NON-ARCHIMEDEAN MONOTONE FUNCTIONS

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Introduction.

In the sequel, $K$ is a non-archimedean, non-trivially valued field, that is complete under the metric induced by the valuation. The residue class field of $K$ is denoted by $k$. $X$ will always be a closed, non empty subset of $K$ without isolated points (except in 2.2, if you want).

Since $K$ admits no ordering in the usual sense it is not possible to define monotone functions $X \to K$ just by taking over the classical definitions. Thus, our procedure will be to try and find statements for functions $\mathbb{R} \to \mathbb{R}$ equivalent to monotony, and formulated in terms that are translatable to $K$. This way we will obtain several definitions of "$f : X \to K$ is monotone", that are, although not equivalent, closely related.

The connections between these various definitions and the properties of the non-archimedean monotone functions can be put together to form a little theory which is interesting in its own right, but of which the relations to the other parts of $p$-adic analysis are yet not very tight.

1. Monotone functions.

For a function $f : \mathbb{R} \to \mathbb{R}$ the following conditions are equivalent:

(a) $f$ is monotone (in the non-strict sense),

(b) If $C \subset \mathbb{R}$ is convex then $f^{-1}(C)$ is convex,

(c) If $x$ is between $y$, $z$ then $f(x)$ is between $f(y)$ and $f(z)$.

Also, the following conditions are equivalent:

(a) $f$ is strictly monotone,

(b) $f$ is injective. If $C \subset \mathbb{R}$ is convex then $f(C)$ is relatively convex in $f(\mathbb{R})$,

(c) If $f(x)$ is between $f(y)$ and $f(z)$ then $x$ is between $y$ and $z$.

Let $x, y \in K$. Then the smallest ball that contains $x, y$ is denoted by $[x, y]$. $z \in K$ is between $x$ and $y$ if $z \in [x, y]$. (If $z \notin [x, y]$, we

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call \( x, y \) at the same side of \( z \). A subset \( C \subseteq K \) is called convex if \( x, y \in C \), \( z \in [x, y] \) implies \( z \in C \). Each convex subset of \( K \) can be written in at least one of the following forms

\[ \{ x : |x - a| < r \}, \{ x : |x - a| \leq r \} \]

for some \( a \in K \), \( r \in (0, \infty) \).

Let \( Z \subseteq Y \subseteq K \). Then \( Z \) is called convex in \( Y \) if \( Z = C \cap Y \), where \( C \) is convex.

With all these definitions we have the following theorem.

**Theorem 1.1.** Let \( f : X \to K \). Then the following conditions are equivalent:

1. If \( x, y, z \in X \), \( x \) is between \( y \) and \( z \) then \( f(x) \) is between \( f(y) \) and \( f(z) \).
2. If \( C \subseteq K \) is convex, then \( f^{-1}(C) \) is convex in \( X \).

We denote the collection of those \( f : X \to K \) satisfying (1) or (2) by \( M_b(X) \), i.e., \( f \in M_b(X) \) if, and only if, for each \( x, y, z \in X \),

\[ |x - y| \leq |y - z| \implies |f(x) - f(y)| \leq |f(y) - f(z)|. \]

Isometries are in \( M_b \) (viz. exp), but also non-trivial locally constant functions (e.g., choose a center in each ball of radius \( r > 0 \), and let \( f \) be the map assigning to \( x \in X \) the center of the ball of radius \( r \) to which \( x \) belongs. Then \( f \in M_b(X) \)).

**Theorem 1.2.** Let \( f : X \to K \). Then the following conditions are equivalent:

1. If \( x, y, z \in X \), \( f(x) \) is between \( f(y) \) and \( f(z) \) then \( x \) is between \( y \) and \( z \),
2. If \( 0 \subseteq X \) is convex in \( X \) then \( f(0) \) is convex in \( f(X) \) if \( f \) is injective.

We denote the collection of those \( f : X \to K \) satisfying (1') or (2') by \( M_s(X) \), i.e., \( f \in M_s(X) \) if, and only if, for each \( x, y, z \in X \),

\[ |x - y| < |y - z| \implies |f(x) - f(y)| < |f(y) - f(z)|. \]

The classical situations suggests the question as to whether \( M_s(X) \subseteq M_b(X) \) and also whether \( f \in M_b(X) \), \( f \) injective implies \( f \in M_s(X) \). In general, both statements are false, but we do have the following:

**Theorem 1.3.** \( f \in M_s(X) \) implies \( f^{-1} \in M_b(f(X)) \). \( f \in M_b(X) \), \( f \) injective implies \( f^{-1} \in M_s(f(X)) \). If \( k \) is finite and \( X \) is convex, then an injective \( M_b \)-function is in \( M_s(X) \).
So we are led to define $M_b(X) := M_b(X) \cap M_s(X)$ as being the more or less natural translation of "the space of the strictly monotone functions".

The following theorem concerns continuity of monotone functions. For a function $f : X \to K$, we define its oscillation function, $\omega_f$, in the usual way:

$$\omega_f(x) := \lim_{n \to \infty} \sup \{|f(x) - f(y)| : |x - a| \leq \frac{1}{n}, |y - a| \leq \frac{1}{n}\}$$

$$\omega_f(a) := \lim_{n \to \infty} \sup \{|f(x) - f(a)| : |x - a| \leq \frac{1}{n}| (a \in X) .

f is continuous at $a$ if, and only if, $\omega_f(a) = 0$.

**Theorem 1.4.** - Let $f$ be either in $M_b(X)$ or in $M_s(X)$. Then

(i) $\omega_f(x) = \inf_{y \neq a} |f(y) - f(a)|$ $(a \in X)$

(ii) $f$ is bounded on compact subsets of $X$,

(iii) For each $a \in X$ we have the following alternative. Either $f$ is continuous at $a$, or for each sequence $x_1, x_2, \ldots \ (x_n \neq a)$ converging to $a$, the sequence $f(x_1), f(x_2), \ldots$ is bounded and has no convergent subsequence.

Let $g \in M_b(X)$. If $Y \subset X$ is spherically complete, then so is $g(Y)$.

Let $h \in M_s(X)$. If $Z \subset h(X)$ is spherically complete, then so is $h^{-1}(Z)$.

**Proof (sketch).** - If $f \in M_b(X) \cup M_s(X)$, then:

$$|x - y| < |y - z| \implies |f(x) - f(y)| \leq |f(y) - f(z)| .$$

So $f$ is locally bounded, and (ii) follows. Of (i), only the $\leq$ part is interesting. Choose $z \neq a$. If $|x - a| < |z - a|$, then

$$|f(x) - f(a)| \leq |f(z) - f(a)| \text{ whence } \omega_f(a) \leq |f(z) - f(a)| .$$

Let $\lim x_n = a \ (x_n \neq a \text{ for all } n)$ and $\lim f(x_n) = \alpha$. Let $\lim y_n = a$. It suffices to show that $\lim f(y_n) = \alpha$. Indeed, let $\varepsilon > 0$, and choose $k$ such that $|f(x_k) - \alpha| < \varepsilon$. Then $|y_n - a| < |x_k - a|$ for large $n$, so $|y_n - x_m| < |x_k - x_m|$ for large $m$ depending on $m$. Hence $|f(y_n) - f(x_n)| \leq |f(x_n) - f(x_m)|$, so $(m \to \infty) |f(y_n) - \alpha| \leq |f(x_k) - \alpha| < \varepsilon$, and we have (iii). The rest of the proof is straightforward.

**Corollary 1.5.** - Let $f : X \to K$ be in $M_b(X) \cup M_s(X)$.

(i) If $K$ is a local field, then $f$ is continuous.

(ii) If $|K|$ is discrete, then $f \in M_b(X) \Rightarrow f$ is a homeomorphism $X \sim f(X)$, and $f \in M_s(X) \Rightarrow f$ is a closed map.

(iii) The graph of $f$ is closed in $K^2$.

(iv) If $f(X)$ has no isolated points, then $f$ is continuous.
An $M_b$-function may be everywhere discontinuous on $K$ (even when $|K|$ is discrete).

**Theorem 1.6.** Let $B$ be the unit ball of $K$.

(i) If $K$ is a local field and $f \in M_b(B) \cup M_s(B)$, then $f$ has bounded difference quotients (i.e., there is $C > 0$ such that $|f(x) - f(y)| \leq C|x - y|$ for all $x \in B$). If, in addition, $f(B)$ is convex, then $f$ is a similarity (i.e., a scalar multiple of an isometry).

(ii) If $K$ has discrete valuation and $f \in M_b(B)$ is bounded, then $f$ has bounded difference quotients. If $f \in M_{bs}(B)$ and if $f(B)$ is convex, then $f$ is a similarity.

2. Monotone functions having a type.

In this section, we want to translate the usual classification of (strictly) monotone functions $\mathbb{R} \to \mathbb{R}$ into two types: the increasing and the decreasing functions. The equivalence relation in $\mathbb{R}^r$: $x \sim y$ if $x$ and $y$ are at the same side of 0, yields $(-\infty, 0)$ and $(0, \infty)$ as equivalence classes. The relation $\sim$ is compatible with the canonical group homomorphism $\mathbb{R}^* \to \mathbb{R}^*/\mathbb{R}^+$, the latter group being $\{1, -1\}$. $\pi(x)$ (usually called $\text{sgn}(x)$) assigns +1 to every positive element and -1 to every negative element. A function $f: \mathbb{R} \to \mathbb{R}$ is strictly monotone if there exists $\sigma: \mathbb{R}^*/\mathbb{R}^+ \to \mathbb{R}^*/\mathbb{R}^+$ such that for all $x \neq y$,

$$\pi(f(x) - f(y)) = \sigma(\pi(x - y)).$$

If $\sigma$ is the identity then $f$ is called increasing; if $\sigma(1) = -1$, $\sigma(-1) = 1$, $f$ is called decreasing. Other maps $\sigma: [-1, 1] \to [-1, 1]$ can not occur (i.e., there is no $f$ such that, for all $x \neq y$,

$$\pi(f(x) - f(y)) = \sigma(\pi(x - y)).$$

This rather weird description of real monotone functions can be used in the non-archimedean case.

For $x, y \in K^*$ define $x \sim y$ if $x, y$ are at the same side of 0. This means: $0 \notin [x, y]$, or $|x - y| > |y|$, or $|xy^{-1} - 1| < 1$. Thus $x \sim y$ if, and only if, $xy^{-1} \in K^+$ where

$$K^+ := \{x \in K; |1 - x| < 1\}.$$

We call the elements of $K^+$ the positive element of $K$.

The relation $\sim$ is compatible with the canonical homomorphism of (multiplicative) groups

$$\pi: K^* \to K^*/K^+ =: \Sigma.$$

We call $\Sigma$ the group of signs and $\pi(x)$ the sign of an element $x \in K^*$ ( $x$ is
positive if, and only if, \( \pi(x) = 1 \).

If \( K \) is a local field, we can make a group embedding \( \rho : \Sigma \rightarrow K^\# \) such that \( \pi \circ \rho \) is the identity on \( \Sigma \). For example, if \( K = \mathbb{Q}_p \), \( \delta \) is a primitive \((p - 1)\)th root of unity, then

\[
\pi(\sum_{n \geq 1} a_n p^n) = a_k p^k \quad (k \in \mathbb{Z}, \ a_k \neq 0)
\]

(Here \( a_n \in \{0, 1, \delta, \ldots, \delta^{p-2}\} \) for each \( n \)).

**Definition 2.1.** - Let \( \sigma : \Sigma \rightarrow \Sigma \) be any map. A function \( f : X \rightarrow K \) is monotone of type \( \sigma \) if, for all \( x, y \in X \), \( x \neq y \),

\[
\pi(f(x) - f(y)) = \sigma(\pi(x - y))
\]

(i.e., if \( x - y \in \alpha \in \Sigma \) then \( f(x) - f(y) \in \sigma(\alpha) \)).

We call \( f \) of type \( \beta \in \Sigma \) if \( f \) is of type \( \sigma \) where \( \sigma \) is the multiplication with \( \beta \), i.e.,

\[
\frac{f(x) - f(y)}{x - y} \in \beta \quad (x, y \in X, x \neq y).
\]

We call \( f \) increasing if \( f \) is of type \( \sigma \) where \( \sigma \) is the identity, i.e.,

\[
\frac{f(x) - f(y)}{x - y} \text{ is positive} \quad (x \neq y).
\]

Clearly, if \( f \) is of type \( \beta \), and if \( b \in \beta \), then \( b^{-1} f \) is increasing.

First, we look at increasing functions, then we discuss more general types \( \sigma \).

Notice that increasing functions are isometries hence are in \( N_{ba}(X) \). If \( f \) is increasing then \( f(x) = x + h(x) \), where \( |h(x) - h(y)| < |x - y| \) \((x, y \in X, x \neq y)\).

Such \( h \) we call pseudo-contractions.

**Lemma 2.2.** - Let \( X \) be an ultrametric space. Then the following are equivalent

(a) \( X \) is spherically complete,

(b) Each pseudocontraction \( X \rightarrow X \) has a (unique) fixed point.

**Proof (sketch).** - (a) \( \rightarrow \) (b). Let \( \sigma : X \rightarrow X \) be a pseudocontraction. A convex set \( C \subset X \) is called invariant if \( \sigma(C) \subset C \). It is easily proved that the invariant convex subsets of \( X \) form a nest. Let \( C_0 \) be the smallest invariant convex set. If \( a \in C_0 \) and \( \sigma(a) \neq a \) then

\[
B_0 := \{ x \in X ; \ d(x, \sigma(a)) < d(a, \sigma(a)) \}
\]

is invariant, convex, and does not contain \( a \). Hence \( \sigma(a) = a \) for all \( a \in C_0 \), and \( C_0 \) is a singleton. (b) \( \rightarrow \) (a). If \( B_1 \neq B_2 \neq \ldots \) are balls in \( X \) with \( \bigcap B_n = \emptyset \) then choose \( x_n \in B_n \cap B_{n+1} \) \((n \in \mathbb{N})\). The map \( \sigma : X \rightarrow X \) defined by

\[
\sigma(x) = x_{n+1} \quad (x \in B_n \cap B_{n+1})
\]

is a pseudocontraction without a fixed point.
COROLLARY 2.3. - Let $X$ be convex, let $K$ be spherically complete, and let $f : X \to K$ be increasing. Then $f(X)$ is convex. If $f(X) \subseteq X$, then $f$ is surjective.

Proof. - Let $f(X) \subseteq X$. Choose $a \in X$. Then $x \mapsto -f(x) + x + a$ is a pseudocontraction mapping $X$ into $X$, hence has a fixed point. So $f(x) = a$ for some $x \in X$.

If $K$ is not spherically complete, we have always increasing $f : K \to K$ that are not surjective. (Let $h : K \to K$ be a pseudocontraction without a fixed point. Let $f(x) = x - h(x)$ ($x \in K$), then $0 \notin \text{im } f$). The inverse $f^{-1} : f(K) \to K$ can, of course, not be extended to an increasing function $K \to K$.

THEOREM 2.4. - Let $K$ be spherically complete, and let $f : X \to K$ be increasing. Then $f$ can be extended to an increasing function $K \to K$.

Proof. - By Zorn's Lemma, it suffices to extend $f$ to an increasing function on $X \cup \{a\}$, where $a \notin X$. We are done if we can find $a \in K$ such that, for all $x \in X$,

$$|\frac{a - f(x)}{a - x} - 1| < 1$$

i.e. $a \in B_{f(x) - (a - x)}(|a - x|)$ for all $x \in X$. These balls form a nest.

Let us call a function $f : X \to K$ positive if $f(X) \subseteq K^+$. 

THEOREM 2.5.

(i) If $f : X \to K$ is increasing then $f'$ is positive,

(ii) If $g : X \to K$ is a positive Baire class one function, then $g$ has an increasing antiderivative,

(iii) If $g : X \to K$ is continuous and positive, then $g$ has a $C^1$-antiderivative,

(iv) If $f \in C^1(X)$ and $f'$ is positive then $f = j + h$ where $j$ is increasing, and $h$ is locally constant.

EXAMPLES.

1° The exponential function (defined on its natural convergence region) is increasing.

2° Let $f \in C(\mathbb{Z}_p)$, and let $e_0 = \xi_{\mathbb{Z}_p}$, for $n \in \mathbb{N}$,

$$e_n(x) = \begin{cases} 1 & \text{if } |x - n| < \frac{1}{n} \\ 0 & \text{elsewhere} \end{cases} \quad (x \in \mathbb{Z}_p).$$

Then $e_0, e_1, \ldots$ form an orthonormal base of $C(\mathbb{Z}_p)$, so there exist $\lambda_0, \lambda_1, \ldots \in \mathbb{Q}_p$ such that $f = \sum_{n=0}^{\infty} \lambda_n e_n$, uniformly.
\[ f \text{ is increasing if, and only if, for all } n \in \mathbb{N}, \]
\[ |\lambda_n - \{n\}| < \{n\} \]

(where, if \( n = a_0 + a_1 p + \ldots + a_k p^k \) (\( a_i \in \{0, 1, \ldots, p - 1\} \) for each \( i \), \( a_k \neq 0 \)), then \( \{n\}_1 = a_k p^k \)).

In other words, \( f = \sum \lambda_n e_n \in \mathcal{C}(\mathbb{Z}_p) \) is increasing if, and only if, \( \lambda_n/\{n\} \) is positive for all \( n \in \mathbb{N} \).

Let \( \alpha, \beta \in \Sigma \). If the set theoretic sum \( \alpha + \beta := \{x + y ; x \in \alpha, y \in \beta\} \) does not contain 0 then \( \alpha + \beta \in \Sigma \), notation \( \alpha \oplus \beta \). It follows that \( \alpha \oplus \beta \) is defined if, and only if, \( \alpha \neq -\beta \).

If \( x, y \in \alpha \in \Sigma \) then \( |x| = |y| \). This defines \( |\alpha| \) in a natural way.

We have the following results.

**Theorem 2.6.** Let \( f : K \to K \) be monotone of type \( \sigma : \Sigma \to \Sigma \). Let \( \alpha, \beta \in \Sigma \),

\begin{itemize}
  \item[(i)] \( \sigma(-\alpha) = -\sigma(\alpha) \),
  \item[(ii)] If \( \sigma(\alpha) \oplus \sigma(\beta) \) is defined then so is \( \alpha \oplus \beta \) and \( \sigma(\alpha \oplus \beta) = \sigma(\alpha) \oplus \sigma(\beta) \),
  \item[(iii)] \( |\alpha| < |\beta| \) implies \( |\sigma(\alpha)| < |\sigma(\beta)| \),
  \item[(iv)] If \( |\beta| = 1 \), \( \beta \) contains an element of the prime field of \( K \) then \( \sigma(\beta) = \beta \sigma(\alpha) \),
  \item[(v)] \( f \in \mathcal{M}_b(K) \),
  \item[(vi)] \( f \) is either nowhere continuous or uniformly continuous.
\end{itemize}

**Theorem 2.7.** Let \( f : K \to K \) be monotone of type \( \sigma : \Sigma \to \Sigma \). Then the following conditions are equivalent,

\begin{itemize}
  \item[(a)] \( \sigma \) is injective,
  \item[(b)] \( f \in \mathcal{M}_b(K) \),
  \item[(c)] If for some \( \alpha, \beta \in \Sigma \), \( \alpha \oplus \beta \) is defined, then so is \( \sigma(\alpha) \oplus \sigma(\beta) \),
  \item[(d)] \( |\sigma(\alpha)| < |\sigma(\beta)| \) implies \( |\alpha| < |\beta| \) (\( \alpha, \beta \in \Sigma \)).
\end{itemize}

**Corollary 2.8.** Let \( k \) be a prime field, and let \( f : K \to K \) be monotone of type \( \sigma : \Sigma \to \Sigma \). Then \( \sigma \) is injective.

(If \( K = \mathbb{Q}_p(\sqrt[p-1]{-1}) \), \( p = 3 \mod 4 \), we can find an example of an \( f : K \to K \) monotone of type \( \sigma \), where \( \sigma \) is not injective).

**Example 2.9.** Let \( K = \mathbb{Q}_p \). Then

\[ \{\sigma : \Sigma \to \Sigma : \text{there is } f : \mathbb{Q}_p \to \mathbb{Q}_p, f \text{ monotone of type } \sigma\} \]
consists of all \( \sigma : \Sigma \to \Sigma \) of the form
\[
\sigma_1 \sigma_2 \ldots \sigma_n \overset{p}{\to} \sigma_1^p \sigma_2^p \ldots \sigma_n^p \lambda(n)
\]
where \( s : \mathbb{Z} \to \{0, 1, 2, \ldots, p - 2\} \) and \( \lambda : \mathbb{Z} \to \mathbb{Z} \) is strictly increasing.

3. Functions of bounded variation.

**Lemma 3.1.** Let \( f : X \to K \) have bounded difference quotients. Then \( f \) is a linear combination of two increasing functions.

**Proof.** Choose \( \lambda \in K \),
\[
|\lambda| > \sup \{\frac{|f(x) - f(y)|}{x - y} ; x \neq y\}.
\]
Then \( \lambda^{-1} f \) is a (pseudo-) contraction, so \( g(x) := -x + \lambda^{-1} f(x) (x \in X) \) is increasing. If \( h(x) := x (x \in X) \), then \( \lambda h + \lambda g = f \).

**Corollary 3.2.** Let \( X \) be the unit ball of a local field \( K \) and let \( f : X \to K \). Then the following are equivalent
(a) \( f \in \mathcal{B}_A(X) \) (i.e., \( \sup \{\frac{|f(x) - f(y)|}{x - y} ; x \neq y\} < \infty \)),
(b) \( f \) is a linear combination of two increasing functions,
(c) \( f \in \mathcal{H}_b(X) \),
(d) \( f \in \mathcal{H}_{b,s}(X) \).

**Proof.** Use 1.6.

**References**
