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EXTRAIT DE

"SYMPOSION DÉDIÉ À A.F. MONNA"

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0. Introduction

Our aim in this paper is to present reasonable definitions for a function $f: K \to K$ to be "monotone", where $K$ is a local field (i.e., a non-archimedean non-trivially valued field that is locally compact in the topology induced by the valuation). The future will tell us whether filling this gap in p-adic analysis has been of any use.

The fact that - until recently - the concept of "monotone function" has been absent in p-adic analysis is not surprising since a decent (partial) ordering in $K$, (compatible with the algebraic and topological structure) is not available. Thus, in the sequel we will look for substitutes for "ordering" in $K$ as a basis for our theory. One of them is the notion of "pseudo-ordering", introduced by A.F. Monna [1].

The theory can be build up in a more general setting, namely for functions $f: X \to K$, where $K$ is any complete non-archimedean field and $X \subset K$, but restriction to local fields avoids a lot of technicalities and enlights the main track. For the extended theory, see [4].

The elementary facts about analysis in local fields can be found in [2].

Notations and definitions

FROM NOW ON IN THIS PAPER $K$ IS A LOCAL FIELD WITH RESIDUE CLASS FIELD $k$. 

P-adic monotone functions

by

W.H. Schikhof
Set $|K| = \{|x| : x \in K\}$

$|K^*| = |K|\{0\}$ (the value group)

$\pi$: a (fixed) element of $K^*$ such that $|\pi|$ generates $|K^*|$, $|\pi| < 1$.

For a prime $p$ we denote by $\mathbb{Q}_p$ the non-archimedean valued field of the $p$-adic numbers, by $\mathbb{Z}_p$ its valuation ring $\{x \in \mathbb{Q}_p : |x| \leq 1\}$. The residue class field, $\mathbb{Z}_p/p\mathbb{Z}_p$ of $\mathbb{Q}_p$ is the field of $p$ elements and is denoted by $\mathbb{F}_p$.

The characteristic of a field $L$ is denoted by $\chi(L)$.

For a $K$-vector space $E$ and a subset $S$ of $E$ we denote its $K$-linear span by $[S]$.

Let $a \in K$, $r \in [0,\infty)$. The ball with center $a$ and radius $r$ is by definition $\{x \in K : |x - a| \leq r\}$. It is easy to see that the intersection of a collection of balls is either empty or again a ball.

Let $x,y \in K$. Then the smallest ball containing $x$ and $y$ is denoted by $[x,y]$. A subset $C$ of $K$ is called convex if $x,y \in C$ implies $[x,y] \subseteq C$. Each ball is convex. A convex set $\neq K$, $\neq \emptyset$ is a ball.

It follows that $K$ is the only unbounded convex set in $K$.

FROM NOW ON $X$ IS A CONVEX SUBSET OF $K$.

1. Two notions of monotony

Interpreting the above notion of convexity also for the real numbers it is quite natural to introduce the following geometric expressions.

Let $x,y,z \in K$. We say that $z$ is between $x$ and $y$ if $z \in [x,y]$.

If $z$ is not between $x$ and $y$ we say that $x,y$ are at the same side of $z$. This yields more or less automatically the following
DEFINITION 1. Let $f: X \to K$. We say that $f \in M_b(X)$ (f respects "betweenness") if for all $x, y, z \in X$

$\ (* \ ) z \in [x,y] \Rightarrow f(z) \in [f(x), f(y)]$.

We say that $f \in M_s(X)$ (f respects "sides") if for all $x, y, z \in X$

$\ (** \ ) z \notin [x,y] \Rightarrow f(z) \notin [f(x), f(y)]$.

REMARKS

1. If we, in the above definition, replace formally $K$ by $\mathbb{R}$ and $X$ by an interval, we see that $f \in M_b(X)$ just means that $f$ is monotone and that $f \in M_s(X)$ becomes "$f$ is strictly monotone".

(These facts can easily be proved). So for the time being we let our intuition be guided by this analogy:

The statements in 2 and 3 below are direct consequences of the definitions, and the proofs are left to the reader.

2. $M_b(X)$ is closed under pointwise limits.

The constant functions are in $M_b(X)$. If $f \in M_b(X)$ and $f(a) = f(b)$ then $f$ is constant on $[a,b]$.

$f \in M_b(X) \Rightarrow$ For each convex $C \subseteq K$ the inverse image $f^{-1}(C)$ is convex. For each $a, b \in K$ the map $x \mapsto ax + b$ is in $M_b(X)$.

$f \in M_s(X) \Rightarrow f$ is injective.

If $a, b \in K$, $a \neq 0$ then $x \mapsto ax + b$ is in $M_s(X)$.

Each isometry $X \to K$ is in $M_{bs}(X)$ where

$M_{bs}(X): = M_b(X) \cap M_s(X)$.

3. Without harm we may replace in Definition 1 (*) by (*)', or (*)'' or (*)''', where

$\ (*)' \ : \ |x-z| \leq |x-y| \Rightarrow |f(x) - f(z)| \leq |f(x) - f(y)|$

$\ (*)'' \ : \ |x-z| = |x-y| \Rightarrow |f(x) - f(z)| = |f(x) - f(y)|$

$\ (*)''' \ : \ |f(x) - f(z)| < |f(x) - f(y)| \Rightarrow |x-z| < |x-y|$
Similarly, we may replace (**) by (**)' or (**)' or (**)'', where

\[ (**)' : |x-z| < |x-y| \rightarrow |f(x) - f(z)| < |f(x) - f(y)| \]
\[ (**)'' : |f(x) - f(z)| = |f(x) - f(y)| \rightarrow |x-z| = |x-y| \]
\[ (**)'''' : |f(x) - f(z)| < |f(x) - f(y)| \rightarrow |x-z| < |x-y| . \]

4. In the next section we will study \( M_b(X) \) and \( M_s(X) \). For example, the natural questions: \( M_s(X) \subset M_b(X) \)? \( f \in M_b(X) \), \( f \) injective \( \Rightarrow \) \( f \in M_s(X) \)? Notice that our definitions do not refer to any "type" of monotony (such as "increasing" and "decreasing" for real functions). In section 4 we will introduce such a concept. It will turn out that monotone functions having a "type" are \( M_{bs} \)-functions, but not conversely.

2. Properties of monotone functions

**Theorem 2.1.** Let \( f \in M_b(X) \). If \( a,b,c \in X \), \( |a-b| < |a-c| \), \( f(a) \neq f(c) \) then \( |f(a) - f(b)| < |f(a) - f(c)| \). In particular, if \( f \in M_b(X) \), \( f \) is injective then \( f \in M_s(X) \).

**Proof.** Without loss, assume \( X = [a,c] \). Since \( f \in M_b(X) \) we have \( \{f(a), f(c)\} \subset f(X) \subset [f(a), f(c)] \), hence the diameter of \( f(X) \) equals \( M := |f(a) - f(c)| \). The ball \([f(a), f(c)]\) has a partition into \( n \) balls \( V_1, \ldots, V_n \) each having radius \( M/\pi \), where \( n \) is the number of elements of \( k \). The sets \( f^{-1}(V_i), \ldots, f^{-1}(V_n) \) form a partition of \( X \), each \( f^{-1}(V_i) \) is convex (since \( f \in M_b(X) \)), at least two of the \( f^{-1}(V_i) \) are non-empty (since \( a \) and \( c \) cannot both lie in the same \( f^{-1}(V_i) \)). It follows that the diameter of each \( f^{-1}(V_i) \) is strictly less than \( |a-c| \). (Otherwise \( f^{-1}(V_i) = X \) for some \( i \) and \( f^{-1}(V_j) = \emptyset \) for \( j \neq i \)). Consequently the partition \( f^{-1}(V_1), \ldots, f^{-1}(V_n) \) of \( X \) must be the partition of \( X \) into balls with radius \( |a-c|/\pi | \).
Now if \( |a-b| < |a-c| \) then \( a, b \in f^{-1}(V_i) \) for some \( i \), so

\[
m|\pi| \leq |f(a) - f(b)|.
\]

EXAMPLE. Let \( p \neq 2 \) and let \( f: \mathbb{Z}_p \to \mathbb{Q}_p \) "tear apart" \( \mathbb{Z}_p \) by sending

\[
k + p\mathbb{Z}_p \text{ into } p^{-k} + p\mathbb{Z}_p \quad (k = 0, 1, 2, \ldots, p-1)
\]

via translations. Then one easily checks that \( f \in M_s(\mathbb{Z}_p) \backslash M_b(\mathbb{Z}_p) \).

Hence, it seems that \( M_{bs} \)-functions are the "translation" of the real strictly monotone functions (rather than \( M_s \)-functions).

In the sequel we often make use of the following observation. If \( f \) is either in \( M_b(X) \) or in \( M_s(X) \) then

\[
(*) \quad |x-z| < |x-y| \Rightarrow |f(x) - f(z)| \leq |f(x) - f(y)| \quad (x, y, z \in X)
\]

(Functions with property (*) are called weakly monotone in \([4]\)).

**Lemma 2.2.** Let \( f \in M_b(X) \) or \( f \in M_s(X) \). Then, if \( Y \subseteq X \) is bounded

then \( f(Y) \) is bounded.

**Proof.** \( Y \) is precompact, so \( \rho = \max\{|x-y| : x, y \in Y\} \) exists. We may assume \( \rho > 0 \). The equivalence relation \( x \sim y \) iff \( |x-y| < \rho \) divides \( Y \) into finitely many classes \( Y_1, \ldots, Y_n \) where \( n \geq 2 \). Choose \( a_i \in Y_i \) for each \( i \) and set \( M = \max_{i} |f(a_i)| \). We prove \( |f| \leq M \) on \( Y \).

In fact, let \( x \in Y \). There is \( i \in \{1, \ldots, n\} \) such that \( |x-a_i| < \rho \).

For \( j \neq i \) we have \( |x-a_j| < \rho = |a_i - a_j| \), so \( |f(x) - f(a_i)| \leq |f(a_i) - f(a_j)| \leq M \). Hence \( |f(x)| \leq M \).

We have a "dual" statement which is only of interest in case \( X = K \):

**Lemma 2.3.** Let \( f \in M_s(K) \) or \( f \in M_b(K) \). If \( f \) is not constant then

for a bounded \( Z \subseteq K \) the inverse image \( f^{-1}(Z) \) is bounded.

**Proof.** We prove: \( Z \subseteq K \) bounded, \( T = f^{-1}(Z) \) is unbounded implies \( f \) is constant. In fact, let \( a, b \in K \). There are \( x_1, x_2, \ldots \) in \( T \) such that
\(*) \max (|a|, |b|) < |x_1| < |x_2| < \ldots \\

The precompactness of \{f(x_1), f(x_2), \ldots\} implies convergence of a subsequence of \{f(x_1), f(x_2), \ldots\}. Without loss, assume that \(\lim_{n \to \infty} f(x_n)\) exists. From (*) we obtain

\[ |a-b| < |x_1-a| < |x_2-x_1| < |x_3 - x_2| < \ldots, \]

so that for all \(n \in \mathbb{N}\)

\[ |f(a) - f(b)| \leq |f(x_{n+1}) - f(x_n)|. \]

Hence,

\[ |f(a) - f(b)| \leq \lim_{n \to \infty} |f(x_{n+1}) - f(x_n)| = 0. \]

It follows that \(f\) is constant.

2.1., 2.2. and 2.3. yield the continuity properties 2.4., 2.5. and 2.6.: 

**THEOREM 2.4.** Let \(X\) be bounded with diameter \(r > 0\) and let \(f \in M_b(X)\) or \(f \in M_s(X)\). Then \(f\) satisfies the Lipschitz-condition

\[ |f(x) - f(y)| \leq M|x-y| \quad (x, y \in X) \]

where \(M = r^{-1}\sup\{ |f(x) - f(y)| : x, y \in X\} < \infty\).

**PROOF.** By 2.2. \(f\) is bounded, so \(M < \infty\). Choose any \(a \in X\). We prove by induction on \(n\):

\[ P(n): |x-a| = |\pi|^n r \text{ then } |f(x) - f(a)| \leq |\pi|^n r M. \quad (x \in X) \]

Clearly \(P(0)\) holds. Suppose \(P(n-1)\). Let \(x \in X\) such that \(|x-a| = |\pi|^n r\) and choose \(b \in X\) with \(|b-a| = |\pi|^{n-1} r\).

Then \(|x-a| < |b-a|\). If \(f(b) = f(a)\) then \(|f(x) - f(a)| \leq |f(b) - f(a)| = 0\), so certainly \(|f(x) - f(a)| \leq |\pi|^{n-1} r M\). If \(f(a) \neq f(b)\) then either by Theorem 2.1. or since \(f \in M_s(X)\) we have \(|f(x) - f(a)| < |f(b) - f(a)| \leq (\text{induction hypothesis}) \leq |\pi|^{n-1} r M\), so that \(|f(x) - f(a)| \leq |\pi|^{n} r M\).

**REMARK.** The map \(\Sigma_{a_n} p^n \to \Sigma_{a_{n} p^{2n}}\) is in \(M_{bs}(Q_p)\) and has unbounded difference quotients.
THEOREM 2.5. Let \( f \in M_b(X) \) or \( f \in M_s(X) \). Then \( f \) is continuous.

Further, if \( Y \subseteq X \) is closed then \( f(Y) \) is closed in \( K \).

In particular, an \( f \in M_s(X) \) is a homeomorphism \( X \cong f(X) \).

PROOF. If \( X \) is bounded then everything follows from 2.4., so let \( X = K \). The continuity of \( f \) is clear (restrict \( f \) to bounded convex subsets). Let \( Y \subseteq K \) be closed and suppose \( \alpha \in \overline{f(Y)} \setminus f(Y) \). There are \( a_1, a_2, \ldots \) in \( Y \) for which \( \lim_{n \to \infty} f(a_n) = \alpha \). \( f \) is not constant, so by 2.3. the sequence \( a_1, a_2, \ldots \) is bounded, assume it converges, say \( a = \lim_{n \to \infty} a_n \). By continuity, \( f(a) = \lim_{n \to \infty} f(a_n) = \alpha \), a contradiction. The last statement is now trivial.

THEOREM 2.6. Let \( f \in M_b(X) \) (or \( f \in M_s(X) \) for that matter). Then the following are equivalent.

(a) \( f \) is a homeomorphism \( X \cong f(X) \).
(b) \( f \) is injective.
(c) \( f \in M_s(X) \).
(d) \( f(X) \) has no isolated points.

PROOF. The implications \((a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)\) are easy (2.1., continuity and injectivity of \( M^s \)-functions). We prove \((d) \Rightarrow (a)\), that is, if \( \lim_{n \to \infty} f(x_n) = f(a) \) then \( \lim_{n \to \infty} x_n = a \). Now since \( f(a) \) is not isolated we can find \( a_1, a_2, \ldots \) such that \( f(a_n) \neq f(a) \) for all \( n \), \( \lim_{n \to \infty} f(a_n) = f(a) \). By 2.3. we may assume that \( b_n = \lim_{n \to \infty} a_n \) exists. If suffices to prove that \( \lim_{n \to \infty} x_n = b \) (apply the result for \( x_n = a \) for all \( n \) and we find \( a = b \)). Let \( \varepsilon > 0 \). There is \( k \in \mathbb{N} \) for which \( |a_k - b| < \varepsilon \). For large \( n \) we have \( |f(x_n) - f(a)| < |f(a_k) - f(a)| \), hence for large \( n \) and \( m(n) \) we get \( |f(x_n) - f(a_m)| < |f(a_k) - f(a_m)| \) whence \( |x_n - a_m| \leq |a_k - a_m| \), so \( |x_n - b| < |a_k - b| < \varepsilon \) for large \( n \).

THEOREM 2.7. Let \( f \in M_b(X) \) or \( f \in M_s(X) \) and assume that \( f(X) \) is convex. Then \( f \) is a scalar multiple of an isometry, or \( f \) is constant.
PROOF. We may assume that \( f \) is not constant, so \( f(X) \) is open, non-empty. Let \( X = f(X) \) be bounded. By 2.6, \( f \) is injective. It is clear that also \( f^{-1} \in M_b(X) \cup M_s(X) \). Applying 2.4 to both \( f \) and \( f^{-1} \) we get (in both cases \( M = 1 \)) for all \( x,y,u,v \in X \) that \(|f(x) - f(y)| \leq |x-y| \) and \(|f^{-1}(u) - f^{-1}(v)| \leq |u-v| \). It follows that \( f \) is an isometry. The general case is now easy. (If \( X, f(X) \) are bounded a transformation of the type \( x \to ax+b \) sends \( f(X) \) into \( X \); if \( X = f(X) = K \), apply 2.7. to \( X_n = \{ x \in K : |x| \leq n \} \) (\( n \in \mathbb{N} \)).

The following theorem describes the functions "of bounded variation":

**THEOREM 2.8.** Let \( X \) be bounded. Then \([M_b(X)] = [M_s(X)] = BA(X)\), where \( BA(X) \) is the linear space of functions \( f: X \to K \) having bounded difference quotients.

**PROOF.** By 2.4 we are done if we can prove that an \( f \in BA(X) \) is the sum of two \( M_{bs} \)-functions. Let \( \lambda \in K \) such that \(|f(x) - f(y)| \leq |\lambda| |x-y| \) for all \( x,y \in X \), \( x \neq y \). Let \( g(x) := \lambda x \) (\( x \in X \)) and let \( h := f - g \).

Then \( g, h \) are in \( M_{bs}(X) \) (scalar multiples of isometries) and \( f = g + h \).

### 3. Monotone sequences

We will not delve deeply into this subject, but content ourselves with presenting definitions and some facts indicating that these notions are not that bad.

**DEFINITION 3.1.** Let \( x_1, x_2, \ldots \) be a sequence in \( K \). It is called

- **b-monotone**, if \( k \leq l \leq m \) implies \( x_l \in [x_k, x_m] \),
- **s-monotone**, if \( l < k, m \) implies \( x_l \notin [x_k, x_m] \).

**THEOREM 3.2.** A sequence \( x_1, x_2, \ldots \) in \( K \) is s-monotone if and only if \(|x_1-x_2| > |x_2-x_3| > \ldots\)

A sequence \( x_1, x_2, \ldots \) in \( K \) is b-monotone if and only if for each \( k, m \in \mathbb{N} \), \( k < m \).
\[ |x_m - x_k| = \max \{ |x_{i+1} - x_i| : k \leq i < m \}. \]

**PROOF.** Let \( x_1, x_2, \ldots \) be s-monotone, and let \( n \in \mathbb{N} \). We have

\[ x_n \notin [x_{n+1}, x_{n+2}] \text{ so } |x_n - x_{n+1}| > |x_{n+1} - x_{n+2}|. \]

Conversely, if

\[ |x_1 - x_2| > |x_2 - x_3| > \ldots \text{, let } 1 < k, m. \]

Then

\[ |x_k - x_1| = |x_{k+1} - x_{k}| \text{ hence } |x_k - x_1| > |x_m - x_k| \text{ i.e., } x_1 \notin [x_k, x_m]. \]

Let \( x_1, x_2, \ldots \) be b-monotone, and let \( k, m \in \mathbb{N} \), \( k < m \), and \( k \leq i < m \).

Then \( x_i \in [x_k, x_m] \) and \( x_{i+1} \in [x_k, x_m] \), so \( [x_i, x_{i+1}] \subset [x_k, x_m] \), hence

\[ |x_{i+1} - x_i| \leq |x_m - x_k|. \]

The rest is obvious. To prove the converse, let \( k < 1 < m \). Then

\[ |x_1 - x_k| = \max \{ |x_{i+1} - x_i| : k < i < l \} \leq \max \{ |x_{i+1} - x_i| : k < i < m \} = |x_m - x_k|. \]

Hence \( x_1 \in [x_k, x_m] \).

**COROLLARY 3.3.** Each s-monotone sequence is b-monotone. An s-monotone sequence \( x_1, x_2, \ldots \) is convergent and \( x_n \neq x_m \) whenever \( n \neq m \). \( M_b \)-functions map b-monotone sequences into b-monotone sequences. \( M_s \)-functions map s-monotone sequences into s-monotone sequences.

A b-monotone sequence need not be convergent. In fact, if

\[ |x_1| < |x_2| < \ldots \text{ then } x_1, x_2, \ldots \text{ is b-monotone. But we have} \]

**THEOREM 3.4.** Let \( x_1, x_2, \ldots \) be b-monotone. Then either

\[ \lim_{n \to \infty} |x_n| = \infty \]

or \( \lim_{n \to \infty} x_n \) exists.

**PROOF.** If not \( \lim_{n \to \infty} |x_n| = \infty \) then the sequence has a bounded, hence a convergent subsequence, say, \( \lim_{i \to \infty} x_i = x \). We show that \( \lim_{n \to \infty} x_n = x \).

Let \( \epsilon > 0 \). Then

\[ |x - x_{n_k}| < \epsilon \text{ for } k \geq k_0. \]

If \( 1, m \geq n_{k_0} \), then if \( n_k \geq 1, m \) we have

\[ |x_{1-m} - x_m| \leq |x_{n_{k_0}} - x_{n_{k}}| \leq \max \{ |x-x_{n_{k_0}}, |x-x_{n_{k}}| \} < \epsilon. \]

Hence \( x_1, x_2, \ldots \) is Cauchy, so it must converge, to \( x \).

**THEOREM 3.5.** Each sequence in \( K \) has a b-monotone subsequence.
PROOF. We may assume that \( x_1, x_2, \ldots \) is bounded, and that it has a convergent subsequence \( y_1, y_2, \ldots \) where \( y_n \neq y_m \) whenever \( n \neq m \). Set \( y = \lim_{n \to \infty} y_n \). Now it is easy to construct a subsequence \( z_1, z_2, \ldots \) of \( y_1, y_2, \ldots \) for which \( |y-z_1| > |y-z_2| > |y-z_3| > \ldots \)
Hence \( |z_1-z_2| > |z_2-z_3| > \ldots \). The sequence \( z_1, z_2, \ldots \) is s-monotone, hence \( b \)-monotone.

EXAMPLE. Let \( x = \sum_{n=0}^{\infty} a_n p^n \in \mathbb{Z}_p \), and let \( x_k = \sum_{n=0}^{k} a_n p^n \). Then \( x_0, x_1, \ldots \) is a \( b \)-monotone sequence, converging to \( x \).

4. Monotone functions of type \( \sigma \)

Following Monna [1] we introduce the concept of "sides of zero" in \( K \) (a generalization of the partition \( \mathbb{R}^* = \mathbb{R}^+ \cup \mathbb{R}^- \)). Let \( K^* = K \setminus \{0\} \). Define

\[
x \sim y \text{ if } x \text{ and } y \text{ are at the same side of } 0 \quad (x, y \in K^*).
\]

Then we have \( x \sim y \Leftrightarrow 0 \not\in [x,y] \Leftrightarrow |x-y| < |y| \Leftrightarrow |xy^{-1}| < 1 \Leftrightarrow xy^{-1} \in K^+ \),
where

\[
K^+ = \{ x \in K : |1-x| < 1 \}.
\]

\( K^+ \) is a multiplicative open compact subgroup of \( K^* \), called the group of the positive elements of \( K \). We see that \( \sim \) is the equivalence relation induced by the canonical group homomorphism, the "sign map":

\[
\text{sgn}: K^* \to K^*/K^+.
\]

The quotient group \( \Sigma = K^*/K^+ \) (comparable with \( \{1, -1\} \) in the real case) is called the group of signs of elements of \( K \), or the group of sides of zero of \( K \). \( \Sigma \) is an infinite group, whose elements are multiplicative cosets of \( K^+ \).

Let \( \alpha \in \Sigma \) and let \( x, y \in \alpha \). Then \( |x-y| < |x| \), so in particular,
\[ |x| = |y|. \] Therefore we may define the absolute value of a sign \( \alpha \in \Sigma \) as

\[ |\alpha| = |x| \quad (x \in \alpha) \]

The map \( \alpha \mapsto |\alpha| \) is a surjective group homomorphism \( \Sigma \to |K^*| \).

Its kernel, \( \{ \alpha \in \Sigma: |\alpha| = 1 \} \) is a multiplicative subgroup of \( \Sigma \), which is naturally isomorphic to the multiplicative group \( k^* \) under \( \alpha \mapsto \overline{\alpha} \) (where \( x \mapsto \overline{x} \) is the canonical map \( \{ x \in K: |x| \leq 1 \} \to k \)).

Let us denote its inverse \( k^* \to \{ \alpha \in \Sigma: |\alpha| = 1 \} \) by

\[ l \mapsto \alpha_l \quad (l \in k^*). \]

For each \( n \in \mathbb{Z} \), let \( \alpha_n = \text{sgn}(\pi^n) \).

Now for each \( \alpha \in \Sigma \) there are unique \( n \in \mathbb{Z}, l \in k^* \) such that

\[ \alpha = \alpha_l \alpha_n. \]

Further, we have

\[ \alpha_l \alpha_{l'} = \alpha_{ll'} \quad (l, l' \in k^*) \]

\[ \alpha_n \alpha_m = \alpha_{n+m} \quad (n, m \in \mathbb{Z}). \]

It follows, that \( \Sigma \) is isomorphic to \( k^* \times \mathbb{Z} \) (or to \( k^* \times |k^*| \) for that matter). It is also possible to identify \( \Sigma \) with a subgroup of \( k^* \) but we shall not need it here.

In the sequel we will use addition of signs. For \( \alpha \in \Sigma \), let

\[ -\alpha = \{-x: x \in \alpha\}. \]

Then clearly \(-\alpha\) is a sign and if \( \alpha = \alpha_l \alpha_n \) (\( l \in k^*, n \in \mathbb{Z} \)) then

\[ -\alpha = \alpha_{-l} \alpha_n. \]

Let \( \alpha, \beta \in \Sigma \). Then \( \alpha + \beta = \{ x+y: x \in \alpha, y \in \beta \} \) is easily seen to be a ball, which turns out to be again a sign iff \( \alpha \neq -\beta \) (iff \( 0 \not\in \alpha + \beta \)).

Therefore we define
\begin{align*}
\alpha \oplus \beta &= \alpha + \beta \quad (\alpha, \beta \in \Sigma, \alpha \neq -\beta).
\end{align*}

We have the following rules that are easy to prove:

RULES 4.1. (i) The operation \( \oplus \) is commutative, associative, distributive, whenever the occurring formulas are defined.

(ii) \( |\alpha \oplus \beta| = \max(|\alpha|, |\beta|) \quad (\alpha, \beta \in \Sigma, \alpha \neq -\beta) \)

(iii) \( |\alpha| < |\beta| \) if and only if \( \alpha \oplus \beta = \beta \quad (\alpha, \beta \in \Sigma) \)

(iv) Let \( n \in \mathbb{N}, 1 \leq n < \chi(k), \alpha \in \Sigma. \) Then \( \oplus \alpha(\ldots \oplus \alpha(n \text{ times})) \) exists and equals \( n\alpha. \)

Thus, we get for \( l, l' \in k^*, m, n \in \mathbb{Z} : \)

\[
\alpha_l^m \oplus \alpha_{l'}^n = \begin{cases} 
\alpha_l^m & \text{if } m < n \\
\alpha_{l'}^n & \text{if } m > n \\
\alpha_{l+1}^m & \text{if } m = n \text{ and } l + l' \neq 0.
\end{cases}
\]

Next, we define a "pseudo-ordering" in \( K. \) Let \( x, y \in K, \alpha \in \Sigma. \) We say that \( x > y \) (\( x \) is greater than \( y \) in the sense of \( \alpha \)) in case \( x - y \in \alpha. \) The following easy consequences obtain.

RULES 4.2. (i) Let \( x, y \in K. \) Then if \( y \neq x \) there is exactly one \( \alpha \in \Sigma \) for which \( x > y. \) If \( y = x \) then \( x > y \) for no \( \alpha. \) ("\( K \) is totally pseudo-ordered").

(ii) If \( x > y, y > z \) for some \( x, y, z \in K; \alpha, \beta \in \Sigma, \) and if \( \alpha \oplus \beta \) exists then \( x > z. \) ("Transitivity").

(iii) If \( x > y \) for some \( x, y \in K; \alpha \in \Sigma \) and if \( z \in K, \alpha \) then \( x + z > y + z \) ("Compatibility with addition").

(iv) If \( x, y, z \in K; \alpha, \beta \in \Sigma, x > y \) and \( z > 0 \) then \( xz > yz \) ("Compatibility with multiplication").

We define:
DEFINITION 4.3. Let $\sigma: \Sigma \to \Sigma$ be an injection and $f: K \to K$. 

$f$ is called monotone of type $\sigma$ if for all $\alpha \in \Sigma$, $x, y \in K$ we have 

$x > y$ implies $f(x) > f(y)$.

$f$ is called increasing if $\sigma$ is the identity map.

REMARKS

1. One can extend easily the above definition for functions $f: X \to K$, where $X$ is a convex subset of $K$, $\sigma: \Sigma(X) \to \Sigma$. Here $\Sigma(X)$ is the collection of signs that "occur in $X" \text{ i.e.} \{\text{sgn}(x-y): x, y \in X, x \neq y\}. \text{ We leave it to the reader to do this and, in case } X \neq K, \text{ to show that there is } \beta \in \Sigma \text{ such that} 

$$\Sigma(X) = \{\alpha \in \Sigma: |\alpha| < |\beta|\}.$$ 

2. Notice that $f: K \to K$ is increasing if and only if the difference quotient

$$\Phi_f(x, y) := \frac{f(x) - f(y)}{x - y} \quad (x \neq y)$$

is positive. In particular, $f$ is an isometry.

3. The requirement made in 4.3. that $\sigma$ be an injection is essential in case $K$ is not a prime field (see [4]).

4. In case $\chi(K) = 0$, the exponential function, defined by the power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is increasing on its convergence region.

If $f$ is an increasing function and $\beta \in \Sigma$ then for each $b \in \beta$ the function $bf$ is of type $\sigma$ where $\sigma$ is the multiplier $a \mapsto a \beta$.

THEOREM 4.4. (i) Let $f: K \to K$ be a monotone of type $\sigma: \Sigma \to \Sigma$. Then

(*) $\sigma(\alpha \oplus \beta) = \sigma(\alpha) \oplus \sigma(\beta) \quad (\alpha, \beta \in \Sigma, \alpha \neq -\beta)$

(ii) Let $\sigma: \Sigma \to \Sigma$ satisfy (*). Then there is a function $g: K \to K$, monotone of type $\sigma$. 

|
PROOF. (i) First we show that \( \sigma(-\alpha) = -\sigma(\alpha) \) \((\alpha \in \Sigma)\). In fact choose \( x, y \in K \) such that \( x - y \in \alpha \). Then \( y - x \in -\alpha \), so \( f(x) - f(y) \in \sigma(\alpha) \cap (-\sigma(-\alpha)) \neq \emptyset \), which implies \( \sigma(\alpha) = -\sigma(-\alpha) \).

Now take \( \alpha, \beta \in \Sigma, \alpha \neq -\beta \). Then by injectivity of \( \sigma \), \( \sigma(\alpha) \neq \sigma(-\beta) = -\sigma(\beta) \) so that \( \sigma(\alpha) \oplus \sigma(\beta) \) exists. Now choose \( x, y, z \in K \) such that \( x - y \in \alpha, y - z \in \beta \). Then \( f(x) - f(z) = f(x) - f(y) + f(y) - f(z) \in \sigma(\alpha) \oplus \sigma(\beta) \). Also, \( x - z \in \alpha \oplus \beta \) so \( f(x) - f(z) \in \sigma(\alpha \oplus \beta) \). As \( \sigma(\alpha) \oplus \sigma(\beta) \) and \( \sigma(\alpha \oplus \beta) \) have a non-empty intersection they are equal.

The proof of (ii) will be furnished by Lemma 4.5. and Lemma 4.6.

**LEMMA 4.5.** Let \( \sigma: \Sigma \to \Sigma \) satisfy \( \sigma(\alpha \oplus \beta) = \sigma(\alpha) \oplus \sigma(\beta) \) \((\alpha, \beta \in \Sigma, \alpha \neq -\beta)\). Then we have

(i) \( \sigma(n\alpha) = n\sigma(\alpha) \) \((n \in \mathbb{N}, 1 \leq n < \chi(k), \alpha \in \Sigma)\).

(ii) \( \sigma(-\alpha) = -\sigma(\alpha) \) \((\alpha \in \Sigma)\).

(iii) For all \( \alpha, \beta \in \Sigma \): \( |\alpha| < |\beta| \) if and only if \( |\sigma(\alpha)| < |\sigma(\beta)| \).

(iv) \( \lim_{|\alpha| \to 0} |\sigma(\alpha)| = 0 \).

**PROOF.** (i) is a direct consequence of 4.1.(iv). To prove (ii), set \( q := \chi(k) \). Then for \( \beta \in \Sigma \), \( (q-1)\beta = -\beta \), so, by (i), \( \sigma(-\alpha) = \sigma(q-1)\alpha = (q-1)\sigma(\alpha) = -\sigma(\alpha) \). Let \( \alpha, \beta \in \Sigma, |\alpha| < |\beta| \). Then \( \alpha \oplus \beta = \beta \) so \( \sigma(\alpha) \oplus \sigma(\beta) = \sigma(\alpha \oplus \beta) = \sigma(\beta) \) whence \( |\sigma(\alpha)| < |\sigma(\beta)| \). Conversely, suppose \( |\sigma(\alpha)| < |\sigma(\beta)| \). Then clearly \( \alpha \neq -\beta \) (otherwise \( |\sigma(\alpha)| = |\sigma(-\beta)| = |-\sigma(\beta)| = |\sigma(\beta)| \)), so \( \alpha \oplus \beta \) exists. Now \( \sigma(\beta) = \sigma(\alpha) \oplus \sigma(\beta) = \sigma(\alpha \oplus \beta) \). By injectivity of \( \sigma \) we obtain \( \beta = \alpha \oplus \beta \) whence \( |\alpha| < |\beta| \).

So we have (iii). (iv) follows from the fact that if \( |\alpha_1| > |\alpha_2| > \ldots \to 0 \) then \( |\sigma(\alpha_1)| > |\sigma(\alpha_2)| > \ldots \), which last sequence tends to 0 due to the discreteness of the valuation.
LEMMA 4.6. (Extension theorem for monotone functions). Let 
\( \phi \neq Y \subset K, f: Y \rightarrow K, \sigma: \Sigma \rightarrow \Sigma \) satisfy \( \sigma(\alpha \oplus \beta) = \sigma(\alpha) \oplus \sigma(\beta) \) (\( \alpha, \beta \in \Sigma, \alpha \neq -\beta \)). Suppose 
\[
\text{if } x > y \text{ then } f(x) > f(y) \quad (x, y \in Y, \alpha \in \Sigma)
\]
Then \( f \) can be extended to a function \( \bar{f}: K \rightarrow K, \) monotone of type \( \sigma. \)

PROOF. By Zorn's lemma it suffices to extend \( f \) to \( Y \cup \{a\} \) (\( a \notin Y \)) such that \( f(x) - \bar{f}(a) \in \sigma(\text{sgn}(x - a)) \) and \( \bar{f}(a) - f(x) \in \sigma(\text{sgn}(a - x)) \) for all \( x \in Y. \) By 4.5.(ii) it suffices to consider only the second case. Let 
\[
B_x := f(x) + \sigma(\text{sgn}(a - x)) \quad (x \in Y)
\]
Then each \( B_x \) is a ball having diameter \( |\pi| \) \(|\sigma(\text{sgn}(a - x))| \neq 0. \)
By the local compactness (in fact, spherical completeness) of \( K \) we are done if we can show that \( B_x \cap B_y \neq \emptyset \) whenever \( x, y \in Y, x \neq y. \)
Set \( \alpha := \text{sgn}(a - x) \) and \( \beta := \text{sgn}(a - y), b \in \sigma(\alpha), c \in \sigma(\beta). \) We have to prove that \( |f(x) + b - (f(y) + c)| \leq |\pi| \max(|\sigma(\alpha)|, |\sigma(\beta)|). \)
Consider two cases.

1) \( \alpha = \beta. \) Then \( a - x \) and \( a - y \) are in \( \alpha, \) so \( |a - x - (a - y)| = |x - y| \leq |\alpha|, \) hence \( |\text{sgn}(x - y)| < |\alpha|. \) By 4.5.(iii) we have 
\[
|\sigma(\text{sgn}(x - y))| < |\sigma(\alpha)|, \text{ so } |\text{sgn}(f(x) - f(y))| < |\sigma(\alpha)| \text{ whence } |f(x) - f(y)| < |\sigma(\alpha)|. \text{ Also } |b - c| < |\sigma(\alpha)| \text{ since both } b \text{ and } c \text{ are in } \sigma(\alpha). \text{ Consequently } |f(x) + b - (f(y) + c)| < |\sigma(\alpha)|.
\]

2) \( \alpha \neq \beta. \) Then \( x - y = a - y - (a - x) \in \beta \oplus -\alpha, \) so \( f(x) - f(y) \in \sigma(\beta \oplus -\alpha) \). Now \( b - c \in \sigma(\alpha) \oplus (-\sigma(\beta)) = \sigma(\alpha) \oplus \sigma(-\beta) = \sigma(\alpha \oplus -\beta) = -\sigma(\beta \oplus -\alpha). \) Therefore \( |f(x) - f(y) - (b - c)| < |\sigma(\beta \oplus -\alpha)| = \max(|\sigma(\beta)|, |\sigma(\alpha)|). \)

(The proof of 4.4.(ii): Choose \( Y := \{0\} \) and let \( f: Y \rightarrow K \) be defined via \( f(0) = 0. \) Extend \( f \) in the way of 4.6.).
COROLLARY 4.7. Let \( f: K \to K \) be monotone of type \( \sigma: \Sigma \to \Sigma \). Then \( f \in M_{bs}(K) \) (see section 2). More than that: there exists a strictly increasing function \( \phi: |K| \to |K| \), continuous at 0, \( \phi(0) = 0 \) such that
\[
|f(x) - f(y)| = \phi(|x - y|). \quad (x, y \in K)
\]

**PROOF.** Let \( x, y, u, v \in K \) and \( x - y \in \alpha \in \Sigma, u - v \in \beta \in \Sigma \). Then we have by 4.5.(iii): \( |x - y| < |u - v| \Rightarrow \alpha < \beta \Rightarrow |\sigma(\alpha)| < |\sigma(\beta)| \Rightarrow |f(x) - f(y)| < |f(u) - f(v)| \). The existence of \( \phi \) is now clear. The continuity follows from 4.5.(iv).

**REMARK.** There exist isometries \( K \to K \) that are monotone of type \( \sigma \) for no \( \sigma \) (see [4]).

**THEOREM 4.8.** Let \( f: K \to K \) be monotone of type \( \sigma: \Sigma \to \Sigma \). Then \( \sigma \) is surjective if and only if \( f \) is a bijection (in fact, \( f \) is a nonzero scalar multiple of an isometry, by 2.7.).

**PROOF.** If \( \sigma \) is surjective then \( \sigma^{-1} \) exists and satisfies the condition of 4.6., so there is a \( g: K \to K \), monotone of type \( \sigma^{-1} \). Then \( f \circ g \) is monotone of type 1, i.e., increasing. It suffices to show that an increasing \( h: K \to K \) is surjective. Let \( a \in K \) and consider the map \( \psi: x \mapsto x - h(x) + a \quad (x \in K) \). Then \( |\psi(x) - \psi(y)| \leq |\pi| |x - y| \) \((x, y \in K)\). By the Banach contraction theorem, \( \psi \) has a fixed point \( t \). Then \( h(t) = a; h \) is surjective. The converse is easy.

**EXAMPLE.** The monotone functions on \( \Phi_p \).

As we have seen in section 4, the group of signs of \( \Phi_p \) is isomorphic to \( \Pi_p^* \times \mathbb{Z} \). Using this interpretation we can describe the sign function as follows. Let \( x \in \Phi_p \), \( x = \sum_{n \geq k} a_n p^n \) be its standard expansion (i.e., \( k \in \mathbb{Z}, a_n \in \{0,1,2,\ldots,p-1\}, a_k \neq 0 \)). Then
\[
\text{sgn}(x) = (a_k, n) \in \Pi_p^* \times \mathbb{Z}.
\]
Let $\sigma: \Sigma \to \Sigma$ be a type of some monotone function. By 4.5, we have $\sigma(1,n) = 1\sigma(1,n), ((1,n) \in \Sigma)$. Set

$$\sigma(1,n) = (s(n), \lambda(n)) \quad (n \in \mathbb{Z})$$

Since $n < m \Leftrightarrow |p^n| > |p^m| \Leftrightarrow |(1,n)| > |(1,m)| \Leftrightarrow |\sigma(1,n)| > |\sigma(1,m)|$
\Leftrightarrow $|p^{\lambda(n)}| > |p^{\lambda(m)}| \Leftrightarrow \lambda(n) < \lambda(m)$, we see that $\lambda: \mathbb{Z} \to \mathbb{Z}$ is strictly increasing. Thus, $\sigma$ has the form

$$(*) (1,n) \mapsto (ls(n), \lambda(n))$$

where $s: \mathbb{Z} \to \mathbb{F}_p^*$ and $\lambda: \mathbb{Z} \to \mathbb{Z}$ is strictly increasing.

Conversely, if we have a map $\sigma$ satisfying $(*)$ where $s: \mathbb{Z} \to \mathbb{F}_p^*$ and $\lambda: \mathbb{Z} \to \mathbb{Z}$ is strictly increasing the function

$$\Sigma a_n p^n \mapsto \Sigma a_n s(n)p^{\lambda(n)}$$

is monotone of type $\sigma$, as can easily be verified.

For a criterion in order that a continuous function $\mathbb{Z}_p \to \mathbb{Q}_p$ be increasing (expressed by means of its coordinates with respect to some orthonormal base), see [4].
Appendix

Differentiation

We briefly consider the relationship between monotony and differentiation. We refer to [4] for the proofs. Although even increasing functions may be nowhere differentiable there are some connections that are similar to those in the real case.

A function $g: K \rightarrow K$ is called positive if $g(K) \subseteq K^+$. A function $h: K \rightarrow K$ is of the first class of Baire if there exists a sequence $h_1, h_2, \ldots$ of continuous functions $K \rightarrow K$ that converges pointwise to $h$.

**THEOREM.** (i) Let $f: K \rightarrow K$ be increasing, differentiable. Then $f'$ is positive, of the first class of Baire.

(ii) A positive function of the first class of Baire has an increasing antiderivative.

**THEOREM.** Let $f: K \rightarrow K$ be continuously differentiable (which means here that $\lim_{x,y \rightarrow a} (x-y)^{-1}(f(x) - f(y))$ exists for $a \in K$), and suppose $f'(a) \neq 0$. Then there is a (convex) neighborhood $X$ of $a$ such that $f|X$ is monotone of type $\sigma$, where $\sigma$ is the map $a \mapsto \text{sgn}(f'(a)) \cdot a$.

**THEOREM.** Let $f: K \rightarrow K$ be monotone of type $\sigma$, differentiable. Then there are two cases.

I. $f'(a) = 0$ for some $a \in K$. Then $f' = 0$ everywhere and $\lim_{|a| \rightarrow 0} \frac{\sigma(a)}{|a|} = 0$.

II. $f'(a) \neq 0$ for some $a \in K$. Then $f' \neq 0$ everywhere. In fact, $f'$ has constant sign ($x \mapsto \text{sgn}(f'(x))$ is constant). For small $|a|$, $\frac{\sigma(a)}{a}$ is constant. $f'(a)^{-1}f$ is locally increasing.
REMARK. One can make an example of an everywhere differentiable \( f: \mathbb{R} \to \mathbb{R} \) with \( f' = 1 \) (so \( f' \) is positive) such that \( f \) is not even locally injective at 0. (\( f \) is, of course, not continuously differentiable).