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NON-ARCHIMEDEAN MONOTONE FUNCTIONS
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Introduction.

In the sequel, \( K \) is a non-archimedean, non-trivially valued field, that is complete under the metric induced by the valuation. The residue class field of \( K \) is denoted by \( k \). \( X \) will always be a closed, non empty subset of \( K \) without isolated points (except in 2.2, if you want).

Since \( K \) admits no ordering in the usual sense it is not possible to define monotone functions \( X \rightarrow K \) just by taking over the classical definitions. Thus, our procedure will be to try and find statements for functions \( \mathbb{R} \rightarrow \mathbb{R} \) equivalent to monotony, and formulated in terms that are translatable to \( K \). This way we will obtain several definitions of "\( f : X \rightarrow K \) is monotone", that are, although not equivalent, closely related.

The connections between these various definitions and the properties of the non-archimedean monotone functions can be put together to form a little theory which is interesting in its own right, but of which the relations to the other parts of \( p \)-adic analysis are yet not very tight.

1. Monotone functions.

For a function \( f : \mathbb{R} \rightarrow \mathbb{R} \) the following conditions are equivalent:

(a) \( f \) is monotone (in the non-strict sense),

(b) If \( C \subset \mathbb{R} \) is convex then \( f^{-1}(C) \) is convex,

(γ) If \( x \) is between \( y, z \) then \( f(x) \) is between \( f(y) \) and \( f(z) \).

Also, the following conditions are equivalent:

(a) \( f \) is strictly monotone,

(b) \( f \) is injective. If \( C \subset \mathbb{R} \) is convex then \( f(C) \) is relatively convex in \( f(\mathbb{R}) \),

(c) If \( f(x) \) is between \( f(y) \) and \( f(z) \) then \( x \) is between \( y \) and \( z \).

Let \( x, y \in K \). Then the smallest ball that contains \( x, y \) is denoted by \([x, y]\). \( z \in K \) is between \( x \) and \( y \) if \( z \in [x, y] \). (If \( z \notin [x, y] \), we

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call $x$, $y$ at the same side of $z$). A subset $C \subset K$ is called convex if $x, y \in C$, $z \in [x, y]$ implies $z \in C$. Each convex subset of $K$ can be written in at least one of the following forms

$$\{x : |x - a| < r\}, \{x : |x - a| \leq r\}$$

for some $a \in K$, $r \in (0, \infty)$.

Let $Z \subset Y \subset K$. Then $Z$ is called convex in $Y$ if $Z = C \cap Y$, where $C$ is convex.

With all these definitions we have the following theorem.

**THEOREM 1.1.** Let $f : X \rightarrow K$. Then the following conditions are equivalent:

1. If $x, y, z \in X$, $x$ is between $y$ and $z$ then $f(x)$ is between $f(y)$ and $f(z)$,
2. If $C \subset K$ is convex, then $f^{-1}(C)$ is convex in $X$.

We denote the collection of those $f : X \rightarrow K$ satisfying (1) or (2) by $M_b(X)$, i.e. $f \in M_b(X)$ if, and only if, for each $x, y, z \in X$,

$$|x - y| \leq |y - z| \quad \text{implies} \quad |f(x) - f(y)| \leq |f(y) - f(z)|.$$

Isometries are in $M_b$ (viz. exp), but also non trivial locally constant functions (e.g., choose a center in each ball of radius $r > 0$, and let $f$ be the map assigning to $x \in X$ the center of the ball of radius $r$ to which $x$ belongs. Then $f \in M_b(X)$).

**THEOREM 1.2.** Let $f : X \rightarrow K$. Then the following conditions are equivalent

1'. If $x, y, z \in X$, $f(x)$ is between $f(y)$ and $f(z)$ then $x$ is between $y$ and $z$,
2'. If $C \subset X$ is convex in $X$ then $f(C)$ is convex in $f(X)$. $f$ is injective.

We denote the collection of those $f : X \rightarrow K$ satisfying (1') or (2') by $M_s(X)$, i.e. $f \in M_s(X)$ if, and only if, for each $x, y, z \in X$,

$$|x - y| < |y - z| \quad \text{implies} \quad |f(x) - f(y)| < |f(y) - f(z)|.$$

The classical situations suggests the question as to wether $M_s(X) \subset M_b(X)$ and also wether $f \in M_b(X)$, $f$ injective implies $f \in M_s(X)$. In general, both statements are false, but we do have the following:

**THEOREM 1.3.** $f \in M_s(X)$ implies $f^{-1} \in M_b(f(X))$. $f \in M_b(X)$, $f$ injective implies $f^{-1} \in M_s(f(X))$. If $k$ is finite and $X$ is convex, then an injective $M_b$-function is in $M_s(X)$. 


So we are led to define $M_{ba}(X) := M_b(X) \cap M_s(X)$ as being the more or less natural translation of "the space of the strictly monotone functions".

The following theorem concerns continuity of monotone functions. For a function $f : X \to K$, we define its oscillation function, $\omega_f$, in the usual way:

$$
\omega_f(a) := \lim_{n \to \infty} \sup\{|f(x) - f(y)| : |x - a| \leq \frac{1}{n} ; |y - a| \leq \frac{1}{n}\}
$$

$$
= \lim_{n \to \infty} \sup\{|f(x) - f(a)| : |x - a| \leq \frac{1}{n}\} \quad (a \in X).
$$

$f$ is continuous at $a$ if, and only if, $\omega_f(a) = 0$.

**Theorem 1.4.** Let $f$ be either in $M_b(X)$ or in $M_s(X)$. Then

(i) $\omega_f(a) = \inf_{z \neq a} |f(z) - f(a)| \quad (a \in X)$

(ii) $f$ is bounded on compact subsets of $X$,

(iii) For each $a \in X$ we have the following alternative. Either $f$ is continuous at $a$, or for each sequence $x_1, x_2, \ldots$ $(x_n \neq a)$ converging to $a$, the sequence $f(x_1), f(x_2), \ldots$ is bounded and has no convergent subsequence.

Let $g \in M_b(X)$. If $Y \subset X$ is spherically complete, then so is $g(Y)$.

Let $h \in M_s(X)$. If $Z \subset h(X)$ is spherically complete, then so is $h^{-1}(Z)$.

**Proof (sketch).** If $f \in M_b(X) \cup M_s(X)$, then:

$$
|x - y| < |y - z| \implies |f(x) - f(y)| \leq |f(y) - f(z)|.
$$

So $f$ is locally bounded, and (ii) follows. Of (i), only the $\leq$ part is interesting. Choose $z \neq a$. If $|x - a| < |z - a|$, then

$$
|f(x) - f(a)| \leq |f(z) - f(a)| \quad \text{whence } \omega_f(a) \leq |f(z) - f(a)|.
$$

Let $\lim x_n = a$ ($x_n \neq a$ for all $n$) and $\lim f(x_n) = \alpha$. Let $\lim y_n = a$. It suffices to show that $\lim f(y_n) = \alpha$. Indeed, let $\varepsilon > 0$, and choose $k$ such that $|f(x_k) - \alpha| < \varepsilon$. Then $|y_n - a| < |x_k - a|$ for large $n$, so

$$
|y_n - x_m| < |x_k - x_m|
$$

for large $m$ depending on $m$. Hence $|f(y_n) - f(x_n)| \leq |f(x_k) - f(x_m)|$, so

$$(m \to \infty) \quad |f(y_n) - \alpha| \leq |f(x_k) - \alpha| < \varepsilon,$$

and we have (iii). The rest of the proof is straightforward.

**Corollary 1.5.** Let $f : X \to K$ be in $M_b(X) \cup M_s(X)$.

(i) If $K$ is a local field, then $f$ is continuous,

(ii) If $|K|$ is discrete, then $f \in M_s(X) \Rightarrow f$ is a homeomorphism $X \sim f(X)$, and $f \in M_b(X) \Rightarrow f$ is a closed map.

(iii) The graph of $f$ is closed in $K^2$.

(iv) If $f(X)$ has no isolated points, then $f$ is continuous.
An $M_b$-function may be everywhere discontinuous on $K$ (even when $|K|$ is discrete).

**Theorem 1.6.** Let $B$ be the unit ball of $K$,

(i) If $K$ is a local field and $f \in M_b(B) \cup M_s(B)$, then $f$ has bounded difference quotients (i.e., there is $C > 0$ such that $|f(x) - f(y)| \leq C|x - y|$ for all $x \in B$). If, in addition, $f(B)$ is convex, then $f$ is a similarity (i.e., a scalar multiple of an isometry).

(ii) If $K$ has discrete valuation and $f \in M_s(B)$ is bounded, then $f$ has bounded difference quotients. If $f \in M_b(B)$ and if $f(B)$ is convex, then $f$ is a similarity.

2. **Monotone functions having a type.**

In this section, we want to translate the usual classification of (strictly) monotone functions $\mathbb{R} \rightarrow \mathbb{R}$ into two types: the increasing and the decreasing functions. The equivalence relation in $\mathbb{R}^r$: $x \sim y$ if $x$ and $y$ are at the same side of 0, yields $(-\infty, 0)$ and $(0, \infty)$ as equivalence classes. The relation $\sim$ is compatible with the canonical group homomorphism $\mathbb{R}^* \twoheadrightarrow \mathbb{R}^*/\mathbb{R}^+$, the latter group being $\{1, -1\}$. $\pi(x)$ (usually called $\text{sgn}(x)$) assigns $+1$ to every positive element and $-1$ to every negative element. A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is strictly monotone if there exists $\sigma: \mathbb{R}^*/\mathbb{R}^+ \rightarrow \mathbb{R}^*/\mathbb{R}^+$ such that for all $x \neq y$

$$\pi(f(x) - f(y)) = \sigma(\pi(x - y)),$$

If $\sigma$ is the identity then $f$ is called increasing; if $\sigma(1) = -1$, $\sigma(-1) = 1$, $f$ is called decreasing. Other maps $\sigma: \{-1, 1\} \rightarrow \{-1, 1\}$ can not occur (i.e., there is no $f$ such that, for all $x \neq y$,

$$\pi(f(x) - f(y)) = \sigma(\pi(x - y))$$

This rather weird description of real monotone functions can be used in the non-archimedean case.

For $x, y \in K^*$ define $x \sim y$ if $x, y$ are at the same side of 0. This means: $0 \not\in [x, y]$, or $|x - y| > |y|$, or $|xy^{-1} - 1| < 1$. Thus $x \sim y$ if, and only if, $xy^{-1} \in K^+$ where

$$K^+ := \{x \in K; |1 - x| < 1\}.$$

We call the elements of $K^+$ the positive element of $K$.

The relation $\sim$ is compatible with the canonical homomorphism of (multiplicative) groups

$$\pi: K^* \rightarrow K^*/K^+ =: \Sigma.$$

We call $\Sigma$ the group of signs and $\pi(x)$ the sign of an element $x \in K^*$ (x is
positive if, and only if, $\pi(x) = 1$.

If $K$ is a local field, we can make a group embedding $\rho : \Sigma \hookrightarrow K^\#$ such that $\pi \circ \rho$ is the identity on $\Sigma$. For example, if $K = \mathbb{Q}_p$, $\delta$ is a primitive $(p - 1)$th root of unity, then

$$\pi(\sum_{n \geq 0} a_n p^n) = a_k p^k \quad (k \in \mathbb{Z}, \text{ } a_k \neq 0)$$

(Here $a_n \in \{0, 1, 2, \ldots, \delta^{p-2}\}$ for each $n$).

**Definition 2.1.** Let $\sigma : \Sigma \rightarrow \Sigma$ be any map. A function $f : X \rightarrow K$ is monotone of type $\sigma$ if, for all $x, y \in X$, $x \neq y$,

$$\pi(f(x) - f(y)) = \sigma(\pi(x - y))$$

(i.e., if $x - y \in \sigma$ then $f(x) - f(y) \in \sigma(a)$).

We call $f$ of type $\beta \in \Sigma$ if $f$ is of type $\sigma$ where $\sigma$ is the multiplication with $\beta$, i.e.,

$$\frac{f(x) - f(y)}{x - y} \in \beta \quad (x, y \in X, x \neq y).$$

We call $f$ increasing if $f$ is of type $\sigma$ where $\sigma$ is the identity, i.e.,

$$\frac{f(x) - f(y)}{x - y}$$

is positive ($x \neq y$).

Clearly, if $f$ is of type $\beta$, and if $b \in \beta$, then $b^{-1} f$ is increasing.

First, we look at increasing functions, then we discuss more general types $\sigma$.

Notice that increasing functions are isometries hence are in $M_{bs}(X)$. If $f$ is increasing then $f(x) = x + h(x)$, where $|h(x) - h(y)| < |x - y|$ ($x, y \in X, x \neq y$).

Such $h$ we call pseudo-contractions.

**Lemma 2.2.** Let $X$ be an ultrametric space. Then the following are equivalent

(a) $X$ is spherically complete,

(b) Each pseudocontraction $X \rightarrow X$ has a (unique) fixed point.

**Proof (sketch).** (a) $\rightarrow$ (b). Let $\sigma : X \rightarrow X$ be a pseudocontraction. A convex set $C \subset X$ is called invariant if $\sigma(C) \subset C$. It is easily proved that the invariant convex subsets of $X$ form a nest. Let $C_0$ be the smallest invariant convex set. If $a \in C_0$ and $\sigma(a) \neq a$ then

$$B_0 := \{x \in X; d(x, \sigma(a)) < d(a, \sigma(a))\}$$

is invariant, convex, and does not contain $a$. Hence $\sigma(a) = a$ for all $a \in C_0$, and $C_0$ is a singleton. (b) $\rightarrow$ (a). If $B_1 \neq B_2 \neq \ldots$ are balls in $X$ with $\cap B_n = \emptyset$ then choose $x_n \in B_n \setminus B_{n+1}$ ($n \in \mathbb{N}$). The map $\sigma : X \rightarrow X$ defined by

$$\sigma(x) = x_{n+1} \quad (x \in B_n \setminus B_{n+1})$$

is a pseudocontraction without a fixed point.
COROLLARY 2.3. - Let X be convex, let K be spherically complete, and let
f : X → K be increasing. Then f(X) is convex. If f(X) ⊂ X, then f is sur-
jective.

Proof. - Let f(X) ⊂ X. Choose α ∈ X. Then x ↦ f(x) + x + α is a pseudo-
contraction mapping X into X, hence has a fixed point. So f(x) = α for some
x ∈ X.

If K is not spherically complete, we have always increasing f : K → K that
are not surjective. (Let h : K → K be a pseudocontraction without a fixed point
Let f(x) = x − h(x) (x ∈ K), then 0 ∉ im f). The inverse f⁻¹ : f(K) → K can,
of course, not be extended to an increasing function K → K.

THEOREM 2.4. - Let K be spherically complete, and let f : X → K be increa-
sing. Then f can be extended to an increasing function K → K.

Proof. - By Zorn's Lemma, it suffices to extend f to an increasing function on
X ∪ {a}, where a ∉ X. We are done if we can find α ∈ K such that, for all
x ∈ X,

\[ \frac{α - f(x)}{α - x} - 1 < 1 \]

i.e. α ∈ B_{f(x)}(a−x)(|a − x|) for all x ∈ X. These balls form a nest.

Let us call a function f : X → K positive if f(X) ⊂ K⁺.

THEOREM 2.5.

(i) If f : X → K is increasing then f' is positive,
(ii) If g : X → K is a positive Baire class one function, then g has an in-
creasing antiderivative,
(iii) If g : X → K is continuous and positive, then g has a C¹-antideriva-
tive,
(iv) If f ∈ C¹(X) and f' is positive then f = j + h where j is increasing,
and h is locally constant.

EXAMPLES.

1° The exponential function (defined on its natural convergence region) is in-
creasing.

2° Let f ∈ C(Z_p), and let e₀ = δ_{Z_p}, for n ∈ Nₙ,\n
\[ e_n(x) = \begin{cases} 1 & \text{if } |x - n| < \frac{1}{n} \\ 0 & \text{elsewhere} \end{cases} \quad (x ∈ Z_p). \]

Then e₀, e₁, ..., form an orthonormal base of C(Z_p), so there exist
λ₀, λ₁, ..., ∈ R such that f = ∑_{n=0}^{∞} λ_n e_n, uniformly.
Let $\sigma : \Sigma \to \Sigma$. Let $\alpha, \beta \in \Sigma$,

(i) $\sigma(-\alpha) = -\sigma(\alpha)$,

(ii) If $\sigma(\alpha) \oplus \sigma(\beta)$ is defined then so is $\alpha \oplus \beta$ and $\sigma(\alpha \oplus \beta) = \sigma(\alpha) \oplus \sigma(\beta)$,

(iii) $|\alpha| < |\beta|$ implies $|\sigma(\alpha)| < |\sigma(\beta)|$,

(iv) If $|\beta| = 1$, $\beta$ contains an element of the prime field of $K$ then $\sigma(\beta) = \beta \sigma(\alpha)$,

(v) $f \in M_a(K)$,

(vi) $f$ is either nowhere continuous or uniformly continuous.

**Theorem 2.6.** Let $f : K \to K$ be monotone of type $\sigma : \Sigma \to \Sigma$. Let $\alpha, \beta \in \Sigma$,

(a) $\sigma$ is injective,

(b) $f \in M_b(X)$,

(c) If for some $\alpha, \beta \in \Sigma$, $\alpha \oplus \beta$ is defined, then so is $\sigma(\alpha) \oplus \sigma(\beta)$,

(d) $|\sigma(\alpha)| < |\sigma(\beta)|$ implies $|\alpha| < |\beta|$ ($\alpha, \beta \in \Sigma$).

**Corollary 2.8.** Let $k$ be a prime field, and let $f : K \to K$ be monotone of type $\sigma : \Sigma \to \Sigma$. Then $\sigma$ is injective.

(If $K = \mathbb{Q}(\sqrt[p]{-1})$, $p = 3 \text{ mod } 4$, we can find an example of an $f : K \to K$ monotone of type $\sigma$, where $\sigma$ is not injective).

**Example 2.9.** Let $K = \mathbb{Q}_p$. Then

$\{\sigma : \Sigma \to \Sigma : \text{there is } f : \mathbb{Q}_p \to \mathbb{Q}_p, f \text{ monotone of type } \sigma\}$
consists of all \( \sigma : \Sigma \to \Sigma \) of the form

\[
\frac{1}{p^n} \to \frac{1}{p^n} \delta(n)p\lambda(n)
\]

where \( s : \mathbb{Z} \to \{0, 1, 2, \ldots, p - 2\} \) and \( \lambda : \mathbb{Z} \to \mathbb{Z} \) is strictly increasing.

3. Functions of bounded variation.

**Lemma 3.1.** Let \( f : X \to K \) have bounded difference quotients. Then \( f \) is a linear combination of two increasing functions.

**Proof.** Choose \( \lambda \in K \),

\[
|\lambda| > \sup\{|f(x) - f(y)| \mid x \neq y\}.
\]

Then \( \lambda^{-1} f \) is a (pseudo-) contraction, so \( g(x) := -x + \lambda^{-1} f(x) \) \((x \in X)\) is increasing. If \( h(x) := x \) \((x \in X)\), then \( \lambda h + \lambda g = f \).

**Corollary 3.2.** Let \( X \) be the unit ball of a local field \( K \) and let \( f : X \to K \). Then the following are equivalent

- (a) \( f \in BA(X) \) \((i.e., \sup\{|f(x) - f(y)| \mid x \neq y\} < \infty\) \),
- (b) \( f \) is a linear combination of two increasing functions,
- (c) \( f \in [G_\delta(X)] \),
- (d) \( f \in [G_\delta(X)] \).

**Proof.** Use 1.6.

**References**
