NON-ARCHIMEDEAN MONOTONE FUNCTIONS

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Introduction.

In the sequel, $K$ is a non-archimedean, non-trivially valued field, that is complete under the metric induced by the valuation. The residue class field of $K$ is denoted by $k$. $X$ will always be a closed, non empty subset of $K$ without isolated points (except in 2.2, if you want).

Since $K$ admits no ordering in the usual sense it is not possible to define monotone functions $X \to K$ just by taking over the classical definitions. Thus, our procedure will be to try and find statements for functions $\mathbb{R} \to \mathbb{R}$ equivalent to monotony, and formulated in terms that are translatable to $K$. This way we will obtain several definitions of "$f : X \to K$ is monotone", that are, although not equivalent, closely related.

The connections between these various definitions and the properties of the non-archimedean monotone functions can be put together to form a little theory which is interesting in its own right, but of which the relations to the other parts of $p$-adic analysis are yet not very tight.

1. Monotone functions.

For a function $f : \mathbb{R} \to \mathbb{R}$ the following conditions are equivalent:

(a) $f$ is monotone (in the non-strict sense),

(b) If $C \subset \mathbb{R}$ is convex then $f^{-1}(C)$ is convex,

(c) If $x$ is between $y, z$ then $f(x)$ is between $f(y)$ and $f(z)$.

Also, the following conditions are equivalent:

(a) $f$ is strictly monotone,

(b) $f$ is injective. If $C \subset \mathbb{R}$ is convex then $f(C)$ is relatively convex in $f(\mathbb{R})$,

(c) If $f(x)$ is between $f(y)$ and $f(z)$ then $x$ is between $y$ and $z$.

Let $x, y \in K$. Then the smallest ball that contains $x, y$ is denoted by $[x, y]$. $z \in K$ is between $x$ and $y$ if $z \in [x, y]$. (If $z \notin [x, y]$, we

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call \( x, y \) at the same side of \( z \). A subset \( C \subseteq K \) is called convex if \( x, y \in C, z \in [x, y] \) implies \( z \in C \). Each convex subset of \( K \) can be written in at least one of the following forms

\[
\{ x : |x - a| < r \}, \{ x : |x - a| \leq r \}
\]

for some \( a \in K, r \in (0, \infty) \).

Let \( Z \subseteq Y \subseteq K \). Then \( Z \) is called convex in \( Y \) if \( Z = C \cap Y \), where \( C \) is convex.

With all these definitions we have the following theorem.

**Theorem 1.1.** Let \( f : X \rightarrow K \). Then the following conditions are equivalent:

1. If \( x, y, z \in X \), \( x \) is between \( y \) and \( z \) then \( f(x) \) is between \( f(y) \) and \( f(z) \),
2. If \( C \subseteq K \) is convex, then \( f^{-1}(C) \) is convex in \( X \).

We denote the collection of those \( f : X \rightarrow K \) satisfying (1) or (2) by \( M_b(x) \), i.e., \( f \in M_b(X) \) if, and only if, for each \( x, y, z \in X \),

\[
|x - y| \leq |y - z| \implies |f(x) - f(y)| \leq |f(y) - f(z)| .
\]

Isometries are in \( M_b \) (viz. exp), but also non trivial locally constant functions (e.g., choose a center in each ball of radius \( r > 0 \), and let \( f \) be the map assigning to \( x \in X \) the center of the ball of radius \( r \) to which \( x \) belongs. Then \( f \in M_b(X) \)).

**Theorem 1.2.** Let \( f : X \rightarrow K \). Then the following conditions are equivalent

1. If \( x, y, z \in X \), \( f(x) \) is between \( f(y) \) and \( f(z) \) then \( x \) is between \( y \) and \( z \),
2. If \( C \subseteq X \) is convex in \( X \) then \( f(C) \) is convex in \( f(X) \). \( f \) is injective.

We denote the collection of those \( f : X \rightarrow K \) satisfying (1') or (2') by \( M_s(X) \), i.e., \( f \in M_s(X) \) if, and only if, for each \( x, y, z \in X \),

\[
|x - y| < |y - z| \implies |f(x) - f(y)| < |f(y) - f(z)| .
\]

The classical situations suggests the question as to wether \( M_s(X) \subseteq M_b(X) \) and also wether \( f \in M_b(X) \), \( f \) injective implies \( f \in M_s(X) \). In general, both statements are false, but we do have the following:

**Theorem 1.3.** \( f \in M_s(X) \implies f^{-1} \in M_b(f(X)) \). \( f \in M_b(X) \), \( f \) injective implies \( f^{-1} \in M_s(f(X)) \). If \( k \) is finite and \( X \) is convex, then an injective \( M_b \)-function is in \( M_s(X) \).
So we are led to define \( M_{ba}(X) := M_b(X) \cap M_s(X) \) as being the more or less natural translation of "the space of the strictly monotone functions".

The following theorem concerns continuity of monotone functions. For a function \( f: X \to K \), we define its oscillation function, \( \omega_x \), in the usual way:

\[
\omega_x(a) := \lim_{n \to \infty} \sup \{|f(x) - f(y)| : |x - a| \leq \frac{1}{n}, |y - a| \leq \frac{1}{n}\}
\]

\[
= \lim_{n \to \infty} \sup \{|f(x) - f(a)| : |x - a| \leq \frac{1}{n}\} \quad (a \in X).
\]

\( f \) is continuous at \( a \) if, and only if, \( \omega_x(a) = 0 \).

**Theorem 1.4.** Let \( f \) be either in \( M_b(X) \) or in \( M_s(X) \). Then

(i) \( \omega_x(a) = \inf_{y \neq a} |f(z) - f(a)| \quad (a \in X) \)

(ii) \( f \) is bounded on compact subsets of \( X \),

(iii) For each \( a \in X \) we have the following alternative. Either \( f \) is continuous at \( a \), or for each sequence \( x_1, x_2, \ldots \) \( (x_n \neq a) \) converging to \( a \), the sequence \( f(x_1), f(x_2), \ldots \) is bounded and has no convergent subsequence.

Let \( g \in M_b(X) \). If \( Y \subset X \) is spherically complete, then so is \( g(Y) \).

Let \( h \in M_s(X) \). If \( Z \subset h(X) \) is spherically complete, then so is \( h^{-1}(Z) \).

**Proof (sketch).** If \( f \in M_b(X) \cup M_s(X) \), then:

\[
|x - y| < |y - z| \implies |f(x) - f(y)| \leq |f(y) - f(z)|.
\]

So \( f \) is locally bounded, and (ii) follows. Of (i), only the \( \leq \) part is interesting. Choose \( z \neq a \). If \( |x - a| < |z - a| \), then

\[
|f(x) - f(a)| \leq |f(z) - f(a)| \quad \text{whence} \quad \omega_x(a) \leq |f(z) - f(a)|.
\]

Let \( \lim x_n = a \) \( (x_n \neq a \text{ for all } n) \) and \( \lim f(x_n) = \alpha \). Let \( \lim y_n = a \). It suffices to show that \( \lim f(y_n) = \alpha \). Indeed, let \( \varepsilon > 0 \), and choose \( k \) such that

\[
|f(x_k) - \alpha| < \varepsilon.
\]

Then \( |y_n - a| < |x_k - a| \) for large \( n \), so

\[
|y_n - x_m| < |x_k - x_m|
\]

for large \( m \) depending on \( m \). Hence

\[
|f(y_n) - f(x_n)| \leq |f(x_k) - f(x_m)|, \quad \text{so} \quad (m \to \infty) \quad |f(y_n) - \alpha| \leq |f(x_k) - \alpha| < \varepsilon,
\]

and we have (iii). The rest of the proof is straightforward.

**Corollary 1.5.** Let \( f: X \to K \) be in \( M_b(X) \cup M_s(X) \).

(i) If \( K \) is a local field, then \( f \) is continuous.

(ii) If \( |K| \) is discrete, then \( f \in M_s(X) \implies f \) is a homeomorphism \( X \sim f(X) \), and \( f \in M_b(X) \implies f \) is a closed map.

(iii) The graph of \( f \) is closed in \( K^2 \).

(iv) If \( f(X) \) has no isolated points, then \( f \) is continuous.
An $M_b$-function may be everywhere discontinuous on $K$ (even when $|K|$ is discrete).

**THEOREM 1.6.** Let $B$ be the unit ball of $K$,

(i) If $K$ is a local field and $f \in M_b(B) \cup M_s(B)$, then $f$ has bounded difference quotients (i.e., there is $C > 0$ such that $|f(x) - f(y)| \leq C|x - y|$ for all $x \in B$). If, in addition, $f(B)$ is convex, then $f$ is a similarity (i.e., a scalar multiple of an isometry).

(ii) If $K$ has discrete valuation and $f \in M_b(B)$ is bounded, then $f$ has bounded difference quotients. If $f \in M_{bs}(B)$ and if $f(B)$ is convex, then $f$ is a similarity.

2. **Monotone functions having a type.**

In this section, we want to translate the usual classification of (strictly) monotone functions $R \to R$ into two types: the increasing and the decreasing functions. The equivalence relation in $R^\ast$: $x \sim y$ if $x$ and $y$ are at the same side of $0$, yields $(-\infty, 0)$ and $(0, \infty)$ as equivalence classes. The relation $\sim$ is compatible with the canonical group homomorphism $R^\ast \to \mathbb{R}^\ast / \mathbb{R}^+$, the latter group being $\{1, -1\}$. $\pi(x)$ (usually called $\text{sgn}(x)$) assigns $+1$ to every positive element and $-1$ to every negative element. A function $f: R \to R$ is strictly monotone if there exists $\sigma: \mathbb{R}^\ast / \mathbb{R}^+ \to \mathbb{R}^\ast / \mathbb{R}^+$ such that for all $x \neq y$

$$\pi(f(x) - f(y)) = \sigma(\pi(x - y)).$$

If $\sigma$ is the identity then $f$ is called increasing; if $\sigma(1) = -1$, $\sigma(-1) = 1$, $f$ is called decreasing. Other maps $\sigma: \{-1, 1\} \to \{-1, 1\}$ cannot occur (i.e., there is no $f$ such that, for all $x \neq y$,

$$\pi(f(x) - f(y)) = \sigma(\pi(x - y)).$$

This rather weird description of real monotone functions can be used in the non-archimedean case.

For $x, y \in K^\ast$, define $x \sim y$ if $x, y$ are at the same side of $0$. This means: $0 \not\in [x, y]$, or $|x - y| > |y|$, or $|xy^{-1} - 1| < 1$. Thus $x \sim y$ if, and only if, $xy^{-1} \in K^+$ where

$$K^+ := \{x \in K; \ |1 - x| < 1\}.$$  

We call the elements of $K^+$ the positive element of $K$.

The relation $\sim$ is compatible with the canonical homomorphism of (multiplicative) groups

$$\pi: K^\ast \to K^\ast / K^+ =: \Sigma.$$  

We call $\Sigma$ the group of signs and $\pi(x)$ the sign of an element $x \in K^\ast$ ($x$ is
positive if, and only if, \( n(x) = 1 \).

If \( K \) is a local field, we can make a group embedding \( \rho : \Sigma \rightarrow K^* \) such that
\[ n \circ \rho \] is the identity on \( \Sigma \). For example, if \( K = \mathbb{Q} \), \( \delta \) is a primitive \((p - 1)\)th root of unity, then
\[ n(\sum_{k \geq 1} a_k p^k) = a_k p^k \quad (k \in \mathbb{Z}, \ a_k \neq 0) \]
(Here \( a_n \in \{0, 1, \delta, \ldots, \delta^{p-2}\} \) for each \( n \)).

**Definition 2.1.** Let \( \sigma : \Sigma \rightarrow \Sigma \) be any map. A function \( f : X \rightarrow K \) is monotone of type \( \sigma \) if, for all \( x, y \in X \), \( x \neq y \),
\[ n(f(x) - f(y)) = \sigma(n(x - y)) \]
(i.e., if \( x - y \in \sigma \in \Sigma \) then \( f(x) - f(y) \in \sigma(x) \)).

We call \( f \) of type \( \beta \in \Sigma \) if \( f \) is of type \( \sigma \) where \( \sigma \) is the multiplication with \( \beta \), i.e.,
\[ \frac{f(x) - f(y)}{x - y} \in \beta \quad (x, y \in X, x \neq y). \]

We call \( f \) increasing if \( f \) is of type \( \sigma \) where \( \sigma \) is the identity, i.e.,
\[ f(x) - f(y) \] is positive \((x \neq y)\).

Clearly, if \( f \) is of type \( \beta \), and if \( b \in \beta \), then \( b^{-1} f \) is increasing.

First, we look at increasing functions, then we discuss more general types \( \sigma \).
Notice that increasing functions are isometries hence are in \( M_{bs}(X) \). If \( f \) is increasing then \( f(x) = x + h(x) \), where \(|h(x) - h(y)| < |x - y| \quad (x, y \in X, x \neq y)\).

Such \( h \) we call pseudo-contractions.

**Lemma 2.2.** Let \( X \) be an ultrametric space. Then the following are equivalent

(a) \( X \) is spherically complete,

(b) Each pseudocontraction \( X \rightarrow X \) has a (unique) fixed point.

**Proof (sketch).** - (a) \( \rightarrow \) (b). Let \( \sigma : X \rightarrow X \) be a pseudocontraction. A convex set \( C \subseteq X \) is called invariant if \( \sigma(C) \subseteq C \). It is easily proved that the invariant convex subsets of \( X \) form a nest. Let \( C_0 \) be the smallest invariant convex set. If \( a \in C_0 \) and \( \sigma(a) \neq a \) then
\[ B_0 := \{x \in X ; \ d(x, \sigma(a)) < d(a, \sigma(a))\} \]
is invariant, convex, and does not contain \( a \). Hence \( \sigma(a) = a \) for all \( a \in C_0 \), and \( C_0 \) is a singleton. (b) \( \rightarrow \) (a). If \( B_1 \neq B_2 \neq \ldots \) are balls in \( X \) with \( \cap B_n = \emptyset \) then choose \( x_n \in B_n \backslash B_{n+1} \quad (n \in \mathbb{N}) \). The map \( \sigma : X \rightarrow X \) defined by
\[ \sigma(x) = x_{n+1} \quad (x \in B_n \backslash B_{n+1}) \]
is a pseudocontraction without a fixed point.
COROLLARY 2.3. - Let $X$ be convex, let $K$ be spherically complete, and let $f : X \to K$ be increasing. Then $f(X)$ is convex. If $f(X) \subseteq X$, then $f$ is surjective.

Proof. - Let $f(X) \subseteq X$. Choose $\alpha \in X$. Then $x \mapsto -f(x) + x + \alpha$ is a pseudocontraction mapping $X$ into $X$, hence has a fixed point. So $f(x) = \alpha$ for some $x \in X$.

If $K$ is not spherically complete, we have always increasing $f : K \to K$ that are not surjective. (Let $h : K \to K$ be a pseudocontraction without a fixed point. Let $f(x) = x - h(x)$ ($x \in K$), then $0 \not\in \text{Im } f$). The inverse $f^{-1} : f(K) \to K$ can, of course, not be extended to an increasing function $K \to K$.

THEOREM 2.4. - Let $K$ be spherically complete, and let $f : X \to K$ be increasing. Then $f$ can be extended to an increasing function $K \to K$.

Proof. - By Zorn's Lemma, it suffices to extend $f$ to an increasing function on $X \cup \{a\}$, where $a \not\in X$. We are done if we can find $\alpha \in K$ such that, for all $x \in X$,

$$|\alpha - f(x)| < 1$$

i.e. $\alpha \in B_{f(x)}(a - x)$ for all $x \in X$. These balls form a nest.

Let us call a function $f : X \to K$ positive if $f(X) \subseteq K^+$.

THEOREM 2.5.

(i) If $f : X \to K$ is increasing then $f'$ is positive,

(ii) If $g : X \to K$ is a positive Baire class one function, then $g$ has an increasing antiderivative,

(iii) If $g : X \to K$ is continuous and positive, then $g$ has a $C^1$-antiderivative,

(iv) If $f \in C^1(X)$ and $f'$ is positive then $f = j + h$ where $j$ is increasing, and $h$ is locally constant.

EXAMPLES.

1° The exponential function (defined on its natural convergence region) is increasing.

2° Let $f \in C(\mathbb{Z}_p)$, and let $e_n = e_n^{\mathbb{Z}_p}$, for $n \in \mathbb{N}$,

$$e_n(x) = \begin{cases} 1 & \text{if } |x - n| < \frac{1}{n} \\ 0 & \text{otherwise} \end{cases} \quad (x \in \mathbb{Z}_p).$$

Then $e_0, e_1, \ldots$ form an orthonormal base of $C(\mathbb{Z}_p)$, so there exist $\lambda_0, \lambda_1, \ldots \in \mathbb{Q}_p$ such that $f = \sum_{n=0}^{\infty} \lambda_n e_n$, uniformly.
f is increasing if, and only if, for all \( n \in \mathbb{N} \),
\[
|\lambda_n - \{n\}| < \{n\}
\]
(where, if \( n = a_0 + a_1 p + \ldots + a_k p^k \) (\( a_i \in \{0, 1, \ldots, p - 1\} \) for each \( i \), \( a_k \neq 0 \)), then \( \{n\}_1 = a_k p^k \)).

In other words, \( f = \sum \lambda_n e_n \in \mathcal{C}(\mathbb{Z}_p) \) is increasing if, and only if, \( \lambda_{n}/\{n\} \) is positive for all \( n \in \mathbb{N} \).

Let \( \alpha, \beta \in \Sigma \). If the set theoretic sum \( \alpha + \beta := \{x + y ; x \in \alpha, y \in \beta\} \) does not contain 0 then \( \alpha + \beta \in \Sigma \), notation \( \alpha \circ \beta \). It follows that \( \alpha \circ \beta \) is defined if, and only if, \( \alpha \neq - \beta \).

If \( x, y \in \alpha \in \Sigma \) then \( |x| = |y| \). This defines \( |\alpha| \) in a natural way.

We have the following results.

**THEOREM 2.6.** Let \( f : \mathbb{K} \rightarrow \mathbb{K} \) be monotone of type \( \sigma : \Sigma \rightarrow \Sigma \). Let \( \alpha, \beta \in \Sigma \),
\[
\begin{align*}
(i) & \quad \sigma(- \alpha) = - \sigma(\alpha), \\
(ii) & \quad \text{If } \sigma(\alpha) \circ \sigma(\beta) \text{ is defined then so is } \alpha \circ \beta \text{ and } \sigma(\alpha \circ \beta) = \sigma(\alpha) \circ \sigma(\beta), \\
(iii) & \quad |\alpha| < |\beta| \text{ implies } |\sigma(\alpha)| < |\sigma(\beta)|, \\
(iv) & \quad \text{If } |\beta| = 1, \beta \text{ contains an element of the prime field of } \mathbb{K} \text{ then } \\
& \quad \sigma(\beta \alpha) = \beta \sigma(\alpha), \\
(v) & \quad f \in \mathbb{M}_1(\mathbb{K}), \\
(vi) & \quad f \text{ is either nowhere continuous or uniformly continuous}.
\end{align*}
\]

**THEOREM 2.7.** Let \( f : \mathbb{K} \rightarrow \mathbb{K} \) be monotone of type \( \sigma : \Sigma \rightarrow \Sigma \). Then the following conditions are equivalent,
\[
\begin{align*}
(a) & \quad \sigma \text{ is injective}, \\
(b) & \quad f \in \mathbb{M}_b(\mathbb{K}), \\
(c) & \quad \text{If for some } \alpha, \beta \in \Sigma, \alpha \circ \beta \text{ is defined, then so is } \sigma(\alpha) \circ \sigma(\beta), \\
(d) & \quad |\sigma(\alpha)| < |\sigma(\beta)| \text{ implies } |\alpha| < |\beta| (\alpha, \beta \in \Sigma).
\end{align*}
\]

**COROLLARY 2.8.** Let \( \mathbb{K} \) be a prime field, and let \( f : \mathbb{K} \rightarrow \mathbb{K} \) be monotone of type \( \sigma : \Sigma \rightarrow \Sigma \). Then \( \sigma \) is injective.

(If \( \mathbb{K} = \mathbb{Q}(\sqrt{-1}) \), \( p = 3 \mod 4 \), we can find an example of an \( f : \mathbb{K} \rightarrow \mathbb{K} \) monotone of type \( \sigma \), where \( \sigma \) is not injective).

**EXAMPLE 2.9.** Let \( \mathbb{K} = \mathbb{Q}_p \). Then
\[
\{\sigma : \Sigma \rightarrow \Sigma : \text{there is } f : \mathbb{Q}_p \rightarrow \mathbb{Q}_p, \text{ } f \text{ monotone of type } \sigma\}
\]
consists of all \( \sigma : \Sigma \to \Sigma \) of the form
\[
\phi p \mapsto \phi \delta^x(n) \lambda(n)
\]
where \( s : \mathbb{Z} \to \{0, 1, 2, \ldots, p-2\} \) and \( \lambda : \mathbb{Z} \to \mathbb{Z} \) is strictly increasing.

3. Functions of bounded variation.

**Lemma 3.1.** Let \( f : X \to K \) have bounded difference quotients. Then \( f \) is a linear combination of two increasing functions.

**Proof.** Choose \( \lambda \in K \),

\[
|\lambda| > \sup \{ |f(x) - f(y)| / (x - y) ; x \neq y \}.
\]

Then \( \lambda^{-1} f \) is a (pseudo-) contraction, so \( g(x) := -x + \lambda^{-1} f(x) \) \( (x \in X) \) is increasing. If \( h(x) := x \) \( (x \in X) \), then \( \lambda h + \lambda g = f \).

**Corollary 3.2.** Let \( X \) be the unit ball of a local field \( K \) and let \( f : X \to K \). Then the following are equivalent

(\( \alpha \)) \( f \in BA(X) \) (i.e., \( \sup \{ |f(x) - f(y)| / (x - y) ; x \neq y \} < \infty \)),

(\( \beta \)) \( f \) is a linear combination of two increasing functions,

(\( \gamma \)) \( f \in [I^b_\varphi(X)] \),

(\( \delta \)) \( f \in [I^b_\varphi(X)] \).

**Proof.** Use 1.6.

**References**
