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EXTRAIT DE

"SYMPOSION DÉDIÉ À A.F. MONNA"

COMMUNICATIONS OF THE MATHEMATICAL INSTITUTE
RIJKSUNIVERSITEIT UTRECHT 12-1980
**P-adic monotone functions**

by

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0. **Introduction**

Our aim in this paper is to present reasonable definitions for a function \( f: K \to K \) to be "monotone", where \( K \) is a local field (i.e., a non-archimedean non-trivially valued field that is locally compact in the topology induced by the valuation). The future will tell us whether filling this gap in p-adic analysis has been of any use.

The fact that - until recently - the concept of "monotone function" has been absent in p-adic analysis is not surprising since a decent (partial) ordering in \( K \), (compatible with the algebraic and topological structure) is not available. Thus, in the sequel we will look for substitutes for "ordering" in \( K \) as a basis for our theory. One of them is the notion of "pseudo-ordering", introduced by A.F. Monna [1].

The theory can be build up in a more general setting, namely for functions \( f: X \to K \), where \( K \) is any complete non-archimedean field and \( X \subseteq K \), but restriction to local fields avoids a lot of technicalities and enlights the main track. For the extended theory, see [4].

The elementary facts about analysis in local fields can be found in [2].

**Notations and definitions**

**FROM NOW ON IN THIS PAPER K IS A LOCAL FIELD WITH RESIDUE CLASS FIELD**\( k \).
Set \( |K| : = \{ |x| : x \in K \} \)
\( |K^*| : = |K| \setminus \{0 \} \) (the value group)
\[ \pi : \text{a (fixed) element of } K^* \text{ such that } |\pi| \text{ generates } |K^*|, \]
\[ |\pi| < 1. \]
For a prime \( p \) we denote by \( \mathbb{Q}_p \) the non-archimedean valued field of the \( p \)-adic numbers, by \( \mathbb{Z}_p \) its valuation ring \( \{ x \in \mathbb{Q}_p : |x| \leq 1 \} \).
The residue class field, \( \mathbb{Z}_p / p \mathbb{Z}_p \) of \( \mathbb{Q}_p \) is the field of \( p \) elements and is denoted by \( \mathbb{F}_p \).
The characteristic of a field \( L \) is denoted by \( \chi(L) \).
For a \( K \)-vector space \( E \) and a subset \( S \) of \( E \) we denote its \( K \)-linear span by \( \langle S \rangle \).
Let \( a \in K \), \( r \in [0, \infty) \). The ball with center \( a \) and radius \( r \) is by definition \( \{ x \in K : |x - a| \leq r \} \). It is easy to see that the intersection of a collection of balls is either empty or again a ball.
Let \( x, y \in K \). Then the smallest ball containing \( x \) and \( y \) is denoted by \( [x, y] \). A subset \( C \) of \( K \) is called convex if \( x, y \in C \) implies \( [x, y] \subseteq C \). Each ball is convex. A convex set \( \neq K \), \( \neq \emptyset \) is a ball.
It follows that \( K \) is the only unbounded convex set in \( K \).

FROM NOW ON \( X \) IS A CONVEX SUBSET OF \( K \).

1. **Two notions of monotony**

Interpreting the above notion of convexity also for the real numbers it is quite natural to introduce the following geometric expressions.

Let \( x, y, z \in K \). We say that \( z \) is between \( x \) and \( y \) if \( z \in [x, y] \).
If \( z \) is not between \( x \) and \( y \) we say that \( x, y \) are at the same side of \( z \). This yields more or less automatically the following
DEFINITION 1. Let $f: X \to K$. We say that $f \in M_b(X)$ (f respects "betweenness") if for all $x, y, z \in X$

(*) $z \in [x, y] \Rightarrow f(z) \in [f(x), f(y)]$.

We say that $f \in M_s(X)$ (f respects "sides") if for all $x, y, z \in X$

(**) $z \notin [x, y] \Rightarrow f(z) \notin [f(x), f(y)]$.

REMARKS

1. If we, in the above definition, replace formally $K$ by $\mathbb{R}$ and $X$ by an interval, we see that $f \in M_b(X)$ just means that $f$ is monotone and that $f \in M_s(X)$ becomes "f is strictly monotone". (These facts can easily be proved). So for the time being we let our intuition be guided by this analogy:
The statements in 2 and 3 below are direct consequences of the definitions, and the proofs are left to the reader.

2. $M_b(X)$ is closed under pointwise limits.
The constant functions are in $M_b(X)$. If $f \in M_b(X)$ and $f(a) = f(b)$ then $f$ is constant on $[a, b]$.

$f \in M_b(X) \Rightarrow$ For each convex $C \subseteq K$ the inverse image $f^{-1}(C)$ is convex. For each $a, b \in K$ the map $x \mapsto ax + b$ is in $M_b(X)$.

$f \in M_s(X) \Rightarrow f$ is injective.
If $a, b \in K$, $a \neq 0$ then $x \mapsto ax + b$ is in $M_s(X)$.
Each isometry $X \to K$ is in $M_{bs}(X)$ where

$$M_{bs}(X) = M_b(X) \cap M_s(X).$$

3. Without harm we may replace in Definition 1 (*) by (*)', or (*)'' or (*)''', where

(*)' : $|x-z| \leqslant |x-y| \Rightarrow |f(x) - f(z)| \leqslant |f(x) - f(y)|$

(*)'' : $|x-z| = |x-y| \Rightarrow |f(x) - f(z)| = |f(x) - f(y)|$

(*)''' : $|f(x) - f(z)| < |f(x) - f(y)| \Rightarrow |x-z| < |x-y|$
Similarly, we may replace (**) by (**)' or (**)' or (**)'', where

\[
(**)' : |x-z| < |x-y| \rightarrow |f(x) - f(z)| < |f(x) - f(y)|
\]

\[
(**)'' : |f(x) - f(z)| = |f(x) - f(y)| \rightarrow |x-z| = |x-y|
\]

\[
(**)''' : |f(x) - f(z)| \leq |f(x) - f(y)| \rightarrow |x-z| \leq |x-y|.
\]

4. In the next section we will study \( M_b(X) \) and \( M_s(X) \). For example, the natural questions: \( M_s(X) \subseteq M_b(X) \)? \( f \in M_b(X) \), \( f \) injective \( \Rightarrow \) \( f \in M_s(X) \)? Notice that our definitions do not refer to any "type" of monotony (such as "increasing" and "decreasing" for real functions). In section 4 we will introduce such a concept. It will turn out that monotone functions having a "type" are \( M_{bs} \)-functions, but not conversely.

2. Properties of monotone functions

THEOREM 2.1. Let \( f \in M_b(X) \). If \( a,b,c \in X \), \( a-b < a-c \), \( f(a) \neq f(c) \) then \( |f(a) - f(b)| < |f(a) - f(c)| \). In particular, if \( f \in M_b(X) \), \( f \) is injective then \( f \in M_s(X) \).

PROOF. Without loss, assume \( X = [a,c] \). Since \( f \in M_b(X) \) we have \( \{f(a), f(c)\} \subseteq f(X) \subseteq [f(a), f(c)] \), hence the diameter of \( f(X) \) equals \( M: = |f(a) - f(c)| \). The ball \([f(a), f(c)]\) has a partition into \( n \) balls \( V_1, \ldots, V_n \) each having radius \( M/|n| \), where \( n \) is the number of elements of \( k \). The sets \( f^{-1}(V_1), \ldots, f^{-1}(V_n) \) form a partition of \( X \), each \( f^{-1}(V_i) \) is convex (since \( f \in M_b(X) \)), at least two of the \( f^{-1}(V_i) \) are non-empty (since \( a \) and \( c \) cannot both lie in the same \( f^{-1}(V_i) \)). It follows that the diameter of each \( f^{-1}(V_i) \) is strictly less than \( |a-c| \). (Otherwise \( f^{-1}(V_i) = X \) for some \( i \) and \( f^{-1}(V_j) = \emptyset \) for \( j \neq i \).)

Consequently the partition \( f^{-1}(V_1), \ldots, f^{-1}(V_n) \) of \( X \) must be the partition of \( X \) into balls with radius \( |a-c||n| \).
Now if \( |a-b| < |a-c| \) then \( a, b \in f^{-1}(V_i) \) for some \( i \), so \( |f(a) - f(b)| \leq M|\pi| < |f(a) - f(c)| \).

EXAMPLE. Let \( p \neq 2 \) and let \( f: \mathbb{Z}_p \to \mathbb{Q}_p \) "tear apart" \( \mathbb{Z}_p \) by sending \( k + p \mathbb{Z}_p \) into \( p^{-k} + p \mathbb{Z}_p \) \( (k = 0, 1, 2, \ldots, p-1) \) via translations. Then one easily checks that \( f \in M_s(\mathbb{Z}_p) \setminus M_b(\mathbb{Z}_p) \).

Hence, it seems that \( M_s \)-functions are the "translation" of the real strictly monotone functions (rather than \( M_b \)-functions).

In the sequel we often make use of the following observation. If \( f \) is either in \( M_b(X) \) or in \( M_s(X) \) then

\[
(*) \quad |x-z| < |x-y| \to |f(x) - f(z)| \leq |f(x) - f(y)| \quad (x,y,z \in X)
\]

(Functions with property \((*)\) are called weakly monotone in \([\cdot, +]\)).

**LEMMA 2.2.** Let \( f \in M_b(X) \) or \( f \in M_s(X) \). Then, if \( Y \subset X \) is bounded then \( f(Y) \) is bounded.

**PROOF.** \( Y \) is precompact, so \( r := \max\{|x-y| : x,y \in Y\} \) exists. We may assume \( r > 0 \). The equivalence relation \( x \sim y \) iff \( |x-y| < r \) \( (x,y \in Y) \) divides \( Y \) into finitely many classes \( Y_1, \ldots, Y_n \) where \( n \geq 2 \). Choose \( a_i \in Y_i \) for each \( i \) and set \( M := \max_i |f(a_i)| \). We prove \( |f| \leq M \) on \( Y \).

In fact, let \( x \in Y \). There is \( i \in \{1, \ldots, n\} \) such that \( |x-a_i| < r \).

For \( j \neq i \) we have \( |x-a_j| < |a_i - a_j| \), so \( |f(x) - f(a_i)| \leq |f(a_i) - f(a_j)| \leq M \). Hence \( |f(x)| \leq M \).

We have a "dual" statement which is only of interest in case \( X = K \):

**LEMMA 2.3.** Let \( f \in M_s(K) \) or \( f \in M_b(K) \). If \( f \) is not constant then for a bounded \( Z \subset K \) the inverse image \( f^{-1}(Z) \) is bounded.

**PROOF.** We prove: \( Z \subset K \) bounded, \( T := f^{-1}(Z) \) is unbounded implies \( f \) is constant. In fact, let \( a,b \in K \). There are \( x_1, x_2, \ldots \) in \( T \) such that
(*) $\max(|a|, |b|) < |x_1| < |x_2| < \ldots$

The precompactness of $\{f(x_1), f(x_2), \ldots\}$ implies convergence of a subsequence of $f(x_1), f(x_2), \ldots$. Without loss, assume that $\lim_{n \to \infty} f(x_n)$ exists. From (*) we obtain

$$|a-b| < |x_1-a| < |x_2-x_1| < |x_3 - x_2| < \ldots,$$

so that for all $n \in \mathbb{N}$

$$|f(a) - f(b)| \leq |f(x_{n+1}) - f(x_n)|.$$

Hence,

$$|f(a) - f(b)| \leq \lim_{n \to \infty} |f(x_{n+1}) - f(x_n)| = 0.$$

It follows that $f$ is constant.

2.1., 2.2. and 2.3. yield the continuity properties 2.4., 2.5. and 2.6.:

**THEOREM 2.4.** Let $X$ be bounded with diameter $r > 0$ and let $f \in M_b(X)$ or $f \in M_s(X)$. Then $f$ satisfies the Lipschitz-condition

$$|f(x) - f(y)| \leq M|x-y| \quad (x,y \in X)$$

where $M = r^{-1}\sup\{|f(x) - f(y)| : x,y \in X\} < \infty$.

**PROOF.** By 2.2. $f$ is bounded, so $M < \infty$. Choose any $a \in X$. We prove by induction on $n$: $P(n)$: "If $|x-a| = \pi^n r$ then $|f(x) - f(a)| \leq |\pi^n r M$. ($x \in X$)". Clearly $P(0)$ holds. Suppose $P(n-1)$. Let $x \in X$ such that $|x-a| = \pi^n r$ and choose $b \in X$ with $|b-a| = |\pi|^{n-1} r$. Then $|x-a| < |b-a|$. If $f(b) = f(a)$ then $|f(x) - f(a)| \leq |f(b) - f(a)| = 0$, so certainly $|f(x) - f(a)| \leq |\pi|^{n-1} r M$. If $f(a) \neq f(b)$ then either by Theorem 2.1. or since $f \in M_s(X)$ we have $|f(x) - f(a)| < |f(b) - f(a)| \leq$ (induction hypothesis) $\leq |\pi|^{n-1} r M$, so that $|f(x) - f(a)| \leq |\pi|^{n} r M$.

**REMARK.** The map $\Sigma_{a_n p^n} \to \Sigma_{a_n p^{2n}}$ is in $M_{bs}(\Phi_p)$ and has unbounded difference quotients.
THEOREM 2.5. Let $f \in M_b(X)$ or $f \in M_s(X)$. Then $f$ is continuous.

Further, if $Y \subseteq X$ is closed then $f(Y)$ is closed in $K$.

In particular, an $f \in M_s(X)$ is a homeomorphism $X \sim f(X)$.

PROOF. If $X$ is bounded then everything follows from 2.4., so let $X = K$. The continuity of $f$ is clear (restrict $f$ to bounded convex subsets). Let $Y \subseteq K$ be closed and suppose $a \in f(Y) \setminus f(Y)$. There are $a_1, a_2, \ldots$ in $Y$ for which $\lim_{n \to \infty} f(a_n) = a$. $f$ is not constant, so by 2.3. the sequence $a_1, a_2, \ldots$ is bounded, assume it converges, say $a = \lim_{n \to \infty} a_n$. By continuity, $f(a) = \lim_{n \to \infty} f(a_n) = a$, a contradiction.

The last statement is now trivial.

THEOREM 2.6. Let $f \in M_b(X)$ (or $f \in M_s(X)$ for that matter). Then the following are equivalent.

(a) $f$ is a homeomorphism $X \sim f(X)$.

(b) $f$ is injective.

(c) $f \in M_s(X)$.

(d) $f(X)$ has no isolated points.

PROOF. The implications $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)$ are easy (2.1., continuity and injectivity of $M_s$-functions). We prove $(d) \Rightarrow (a)$, that is, if $\lim_{n \to \infty} f(x_n) = f(a)$ then $\lim_{n \to \infty} x_n = a$. Now since $f(a)$ is not isolated we can find $a_1, a_2, \ldots$ such that $f(a_n) \neq f(a)$ for all $n$, $\lim_{n \to \infty} f(a_n) = f(a)$. By 2.3. we may assume that $b = \lim_{n \to \infty} a_n$ exists. If suffices to prove that $\lim_{n \to \infty} x_n = b$ (apply the result for $x_n = a$ for all $n$ and we find $a = b$). Let $\varepsilon > 0$. There is $k \in \mathbb{N}$ for which $|a_k - b| < \varepsilon$. For large $n$ we have $|f(x_n) - f(a)| < |f(a_k) - f(a)|$, hence for large $n$ and $m(n)$ we get $|f(x_n) - f(a_m)| < |f(a_k) - f(a_m)|$ whence $|x_n - a_m| \leq |a_k - a_m|$, so $|x_n - b| \leq |a_k - b| < \varepsilon$ for large $n$.

THEOREM 2.7. Let $f \in M_b(X)$ or $f \in M_s(X)$ and assume that $f(X)$ is convex. Then $f$ is a scalar multiple of an isometry, or $f$ is constant.
PROOF. We may assume that \( f \) is not constant, so \( f(X) \) is open, non-empty. Let \( X = f(X) \) be bounded. By 2.6, \( f \) is injective. It is clear that also \( f^{-1} \in M_b(X) \cup M_s(X) \). Applying 2.4 to both \( f \) and \( f^{-1} \) we get (in both cases \( M = 1 \)) for all \( x, y, u, v \in X \) that \( |f(x) - f(y)| \leq |x-y| \) and \( |f^{-1}(u) - f^{-1}(v)| \leq |u-v| \). It follows that \( f \) is an isometry. The general case is now easy. (If \( X, f(X) \) are bounded a transformation of the type \( x \mapsto ax+b \) sends \( f(X) \) into \( X \); if \( X = f(X) = K \), apply 2.7. to \( X = \{x \in K: |x| \leq n\} \ (n \in \mathbb{N}) \).

The following theorem describes the functions "of bounded variation":

**THEOREM 2.8.** Let \( X \) be bounded. Then \( [M_s(X)] = [M_b(X)] = \text{BA}(X) \), where \( \text{BA}(X) \) is the linear space of functions \( f: X \to K \) having bounded difference quotients.

**PROOF.** By 2.4 we are done if we can prove that an \( f \in \text{BA}(X) \) is the sum of two \( M_{bs} \)-functions. Let \( \lambda \in K \) such that \( |f(x) - f(y)| < |\lambda| |x-y| \) for all \( x, y \in X, x \neq y \). Let \( g(x) = \lambda x \ (x \in X) \) and let \( h = f - g \). Then \( g, h \) are in \( M_{bs}(X) \) (scalar multiples of isometries) and \( f = g + h \).

3. Monotone sequences

We will not delve deeply into this subject, but content ourselves with presenting definitions and some facts indicating that these notions are not that bad.

**DEFINITION 3.1.** Let \( x_1, x_2, \ldots \) be a sequence in \( K \). It is called \( b \)-monotone, if \( k \leq l \leq m \) implies \( x_l \in [x_k, x_m] \), \( s \)-monotone, if \( 1 \leq k, m \) implies \( x_k \notin [x_1, x_m] \).

**THEOREM 3.2.** A sequence \( x_1, x_2, \ldots \) in \( K \) is \( s \)-monotone if and only if \( |x_1-x_2| > |x_2-x_3| > \ldots \) A sequence \( x_1, x_2, \ldots \) in \( K \) is \( b \)-monotone if and only if for each \( k, m \in \mathbb{N}, k < m \):
\[ |x_m - x_k| = \max \{|x_{i+1} - x_i| : k \leq i < m\}. \]

**PROOF.** Let \( x_1, x_2, \ldots \) be \( s \)-monotone, and let \( n \in \mathbb{N} \). We have
\[ x_n \notin [x_{n+1}, x_{n+2}] \text{ so } |x_n - x_{n+1}| > |x_{n+1} - x_{n+2}|. \]
Conversely, if
\[ |x_1 - x_2| > |x_2 - x_3| > \ldots, \]
let \( 1 \leq k, m \). Then
\[ |x_k - x_1| = |x_{k+1} - x_1| \]
and
\[ |x_m - x_k| = |x_{k+1} - x_k| \]

**COROLLARY 3.3.** Each \( s \)-monotone sequence is \( b \)-monotone. An \( s \)-monotone sequence \( x_1, x_2, \ldots \) is convergent and \( x_n \neq x_m \) whenever \( n \neq m \). \( M_b \)-functions map \( b \)-monotone sequences into \( b \)-monotone sequences. \( M_s \)-functions map \( s \)-monotone sequences into \( s \)-monotone sequences.

**THEOREM 3.4.** Let \( x_1, x_2, \ldots \) be \( b \)-monotone. Then either \( \lim_{n \to \infty} |x_n| = 0 \) or \( \lim_{n \to \infty} x_n \) exists.

**PROOF.** If not \( \lim_{n \to \infty} |x_n| = \infty \) then the sequence has a bounded, hence a convergent subsequence, say, \( \lim_{i \to \infty} x_i = x \). We show that \( \lim_{n \to \infty} x_n = x \).

Let \( \varepsilon > 0 \). Then
\[ |x - x_{n_k}| < \varepsilon \text{ for } k \geq k_0. \]
If \( l, m \geq n_{k_0} \) then if
\[ n_k \geq 1, m \]
we have
\[ |x_{l} - x_{m}| \leq |x_{n_{k}} - x_{n_{k}}| \leq \max (|x - x_{n_{k_0}}|, |x - x_{n_{k}}|) < \varepsilon. \]

Hence \( x_1, x_2, \ldots \) is Cauchy, so it must converge, to \( x \).

**THEOREM 3.5.** Each sequence in \( K \) has a \( b \)-monotone subsequence.
PROOF. We may assume that $x_1, x_2, \ldots$ is bounded, and that it has a convergent subsequence $y_1, y_2, \ldots$ where $y_n \neq y_m$ whenever $n \neq m$. Set $y = \lim_{n \to \infty} y_n$. Now it is easy to construct a subsequence $z_1, z_2, \ldots$ of $y_1, y_2, \ldots$ for which $|y - z_1| > |y - z_2| > |y - z_3| > \ldots$ Hence $|z_1 - z_2| > |z_2 - z_3| > \ldots$ The sequence $z_1, z_2, \ldots$ is $s$-monotone, hence $b$-monotone.

EXAMPLE. Let $x = \sum_{n=0}^{\infty} a_n p^n \in \mathbb{Z}_p$, and let $x_k = \sum_{n=0}^{k} a_n p^n$. Then $x_0, x_1, \ldots$ is a $b$-monotone sequence, converging to $x$.

4. Monotone functions of type $\sigma$

Following Monna [1] we introduce the concept of "sides of zero" in $K$ (a generalization of the partition $\mathbb{R}^* = \mathbb{R}_+ \cup \mathbb{R}_-$). Let $K^* = K \setminus \{0\}$. Define

$$x \sim y \text{ if } x \text{ and } y \text{ are at the same side of } 0 \quad (x, y \in K^*).$$

Then we have $x \sim y \Rightarrow 0 \notin [x, y] \Leftrightarrow |x - y| < |y| \Leftrightarrow |xy^{-1} - 1| < 1 \Leftrightarrow xy^{-1} \in K^+$, where

$$K^* = \{x \in K: |1-x| < 1\}.$$

$K^*$ is a multiplicative open compact subgroup of $K^*$, called the group of the positive elements of $K$. We see that $\sim$ is the equivalence relation induced by the canonical group homomorphism, the "sign map":

$$\text{sgn}: K^* \to K^*/K^+.$$

The quotient group $\Sigma = K^*/K^+$ (comparable with $\{1, -1\}$ in the real case) is called the group of signs of elements of $K$, or the group of sides of zero of $K$. $\Sigma$ is an infinite group, whose elements are multiplicative cosets of $K^+$.

Let $\alpha \in \Sigma$ and let $x, y \in \alpha$. Then $|x - y| < |x|$, so in particular,
$|x| = |y|$. Therefore we may define the absolute value of a sign $\alpha \in \Sigma$ as

$$|\alpha| = |x| \quad (x \in \alpha)$$

The map $\alpha \mapsto |\alpha|$ is a surjective group homomorphism $\Sigma \to |K^*|$. Its kernel, $\{\alpha \in \Sigma: |\alpha| = 1\}$ is a multiplicative subgroup of $\Sigma$, which is naturally isomorphic to the multiplicative group $K^*$ under $\alpha \mapsto \overline{\alpha}$ (where $x \mapsto \overline{x}$ is the canonical map $\{x \in K: |x| \leq 1\} \to k$). Let us denote its inverse $k^* \cong \{\alpha \in \Sigma: |\alpha| = 1\}$ by

$$1 \mapsto \alpha_1 \quad (1 \in k^*).$$

For each $n \in \mathbb{Z}$, let $\alpha_n := \text{sgn}(\pi^n)$.

Now for each $\alpha \in \Sigma$ there are unique $n \in \mathbb{Z}$, $l \in k^*$ such that

$$\alpha = \alpha_1 \alpha_n.$$ 

Further, we have

$$\alpha_1 \alpha_1' = \alpha_{1l}' \quad (l, l' \in k^*)$$

$$\alpha_n \alpha_m = \alpha_{n+m} \quad (n, m \in \mathbb{Z}).$$

It follows, that $\Sigma$ is isomorphic to $k^* \times \mathbb{Z}$ (or to $k^* \times |k^*|$ for that matter). It is also possible to identify $\Sigma$ with a subgroup of $k^*$ but we shall not need it here.

In the sequel we will use addition of signs. For $\alpha \in \Sigma$, let

$$-\alpha = \{-x: x \in \alpha\}.$$ 

Then clearly $-\alpha$ is a sign and if $\alpha = \alpha_1 \alpha_n$ ($l \in k^*, n \in \mathbb{Z}$) then

$$-\alpha = \alpha_{-1} \alpha_n.$$ 

Let $\alpha, \beta \in \Sigma$. Then $\alpha + \beta = \{x+y: x \in \alpha, y \in \beta\}$ is easily seen to be a ball, which turns out to be again a sign iff $\alpha \neq -\beta$ (iff $0 \not\in \alpha + \beta$).

Therefore we define
\[ \alpha \oplus \beta: = \alpha + \beta \quad (\alpha, \beta \in \Sigma, \alpha \neq -\beta). \]

We have the following rules that are easy to prove:

RULES 4.1. (i) The operation \( \oplus \) is commutative, associative, distributive, whenever the occurring formulas are defined.

(ii) \( |\alpha \oplus \beta| = \max(|\alpha|, |\beta|) \quad (\alpha, \beta \in \Sigma, \alpha \neq -\beta) \)

(iii) \( |\alpha| < |\beta| \) if and only if \( \alpha \oplus \beta = \beta \quad (\alpha, \beta \in \Sigma) \)

(iv) Let \( n \in \mathbb{N}, 1 \leq n < \chi(k), \alpha \in \Sigma \). Then \( \oplus \alpha(= \alpha \oplus \alpha \oplus \ldots \oplus \alpha(n \text{ times})) \) exists and equals \( n\alpha \).

Thus, we get for \( 1, 1' \in k^*, m, n \in \mathbb{Z} : \)

\[
1_m \alpha \oplus 1_n \alpha = \begin{cases} 
\alpha m \alpha & \text{if } m < n \\
\alpha n \alpha & \text{if } m > n \\
\alpha 1+1 \alpha n & \text{if } m = n \text{ and } 1 + 1' \neq 0.
\end{cases}
\]

Next, we define a "pseudo-ordering" in \( k \). Let \( x, y \in k, \alpha \in \Sigma \). We say that \( x > y \) (\( x \) is greater than \( y \) in the sense of \( \alpha \)) in case \( \alpha x - y \in \alpha \). The following easy consequences obtain.

RULES 4.2. (i) Let \( x, y \in k \). Then if \( y \neq x \) there is exactly one \( \alpha \in \Sigma \) for which \( x > y \). If \( y = x \) then \( x > y \) for no \( \alpha \). ("\( k \) is totally pseudo-ordered").

(ii) If \( x > y, y > z \) for some \( x, y, z \in k; \alpha, \beta \in \Sigma, \alpha \oplus \beta \) exists then \( x > z \). ("Transitivity").

(iii) If \( x > y \) for some \( x, y \in k; \alpha \in \Sigma \) and if \( z \in k, \alpha \) then \( x + z > y + z \) ("Compatibility with addition").

(iv) If \( x, y, z \in k; \alpha, \beta \in \Sigma, x > y \) and \( z > 0 \) then \( xz > yz \) ("Compatibility with multiplication").

We define:
DEFINITION 4.3. Let $\sigma: \Sigma \rightarrow \Sigma$ be an injection and $f: K \rightarrow K$. 

$f$ is called monotone of type $\sigma$ if for all $\alpha \in \Sigma$, $x, y \in K$ we have 

$$x > y \text{ implies } f(x) > f(y).$$

$f$ is called increasing if $\sigma$ is the identity map.

REMARKS

1. One can extend easily the above definition for functions $f: X \rightarrow K$, where $X$ is a convex subset of $K$, $\sigma: \Sigma(X) \rightarrow \Sigma$. 

Here $\Sigma(X)$ is the collection of signs that "occur in $X" i.e. 

$$\{\text{sgn}(x-y): x, y \in X, x \neq y\}.$$ 

We leave it to the reader to do this and, in case $X \neq K$, to show that there is $\beta \in \Sigma$ such that 

$$\Sigma(X) = \{\alpha \in \Sigma: |\alpha| < |\beta|\}.$$ 

2. Notice that $f: K \rightarrow K$ is increasing if and only if the difference quotient 

$$\phi_f(x, y) := \frac{f(x) - f(y)}{x - y} \quad (x \neq y)$$

is positive. In particular, $f$ is an isometry.

3. The requirement made in 4.3. that $\sigma$ be an injection is essential in case $k$ is not a prime field (see [4]).

4. In case $\chi(K) = 0$, the exponential function, defined by the power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is increasing on its convergence region.

If $f$ is an increasing function and $\beta \in \Sigma$ then for each $b \in \beta$ the function $bf$ is of type $\sigma$ where $\sigma$ is the multiplier $a \mapsto a\beta$.

THEOREM 4.4. (i) Let $f: K \rightarrow K$ be a monotone of type $\sigma: \Sigma \rightarrow \Sigma$. Then

$$(*) \quad \sigma(\alpha \oplus \beta) = \sigma(\alpha) \oplus \sigma(\beta) \quad (\alpha, \beta \in \Sigma, \alpha \neq -\beta)$$

(ii) Let $\sigma: \Sigma \rightarrow \Sigma$ satisfy $(*)$. Then there is a function $g: K \rightarrow K$, monotone of type $\sigma$. 
PROOF. (i) First we show that $\sigma(-\alpha) = -\sigma(\alpha)$ ($\alpha \in \Sigma$). In fact choose $x, y \in K$ such that $x - y \in \alpha$. Then $y - x \in -\alpha$, so $f(x) - f(y) \in \sigma(\alpha) \cap (-\sigma(-\alpha)) \neq \emptyset$, which implies $\sigma(\alpha) = -\sigma(-\alpha)$.

Now take $\alpha, \beta \in \Sigma$, $\alpha \neq -\beta$. Then by injectivity of $\sigma$, $\sigma(\alpha) \neq \sigma(-\beta) = -\sigma(\beta)$ so that $\sigma(\alpha) \oplus \sigma(\beta)$ exists. Now choose $x, y, z \in K$ such that $x - y \in \alpha$, $y - z \in \beta$. Then $f(x) - f(z) = f(x) - f(y) + f(y) - f(z) \in \sigma(\alpha) \oplus \sigma(\beta)$. Also, $x - z \in \alpha \oplus \beta$ so $f(x) - f(z) \in \sigma(\alpha \oplus \beta)$. As $\sigma(\alpha) \oplus \sigma(\beta)$ and $\sigma(\alpha \oplus \beta)$ have a non-empty intersection they are equal.

The proof of (ii) will be furnished by Lemma 4.5. and Lemma 4.6.

LEMMA 4.5. Let $\sigma: \Sigma \to \Sigma$ satisfy $\sigma(\alpha \oplus \beta) = \sigma(\alpha) \oplus \sigma(\beta)$ ($\alpha, \beta \in \Sigma$, $\alpha \neq -\beta$). Then we have

(i) $\sigma(n\alpha) = n\sigma(\alpha)$ ($n \in \mathbb{N}, 1 \leq n < \chi(k)$, $\alpha \in \Sigma$).

(ii) $\sigma(-\alpha) = -\sigma(\alpha)$ ($\alpha \in \Sigma$)

(iii) For all $\alpha, \beta \in \Sigma$: $|\alpha| < |\beta|$ if and only if $|\sigma(\alpha)| < |\sigma(\beta)|$.

(iv) $\lim_{|\alpha| \to 0} |\sigma(\alpha)| = 0$.

PROOF. (i) is a direct consequence of 4.1.(iv). To prove (ii), set $q := \chi(k)$. Then for $\beta \in \Sigma$, $(q-1)\beta = -\beta$, so, by (i), $\sigma(-\alpha) = \sigma((q-1)\alpha) = (q-1)\sigma(\alpha) = -\sigma(\alpha)$. Let $\alpha, \beta \in \Sigma$, $|\alpha| < |\beta|$. Then $\alpha \oplus \beta = \beta$ so $\sigma(\alpha) \oplus \sigma(\beta) = \sigma(\alpha \oplus \beta) = \sigma(\beta)$ whence $|\sigma(\alpha)| < |\sigma(\beta)|$. Conversely, suppose $|\sigma(\alpha)| < |\sigma(\beta)|$. Then clearly $\alpha \neq -\beta$ (otherwise $|\sigma(\alpha)| = |\sigma(-\beta)| = |\sigma(\beta)| = |\sigma(\beta)|$), so $\alpha \oplus \beta$ exists. Now $\sigma(\beta) = \sigma(\alpha) \oplus \sigma(\beta) = \sigma(\alpha \oplus \beta)$. By injectivity of $\sigma$ we obtain $\beta = \alpha \oplus \beta$ whence $|\alpha| < |\beta|$. So we have (iii). (iv) follows from the fact that if $|\alpha_1| > |\alpha_2| > ... \to 0$ then $|\sigma(\alpha_1)| > |\sigma(\alpha_2)| > ...$, which last sequence tends to 0 due to the discreteness of the valuation.
LEMMA 4.6. (Extension theorem for monotone functions). Let 
\[ \phi \neq Y \subset K, f: Y \to K, \sigma: \Sigma \to \Sigma \text{ satisfy } \sigma(\alpha \oplus \beta) = \sigma(\alpha) \oplus \sigma(\beta) \quad (\alpha, \beta \in \Sigma, \alpha \neq -\beta). \]
Suppose 
\[ x > y \text{ implies } f(x) > f(y) \quad (x, y \in Y, \alpha \in \Sigma) \]
Then \( f \) can be extended to a function \( \overline{f}: K \to K \), monotone of type \( \sigma \).

PROOF. By Zorn's lemma it suffices to extend \( f \) to \( Y \cup \{a\} \) (\( a \not\in Y \)) such that \( f(x) - \overline{f}(a) \in \sigma(\text{sgn}(x - a)) \) and \( \overline{f}(a) - f(x) \in \sigma(\text{sgn}(a - x)) \) for all \( x \in Y \). By 4.5.(ii) it suffices to consider only the second case. Let 
\[ B_x = f(x) + \sigma(\text{sgn}(a - x)) \quad (x \in Y) \]
Then each \( B_x \) is a ball having diameter \( |\pi| \cdot |\sigma(\text{sgn}(a - x))| \neq 0 \).
By the local compactness (in fact, spherical completeness) of \( K \) we are done if we can show that \( B_x \cap B_y \neq \emptyset \) whenever \( x, y \in Y, x \neq y \).

Set \( \alpha = \text{sgn}(a - x) \) and \( \beta = \text{sgn}(a - y), b \in \sigma(\alpha), c \in \sigma(\beta) \). We have to prove that \( |f(x) + b - (f(y) + c)| \leq |\pi| \cdot \max(|\sigma(\alpha)|, |\sigma(\beta)|) \).
Consider two cases.
1) \( \alpha = \beta \). Then \( a - x \) and \( a - y \) are in \( \alpha \), so \( |a - x - (a - y)| = |x - y| < |\alpha| \), hence \( |\text{sgn}(x - y)| < |\alpha| \). By 4.5.(iii) we have 
\[ |\sigma(\text{sgn}(x - y))| < |\sigma(\alpha)|, \text{ so } |\text{sgn}(f(x) - f(y))| < |\sigma(\alpha)| \text{ whence } |f(x) - f(y)| < |\sigma(\alpha)|. \text{ Also } |b - c| < |\sigma(\alpha)| \text{ since both } b \text{ and } c \text{ are in } \sigma(\alpha). \text{ Consequently } |f(x) + b - (f(y) + c)| < |\sigma(\alpha)|. \]
2) \( \alpha \neq \beta \). Then \( x - y = a - y - (a - x) \in \beta \oplus -\alpha \), so \( f(x) - f(y) \in \sigma(\beta \oplus -\alpha) \). Now \( b - c \in \sigma(\alpha) \oplus (\sigma(\beta)) = \sigma(\alpha) \oplus \sigma(-\beta) = \sigma(\alpha \oplus -\beta) = -\sigma(\beta \oplus -\alpha) \). Therefore \( |f(x) - f(y) - (b - c)| < |\sigma(\beta \oplus -\alpha)| = \max(|\sigma(\beta)|, |\sigma(\alpha)|) \).

(The proof of 4.4.(ii): Choose \( Y = \{0\} \) and let \( f: Y \to K \) be defined via \( f(0) = 0 \). Extend \( f \) in the way of 4.6.).
COROLLARY 4.7. Let \( f: K \to K \) be monotone of type \( \sigma: \Sigma \to \Sigma \). Then \( f \in M_{bs}(K) \) (see section 2). More than that: there exists a strictly increasing function \( \phi: |K| \to |K| \), continuous at 0, \( \phi(0) = 0 \) such that
\[
|f(x) - f(y)| = \phi(|x - y|). \quad (x,y \in K)
\]

PROOF. Let \( x, y, u, v \in K \) and \( x - y \in \alpha \in \Sigma, u - v \in \beta \in \Sigma \). Then we have by 4.5.(iii):
\[
|x - y| < |u - v| \iff |\alpha| < |\beta| \iff |\sigma(\alpha)| < |\sigma(\beta)| \iff
|f(x) - f(y)| < |f(u) - f(v)|. \quad \text{The existence of } \phi \text{ is now clear. The continuity follows from } 4.5.(iv).
\]

REMARK. There exist isometries \( K \to K \) that are monotone of type \( \sigma \) for no \( \sigma \) (see [4]).

THEOREM 4.8. Let \( f: K \to K \) be monotone of type \( \sigma: \Sigma \to \Sigma \). Then \( \sigma \) is surjective if and only if \( f \) is a bijection (in fact, \( f \) is a nonzero scalar multiple of an isometry, by 2.7.).

PROOF. If \( \sigma \) is surjective then \( \sigma^{-1} \) exists and satisfies the condition of 4.6., so there is a \( g: K \to K \), monotone of type \( \sigma^{-1} \). Then \( f \circ g \) is monotone of type 1, i.e., increasing. It suffices to show that an increasing \( h: K \to K \) is surjective. Let \( a \in K \) and consider the map \( \psi: x \mapsto x - h(x) + a \quad (x \in K) \). Then
\[
|\psi(x) - \psi(y)| \leq |\pi||x - y| \quad (x, y \in K). \quad \text{By the Banach contraction theorem, } \psi \text{ has a fixed point } t.
\]
Then \( h(t) = a: h \) is surjective. The converse is easy.

EXAMPLE. The monotone functions on \( \mathcal{P}_p \).

As we have seen in section 4, the group of signs of \( \mathcal{P}_p \) is isomorphic to \( \mathbb{P}_p^* \times \mathbb{Z} \). Using this interpretation we can describe the sign function as follows. Let \( x \in \mathcal{P}_p \), \( x = \sum_{n \geq k} a_n p^n \) be its standard expansion (i.e., \( k \in \mathbb{Z}, a_n \in \{0,1,2,...,p-1\}, a_k \neq 0 \)). Then
\[
\text{sgn}(x) = (a_k, n) \in \mathbb{P}_p^* \times \mathbb{Z}.
\]
Let $\sigma: \Sigma \to \Sigma$ be a type of some monotone function. By 4.5. we have $\sigma(1,n) = l\sigma(1,n), ((1,n) \in \Sigma)$. Set

$$\sigma(1,n) = (s(n), \lambda(n)) \quad (n \in \mathbb{Z})$$

Since $n < m \leftrightarrow |p^n| > |p^m| \leftrightarrow |(1,n)| > |(1,m)| \leftrightarrow |\sigma(1,n)| > |\sigma(1,m)| \leftrightarrow |p^\lambda(n)| > |p^\lambda(m)| \leftrightarrow \lambda(n) < \lambda(m)$, we see that $\lambda: \mathbb{Z} \to \mathbb{Z}$ is strictly increasing. Thus, $\sigma$ has the form

$$(*) \quad (1,n) \mapsto (l s(n), \lambda(n))$$

where $s: \mathbb{Z} \to \mathbb{F}_p^*$ and $\lambda: \mathbb{Z} \to \mathbb{Z}$ is strictly increasing.

Conversely, if we have a map $\sigma$ satisfying $(*)$ where $s: \mathbb{Z} \to \mathbb{F}_p^*$ and $\lambda: \mathbb{Z} \to \mathbb{Z}$ is strictly increasing the function

$$\Sigma a_n p^n \mapsto \Sigma a_n s(n)p^\lambda(n)$$

is monotone of type $\sigma$, as can easily be verified.

For a criterion in order that a continuous function $\mathbb{Z}_p \to \mathbb{Q}_p$ be increasing (expressed by means of its coordinates with respect to some orthonormal base), see [4].
Differentiation

We briefly consider the relationship between monotony and differentiation. We refer to [4] for the proofs. Although even increasing functions may be nowhere differentiable there are some connections that are similar to those in the real case.

A function \( g: K \to K \) is called positive if \( g(K) \subseteq K^+ \).

A function \( h: K \to K \) is of the first class of Baire if there exists a sequence \( h_1, h_2, \ldots \) of continuous functions \( K \to K \) that converges pointwise to \( h \).

**Theorem.** (i) Let \( f: K \to K \) be increasing, differentiable. Then \( f' \) is positive, of the first class of Baire.

(ii) A positive function of the first class of Baire has an increasing antiderivative.

**Theorem.** Let \( f: K \to K \) be continuously differentiable (which means here that \( \lim_{x \to a} (x-y)^{-1}(f(x) - f(y)) \) exists for \( a \in K \)), and suppose \( f'(a) \neq 0 \). Then there is a (convex) neighborhood \( X \) of \( a \) such that \( f|X \) is monotone of type \( \sigma \), where \( \sigma \) is the map \( a \mapsto \text{sgn}(f'(a))a \).

**Theorem.** Let \( f: K \to K \) be monotone of type \( \sigma \), differentiable. Then there are two cases.

I. \( f'(a) = 0 \) for some \( a \in K \). Then \( f' = 0 \) everywhere and \( \lim_{|a| \to 0} \frac{\sigma(a)}{|a|} = 0 \).

II. \( f'(a) \neq 0 \) for some \( a \in K \). Then \( f' \neq 0 \) everywhere.

In fact, \( f' \) has constant sign (\( x \mapsto \text{sgn}(f'(x)) \) is constant). For small \( |a| \), \( \frac{\sigma(a)}{a} \) is constant. \( f'(a)^{-1}f \) is locally increasing.
REMARK. One can make an example of an everywhere differentiable $f: K \to K$ with $f' = 1$ (so $f'$ is positive) such that $f$ is not even locally injective at $0$. ($f$ is, of course, not continuously differentiable).