nilpotents.

Next we turn to K-algebras \( A \) without unit. Application of 4.3 to \( A_1 \) where \( A_1 \) is the usual unitary extension of \( A \) does not seem to lead to interesting results. We follow a different path.

An element \( x \) of a normed K-algebra \( A \) is called a two-sided topological zero divisor if there are sequences \( s_1, s_2, \ldots \), \( t_1, t_2, \ldots \) such that \( \inf \| s_n \| > 0, \inf \| t_n \| > 0, \lim s_n x = \lim t_n = 0 \).

We have the following analog of 4.3

4.5 THEOREM. Let \((A, \| \|_1', \| \|_2)\) be a bicomplete K-algebra without a unit, and with separating seminorm \( \Delta \). Then we have either (i) or (ii):

(i) \( \ker \Delta = A \). \( A \) has a one-sided unit.
(ii) Ker $\Delta$ consists only of two-sided topological zero divisors with respect to both norms.

PROOF. We prove that if we have not (ii) then we have (i). Hence suppose we have $s \in A$ for which $\Delta(s) = 0$ and such that $s$ is not a two-sided topological zero divisor with respect to both norms. Without loss, assume that the map $x \mapsto xs$ ($x \in A$) is a homeomorphism of $A$ onto $A_s$ with respect to $\| \|_1$. Now let $A_1$ be the usual unitary extension of $A$.

Define for $i = 1, 2$

$$\| (\lambda, x) \|_i := \max(|\lambda|, \| x \|_i) \quad (\lambda \in K, x \in A)$$

Then $(A_1, \| \|_1, \| \|_2)$ is bicomplete and, by 3.5, $\Delta_{A_1}(s) = 0$. Since $A$ is a maximal ideal in $A_1$ of codimension 1, Ker $\Delta_{A_1} \neq A_1$ (3.6). Hence by 4.3 (ii) there are $(\lambda_n, x_n) \in A_1$ such that $\lim_{n \to \infty} s(\lambda_n, x_n) = 0$ and such that

$$\lim (\lambda_n, x_n) s = 0 \text{ in the sense of } \| \|_1$$

and such that $c := \inf \| (\lambda_n, x_n) \|_1 > 0$. If for some subsequence $\mu_1, \mu_2, \ldots$ of $\lambda_1, \lambda_2, \ldots$ we had $\lim_{n \to \infty} \mu_n = 0$ then $\lim_{n \to \infty} y_n \|_1 \to 0$, $\lim_{n \to \infty} y_n \|_1 \to 0$, $\| y_n \| \geq c$ for some subsequence $y_1, y_2, \ldots$ of $x_1, x_2, \ldots$, contradicting our assumption on $s$.

Hence we may assume $\inf_{n} |\lambda_n| > 0$. From

$$\lim_{n \to \infty} (\lambda_n s + x_n s) = 0 \quad \text{(in the sense of } \| \|_1)$$

we arrive at

$$\lim_{n \to \infty} (s + x_n \lambda_n s) = 0 \quad \text{(in the sense of } \| \|_1)$$

It follows that $s \in A_s$ (here the closure if meant with respect to $\| \|_1$).

But $A_s$ is closed, hence there is $e \in A$ for which $s = es$. For each $x \in A$ we have $(xe-x)s = 0$ and since $s$ is no left zero divisor, $xe-x = 0$.

We conclude that $e$ is a one-sided unit for $A$. We proceed to prove that $\Delta(e) = 0$ which will finish the proof. The algebra $eAe$ is universally
closed in $\mathbb{C}$, $e$ is a unit in $e\mathbb{A}$ and $s = es = ese \in e\mathbb{A}$. We have
\[ \Delta(s) = 0, \text{ so by 3.7, } \Delta_{e\mathbb{A}}(s) = 0. \] Since $s$ is not a left topological zero divisor in $\mathbb{A}$ it is certainly not in $e\mathbb{A}$. Applying 4.3 to $e\mathbb{A}$ we see that we are in case (i): $\Delta_{e\mathbb{A}} = 0$. It follows that $\Delta(e) = 0$.

In order to be able to conclude for certain algebras to be in case (i), we briefly look at $K$-algebras $\mathbb{A}$ without unit but having a one-sided unit $e$, say $xe = x$ for all $x \in \mathbb{A}$. Consider $e^\perp := \{y \in \mathbb{A} : ey = 0\}$. It is perfectly easy to see from $x = (x-ex) + ex (x \in \mathbb{A})$ that $\mathbb{A} = e^\perp \oplus e\mathbb{A}$. Since $e\mathbb{A} = e\mathbb{A}e$ is an algebra with a two-sided unit, we have $e^\perp \neq (0)$. $e^\perp$ is a two-sided ideal for which $e\mathbb{A}^\perp = (0)$. In particular all products in $e^\perp$ are zero. Therefore:

4.6 COROLLARY. Let $(\mathbb{A}, || \ ||_1, || \ ||_2)$ be a bicomplete $K$-algebra without unit. Suppose one of the following conditions holds.

(i) $\mathbb{A}$ is commutative.

(ii) $\mathbb{A}$ has no one-sided unit.

(iii) For a two-sided ideal $J$ in $\mathbb{A}$, $J^2 = (0)$ implies $J = (0)$.

(iv) $\mathbb{A}^\perp = (0)$.

Then $\ker \Delta$ contains only two-sided topological zero divisors with respect to both norms.

An application:

4.7 THEOREM. Let $(\mathbb{A}, || \ ||)$ be a $K$-Banach algebra whose norm is multiplicative. If $\mathbb{A}$ is not a (skew) field then $\mathbb{A}$ has UBAT.

PROOF. Let $|| \ ||'$ be some Banach algebra norm on $\mathbb{A}$ and let $\Delta$ be the separating seminorm of $|| \ ||$ and $|| \ ||'$. Since $|| \ ||$ is multiplicative, $\mathbb{A}$ has no topological zero divisors with respect to $|| \ ||$, except $0$. If $\mathbb{A}$ has no unit, apply 4.6 (use (iii) or (iv)) to arrive at $\ker \Delta = (0)$. 


If $A$ has a unit we may use 4.3: case (i) would imply that $A$ is a (skew) field which is forbidden and case (ii) leads again to $\operatorname{Ker} A = (0)$.

5. The uniqueness of the norm topology of $L(E)$.

In this section $E$ is a $K$-Banach space, $L(E)$ is the $K$-algebra of all continuous linear operators $E \to E$, and $A$ is a $K$-Banach algebra.

Let $E$ be a (left) $A$-module with structure map $(a,\xi) \mapsto a\xi$ ($a \in A$, $\xi \in E$). We say that $E$ is 2-fold transitive if for each $\xi_1, \xi_2, \eta_1, \eta_2 \in E$, where $\xi_1, \xi_2$ are linearly independent, there is a $c \in A$ such that $a\xi_1 = \eta_1$, $a\xi_2 = \eta_2$.

By the density lemma of Jacobson we then have $n$-fold transitivity for each $n \in \mathbb{N}$, i.e., if $\xi_1, \ldots, \xi_n \in E$ are linearly independent and $\eta_1, \ldots, \eta_n \in E$ then there exists a $c \in A$ such that $a\xi_i = \eta_i$ ($i = 1, \ldots, n$).

The following is essentially what remains of the proof of Johnson's theorem [2] in the non-archimedean case.

5.1 Lemma. Let $E$ be a 2-fold transitive $A$-module such that the maps $\xi \mapsto a\xi$ ($\xi \in E$) are continuous for each $a \in A$. (Or, equivalently, in the corresponding representation $a \mapsto T_a$, all the $T_a$ are in $L(E)$). Then there exists $M > 0$ such that

$$||a\xi|| \leq M||a|| ||\xi||$$

(a $\in A$, $\xi \in E$)

Proof. By the uniform boundedness principle it suffices to show that the structure map $(a,\xi) \mapsto a\xi$ ($a \in A$, $\xi \in E$) is separately continuous, i.e., we have to show that for each $\xi \in E$ the map $a \mapsto a\xi$ ($a \in A$) is continuous. By 2-fold transitivity (in fact, irreducibility) these maps are continuous either for all $\xi \in E$, $\xi \neq 0$ or for no such $\xi$.

First assume $\dim_K E = \infty$. We assume that $a \mapsto a\xi$ is continuous only in
case \( \xi = 0 \) and shall derive a contradiction. Choose independent 
\( \xi_1, \xi_2, \ldots \in E \) such that \( 1 \leq \| \xi_i \| \leq 2 \) for all \( i \), and set 
\( J_i := \{ a \in A : a \xi_i = 0 \} \) (\( i = 1, 2, \ldots \)). Each \( J_i \) is a maximal modular left 
ideal of \( A \) (if \( x \xi_i = \xi_i \) then \( x \) is an identity modulo \( J_i \)), hence closed 
in \( A \). For each \( m \geq 2 \) we have

\[
(*) \quad A = (J_1 \cap J_2 \cap \ldots \cap J_{m-1}) + J_m
\]

(By the \( m \)-fold transitivity there is \( x \in A \) such that \( x \xi_1 = x \xi_2 = \ldots = x \xi_{m-1} = 0 \), 
\( x \xi_m \neq 0 \), hence \( x \in (J_1 \cap J_2 \cap \ldots \cap J_{m-1}) \), \( x \notin J_m \). Now \( J_m \) is maximal and 
\( (*) \) follows). The addition map \( (J_1 \cap \ldots \cap J_{m-1}) \times J_m \to A \) is continuous and 
surjective hence open by Banach's open mapping theorem. So there is \( \gamma > 0 \) 
such that we can write each \( a \in A \) as \( b+c \) where \( b \in J_1 \cap \ldots \cap J_{m-1}, \ c \in J_m \)

\[
\|b\| \leq \gamma \|a\|, \quad \|c\| \leq \gamma \|a\|.
\]

With the help of this one can choose inductively 
\( \xi_1, \xi_2, \ldots \in A \) such that for each \( n \in \mathbb{N}, \ n \geq 2 \)

\[
\|x^n\| \leq 2^{-n}; \quad x^n \in J_1 \cap \ldots \cap J_{n-1}; \quad \|x^n \xi_i\| \geq n + \| \sum_{i=2}^{n-1} x_i \xi_i \|
\]

using also the discontinuity at 0 of \( x \mapsto x \xi_n \).

Set \( z := \sum _{i=2}^{\infty} x_i \in A \). Since for \( n \in \mathbb{N}, \ n \geq 2 \) we have \( \sum_{i=2}^{n-1} x_i \in J \) we get

\[
\|z\xi_n\| = \|(x_2 + \ldots + x_n) \xi_n\| \geq \|x_n \xi_n\| - \| \sum_{i=2}^{n-1} x_i \xi_i \| \geq n.
\]

Thus, \( \lim_{n \to \infty} \|z\xi_n\| = \infty \). But the sequence \( \xi_1, \xi_2, \ldots \) is bounded, so this 
conflicts with the continuity of \( x \mapsto z \xi (\xi \in E) \).

If, finally, \( \dim K E < \infty \) the map \( a \mapsto a \xi (a \in A) \) can be decomposed:

\[
A + A/I \sim E
\]

where \( A/I \) is equipped with the quotient norm and where \( I := \{ x \in A : \) 
\( x \xi = 0 \} \). It follows that \( a \mapsto a \xi (a \in A) \) is continuous.

5.2 THEOREM. Let \( (B, \| \cdot \|_1, \| \cdot \|_2) \) be a bicomplete \( K \)-algebra, and 
suppose \( E \) is a 2-fold transitive \( B \)-module such that the map
\( \xi \to b\xi \, (\xi \in E) \) is \emph{continuous} for each \( b \in B \). Set
\[
I_E := \{ x \in B : x\xi = 0 \text{ for all } \xi \in E \}.
\]

Then \( \ker \Delta \subseteq I_E \) where \( \Delta \) is the \emph{separating seminorm} of \( \| \|_1 \) and \( \| \|_2 \).

**PROOF.** Let \( b \notin I_E \). Then there is \( \xi \in E \) such that \( b\xi \neq 0 \).

Lemma 5.1 yields the existence of \( M > 0 \) such that
\[
\begin{align*}
\| x\xi \| &\leq M \| x \|_1 \| \xi \| \\
\| x\xi \| &\leq M \| x \|_2 \| \xi \|
\end{align*}
\]
\( (x \in B, \xi \in E) \).

The seminorm \( p: x \mapsto M^{-1} \| \xi \|^{-1} \| x\xi \| \) (\( x \in B \)) satisfies \( p \leq \| \|_1 \), \( p \leq \| \|_2 \), \( p(b) \neq 0 \). So \( 0 < p(b) \leq \Delta(b) \). It follows that \( \ker \Delta \subseteq I_E \).

5.3 **COROLLARY.** Let \( E \) have the property that for each independent \( \xi_1, \xi_2 \in E \) and \( \eta_1, \eta_2 \) there exists \( T \in L(E) \) such that \( T\xi_1 = \eta_1 \), \( T\xi_2 = \eta_2 \). Then \( L(E) \) has UB\( \Lambda \).

**PROOF.** \( E \) is a 2-fold transitive \( L(E) \)-module under \( (T, \xi) \mapsto T\xi \, (T \in L(E), \xi \in E) \), satisfying the continuity condition of 5.2. \( I_E = \{ T \in L(E) : T\xi = 0 \text{ for all } \xi \in E \} = \{0\} \). Hence for each two Banach algebra norms the separating seminorm is a norm, so the norms are equivalent.

Finally we indicate a class of Banach spaces \( E \) for which \( L(E) \) has UB\( \Lambda \).

For the notions used below see [5].

5.4 **THEOREM.** Let \( E \) be a \( K \)-Banach space. Each of the following conditions implies that \( L(E) \) has a unique Banach algebra topology.

(i) \( K \) is spherically complete.

(ii) \( E \) has a base. (In particular, \( E \) is of countable type.)

(iii) \( E \) is the dual of some \( K \)-Banach space.

(iv) \( E \) is spherically complete.
PROOF. We shall first prove: if the elements of the dual $E'$ separate the points of $E$ then $l(E)$ has UBAT, which takes care of (i), (ii) and (iii). Let $\xi_1, \xi_2 \in E$ be independent and $\eta_1, \eta_2 \in E$. There are $f, g \in E'$ such that $f_i(\xi_j) = \delta_{1j}$ ($i,j \in \{1,2\}$). The map

$$\xi \mapsto f_1(\xi)\eta_1 + f_2(\xi)\eta_2$$

is in $l(E)$ and sends $\xi_i$ into $\eta_i$ ($i = 1, 2$). Now apply 5.3.

Finally, we prove (v). Let $\xi_1, \xi_2 \in E$ be independent and $\eta_1, \eta_2 \in E$. Let $A$ be the map $\lambda_1 \xi_1 + \lambda_2 \xi_2 \mapsto \lambda_1 \eta_1 + \lambda_2 \eta_2$ ($\lambda_1, \lambda_2 \in K$), $A : D \to E$ where $D$ is the subspace of $E$ spanned by $\xi_1$ and $\xi_2$. By the spherical completeness of $E$, $A$ can be extended to an element of $l(E)$. Now apply 5.3.

PROBLEM: Do there exist K-Banach spaces $E$ for which $l(E)$ admits inequivalent Banach algebra norms?

REFERENCES


