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EXTRAIT DE

"SYMPOSION DÉDIÉ À A.F. MONNA"

COMMUNICATIONS OF THE MATHEMATICAL INSTITUTE
RIJKSUNIVERSITEIT UTRECHT 12-1980
P-adic monotone functions

by

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0. Introduction

Our aim in this paper is to present reasonable definitions for a function \( f: K \to K \) to be "monotone", where \( K \) is a local field (i.e., a non-archimedean non-trivially valued field that is locally compact in the topology induced by the valuation). The future will tell us whether filling this gap in p-adic analysis has been of any use.

The fact that - until recently - the concept of "monotone function" has been absent in p-adic analysis is not surprising since a decent (partial) ordering in \( K \), (compatible with the algebraic and topological structure) is not available. Thus, in the sequel we will look for substitutes for "ordering" in \( K \) as a basis for our theory. One of them is the notion of "pseudo-ordering", introduced by A.F. Monna [1].

The theory can be build up in a more general setting, namely for functions \( f: X \to K \), where \( K \) is any complete non-archimedean field and \( X \subseteq K \), but restriction to local fields avoids a lot of technicalities and enlights the main track. For the extended theory, see [4].

The elementary facts about analysis in local fields can be found in [2].

Notations and definitions

FROM NOW ON IN THIS PAPER \( K \) IS A LOCAL FIELD WITH RESIDUE CLASS FIELD \( k \).
Set \( |K| = \{|x| : x \in K\} \)
\( |K^*| = |K| \setminus \{0\} \) (the value group)

\( \pi: \) a (fixed) element of \( K^* \) such that \( |\pi| \) generates \( |K^*| \), \( |\pi| < 1 \).

For a prime \( p \) we denote by \( \mathbb{Q}_p \) the non-archimedean valued field of the \( p \)-adic numbers, by \( \mathbb{Z}_p \) its valuation ring \( \{x \in \mathbb{Q}_p : |x| \leq 1\} \).

The residue class field, \( \mathbb{Z}_p / p \mathbb{Z}_p \) of \( \mathbb{Q}_p \) is the field of \( p \) elements and is denoted by \( \mathbb{F}_p \).

The characteristic of a field \( L \) is denoted by \( \chi(L) \).

For a \( K \)-vector space \( E \) and a subset \( S \) of \( E \) we denote its \( K \)-linear span by \( \langle S \rangle \).

Let \( a \in K, r \in [0,\infty) \). The ball with center \( a \) and radius \( r \) is by definition \( \{x \in K : |x - a| \leq r\} \). It is easy to see that the intersection of a collection of balls is either empty or again a ball.

Let \( x, y \in K \). Then the smallest ball containing \( x \) and \( y \) is denoted by \( [x, y] \). A subset \( C \) of \( K \) is called convex if \( x, y \in C \) implies \( [x, y] \subseteq C \). Each ball is convex. A convex set \( \neq K, \neq \emptyset \) is a ball.

It follows that \( K \) is the only unbounded convex set in \( K \).

FROM NOW ON \( X \) IS A CONVEX SUBSET OF \( K \).

1. Two notions of monotony

Interpreting the above notion of convexity also for the real numbers it is quite natural to introduce the following geometric expressions.

Let \( x, y, z \in K \). We say that \( z \) is between \( x \) and \( y \) if \( z \in [x, y] \).

If \( z \) is not between \( x \) and \( y \) we say that \( x, y \) are at the same side of \( z \). This yields more or less automatically the following
DEFINITION 1. Let \( f: X \to K \). We say that \( f \in M_b(X) \) (\( f \) respects "betweenness") if for all \( x, y, z \in X \)

\[ (\ast) \quad z \in [x,y] \to f(z) \in [f(x), f(y)]. \]

We say that \( f \in M_s(X) \) (\( f \) respects "sides") if for all \( x, y, z \in X \)

\[ (\ast\ast) \quad z \notin [x,y] \to f(z) \notin [f(x), f(y)]. \]

REMARKS

1. If we, in the above definition, replace formally \( K \) by \( \mathbb{R} \) and \( X \) by an interval, we see that \( f \in M_b(X) \) just means that \( f \) is monotone and that \( f \in M_s(X) \) becomes "\( f \) is strictly monotone". (These facts can easily be proved). So for the time being we let our intuition be guided by this analogy:

   The statements in 2 and 3 below are direct consequences of the definitions, and the proofs are left to the reader.

2. \( M_b(X) \) is closed under pointwise limits.

   The constant functions are in \( M_b(X) \). If \( f \in M_b(X) \) and \( f(a) = f(b) \) then \( f \) is constant on \([a,b]\).

   \( f \in M_b(X) \Rightarrow \) For each convex \( C \subseteq K \) the inverse image \( f^{-1}(C) \) is convex. For each \( a,b \in K \) the map \( x \mapsto ax + b \) is in \( M_b(X) \).

   \( f \in M_s(X) \Rightarrow f \) is injective.

   If \( a,b \in K, a \neq 0 \) then \( x \mapsto ax + b \) is in \( M_s(X) \).

   Each isometry \( X \to K \) is in \( M_{bs}(X) \) where

\[ M_{bs}(X): = M_b(X) \cap M_s(X). \]

3. Without harm we may replace in Definition 1 (\( \ast \)) by (\( \ast \))' or (\( \ast \))'' or (\( \ast \))''', where

\[
(\ast)': \ |x-z| \leq |x-y| \to |f(x) - f(z)| \leq |f(x) - f(y)|
\]

\[
(\ast)''': \ |f(x) - f(z)| \leq |f(x) - f(y)| \to |x-z| < |x-y|
\]

\[
(\ast)'''': \ |f(x) - f(z)| \leq |f(x) - f(y)| \to |x-z| < |x-y|
\]
Similarly, we may replace \((**)\) by \((**)'\) or \((**)'\)' or \((**)'\)'', where:

\[
(**)' : |x-z| < |x-y| \rightarrow |f(x) - f(z)| < |f(x) - f(y)|
\]

\[
(**)'' : |f(x) - f(z)| = |f(x) - f(y)| \rightarrow |x-z| = |x-y|
\]

\[
(**)'''' : |f(x) - f(z)| \leq |f(x) - f(y)| \rightarrow |x-z| \leq |x-y|.
\]

4. In the next section we will study \(M_b(X)\) and \(M_s(X)\). For example, the natural questions: \(M_s(X) \subseteq M_b(X)\)? \(f \in M_b(X)\), \(f\) injective \(\Rightarrow f \in M_s(X)\)? Notice that our definitions do not refer to any "type" of monotonity (such as "increasing" and "decreasing" for real functions). In section 4 we will introduce such a concept. It will turn out that monotone functions having a "type" are \(M_b\) functions, but not conversely.

2. Properties of monotone functions

**Theorem 2.1.** Let \(f \in M_b(X)\). If \(a,b,c \in X\), \(|a-b| < |a-c|\), \(f(a) \neq f(c)\) then \(|f(a) - f(b)| < |f(a) - f(c)|\). In particular, if \(f \in M_b(X)\), \(f\) is injective then \(f \in M_s(X)\).

**Proof.** Without loss, assume \(X = [a,c]\). Since \(f \in M_b(X)\) we have \([f(a), f(c)] \subseteq f(X) \subseteq [f(a), f(c)]\), hence the diameter of \(f(X)\) equals \(M = |f(a) - f(c)|\). The ball \([f(a), f(c)]\) has a partition into \(n\) balls \(V_1, \ldots, V_n\) each having radius \(M/\pi\), where \(n\) is the number of elements of \(k\). The sets \(f^{-1}(V_1), \ldots, f^{-1}(V_n)\) form a partition of \(X\), each \(f^{-1}(V_i)\) is convex (since \(f \in M_b(X)\)), at least two of the \(f^{-1}(V_i)\) are non-empty (since \(a\) and \(c\) cannot both lie in the same \(f^{-1}(V_i)\)). It follows that the diameter of each \(f^{-1}(V_i)\) is strictly less than \(|a-c|\). (Otherwise \(f^{-1}(V_i) = X\) for some \(i\) and \(f^{-1}(V_j) = \emptyset\) for \(j \neq i\)). Consequently the partition \(f^{-1}(V_1), \ldots, f^{-1}(V_n)\) of \(X\) must be the partition of \(X\) into balls with radius \(|a-c|/\pi|\).
Now if |a - b| < |a - c| then a, b \in f^{-1}(V_i) for some i, so |f(a) - f(b)| \leq M|\pi| < |f(a) - f(c)|.

EXAMPLE. Let p \neq 2 and let f: \mathbb{Z}_p \to \mathbb{Q}_p "tear apart" \mathbb{Z}_p by sending
k + p\mathbb{Z}_p \text{ into } p^{-k} + p\mathbb{Z}_p \text{ (k = 0, 1, 2, ..., p-1) via translations. Then one easily checks that } f \in M_s(\mathbb{Z}_p) \setminus M_b(\mathbb{Z}_p).

Hence, it seems that M_{bs} - functions are the "translation" of the real strictly monotone functions (rather than M_s -functions).

In the sequel we often make use of the following observation. If f is either in M_b(X) or in M_s(X) then

\((*)\) \quad |x - z| < |x - y| \to |f(x) - f(z)| \leq |f(x) - f(y)| \quad (x, y, z \in X)

(Functions with property \((*)\) are called weakly monotone in \([\cdot]_+\)).

**LEMMA 2.2.** Let f \in M_b(X) or f \in M_s(X). Then, if Y \subset X is bounded

\text{then } f(Y) \text{ is bounded.}

**PROOF.** Y is precompact, so r: = \max\{|x - y|: x, y \in Y\} exists. We may assume r > 0. The equivalence relation x \sim y iff |x - y| < r (x, y \in Y) divides Y into finitely many classes Y_1, \ldots, Y_n where n \geq 2. Choose a_i \in Y_i for each i and set M: = \max_i |f(a_i)|. We prove |f| \leq M on Y.

In fact, let x \in Y. There is i \in \{1, \ldots, n\} such that |x - a_i| < r.

For j \neq i we have |x - a_i| < r = |a_i - a_j|, so |f(x) - f(a_i)| \leq |f(a_i) - f(a_j)| \leq M. Hence |f(x)| \leq M.

We have a "dual" statement which is only of interest in case X = K:

**LEMMA 2.3.** Let f \in M_s(K) or f \in M_b(K). If f is not constant then

\text{for a bounded } Z \subset K \text{ the inverse image } f^{-1}(Z) \text{ is bounded.}

**PROOF.** We prove: Z \subset K bounded, T: = f^{-1}(Z) is unbounded implies f is constant. In fact, let a, b \in K. There are x_1, x_2, \ldots \text{ in } T \text{ such that}
(*) \( \max (|a|, |b|) < |x_1| < |x_2| < \ldots \)

The precompactness of \( \{f(x_1), f(x_2), \ldots \} \) implies convergence of a subsequence of \( f(x_1), f(x_2), \ldots \) Without loss, assume that \( \lim_{n \to \infty} f(x_n) \) exists. From (*) we obtain

\[
|a-b| < |x_1-a| < |x_2-x_1| < |x_3-x_2| < \ldots ,
\]

so that for all \( n \in \mathbb{N} \)

\[
|f(a) - f(b)| \leq |f(x_{n+1}) - f(x_n)|.
\]

Hence,

\[
|f(a) - f(b)| \leq \lim_{n \to \infty} |f(x_{n+1}) - f(x_n)| = 0.
\]

It follows that \( f \) is constant.

2.1., 2.2. and 2.3. yield the continuity properties 2.4., 2.5. and 2.6.

**THEOREM 2.4.** Let \( X \) be bounded with diameter \( r > 0 \) and let \( f \in \mathcal{M}_b(X) \) or \( f \in \mathcal{M}_s(X) \). Then \( f \) satisfies the Lipschitz-condition

\[
|f(x) - f(y)| \leq M|x-y| \quad (x,y \in X)
\]

where \( M = r^{-1}\sup\{|f(x) - f(y)| : x,y \in X\} < \infty \).

**PROOF.** By 2.2. \( f \) is bounded, so \( M < \infty \). Choose any \( a \in X \). We prove by induction on \( n \): \( P(n):" \)If \( |x-a| = |\pi|^n r \) then \( |f(x) - f(a)| \leq |\pi|^n r M \). \((x \in X)" \) Clearly \( P(0) \) holds. Suppose \( P(n-1) \). Let \( x \in X \) such that \( |x-a| = |\pi|^{n-1} r \) and choose \( b \in X \) with \( |b-a| = |\pi|^{n-1} r \).

Then \( |x-a| < |b-a| \). If \( f(b) = f(a) \) then \( |f(x) - f(a)| \leq |f(b) - f(a)| = 0 \), so certainly \( |f(x) - f(a)| \leq |\pi|^{n-1} r M \). If \( f(a) \neq f(b) \) then either by Theorem 2.1. or since \( f \in \mathcal{M}_s(X) \) we have \( |f(x) - f(a)| < |f(b) - f(a)| \leq (\text{induction hypothesis}) \leq |\pi|^{n-1} r M \), so that \( |f(x) - f(a)| \leq |\pi|^{n-1} r M \).

**REMARK.** The map \( \Sigma_n p^n \to \Sigma_n p^{2n} \) is in \( \mathcal{M}_b(\mathbb{Q}_p) \) and has unbounded difference quotients.
THEOREM 2.5. Let \( f \in M_b(X) \) or \( f \in M_s(X) \). Then \( f \) is continuous.

Further, if \( Y \subset X \) is closed then \( f(Y) \) is closed in \( K \).

In particular, an \( f \in M_s(X) \) is a homeomorphism \( X \sim f(X) \).

PROOF. If \( X \) is bounded then everything follows from 2.4., so let \( X = K \). The continuity of \( f \) is clear (restrict \( f \) to bounded convex subsets). Let \( Y \subset K \) be closed and suppose \( a \in f(Y) \setminus f(Y) \). There are \( a_1, a_2, \ldots \) in \( Y \) for which \( \lim_{n \to \infty} f(a_n) = a \). \( f \) is not constant, so by 2.3. the sequence \( a_1, a_2, \ldots \) is bounded, assume it converges, say \( a = \lim_{n \to \infty} a_n \). By continuity, \( f(a) = \lim_{n \to \infty} f(a_n) = a \), a contradiction. The last statement is now trivial.

THEOREM 2.6. Let \( f \in M_b(X) \) (or \( f \in M_s(X) \) for that matter). Then the following are equivalent.

(a) \( f \) is a homeomorphism \( X \sim f(X) \).

(b) \( f \) is injective.

(y) \( f \in M_s(X) \).

(d) \( f(X) \) has no isolated points.

PROOF. The implications \( (a) \Rightarrow (b) \Rightarrow (y) \Rightarrow (d) \) are easy (2.1., continuity and injectivity of \( M_s \)-functions). We prove \( (d) \Rightarrow (a) \), that is, if \( \lim_{n \to \infty} f(x_n) = f(a) \) then \( \lim_{n \to \infty} x_n = a \). Now since \( f(a) \) is not isolated we can find \( a_1, a_2, \ldots \) such that \( f(a_n) \neq f(a) \) for all \( n \), \( \lim_{n \to \infty} f(a_n) = f(a) \). By 2.3. we may assume that \( b = \lim_{n \to \infty} a_n \) exists. If suffices to prove that \( \lim_{n \to \infty} x_n = b \) (apply the result for \( x_n = a \) for all \( n \) and we find \( a = b \)). Let \( \epsilon > 0 \). There is \( k \in \mathbb{N} \) for which \( |a_k - b| < \epsilon \). For large \( n \) we have \( |f(x_n) - f(a)| < |f(a_k) - f(a)| \), hence for large \( n \) and \( m(n) \) we get \( |f(x_n) - f(a_m)| < |f(a_k) - f(a_m)| \) whence \( |x_n - a_m| \leq |a_k - a_m| \), so \( |x_n - b| \leq |a_k - b| < \epsilon \) for large \( n \).

THEOREM 2.7. Let \( f \in M_b(X) \) or \( f \in M_s(X) \) and assume that \( f(X) \) is convex. Then \( f \) is a scalar multiple of an isometry, or \( f \) is constant.
PROOF. We may assume that \( f \) is not constant, so \( f(X) \) is open, non-empty. Let \( X = f(X) \) be bounded. By 2.6, \( f \) is injective. It is clear that also \( f^{-1} \in M_b(X) \cup M_s(X) \). Applying 2.4 to both \( f \) and \( f^{-1} \) we get (in both cases \( M = 1 \)) for all \( x, y, u, v \in X \) that \( |f(x) - f(y)| \leq |x - y| \) and \( |f^{-1}(u) - f^{-1}(v)| \leq |u - v| \). It follows that \( f \) is an isometry. The general case is now easy. (If \( X, f(X) \) are bounded, a transformation of the type \( x \rightarrow ax + b \) sends \( f(X) \) into \( X \); if \( X = f(X) = K \), apply 2.7. to \( X_n : = \{ x \in K : |x| \leq n \} \) \( (n \in \mathbb{N}) \).

The following theorem describes the functions "of bounded variation":

**THEOREM 2.8.** Let \( X \) be bounded. Then \( \| M_s(X) \| = \| M_b(X) \| = \text{BA}(X) \), where \( \text{BA}(X) \) is the linear space of functions \( f: X \rightarrow K \) having bounded difference quotients.

PROOF. By 2.4 we are done if we can prove that an \( f \in \text{BA}(X) \) is the sum of two \( M_{bs} \)-functions. Let \( \lambda \in K \) such that \( |f(x) - f(y)| \leq |\lambda| |x - y| \) for all \( x, y \in X, x \neq y \). Let \( g(x) : = \lambda x \ (x \in X) \) and let \( h : = f - g \). Then \( g, h \) are in \( M_{bs}(X) \) (scalar multiples of isometries) and \( f = g + h \).

3. Monotone sequences

We will not delve deeply into this subject, but content ourselves with presenting definitions and some facts indicating that these notions are not that bad.

**DEFINITION 3.1.** Let \( x_1, x_2, \ldots \) be a sequence in \( K \). It is called \( b \)-monotone, if \( k \leq l \leq m \) implies \( x_k \in [x_k, x_m] \), \( s \)-monotone, if \( l < k, m \) implies \( x_1 \notin [x_k, x_m] \).

**THEOREM 3.2.** A sequence \( x_1, x_2, \ldots \) in \( K \) is \( s \)-monotone if and only if \( |x_1 - x_2| > |x_2 - x_3| > \ldots \)

A sequence \( x_1, x_2, \ldots \) in \( K \) is \( b \)-monotone if and only if for each \( k, m \in \mathbb{N}, k < m \):
\[ |x_m - x_k| = \max \{|x_{i+1} - x_i|: k \leq i < m\}. \]

**Proof.** Let \( x_1, x_2, \ldots \) be \( s \)-monotone, and let \( n \in \mathbb{N} \). We have \( x_n \not\in [x_{n+1}, x_{n+2}] \) so \( |x_n - x_{n+1}| > |x_{n+1} - x_{n+2}| \). Conversely, if \( |x_1 - x_2| > |x_2 - x_3| > \ldots \), let \( 1 \leq k,m \). Then \( |x_k - x_1| = |x_{k+1} - x_1| \) and \( |x_m - x_k| = |x_{k+1} - x_k| \) hence \( |x_k - x_1| > |x_m - x_k| \) i.e., \( x_1 \not\in [x_k, x_m] \).

Let \( x_1, x_2, \ldots \) be \( b \)-monotone, and let \( k, m \in \mathbb{N} \), \( k < m \), and \( k \leq i < m \). Then \( x_i \in [x_k, x_m] \) and \( x_{i+1} \in [x_k, x_m] \), so \( [x_i, x_{i+1}] \subseteq [x_k, x_m] \), hence \( |x_{i+1} - x_i| \leq |x_m - x_k| \). The rest is obvious. To prove the converse, let \( k < l < m \). Then \( |x_1 - x_k| = \max \{|x_{i+1} - x_i|: k \leq i < l\} \leq \max \{|x_{i+1} - x_i|: k \leq i < m\} = |x_m - x_k| \). Hence \( x_1 \in [x_k, x_m] \).

**Corollary 3.3.** Each \( s \)-monotone sequence is \( b \)-monotone. An \( s \)-monotone sequence \( x_1, x_2, \ldots \) is convergent and \( x_n \neq x_m \) whenever \( n \neq m \). \( M_b \)-functions map \( b \)-monotone sequences into \( b \)-monotone sequences. \( M_s \)-functions map \( s \)-monotone sequences into \( s \)-monotone sequences.

A \( b \)-monotone sequence need not be convergent. In fact, if \( |x_1| < |x_2| < \ldots \) then \( x_1, x_2, \ldots \) is \( b \)-monotone. But we have

**Theorem 3.4.** Let \( x_1, x_2, \ldots \) be \( b \)-monotone. Then either \( \lim_{n \to \infty} |x_n| = \infty \) or \( \lim_{n \to \infty} x_n \) exists.

**Proof.** If not \( \lim_{n \to \infty} |x_n| = \infty \) then the sequence has a bounded, hence a convergent subsequence, say, \( \lim_{i \to \infty} x_i = x \). We show that \( \lim_{n \to \infty} x_n = x \). Let \( \varepsilon > 0 \). Then \( |x - x_{n_k}| < \varepsilon \) for \( k \geq k_0 \). If \( l, m \geq n_{k_0} \) then if \( n_k \geq 1, m \) we have \( |x_l - x_m| \leq |x_{n_{k_0}} - x_{n_{k_0}}| \leq \max (|x - x_{n_{k_0}}|, |x - x_{n_{k_0}}|) < \varepsilon \).

Hence \( x_1, x_2, \ldots \) is Cauchy, so it must converge, to \( x \).

**Theorem 3.5.** Each sequence in \( K \) has a \( b \)-monotone subsequence.
PROOF. We may assume that $x_1, x_2, \ldots$ is bounded, and that it has a convergent subsequence $y_1, y_2, \ldots$ where $y_n \neq y_m$ whenever $n \neq m$. Set $y = \lim_{n \to \infty} y_n$. Now it is easy to construct a subsequence $z_1, z_2, \ldots$ of $y_1, y_2, \ldots$ for which $|y-z_1| > |y-z_2| > |y-z_3| > \ldots$ Hence $|z_1-z_2| > |z_2-z_3| > \ldots$ The sequence $z_1, z_2, \ldots$ is $s$-monotone, hence $b$-monotone.

EXAMPLE. Let $x = \sum_{n=0}^{\infty} a_n p^n \in \mathbb{Z}_p$, and let $x_k = \sum_{n=0}^{k} a_n p^n$. Then $x_0, x_1, \ldots$ is a $b$-monotone sequence, converging to $x$.

4. Monotone functions of type $\sigma$

Following Monna [1] we introduce the concept of "sides of zero" in $K$ (a generalization of the partition $\mathbb{R}^* = \mathbb{R}^+ \cup \mathbb{R}^-$). Let $K^* = K \setminus \{0\}$. Define

$$x \sim y \; \text{if} \; x \; \text{and} \; y \; \text{are at the same side of} \; 0 \; (x, y \in K^*).$$

Then we have $x \sim y \Leftrightarrow 0 \notin [x, y] \Leftrightarrow |x-y| < |y| \Leftrightarrow |xy^{-1}-1| < 1 \Leftrightarrow xy^{-1} \in K^+$, where

$$K^+ = \{x \in K : |1-x| < 1\}.$$

$K^+$ is a multiplicative open compact subgroup of $K^*$, called the group of the positive elements of $K$. We see that $\sim$ is the equivalence relation induced by the canonical group homomorphism, the "sign map":

$$\text{sgn}: K^* \to K^*/K^+.$$ 

The quotient group $\Sigma = K^*/K^+$ (comparable with $\{1, -1\}$ in the real case) is called the group of signs of elements of $K$, or the group of sides of zero of $K$. $\Sigma$ is an infinite group, whose elements are multiplicative cosets of $K^+$.

Let $\alpha \in \Sigma$ and let $x, y \in \alpha$. Then $|x-y| < |x|$, so in particular,
\[ |x| = |y| \]. Therefore we may define the **absolute value of a sign** \( \alpha \in \Sigma \) as

\[ |\alpha| = |x| \quad (x \in \alpha) \]

The map \( \alpha \mapsto |\alpha| \) is a surjective group homomorphism \( \Sigma \to |K^*| \).

Its kernel, \( \{ \alpha \in \Sigma: |\alpha| = 1 \} \) is a multiplicative subgroup of \( \Sigma \), which is naturally isomorphic to the multiplicative group \( K^* \) under \( \alpha \mapsto \bar{\alpha} \) (where \( x \mapsto \bar{x} \) is the canonical map \( \{ x \in K: |x| \leq 1 \} \to K \)).

Let us denote its inverse \( K^* \sim \{ \alpha \in \Sigma: |\alpha| = 1 \} \) by

\[ l \mapsto \alpha_l \quad (l \in K^*) \]

For each \( n \in \mathbb{Z} \), let \( \alpha_n = \text{sgn}(\pi^n) \).

Now for each \( \alpha \in \Sigma \) there are unique \( n \in \mathbb{Z}, l \in K^* \) such that

\[ \alpha = \alpha_l \alpha_n \]

Further, we have

\[ \alpha_l \alpha_{l'} = \alpha_{l+l'} \quad (l,l' \in K^*) \]
\[ \alpha_n \alpha_m = \alpha_{n+m} \quad (n,m \in \mathbb{Z}) \]

It follows, that \( \Sigma \) is isomorphic to \( K^* \times \mathbb{Z} \) (or to \( K^* \times |K^*| \) for that matter). It is also possible to identify \( \Sigma \) with a subgroup of \( K^* \) but we shall not need it here.

In the sequel we will use addition of signs. For \( \alpha \in \Sigma \), let

\[ -\alpha = \{ -x: x \in \alpha \} \]

Then clearly \(-\alpha\) is a sign and if \( \alpha = \alpha_l \alpha_n \) \((l \in K^*, n \in \mathbb{Z}) \) then

\[ -\alpha = \alpha_{-l} \alpha_n \]

Let \( \alpha, \beta \in \Sigma \) Then \( \alpha + \beta = \{ x+y: x \in \alpha, y \in \beta \} \) is easily seen to be a ball, which turns out to be again a sign iff \( \alpha \neq -\beta \) (iff \( 0 \notin \alpha + \beta \)). Therefore we define
\[ \alpha \oplus \beta = \alpha + \beta \quad (\alpha, \beta \in \Sigma, \alpha \neq -\beta). \]

We have the following rules that are easy to prove:

RULES 4.1. (i) The operation \( \oplus \) is commutative, associative, distributive, whenever the occurring formulas are defined.

(ii) \(|\alpha \oplus \beta| = \max(|\alpha|, |\beta|) \quad (\alpha, \beta \in \Sigma, \alpha \neq -\beta)\)

(iii) \(|\alpha| < |\beta| \iff \alpha \oplus \beta = \beta \quad (\alpha, \beta \in \Sigma)\)

(iv) Let \( n \in \mathbb{N}, \ 1 \leq n < \chi(k) \), \( \alpha \in \Sigma \). Then \( \alpha^\oplus_n = \alpha \oplus \alpha \oplus \ldots \oplus \alpha (n \text{ times}) \) exists and equals \( n\alpha \).

Thus, we get for \( l, l' \in k^* \), \( m, n \in \mathbb{Z} \):

\[
\alpha_{l\oplus_m} \oplus \alpha_{l'\oplus_n} = \begin{cases} 
\alpha_{l\oplus_m} & \text{if } m < n \\
\alpha_{l'\oplus_n} & \text{if } m > n \\
\alpha_{l+l'} & \text{if } m = n \text{ and } l + l' \neq 0.
\end{cases}
\]

Next, we define a "pseudo-ordering" in \( K \). Let \( x, y \in K \), \( \alpha \in \Sigma \). We say that \( x > y \) (\( x \) is greater than \( y \) in the sense of \( \alpha \)) in case \( x - y \in \alpha \). The following easy consequences obtain.

RULES 4.2. (i) Let \( x, y \in K \). Then if \( y \neq x \) there is exactly one \( \alpha \in \Sigma \) for which \( x > y \). If \( y = x \) then \( x > y \) for no \( \alpha \). ("\( K \) is totally pseudo-ordered").

(ii) If \( x > y \), \( y > z \) for some \( x, y, z \in K \); \( \alpha, \beta \in \Sigma \), and if \( \alpha \oplus \beta \) exists then \( x > z \). ("Transitivity").

(iii) If \( x > y \) for some \( x, y \in K \); \( \alpha \in \Sigma \) and if \( z \in K \), then \( x + z > y + z \). ("Compatibility with addition").

(iv) If \( x, y, z \in K \); \( \alpha, \beta \in \Sigma \), \( x > y \) and \( z > 0 \) then \( xz > yz \). ("Compatibility with multiplication").

We define:
DEFINITION 4.3. Let $\sigma: \Sigma \to \Sigma$ be an injection and $f: K \to K$. $f$ is called monotone of type $\sigma$ if for all $\alpha \in \Sigma$, $x, y \in K$ we have $x > y$ implies $f(x) > f(y)$. $f$ is called increasing if $\sigma$ is the identity map.

REMARKS
1. One can extend easily the above definition for functions $f: X \to K$, where $X$ is a convex subset of $K$, $\sigma: \Sigma(X) \to \Sigma$. Here $\Sigma(X)$ is the collection of signs that "occur in $X"$ i.e. $\{\text{sgn}(x-y): x, y \in X, x \neq y\}$. We leave it to the reader to do this and, in case $X \neq K$, to show that there is $\beta \in \Sigma$ such that

$$\Sigma(X) = \{\alpha \in \Sigma: |\alpha| < |\beta|\}.$$  

2. Notice that $f: K \to K$ is increasing if and only if the difference quotient

$$\phi f(x, y) := \frac{f(x) - f(y)}{x - y} \quad (x \neq y)$$

is positive. In particular, $f$ is an isometry.

3. The requirement made in 4.3. that $\sigma$ be an injection is essential in case $k$ is not a prime field (see [4]).

4. In case $\chi(K) = 0$, the exponential function, defined by the power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is increasing on its convergence region. If $f$ is an increasing function and $\beta \in \Sigma$ then for each $b \in \beta$ the function $bf$ is of type $\sigma$ where $\sigma$ is the multiplier $a \mapsto \alpha \beta$.

THEOREM 4.4. (i) Let $f: K \to K$ be a monotone of type $\sigma: \Sigma \to \Sigma$. Then

$$\sigma(\alpha + \beta) = \sigma(\alpha) + \sigma(\beta) \quad (\alpha, \beta \in \Sigma, \alpha \neq -\beta)$$

(ii) Let $\sigma: \Sigma \to \Sigma$ satisfy $\ast$. Then there is a function $g: K \to K$, monotone of type $\sigma$. 

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PROOF. (i) First we show that $\sigma(-\alpha) = -\sigma(\alpha)$ ($\alpha \in \Sigma$). In fact choose $x, y \in K$ such that $x - y \in \alpha$. Then $y - x \in -\alpha$, so $f(x) - f(y) \in \sigma(\alpha) \cap (-\sigma(-\alpha)) \neq \emptyset$, which implies $\sigma(\alpha) = -\sigma(-\alpha)$.

Now take $\alpha, \beta \in \Sigma$, $\alpha \neq -\beta$. Then by injectivity of $\sigma$, $\sigma(\alpha) \neq \sigma(-\beta) = -\sigma(\beta)$ so that $\sigma(\alpha) \oplus \sigma(\beta)$ exists. Now choose $x, y, z \in K$ such that $x - y \in \alpha$, $y - z \in \beta$. Then $f(x) - f(z) = f(x) - f(y) + f(y) - f(z) \in \sigma(\alpha) \oplus \sigma(\beta)$. Also, $x - z \in \alpha \oplus \beta$ so $f(x) - f(z) \in \sigma(\alpha \oplus \beta)$. As $\sigma(\alpha) \oplus \sigma(\beta)$ and $\sigma(\alpha \oplus \beta)$ have a non-empty intersection they are equal.

The proof of (ii) will be furnished by Lemma 4.5. and Lemma 4.6.: 

LEMMA 4.5. Let $\sigma: \Sigma \to \Sigma$ satisfy $\sigma(\alpha \oplus \beta) = \sigma(\alpha) \oplus \sigma(\beta)$ ($\alpha, \beta \in \Sigma$, $\alpha \neq -\beta$). Then we have

(i) $\sigma(n\alpha) = n\sigma(\alpha)$ ($n \in \mathbb{N}$, $1 \leq n < \chi(k)$, $\alpha \in \Sigma$).

(ii) $\sigma(-\alpha) = -\sigma(\alpha)$ ($\alpha \in \Sigma$)

(iii) For all $\alpha, \beta \in \Sigma$: $|\alpha| < |\beta|$ if and only if $|\sigma(\alpha)| < |\sigma(\beta)|$.

(iv) $\lim_{|\alpha| \to 0} |\sigma(\alpha)| = 0$.

PROOF. (i) is a direct consequence of 4.1.(iv). To prove (ii), set $q := \chi(k)$. Then for $\beta \in \Sigma$, $(q-1)\beta = -\beta$, so, by (i), $\sigma(-\alpha) = \sigma(q-1)\alpha = (q-1)\sigma(\alpha) = -\sigma(\alpha)$. Let $\alpha, \beta \in \Sigma$, $|\alpha| < |\beta|$. Then $\alpha \oplus \beta = \beta$ so $\sigma(\alpha) \oplus \sigma(\beta) = \sigma(\alpha \oplus \beta) = \sigma(\beta)$ whence $|\sigma(\alpha)| < |\sigma(\beta)|$. Conversely, suppose $|\sigma(\alpha)| < |\sigma(\beta)|$. Then clearly $\alpha \neq -\beta$ (otherwise $|\sigma(\alpha)| = |\sigma(-\beta)| = |\sigma(\beta)| = |\sigma(\beta)|$, so $\alpha \oplus \beta$ exists. Now $\sigma(\beta) = \sigma(\alpha) \oplus \sigma(\beta) = \sigma(\alpha \oplus \beta)$. By injectivity of $\sigma$ we obtain $\beta = \alpha \oplus \beta$ whence $|\alpha| < |\beta|$. So we have (iii). (iv) follows from the fact that if $|\alpha_1| > |\alpha_2| > \ldots \to 0$ then $|\sigma(\alpha_1)| > |\sigma(\alpha_2)| > \ldots$, which last sequence tends to 0 due to the discreteness of the valuation.
LEMMA 4.6. (Extension theorem for monotone functions). Let $\phi \neq Y \subset K$, $f: Y \to K$, $\sigma: \Sigma \to \Sigma$ satisfy $\sigma(\alpha \oplus \beta) = \sigma(\alpha) \oplus \sigma(\beta)$ ($\alpha, \beta \in \Sigma, \alpha \neq -\beta$). Suppose

$x > y$ implies $f(x) > f(y)$ \((x, y \in Y, \alpha \in \Sigma)\)

Then $f$ can be extended to a function $\overline{f}: K \to K$, monotone of type $\sigma$.

PROOF. By Zorn's lemma it suffices to extend $f$ to $Y \cup \{a\}$ \((a \not\in Y)\) such that $f(x) - \overline{f}(a) \in \sigma(\text{sgn}(x - a))$ and $\overline{f}(a) - f(x) \in \sigma(\text{sgn}(a - x))$ for all $x \in Y$. By 4.5.(ii) it suffices to consider only the second case. Let

$$B_x = f(x) + \sigma(\text{sgn}(a - x)) \quad (x \in Y)$$

Then each $B_x$ is a ball having diameter $|\pi| |\sigma(\text{sgn}(a - x))| \neq 0$.

By the local compactness (in fact, spherical completeness) of $K$ we are done if we can show that $B_x \cap B_y \neq \emptyset$ whenever $x, y \in Y, x \neq y$.

Set $\alpha = \text{sgn}(a - x)$ and $\beta = \text{sgn}(a - y), b \in \sigma(\alpha), c \in \sigma(\beta)$. We have to prove that $|f(x) + b - (f(y) + c)| < |\pi| \max(|\sigma(\alpha)|, |\sigma(\beta)|)$.

Consider two cases.

1) $\alpha = \beta$. Then $a - x$ and $a - y$ are in $\alpha$, so $|a - x - (a - y)| = |x - y| < |\alpha|$, hence $|\text{sgn}(x - y)| < |\alpha|$. By 4.5.(iii) we have $|\sigma(\text{sgn}(x - y))| < |\sigma(\alpha)|$, so $|\text{sgn}(f(x) - f(y))| < |\sigma(\alpha)|$ whence $|f(x) - f(y)| < |\sigma(\alpha)|$. Also $|b - c| < |\sigma(\alpha)|$ since both $b$ and $c$ are in $\sigma(\alpha)$. Consequently $|f(x) + b - (f(y) + c)| < |\sigma(\alpha)|$.

2) $\alpha \neq \beta$. Then $x - y = a - y - (a - x) \in \beta \oplus -\alpha$, so $f(x) - f(y) \in \sigma(\beta \oplus -\alpha)$. Now $b - c \in \sigma(\alpha) \oplus (-\sigma(\beta)) = \sigma(\alpha) \oplus \sigma(-\beta) = \sigma(\alpha \oplus -\beta) = -\sigma(\beta \oplus -\alpha)$. Therefore $|f(x) - f(y) - (b - c)| < |\sigma(\beta \oplus -\alpha)| = \max(|\sigma(\beta)|, |\sigma(\alpha)|)$.

(The proof of 4.4.(ii): Choose $Y = \{0\}$ and let $f: Y \to K$ be defined via $f(0) = 0$. Extend $f$ in the way of 4.6.).
COROLLARY 4.7. Let \( f: K \to K \) be monotone of type \( \sigma: \Sigma \to \Sigma \). Then
\( f \in M_{bs}(K) \) (see section 2). More than that: there exists a strictly increasing function \( \phi: |K| \to |K| \), continuous at 0, \( \phi(0) = 0 \) such that
\[ |f(x) - f(y)| = \phi(|x - y|). \]
\( (x, y \in K) \)

PROOF. Let \( x, y, u, v \in K \) and \( x - y \in \alpha \in \Sigma, u - v \in \beta \in \Sigma \). Then we have by 4.5.(iii): \( |x - y| < |u - v| \iff |\alpha| < |\beta| \iff |\sigma(\alpha)| < |\sigma(\beta)| \iff |f(x) - f(y)| < |f(u) - f(v)| \). The existence of \( \phi \) is now clear. The continuity follows from 4.5.(iv).

REMARK. There exist isometries \( K \to K \) that are monotone of type \( \sigma \) for no \( \sigma \) (see [4]).

THEOREM 4.8. Let \( f: K \to K \) be monotone of type \( \sigma: \Sigma \to \Sigma \). Then \( \sigma \) is surjective if and only if \( f \) is a bijection (in fact, \( f \) is a nonzero scalar multiple of an isometry, by 2.7.).

PROOF. If \( \sigma \) is surjective then \( \sigma^{-1} \) exists and satisfies the condition of 4.6., so there is a \( g: K \to K \), monotone of type \( \sigma^{-1} \). Then \( f \circ g \) is monotone of type 1, i.e., increasing. It suffices to show that an increasing \( h: K \to K \) is surjective. Let \( a \in K \) and consider the map
\[ \psi: x \mapsto x - h(x) + a \quad (x \in K). \]
Then \( |\psi(x) - \psi(y)| < |\pi||x - y| \)
\( (x, y \in K) \). By the Banach contraction theorem, \( \psi \) has a fixed point \( t \). Then \( h(t) = a \): \( h \) is surjective. The converse is easy.

EXAMPLE. The monotone functions on \( \mathbb{Q}_p \).
As we have seen in section 4, the group of signs of \( \mathbb{Q}_p \) is isomorphic to \( \mathbb{F}_p^* \times \mathbb{Z} \). Using this interpretation we can describe the sign function as follows. Let \( x \in \mathbb{Q}_p \), \( x = \sum_{n \geq k} a_n p^n \) be its standard expansion (i.e., \( k \in \mathbb{Z}, a_n \in \{0,1,2,\ldots,p-1\}, a_k \neq 0 \)). Then
\[ \text{sgn}(x) = (a_k, n) \in \mathbb{F}_p^* \times \mathbb{Z}. \]
Let $\sigma: \Sigma \to \Sigma$ be a type of some monotone function. By 4.5. we have $\sigma(1,n) = 1\sigma(1,n), ((1,n) \in \Sigma)$. Set

$$\sigma(1,n) = (s(n), \lambda(n)) \quad (n \in \mathbb{Z})$$

Since $n < m \iff |p^n| > |p^m| \iff |(1,n)| > |(1,m)| \iff |\sigma(1,n)| > |\sigma(1,m)| \iff |p^{\lambda(n)}| > |p^{\lambda(m)}| \iff \lambda(n) < \lambda(m)$, we see that $\lambda: \mathbb{Z} \to \mathbb{Z}$ is strictly increasing. Thus, $\sigma$ has the form

(*) $(1,n) \mapsto (ls(n), \lambda(n))$

where $s: \mathbb{Z} \to \mathbb{F}_p^*$ and $\lambda: \mathbb{Z} \to \mathbb{Z}$ is strictly increasing.

Conversely, if we have a map $\sigma$ satisfying (*) where $s: \mathbb{Z} \to \mathbb{F}_p^*$ and $\lambda: \mathbb{Z} \to \mathbb{Z}$ is strictly increasing the function

$$\Sigma a_n p^n \mapsto \Sigma a_n s(n)p^{\lambda(n)}$$

is monotone of type $\sigma$, as can easily be verified.

For a criterion in order that a continuous function $\mathbb{Z}_p \to \mathbb{Q}_p$ be increasing (expressed by means of its coordinates with respect to some orthonormal base), see [4].
Appendix
Differentiation

We briefly consider the relationship between monotony and differentiation. We refer to [4] for the proofs. Although even increasing functions may be nowhere differentiable there are some connections that are similar to those in the real case.

A function \( g: K \to K \) is called positive if \( g(K) \subseteq K^+ \).

A function \( h: K \to K \) is of the first class of Baire if there exists a sequence \( h_1, h_2, \ldots \) of continuous functions \( K \to K \) that converges pointwise to \( h \).

**THEOREM.** (i) Let \( f: K \to K \) be increasing, differentiable. Then \( f' \) is positive, of the first class of Baire.

(ii) A positive function of the first class of Baire has an increasing antiderivative.

**THEOREM.** Let \( f: K \to K \) be continuously differentiable (which means here that \( \lim_{x,y \to a} (x-y)^{-1}(f(x) - f(y)) \) exists for \( a \in K \)),

and suppose \( f'(a) \neq 0 \). Then there is a (convex) neighborhood \( X \) of \( a \) such that \( f|X \) is monotone of type \( \sigma \),

where \( \sigma \) is the map \( \alpha \mapsto \text{sgn}(f'(a)) \cdot \alpha \).

**THEOREM.** Let \( f: K \to K \) be monotone of type \( \sigma \), differentiable. Then there are two cases.

I. \( f'(a) = 0 \) for some \( a \in K \). Then \( f' = 0 \) everywhere and

\[
\lim_{|a| \to 0} \frac{\sigma(a)}{|a|} = 0.
\]

II. \( f'(a) \neq 0 \) for some \( a \in K \). Then \( f' \neq 0 \) everywhere.

In fact, \( f' \) has constant sign (\( x \mapsto \text{sgn}(f'(x)) \) is constant). For small \( |a| \), \( \frac{\sigma(a)}{a} \) is constant. \( f'(a)^{-1}f \) is locally increasing.
REMARK. One can make an example of an everywhere differentiable \( f: K \to K \) with \( f' = 1 \) (so \( f' \) is positive) such that \( f \) is not even locally injective at 0. (\( f \) is, of course, not continuously differentiable).