UNIQUENESS OF THE BANACH ALGEBRA TOPOLOGY
FOR NON-ARCHIMEDEAN ALGEBRAS

W.H. Schikhof

1. Introduction.

An algebra $A$ over $\mathbb{R}$ or $\mathbb{C}$ is said to have a unique Banach algebra topology if any two Banach algebra norms on $A$ are equivalent. Johnson's theorem \cite{2} is very satisfactory; it states that a semi-simple algebra over $\mathbb{R}$ or $\mathbb{C}$ has this property.

In this note we are concerned with the non-archimedean analogue. Thus, let $A$ be an algebra over a complete non-archimedean valued field $K$. We say that $A$ has UBAT (a unique Banach algebra topology) if each two (non-archimedean) Banach algebra norms on $A$ are equivalent. Our problem is to find reasonable conditions on $A$ implying the UBAT property.

It is known \cite{3} that in the non-archimedean case semi-simple algebras (even commutative fields) may fail to have UBAT. In fact, we have

1.1 Example. Let $p$ be a prime. Let $\mathbb{C}_p$ be the completion (with respect to the natural valuation $\mid \cdot \mid$) of the algebraic closure of the field $\mathbb{Q}_p$ of the $p$-adic numbers. Then $(\mathbb{C}_p, \mid \cdot \mid)$ is a valued field and a $\mathbb{Q}_p$-Banach algebra. There exists a valuation $\mid \cdot \mid'$ on $\mathbb{C}_p$, not equivalent to $\mid \cdot \mid$, for which $(\mathbb{C}_p, \mid \cdot \mid')$ is also a $\mathbb{Q}_p$-Banach algebra.
PROOF. It is well known that \( \mathbb{C} \) is algebraically closed. Let \( I \) be a maximal set of algebraically independent elements over \( \mathbb{Q} \). Then

\[ \mathbb{Q}_p \subset \mathbb{Q}_p(I) \subset \mathbb{C}_p, \mathbb{C}_p \]

is the algebraic closure of \( \mathbb{Q}_p(I) \), \( I \neq \emptyset \). Fix \( x \in I \), and define \( \sigma : \mathbb{Q}_p(I) \to \mathbb{Q}_p(I) \) by \( \sigma(x) = px \) and \( \sigma(y) = y \) for \( y \in I \), \( y \neq x \). Then \( \sigma \) is an endomorphism \( \mathbb{Q}_p(I) \to \mathbb{Q}_p(I) \) that can be extended to an endomorphism \( \mathbb{C}_p(I) \to \mathbb{C}_p(I) \). It is easy to see that \( \sigma \) is also a \( \mathbb{Q}_p \)-algebra homomorphism. Define \( |\cdot|' \) via

\[ |x|' := |\sigma(x)| \quad (x \in \mathbb{C}_p). \]

Then \( |\cdot|' \) is not equivalent to \( |\cdot| \) since \( |x^n|' = |p|^n|x^n| \), so there is no \( c > 0 \) for which \( |\cdot|' \geq c|\cdot| \). The rest is obvious.

With 1.1 in mind it is rather surprising that we can prove that a \( K \)-Banach algebra whose norm is multiplicative and that is not a field has UBAT (see 4.7). Further results are:

Tate algebras without nilpotents \( \neq 0 \) have UBAT. (4.4)

\( L(E) \) has UBAT if \( E \) is a well-behaved Banach space. (5.4).

For background information on non-archimedean fields, Banach spaces and algebras we refer to [5].

In the sequel \( K \) is a non-archimedean non-trivially valued complete field.

Instead of "\( A \) is a \( K \)-Banach algebra with respect to the norms \( \|\cdot\|_1 \) and \( \|\cdot\|_2 \)" we will sometimes use the expression

"(\( A, \|\cdot\|_1, \|\cdot\|_2 \) is bicomplete)."

2. Algebras of functions.

Theorem 2.1 is more or less contained in [3].

Let \( X \) be a nonempty set. For \( \mathbb{K} \)

\[ \| f \|_\infty := \sup \{|f(x)| : x \in X \} \] (possibly \( \infty \)). A function algebra is a K-algebra that is, for some X, (algebraically isomorphic to) a subalgebra of \( \mathcal{K} \). Without much effort we can prove

2.1 THEOREM. Let \( F \) be a function algebra. Then

(i) \( F \) has UBAT

(ii) If \( \| \| \) is a Banach algebra norm on \( F \) then
\[ \| \| \geq \| \|_\infty . \]

PROOF. Let \( \| \| \) be a Banach algebra norm on \( F \). Let \( a \in X \). The map 
\[ f \mapsto f(a) \quad (f \in F) \]
is a homomorphism: \( F \to K \), so by [5] it has norm \( \leq 1 \): 
\[ \| f(a) \| \leq \| f \| . \] It follows that \( \| \| \|_\infty \leq \| \| \). Now let \( \| \| \) and 
\[ \| \|_1 \] \( \| \|_2 \) be two Banach algebra norms on \( F \). We prove that the identity:
\[(F, \| \|_1) \to (F, \| \|_2) \]
is continuous. Let \( f, f_1, f_2, \ldots \in F \) such that 
\[ \| f_n \|_1 \to 0, \| f_n - f \|_2 \to 0. \] By the foregoing, \( \| f_n \|_\infty \to 0, \| f_n - f \|_\infty \to 0 \), so \( f = 0 \). Continuity follows after applying the closed graph theorem.

3. The separating seminorm.

(This is a non-archimedean version of [4], (2.5.1))

3.1 DEFINITION. Let \( \| \|_1 \) and \( \| \|_2 \) be norms on a K-vector space \( E \).

The function \( \Delta : E \to \mathbb{R} \) defined by
\[ \Delta(s) := \inf \{ \max(\| x \|_1, \| y \|_2) : x + y = s \} \quad (s \in E) \]
is called the separating seminorm of \( \| \|_1 \) and \( \| \|_2 \).

One easily checks that \( \Delta \) is the largest among the (non-archimedean) seminorms that are \( \leq \| \|_1 \) and \( \leq \| \|_2 \). As in [4] we have

3.2 LEMMA In case \( \| \|_1 \) and \( \| \|_2 \) are complete norms on \( E \) then:
\[ \Delta \] is a norm \( \leftrightarrow \| \|_1 \vee \| \|_2 \).
3.3 **LEMMA.** Let $A$ be a normed $K$-algebra with respect to $\| \cdot \|_1$ and $\| \cdot \|_2$, and let $\Delta$ be its separating seminorm. Then Ker $\Delta$ is a two-sided ideal that is closed with respect to both norms. In fact, we have for $s,t \in A$:

\[
\Delta(st) \leq \Delta(s) \max(\|t\|_1, \|t\|_2)
\]

\[
\Delta(st) \leq \Delta(t) \max(\|s\|_1, \|s\|_2).
\]

The proofs of 3.2 and 3.3 are elementary and similar to the ones in [4], (2.5) and are omitted.

3.4 **DEFINITION.** For a linear subspace $D$ of an algebra $A$ that is normed by $\| \cdot \|_1$, $\| \cdot \|_2$ we set for $d \in D$:

\[
\Delta_D(d) := \inf\{\max(\|x\|_1, \|y\|_2) : x,y \in D, x+y = d\}
\]

($\Delta_D$ is the separating seminorm of the restriction of $\| \cdot \|_1$ and $\| \cdot \|_2$ to $D$).

We have the following elementary facts concerning the behaviour of $\Delta$ with respect to subalgebras and quotients:

3.5 **LEMMA.** With the notations as above we have

(i) $\Delta_D \geq \Delta|_D$, so Ker $\Delta_D \subseteq$ Ker $\Delta \cap D$.

(ii) Let $D$ be a left ideal, then for $x \in A$, $t \in D$:

\[
\Delta_D(xt) \leq \Delta(x) \max(\|t\|_1, \|t\|_2)
\]

\[
\Delta_D(xt) \leq \max(\|x\|_1, \|x\|_2) \Delta_D(t), \text{ so}
\]

Ker $\Delta_D$ is a left ideal in $A$, satisfying

$\text{Ker} \Delta \cdot D \subseteq \text{Ker} \Delta_D \subseteq \text{Ker} \Delta \cap D$.

**PROOF.** (i) For $t \in D$ we have $\Delta(t) = \inf\{\max(\|x\|_1, \|y\|_2) : x+y = t, x,y \in A\} \leq \inf\{\max(\|x\|_1, \|y\|_2) : x+y = t, x,y \in D\} = \Delta_D(t)$. 

(ii) $\Delta_D(x_t) = \inf \max_{z \in D} (\|z\|_1, \|xt-z\|_2) \leq \inf \max_{y \in A} (\|yt\|_1, ||xt-yt||_2)$

\[ \leq \inf \max_{y \in A} (\|y\|_1, t\|_1, ||x-y\|_2, t\|_2) \leq \Delta(x) \max (\|t\|_1, ||t||_2). \]

Also, $\Delta_D(x_t) = \inf \max_{z \in D} (\|z\|_1, ||xt-z||_2) \leq \inf \max_{d \in D} (\|xd\|_1, ||xt-xd||_2) \leq \max (\|x\|_1, ||x||_2) \cdot \Delta_D(t).$

3.6 LEMMA. Let $(A, \| \|_1, \| \|_2)$ be bicomplete. Suppose $D$ is a closed linear subspace with respect to both $\| \|_1$ and $\| \|_2$, and suppose that the quotient norms on $A/D$ are equivalent. Then $\ker \Delta \subset D$.

PROOF. Let $\Delta(x) = 0$ for some $x \in A$. Then there are $x_1, x_2, \ldots$ in $A$ such that $\|x-x_n\|_1 \to 0$, $\|x_n\|_2 \to 0$. Let $\pi: A \to A/D$ be the quotient map. Then $\lim \pi(x_n) = \pi(x)$ for the first quotient norm and $\lim \pi(x_n) = 0$ for the second one. Hence, $\pi(x) = 0$ i.e., $x \in D$.

A subset of a K-algebra $A$ is called universally closed if it is closed with respect to each Banach algebra topology on $A$. (In case $A$ has no Banach algebra topology then, by definition, each subset of $A$ is universally closed). Examples of universally closed sets are

(i) $\emptyset$, $A$, singletons, finite dimensional linear subspaces.

(ii) For each set $X \subset A$ its commutant $X' := \{y \in A : yx = xy \text{ for all } x \in X\}$, in particular, the center of $A$.

(iii) For each $X \subset A$ the left and right annihilator of $X$:

\[ X^\perp := \{y \in A : yx = 0 \text{ for all } x \in X\} \]

\[ X'^\perp := \{y \in A : xy = 0 \text{ for all } x \in X\}. \]

(iv) For each idempotent $e$ of $A$ the left ideal $eA$, the right ideal $e^\perp A$, the subalgebra $eAe$. 

(v) Maximal modular left, right, two-sided ideals.

(vi) If $A$ is unitary, the set of the non-invertible elements of $A$.

We proceed by stating some corollaries of the lemmas 3.5, and 3.6.

3.7 LEMMA. Let $(A, \| \cdot \|_1, \| \cdot \|_2)$ be bicomplete and let $e$ be an idempotent in $A$. Then $\Delta|eA$ is equivalent to $\Delta_eA$

$\Delta|eAe$ is equivalent to $\Delta_{eA}$

$\Delta|eAe$ is equivalent to $\Delta_{eAe}$.

PROOF. For $s \in eA$ we have $\Delta_e (s) = \Delta_e (es) \leq \max (\| e\|_1, \| e\|_2) \Delta_e (s) \leq \max (\| e\|_1, \| e\|_2) \Delta_e (s)$. The other proofs are similar.

3.8 LEMMA. Let $(A, \| \cdot \|_1, \| \cdot \|_2)$ be bicomplete, and let $I$ be a universally closed left ideal, that, as a $K$-algebra, has UBAT. Then $\text{Ker} \Delta \subset \frac{1}{I}$.

PROOF. $(I, \| \cdot \|_1, \| \cdot \|_2)$ is bicomplete, $I$ has UBAT, so $\| \|_1 \| \|_2$ on $I$. Thus $\frac{1}{I}$ is a norm: $\text{Ker} \frac{1}{I} = \{0\}$. By 3.5, $(\text{Ker} \Delta) \cdot I = \{0\}$ so $\text{Ker} \Delta \subset \frac{1}{I}$.

3.9 THEOREM. Let $I$ be a universally closed two-sided ideal in a $K$-algebra $A$. Suppose that $I \cap I = (0)$ (this is true, for example, if for any two-sided ideal $J$ in $A$, $J^2 = (0)$ implies $J = (0)$). Then, if $I$, $A/I$ have UBAT then so has $A$.

PROOF. By 3.6 and 3.5, if $(A, \| \cdot \|_1, \| \cdot \|_2)$ is bicomplete then $\text{Ker} \Delta \subset \frac{1}{I} \cap \frac{1}{I} = (0) : \Delta$ is a norm, so $\| \|_1 \| \|_2$.

3.10 THEOREM. ([3], (1.1)) Let $A$ be a $K$-algebra. Suppose the intersection of the maximal modular left (right, two-sided) ideals
with finite codimension is zero. Then $A$ has UBAT.

**Proof.** Maximal modular left (right, two-sided) ideals are universally closed. Now apply 3.6. $\mathbb{E}$

4. **Topological zero divisors.**

In this section we will show that in many cases, for a bicomplete algebra, the ideal $\text{Ker} \ A$ consists only of topological zero divisors. (Compare [4] (2.5.6)). We first consider unitary algebras.

Let $A$ be a $K$-Banach algebra with identity $1$. Set

$$A^i := \{x \in A : x^{-1} \text{ exists} \}$$

Then $A^i$ is open.

Let us call $T(A) := A^i$.

An element $x \in A$ is called a **strong two-sided topological zero divisor** iff there exist $s_1, s_2, \ldots \in A$ such that $\inf_{n} \|s_n\| > 0$ and

$$\lim_{n} s_n x = \lim_{n} x s_n = 0.$$ 

4.1 **Lemma.** Let $A$ be a $K$-Banach algebra with unit. Then

$$x \in T(A) \setminus A^i \Rightarrow x \text{ is a strong two-sided topological zero divisor.}$$

**Proof:** Let $x \in T(A) \setminus A^i$. Then there are $x_n \in A^i$ such that $\lim_{n} x_n = x$. Then we claim that $\|x_n^{-1}\|$ is unbounded. Suppose namely that

$$\sup_{n} \|x_n^{-1}\| = M < \infty \text{ then for } n, m \in \mathbb{N}.$$ 

$$\|x_n^{-1} x_m^{-1}\| = \|x_n^{-1}(x_n^{-1} x_m^{-1})\| \leq M^2 \|x_n^{-1} x_m^{-1}\|,$$ 

so $y := \lim_{n} x_n^{-1}$ exists.

But then $xy = yx = 1 : x$ would be invertible, a contradiction.

By taking a suitable subsequence, assume $\lim_{n} \|x_n^{-1}\| = \infty$.

There are $\lambda_n \in K, c_1, c_2 \in \mathbb{R}^+$ such that
Then \(\lim_{n \to \infty} |\lambda_n| = \infty\) and
\[
\frac{(x - x_n)x_n^{-1}}{\lambda_n} = \frac{1}{\lambda_n} + 0 \quad (\text{if } n \to \infty)
\]
hence \(x_n \to 0\), where \(s_n := \frac{1}{\lambda_n} x_n^{-1}\).

Analogously,
\[
s_n x = \frac{x_n^{-1} x}{\lambda_n} = \frac{1}{\lambda_n} + 0 \quad (\text{if } n \to \infty).
\]

Thus indeed, \(x\) is a strong two-sided topological zero divisor in the above sense.

For a bicomplete algebra with unit \((A, \|\cdot\|_1, \|\cdot\|_2)\) let us define \(T_1(A)\) (resp. \(T_2(A)\)) to be the closure of \(A\) with respect to \(\|\cdot\|_1\) (resp. \(\|\cdot\|_2\)). We have

4.2 LEMMA. Let \((A, \|\cdot\|_1, \|\cdot\|_2)\) be a bicomplete algebra with unit. Then \(\text{Ker} \Delta \subset T_1(A) \cap T_2(A)\).

PROOF. Choose \(\lambda_1, \lambda_2, \ldots \in \mathbb{K}\) such that \(|\lambda_n| \geq n\) \((n \in \mathbb{N})\). Let \(\Delta(x) = 0\) for some \(x \in A\). Let \(n \in \mathbb{N}\). Then \(\Delta(\lambda_n x) = 0\), so there is a sequence \(x_1, x_2, \ldots \in A\) such that \(\lim_{k \to \infty} \|x_k\|_1 = 0\), \(\lim_{k \to \infty} \|\lambda_n x - x_k\|_2 = 0\) So \(1 - x_k\) is invertible for large \(k\). It follows that \(1 - \lambda_n x \in T_2(A)\), hence so is \(x_n^{-1}\). Now \(x = \lim_{n \to \infty} (x - \lambda_n^{-1})\) (with respect to \(\|\cdot\|_2\)), so \(x \in T_2(A)\).

Similarly, \(x \in T_1(A)\).

Thus we have the following alternative.

4.3 THEOREM. Let \((A, \|\cdot\|_1, \|\cdot\|_2)\) be a bicomplete algebra with unit.

and with separating seminorm \(\Delta\). Then we have either (i) or (ii):
(i) \( \text{Ker } \Delta = A, A = T_1(A) = T_2(A) \). If an element of \( A \) is not invertible then it is a strong two-sided topological zero divisor with respect to both norms.

(ii) \( \text{Ker } \Delta \) is a proper ideal. \( \text{Ker } \Delta \) consists only of strong two-sided topological zero divisors with respect to both norms.

NOTE. In contrast to the classical theory, case (i) can occur. In fact the separating seminorm of \( | \cdot | \) and \( | \cdot |' \) in Example (1.1) must be zero. An example of case (i) in which \( A \) is not a field can easily be made. Let \( A := \mathbb{C}_p \times \mathbb{C}_p \) with pointwise operations. Let

\[
\| (a_1, a_2) \| := \max(|a_1|, |a_2|) \quad ((a_1, a_2) \in A)
\]

\[
\| (a_1, a_2)' \|' := \max(|a_1|', |a_2|')
\]

Then \( (A, \| \cdot \|, \| \cdot \|') \) is a bicomplete \( \mathbb{Q}_p \)-algebra, is not a field. The separating seminorm is zero. (Bij 3.5 (i), with \( D = (0) \times \mathbb{C}_p \), we have \( \Delta(0, 0) = 0 \). Similarly, \( \Delta(0, 0) = 0 \) so \( \Delta = 0 \).

A Tate algebra is a quotient of \( K\{ X_1, \ldots, X_n \} \), where the latter is the algebra of formal power series in \( X_1, \ldots, X_n \) of which the coefficients tend to zero. (see [1] and [6]). We have the following application of 4.3.

4.4 THEOREM. Let \( (A, \| \cdot \|, \| \cdot \|') \) be a bicomplete Tate algebra with separating seminorm \( \Delta \). Then \( \text{Ker } \Delta \) consists of only nilpotent elements. In particular, a Tate algebra without nilpotents \( \neq 0 \) has UBAT.

PROOF. Since \( A \) is noetherian ([3] 1.5) each ideal in \( A \) is universally closed. Let \( P \) be a prime ideal of \( A \). Then \( A/P \) is a noetherian Banach
algebra with respect to both quotient norms, (again denoted by $\| \cdot \|_1$ and $\| \cdot \|_2$). Now $A/P$ has maximal ideals of finite codimension ([6] (4.5)), so the separating seminorm of the norms on $A/P$ is nonzero by 3.6. Theorem 4.2 (ii) tells us that its kernel consists only of topological zero divisors with respect to both norms.

On the other hand for any $x \in A/P$, $x \neq 0$ the map $t \mapsto tx$ ($t \in A/P$) is a bijection of $A/P$ onto the principal ideal $I$ generated by $x$ ($A/P$ has no zero divisors). $I$ is universally closed in $A/P$, the norms $tx \mapsto \| t \|_1$ and $tx \mapsto \| tx \|_1$ ($t \in A/P$) on $I$ are complete, the latter is majorized by the first. By the open mapping theorem they are equivalent: there is a $c > 0$ such that $\| tx \|_1 \geq c \| t \|_1$ ($t \in A/P$). It follows that $x$ is not a topological zero with respect to $\| \cdot \|_1$.

Combining the results of the two previous paragraphs we conclude that $\| \cdot \|_1$ and $\| \cdot \|_2$ induce equivalent quotient norms on $A/P$. By 3.6, $\text{Ker} \Delta$ is contained in the intersection of all prime ideals of $A$, hence consists only of nilpotents.

Next we turn to $K$-algebras $A$ without unit. Application of 4.3 to $A^*_1$ where $A^*_1$ is the usual unitary extension of $A$ does not seem to lead to interesting results. We follow a different path.

An element $x$ of a normed $K$-algebra $A$ is called a two-sided topological zero divisor if there are sequences $s_1, s_2, \ldots, t_1, t_2, \ldots$ such that $\inf \| s_n \| > 0$, $\inf \| t_n \| > 0$, $\lim s_n x = \lim xt_n = 0$.

We have the following analog of 4.3

4.5 THEOREM. Let $(A, \| \cdot \|_1', \| \cdot \|_2)$ be a bicomplete $K$-algebra without a unit, and with separating seminorm $\Delta$. Then we have either (i) or (ii):

(i) $\text{Ker} \Delta = A$. $A$ has a one-sided unit.
(ii) Ker A consists only of two-sided topological zero divisors with respect to both norms.

PROOF. We prove that if we have not (ii) then we have (i). Hence suppose we have \( s \in A \) for which \( \Delta(s) = 0 \) and such that \( s \) is not a two-sided topological zero divisor with respect to both norms. Without loss, assume that the map \( x \mapsto xs \ (x \in A) \) is a homeomorphism of \( A \) onto \( A_s \) with respect to \( \| \| \). Now let \( A_1 \) be the usual unitary extension of \( A \).

Define for \( i = 1,2 \)
\[
\| (\lambda, x) \|_i := \max(\| \lambda \|_i, \| x \|_i) \quad (\lambda \in K, x \in A)
\]

Then \((A_i, \| \|_1, \| \|_2)\) is bicomplete and, by 3.5, \( \Delta_{A_1}(s) = 0 \). Since \( A \) is a maximal ideal in \( A_1 \) of codimension 1, \( \text{Ker } A_{A_1} \neq A_1 \) (3.6). Hence by 4.3 (ii) there are \( (\lambda_n, x_n) \in A_1 \) such that \( \lim s(\lambda_n, x_n) = \lim s(x_n) = 0 \) in the sense of \( \| \|_1 \) and such that
\[
c := \inf \| (\lambda_n, x_n) \|_1 > 0.
\]
If for some subsequence \( \nu_1, \nu_2, \ldots \) of \( \lambda_1, \lambda_2, \ldots \)
we had \( \lim \nu_n = 0 \) then \( \| s y_n \|_1 \to 0, \| y_n s \|_1 \to 0, \| y_n \| \geq c \) for some subsequence \( y_1, y_2, \ldots \) of \( x_1, x_2, \ldots \), contradicting our assumption on \( s \).

Hence we may assume \( \inf \| \lambda_n \| > 0 \). From
\[
\lim_{n \to \infty} (\lambda_n + s) = 0 \quad \text{(in the sense of } \| \|_1)
\]
we arrive at
\[
\lim_{n \to \infty} (s + \frac{x_n}{\lambda_n}) = 0 \quad \text{(in the sense of } \| \|_1)
\]
It follows that \( s \in A_s \) (here the closure if meant with respect to \( \| \|_1 \)). But \( A_s \) is closed, hence there is \( e \in A \) for which \( s = es \). For each \( x \in A \) we have \( (xe-x)s = 0 \) and since \( s \) is no left zero divisor, \( xe-x = 0 \).

We conclude that \( e \) is a one-sided unit for \( A \). We proceed to prove that \( \Delta(e) = 0 \) which will finish the proof. The algebra \( eAe \) is universally
closed in $A$, $e$ is a unit in $eAe$ and $s = es = ese \in eAe$. We have

$$\Delta(s) = 0,$$

so by 3.7, $\Delta_{eAe}(s) = 0$. Since $s$ is not a left topological

zero divisor in $A$ it is certainly not in $eA$. Applying 4.3 to $eAe$ we

see that we are in case (i): $\Delta_{eAe} = 0$. It follows that $\Delta(e) = 0$.

In order to be able to conclude for certain algebras to be in

case (i), we briefly look at $K$-algebras $A$ without unit but having a one-

sided unit $e$, say $xe = x$ for all $x \in A$. Consider $e^\perp := \{ y \in A : ey = 0 \}$.

It is perfectly easy to see from $x = (x-ex) + ex (x \in A)$ that

$A = e^\perp \oplus eA$. Since $eA = eAe$ is an algebra with a two-sided unit, we

have $e^\perp \neq (0)$. $e^\perp$ is a two-sided ideal for which $Ae^\perp = (0)$. In parti-

cular all products in $e^\perp$ are zero. Therefore:

4.6 COROLLARY. Let $(A, ||_1, ||_2)$ be a bicomplete $K$-algebra with-

out unit. Suppose one of the following conditions holds.

(i) $A$ is commutative.

(ii) $A$ has no one-sided unit.

(iii) For a two-sided ideal $J$ in $A$, $J^2 = (0)$ implies $J = (0)$.

(iv) $\downarrow A = (0), A^\perp = (0)$.

Then $\ker \Delta$ contains only two-sided topological zero divisors

with respect to both norms.

An application:

4.7 THEOREM. Let $(A, ||, ||')$ be a K-Banach algebra whose norm is multi-

plicative. If $A$ is not a (skew) field then $A$ has UBAT.

PROOF. Let $||, ||'$ be some Banach algebra norm on $A$ and let $\Delta$ be the

separating seminorm of $||, ||$ and $||, ||'$. Since $||, ||$ is multiplicative,

$A$ has no topological zero divisors with respect to $||, ||$, except 0. If

$A$ has no unit, apply 4.6 (use (iii) or (iv)) to arrive at $\ker \Delta = (0)$. 

If \( A \) has a unit we may use 4.3: case (i) would imply that \( A \) is a (skew) field which is forbidden and case (ii) leads again to \( \text{Ker} \ A = \{0\} \).

5. The uniqueness of the norm topology of \( L(E) \).

In this section \( E \) is a \( K \)-Banach space, \( L(E) \) is the \( K \)-algebra of all continuous linear operators \( E \to E \), and \( A \) is a \( K \)-Banach algebra.

Let \( E \) be a (left) \( A \)-module with structure map \( (a, \xi) \mapsto a\xi \) \((a \in A, \xi \in E)\). We say that \( E \) is 2-fold transitive if for each \( \xi_1, \xi_2, \eta_1, \eta_2 \in E \), where \( \xi_1, \xi_2 \) are linearly independent, there is a \( c \in A \) such that \( a\xi_1 = \eta_1 \), \( a\xi_2 = \eta_2 \).

By the density lemma of Jacobson we then have \( n \)-fold transitivity for each \( n \in \mathbb{N} \) i.e., if \( \xi_1, \ldots, \xi_n \in E \) are linearly independent and \( \eta_1, \ldots, \eta_n \in E \) then there exists a \( c \in A \) such that \( a\xi_i = \eta_i \) \((i = 1, \ldots, n)\).

The following is essentially what remains of the proof of Johnson's theorem [2] in the non-archimedean case.

5.1 LEMMA. Let \( E \) be a 2-fold transitive \( A \)-module such that the maps \( \xi \mapsto a\xi \) \((\xi \in E)\) are continuous for each \( a \in A \). (Or, equivalently, in the corresponding representation \( a \mapsto T_a \) all the \( T_a \) are in \( L(E) \)). Then there exists \( M > 0 \) such that

\[
||a\xi|| \leq M||a||\ ||\xi||
\]

\((a \in A, \xi \in E)\)

PROOF. By the uniform boundedness principle it suffices to show that the structure map \( (a, \xi) \mapsto a\xi \) \((a \in A, \xi \in E)\) is separately continuous i.e., we have to show that for each \( \xi \in E \) the map \( a \mapsto a\xi \) \((a \in A)\) is continuous. By 2-fold transitivity (in fact, irreducibility) these maps are continuous either for all \( \xi \in E, \xi \neq 0 \) or for no such \( \xi \).

First assume \( \dim_k E = \infty \). We assume that \( a \mapsto a\xi \) is continuous only in
case $\xi = 0$ and shall derive a contradiction. Choose independent
$\xi_1, \xi_2, \ldots \in E$ such that $1 \leq ||\xi_i|| \leq 2$ for all $i$, and set
$J_i := \{ a \in A : a\xi_i = 0 \}$ (i = 1, 2, \ldots). Each $J_i$ is a maximal modular left
ideal of $A$ (if $x\xi_i = \xi_i$ then $x$ is an identity modulo $J_i$), hence closed
in $A$. For each $m \geq 2$ we have

$$A = (J_1 \cap J_2 \cap \ldots \cap J_{m-1}) + J_m$$

(By the $m$-fold transitivity there is $x \in A$ such that $x\xi_1 = x\xi_2 = \ldots = x\xi_{m-1} = 0$,
$x\xi_m \neq 0$, hence $x \in (J_1 \cap J_2 \cap \ldots \cap J_{m-1})$, $x \notin J_m$. Now $J_m$ is maximal and
(*) follows). The addition map $(J_1 \cap \ldots \cap J_{m-1}) \times J_m \to A$ is continuous and
surjective hence open by Banach's open mapping theorem. So there is $\gamma > 0$
such that we can write each $a \in A$ as $b + c$ where $b \in J_1 \cap \ldots \cap J_{m-1}$, $c \in J_m$
$||b|| \leq \gamma ||a||$, $||c|| \leq \gamma ||a||$. With the help of this one can choose inductively
$x_1, x_2, \ldots \in A$ such that for each $n \in N$, $n \geq 2$

$$||x_n|| \leq 2^{-n}; x_n \in J_1 \cap \ldots \cap J_{n-1}; ||x_n\xi_n|| \geq n + \sum_{i=2}^{n-1} ||x_i\xi_n||$$

using also the discontinuity at 0 of $x \mapsto x\xi_n$.

Set $z := \sum x_i \in A$. Since for $n \in N$, $n \geq 2$ we have
$\sum x_i \in J$ we get
$$||z\xi_n|| = \sum (x_2 + \ldots + x_n)\xi_n \geq ||x_n\xi_n|| \geq \sum_{i=2}^{n-1} ||x_i\xi_n|| \geq \gamma n.$$  

Thus, $\lim_{n \to \infty} ||z\xi_n|| = \infty$. But the sequence $\xi_1, \xi_2, \ldots$ is bounded, so this
conflicts with the continuity of $x \mapsto z\xi$ ($\xi \in E$).

If, finally, $\dim K E < \infty$ then the map $a \mapsto a\xi$ ($a \in A$) can be decomposed:

$$A + A/I \cong E$$

where $A/I$ is equipped with the quotient norm and where $I := \{ x \in A : x\xi = 0 \}$. It follows that $a \mapsto a\xi$ ($a \in A$) is continuous.

5.2 THEOREM. Let $(B, \| \|_1, \| \|_2)$ be a bicomplete $K$-algebra, and
suppose $E$ is a 2-fold transitive $B$-module such that the map

\( \xi \mapsto b\xi \) (\( \xi \in E \)) is continuous for each \( b \in B \). Set

\[
I_E := \{ x \in B : x\xi = 0 \text{ for all } \xi \in E \}.
\]

Then \( \ker \Delta \subseteq I_E \) where \( \Delta \) is the separating seminorm of \( \| \|_1 \) and \( \| \|_2 \).

PROOF. Let \( b \not\in I_E \). Then there is \( \xi \in E \) such that \( b\xi \neq 0 \).

Lemma 5.1 yields the existence of \( M > 0 \) such that

\[
\begin{align*}
||x\xi|| &\leq M||x||_1 ||\xi|| \\
||x\xi|| &\leq M||x||_2 ||\xi||
\end{align*}
\]

(\( x \in B, \xi \in E \)).

The seminorm \( p : x \mapsto M^{-1}||\xi||^{-1}||x\xi|| \) (\( x \in B \)) satisfies \( p \leq || \|_1, p \leq || \|_2 \), \( p(b) \neq 0 \). So \( 0 < p(b) \leq \Delta(b) \). It follows that \( \ker \Delta \subseteq I_E \).

5.3 COROLLARY. Let \( E \) have the property that for each independent \( \xi_1, \xi_2 \in E \) and \( \eta_1, \eta_2 \) there exists \( T \in L(E) \) such that \( T\xi_1 = \eta_1 \), \( T\xi_2 = \eta_2 \). Then \( L(E) \) has UBAT.

PROOF. \( E \) is a 2-fold transitive \( L(E) \)-module under \( (T, \xi) \mapsto T\xi \) (\( T \in L(E) \), \( \xi \in E \)), satisfying the continuity condition of 5.2. \( I_E = \{ T \in L(E) : T\xi = 0 \text{ for all } \xi \in E \} = \emptyset \). Hence for each two Banach algebra norms the separating seminorm is a norm, so the norms are equivalent.

Finally we indicate a class of Banach spaces \( E \) for which \( L(E) \) has UBAT.

For the notions used below see [5].

5.4 THEOREM. Let \( E \) be a K-Banach space. Each of the following conditions implies that \( L(E) \) has a unique Banach algebra topology.

(i) \( K \) is spherically complete.

(ii) \( E \) has a base. (In particular, \( E \) is of countable type.)

(iii) \( E \) is the dual of some K-Banach space.

(iv) \( E \) is spherically complete.
PROOF. We shall first prove: if the elements of the dual $E'$ separate the points of $E$ then $l(E)$ has UBAT, which takes care of (i), (ii) and (iii). Let $\xi_1, \xi_2 \in E$ be independent and $\eta_1, \eta_2 \in E$. There are $f, g \in E'$ such that $f_1(\xi_j) = \delta_{1j}$ $(i, j \in \{1, 2\})$. The map 

$$\xi \mapsto f_1(\xi)\eta_1 + f_2(\xi)\eta_2$$

is in $l(E)$ and sends $\xi$ into $\eta_i$ $(i = 1, 2)$. Now apply 5.3.

Finally we prove (v). Let $\xi_1, \xi_2 \in E$ be independent and $\eta_1, \eta_2 \in E$. Let $A$ be the map $\lambda_1 \xi_1 + \lambda_2 \xi_2 \mapsto \lambda_1 \eta_1 + \lambda_2 \eta_2$ $(\lambda_1, \lambda_2 \in K)$, $A : D \to E$ where $D$ is the subspace of $E$ spanned by $\xi_1$ and $\xi_2$. By the spherical completeness of $E$, $A$ can be extended to an element of $l(E)$. Now apply 5.3. 

PROBLEM: Do there exist $K$-Banach spaces $E$ for which $l(E)$ admits inequivalent Banach algebra norms?

REFERENCES


