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Introduction.

In the sequel, $K$ is a non-archimedean, non-trivially valued field, that is complete under the metric induced by the valuation. The residue class field of $K$ is denoted by $k$. $X$ will always be a closed, non empty subset of $K$ without isolated points (except in 2.2, if you want).

Since $K$ admits no ordering in the usual sense it is not possible to define monotone functions $X \rightarrow K$ just by taking over the classical definitions. Thus, our procedure will be to try and find statements for functions $\mathbb{R} \rightarrow \mathbb{R}$ equivalent to monotony, and formulated in terms that are translatable to $K$. This way we will obtain several definitions of "$f : X \rightarrow K$ is monotone", that are, although not equivalent, closely related.

The connections between these various definitions and the properties of the non-archimedean monotone functions can be put together to form a little theory which is interesting in its own right, but of which the relations to the other parts of $p$-adic analysis are yet not very tight.

1. Monotone functions.

For a function $f : \mathbb{R} \rightarrow \mathbb{R}$ the following conditions are equivalent:

(a) $f$ is monotone (in the non-strict sense),
(b) If $C \subseteq \mathbb{R}$ is convex then $f^{-1}(C)$ is convex,
(c) If $x$ is between $y$ and $z$ then $f(x)$ is between $f(y)$ and $f(z)$.

Also, the following conditions are equivalent:

(a) $f$ is strictly monotone,
(b) $f$ is injective. If $C \subseteq \mathbb{R}$ is convex then $f(C)$ is relatively convex in $f(\mathbb{R})$,
(c) If $f(x)$ is between $f(y)$ and $f(z)$ then $x$ is between $y$ and $z$.

Let $x, y \in K$. Then the smallest ball that contains $x$ and $y$ is denoted by $[x, y]$. $z \in K$ is between $x$ and $y$ if $z \in [x, y]$. (If $z \notin [x, y]$, we

(*) Texte reçu le 12 mars 1979.
call $x, y$ at the same side of $z$). A subset $C \subseteq K$ is called convex if $x, y \in C$, $z \in [x, y]$ implies $z \in C$. Each convex subset of $K$ can be written in at least one of the following forms

$$\{x : |x - a| < r\}, \{x : |x - a| \leq r\}$$

for some $a \in K$, $r \in (0, \infty)$.

Let $Z \subseteq Y \subseteq K$. Then $Z$ is called convex in $Y$ if $Z = C \cap Y$, where $C$ is convex.

With all these definitions we have the following theorem.

**Theorem 1.1.** Let $f : X \rightarrow K$. Then the following conditions are equivalent:

1. If $x, y, z \in X$, $x$ is between $y$ and $z$ then $f(x)$ is between $f(y)$ and $f(z)$,

2. If $C \subseteq K$ is convex, then $f^{-1}(C)$ is convex in $X$.

We denote the collection of those $f : X \rightarrow K$ satisfying (1) or (2) by $M_b(X)$, i.e., $f \in M_b(X)$ if, and only if, for each $x, y, z \in X$,

$$|x - y| \leq |y - z| \text{ implies } |f(x) - f(y)| \leq |f(y) - f(z)|.$$

Isometries are in $M_b$ (viz. exp), but also non trivial locally constant functions (e.g., choose a center in each ball of radius $r > 0$, and let $f$ be the map assigning to $x \in X$ the center of the ball of radius $r$ to which $x$ belongs. Then $f \in M_b(X)$).

**Theorem 1.2.** Let $f : X \rightarrow K$. Then the following conditions are equivalent:

1. If $x, y, z \in X$, $f(x)$ is between $f(y)$ and $f(z)$ then $x$ is between $y$ and $z$,

2. If $0 \subseteq X$ is convex in $X$ then $f(0)$ is convex in $f(X)$. $f$ is injective.

We denote the collection of those $f : X \rightarrow K$ satisfying (1') or (2') by $M_s(X)$, i.e., $f \in M_s(X)$ if, and only if, for each $x, y, z \in X$,

$$|x - y| < |y - z| \text{ implies } |f(x) - f(y)| < |f(y) - f(z)|.$$

The classical situations suggests the question as to wether $M_s(X) \subseteq M_b(X)$ and also wether $f \in M_b(X)$, $f$ injective implies $f \in M_s(X)$. In general, both statements are false, but we do have the following:

**Theorem 1.3.** $f \in M_s(X)$ implies $f^{-1} \in M_b(f(X)) \cdot f \in M_b(X)$, $f$ injective implies $f^{-1} \in M_s(f(X))$. If $k$ is finite and $X$ is convex, then an injective $M_b$-function is in $M_s(X)$. 


So we are led to define $M_{ba}(X) := M_b(X) \cap M_s(X)$ as being the more or less natural translation of "the space of the strictly monotone functions".

The following theorem concerns continuity of monotone functions. For a function $f : X \to K$, we define its oscillation function, $\omega_f$, in the usual way:

$$\omega_f(a) := \lim_{n \to \infty} \sup \{|f(x) - f(y)| ; |x - a| \leq \frac{1}{n} , |y - a| \leq \frac{1}{n}\}$$

or

$$\omega_f(a) = \lim_{n \to \infty} \sup \{|f(x) - f(a)| ; |x - a| \leq \frac{1}{n}\} \quad (a \in X).$$

$f$ is continuous at $a$ if, and only if, $\omega_f(a) = 0$.

**Theorem 1.4.** Let $f$ be either in $M_b(X)$ or in $M_s(X)$. Then

(i) $\omega_f(a) = \inf_{z \neq a} |f(z) - f(a)| \quad (a \in X)$

(ii) $f$ is bounded on compact subsets of $X$,

(iii) For each $a \in X$ we have the following alternative. Either $f$ is continuous at $a$, or for each sequence $x_1, x_2, \ldots$ $(x_n \neq a)$ converging to $a$, the sequence $f(x_1), f(x_2), \ldots$ is bounded and has no convergent subsequence.

Let $g \in M_b(X)$. If $Y \subseteq X$ is spherically complete, then so is $g(Y)$.

Let $h \in M_s(X)$. If $Z \subseteq h(X)$ is spherically complete, then so is $h^{-1}(Z)$.

**Proof (sketch).** If $f \in M_b(X) \cup M_s(X)$, then:

$$|x - y| < |y - a| \implies |f(x) - f(y)| \leq |f(y) - f(z)|.$$

So $f$ is locally bounded, and (ii) follows. Of (i), only the $< \leq$ part is interesting. Choose $z \neq a$. If $|x - a| < |z - a|$, then

$$|f(x) - f(a)| \leq |f(z) - f(a)| \quad \text{whence } \omega_f(a) \leq |f(z) - f(a)|.$$

Let $\lim x_n = a$ $(x_n \neq a$ for all $n)$ and $\lim f(x_n) = \alpha$. Let $\lim y_n = a$. It suffices to show that $\lim f(y_n) = \alpha$. Indeed, let $\varepsilon > 0$, and choose $k$ such that $|f(x_k) - \alpha| < \varepsilon$. Then $|y_n - a| < |x_n - a|$ for large $n$, so

$$|y_n - x_m| < |x_n - x_m|$$

for large $m$ depending on $m$. Hence $|f(y_n) - f(x_n)| \leq |f(x_n) - f(x_m)|$, so

$$|f(y_n) - \alpha| \leq |f(x_k) - \alpha| < \varepsilon,$$

and we have (iii). The rest of the proof is straightforward.

**Corollary 1.5.** Let $f : X \to K$ be in $M_b(X) \cup M_s(X)$.

(i) If $K$ is a local field, then $f$ is continuous,

(ii) If $|X|$ is discrete, then $f \in M_s(X) \Rightarrow f$ is a homeomorphism $X \sim f(X)$, and $f \in M_b(X) \Rightarrow f$ is a closed map.

(iii) The graph of $f$ is closed in $K^2$.

(iv) If $f(X)$ has no isolated points, then $f$ is continuous.
An \( M_b \)-function may be everywhere discontinuous on \( K \) (even when \(|K|\) is discrete).

**Theorem 1.6.** Let \( B \) be the unit ball of \( K \),

(i) If \( K \) is a local field and \( f \in M_b(B) \cup M_s(B) \), then \( f \) has bounded difference quotients (i.e., there is \( C > 0 \) such that \( |f(x) - f(y)| \leq C|x - y| \) for all \( x \in B \)). If, in addition, \( f(B) \) is convex, then \( f \) is a similarity (i.e., a scalar multiple of an isometry).

(ii) If \( K \) has discrete valuation and \( f \in M_s(B) \) is bounded, then \( f \) has bounded difference quotients. If \( f \in M_{bs}(B) \) and if \( f(B) \) is convex, then \( f \) is a similarity.

2. Monotone functions having a type.

In this section, we want to translate the usual classification of (strictly) monotone functions \( \mathbb{R} \to \mathbb{R} \) into two types: the increasing and the decreasing functions. The equivalence relation in \( \mathbb{R}^r \): \( x \sim y \) if \( x \) and \( y \) are at the same side of \( 0 \), yields \((-\infty, 0)\) and \((0, \infty)\) as equivalence classes. The relation \( \sim \) is compatible with the canonical group homomorphism \( \mathbb{R}^* \to \mathbb{R}^*/\mathbb{R}^+ \), the latter group being \( \{1, -1\} \). \( \pi(x) \) (usually called sgn(\( x \)) ) assigns \( +1 \) to every positive element and \( -1 \) to every negative element. A function \( f : \mathbb{R} \to \mathbb{R} \) is strictly monotone if there exists \( \sigma : \mathbb{R}^*/\mathbb{R}^+ \to \mathbb{R}^*/\mathbb{R}^+ \) such that for all \( x \neq y \)

\[ \pi(f(x) - f(y)) = \sigma(\pi(x - y)). \]

If \( \sigma \) is the identity then \( f \) is called increasing; if \( \sigma(1) = -1, \sigma(-1) = 1 \), \( f \) is called decreasing. Other maps \( \sigma : [-1, 1] \to [-1, 1] \) cannot occur (i.e., there is no \( f \) such that, for all \( x \neq y \),

\[ \pi(f(x) - f(y)) = \sigma(\pi(x - y)). \]

This rather weird description of real monotone functions can be used in the non-archimedean case.

For \( x, y \in K^h \) define \( x \sim y \) if \( x, y \) are at the same side of \( 0 \). This means: \( 0 \not\in [x, y] \), or \( |x - y| > |y| \), or \( |x y^{-1} - 1| < 1 \). Thus \( x \sim y \) if, and only if, \( x y^{-1} \in K^+ \) where

\[ K^+ := \{x \in K ; |1 - x| < 1 \}. \]

We call the elements of \( K^+ \) the positive element of \( K \).

The relation \( \sim \) is compatible with the canonical homomorphism of (multiplicative) groups

\[ \pi : K^* \to K^*/K^+ =: \Sigma. \]

We call \( \Sigma \) the group of signs and \( \pi(x) \) the sign of an element \( x \in K^* \) ( \( x \) is
positive if, and only if, \( n(x) = 1 \).

If \( K \) is a local field, we can make a group embedding \( \rho : \Sigma \rightarrow K^* \) such that 
\[ \pi \circ \rho \] is the identity on \( \Sigma \). For example, if \( K = \mathbb{Q}_p \), \( \delta \) is a primitive 
\((p - 1)\)th root of unity, then 
\[ \pi(\sum_{n \geq k} a_n p^n) = a_k p^k \quad (k \in \mathbb{Z}, \ a_k \not= 0) \]
(Here \( a_n \in \{0, 1, 2, \ldots, 6p-2\} \) for each \( n \)).

**Definition 2.1.** Let \( \sigma : \Sigma \rightarrow \Sigma \) be any map. A function \( f : X \rightarrow K \) is monotone of type \( \sigma \) if, for all \( x, y \in X \), \( x \neq y \),
\[ n(f(x) - f(y)) = \sigma(n(x - y)) \]
(i.e., if \( x - y \in \alpha \in \Sigma \) then \( f(x) - f(y) \in \sigma(\alpha) \)).

We call \( f \) of type \( \beta \in \Sigma \) if \( f \) is of type \( \sigma \) where \( \sigma \) is the multiplication with \( \beta \), i.e.,
\[ \frac{f(x) - f(y)}{x - y} \in \beta \quad (x, y \in X, x \neq y). \]

We call \( f \) increasing if \( f \) is of type \( \sigma \) where \( \sigma \) is the identity, i.e.,
\[ \frac{f(x) - f(y)}{x - y} \] is positive \((x \neq y)\).

Clearly, if \( f \) is of type \( \beta \), and if \( b \in \beta \), then \( b^{-1} f \) is increasing.

First, we look at increasing functions, then we discuss more general types \( \sigma \).
Notice that increasing functions are isometries hence are in \( \mathbb{M}_+^\infty(X) \). If \( f \) is increasing then \( f(x) = x + h(x) \), where \( |h(x) - h(y)| < |x - y| \) \((x, y \in X, x \neq y)\).
Such \( h \) we call pseudo-contractions.

**Lemma 2.2.** Let \( X \) be an ultrametric space. Then the following are equivalent

(\( \alpha \)) \( X \) is spherically complete,

(\( \beta \)) Each pseudocontraction \( X \rightarrow X \) has a (unique) fixed point.

**Proof (sketch).** (\( \alpha \)) \( \rightarrow \) (\( \beta \)). Let \( \sigma : X \rightarrow X \) be a pseudocontraction. A convex set \( C \subset X \) is called invariant if \( \sigma(C) \subset C \). It is easily proved that the invariant convex subsets of \( X \) form a nest. Let \( C_0 \) be the smallest invariant convex set. If \( a \in C_0 \) and \( \sigma(a) \neq a \) then
\[ B_0 := \{x \in X; d(x, \sigma(a)) < d(a, \sigma(a))\} \]
is invariant, convex, and does not contain \( a \). Hence \( \sigma(a) = a \) for all \( a \in C_0 \), and \( C_0 \) is a singleton. (\( \beta \)) \( \rightarrow \) (\( \alpha \)). If \( B_1 \neq B_2 \neq \ldots \) are balls in \( X \) with \( \bigcap B_n = \emptyset \) then choose \( x_n \in B_n \setminus B_{n+1} \) \((n \in \mathbb{N})\). The map \( \sigma : X \rightarrow X \) defined by
\[ \sigma(x) = x_{n+1} \quad (x \in B_n \setminus B_{n+1}) \]
is a pseudocontraction without a fixed point.
COROLLARY 2.3. - Let $X$ be convex, let $K$ be spherically complete, and let $f : X \to K$ be increasing. Then $f(X)$ is convex. If $f(X) \subset X$, then $f$ is surjective.

Proof. - Let $f(X) \subset X$. Choose $\alpha \in X$. Then $x \mapsto -f(x) + x + \alpha$ is a pseudo-contraction mapping $X$ into $X$, hence has a fixed point. So $f(x) = \alpha$ for some $x \in X$.

If $K$ is not spherically complete, we have always increasing $f : K \to K$ that are not surjective. Let $h : K \to K$ be a pseudocontraction without a fixed point. Let $f(x) = x - h(x)$ ($x \in K$), then $0 \not\in f'$. The inverse $f^{-1} : f(K) \to K$ can, of course, not be extended to an increasing function $K \to K$.

THEOREM 2.4. - Let $K$ be spherically complete, and let $f : X \to K$ be increasing. Then $f$ can be extended to an increasing function $K \to K$.

Proof. - By Zorn's Lemma, it suffices to extend $f$ to an increasing function on $X \cup \{a\}$, where $a \not\in X$. We are done if we can find $\alpha \in K$ such that, for all $x \in X$,

$$\left| \frac{\alpha - f(x)}{a - x} - 1 \right| < 1$$

i. e. $\alpha \in B_{f(x)}(a-x)(|a - x|^{-1})$ for all $x \in X$. These balls form a nest.

Let us call a function $f : X \to K$ positive if $f(X) \subset K^+$.

THEOREM 2.5.

(i) If $f : X \to K$ is increasing then $f'$ is positive.

(ii) If $g : X \to K$ is a positive Baire class one function, then $g$ has an increasing antiderivative.

(iii) If $g : X \to K$ is continuous and positive, then $g$ has a $C^1$-antiderivative.

(iv) If $f \in C^1(X)$ and $f'$ is positive then $f = j + h$ where $j$ is increasing, and $h$ is locally constant.

EXAMPLES.

1° The exponential function (defined on its natural convergence region) is increasing.

2° Let $f \in C(\mathbb{Z}_p)$, and let $e_0 = \xi_{\mathbb{Z}_p}$, for $n \in \mathbb{N}$,

$$e_n(x) = \begin{cases} 1 & \text{if } |x - n| < \frac{1}{n} \\ 0 & \text{elsewhere} \end{cases} \quad (x \in \mathbb{Z}_p).$$

Then $e_0, e_1, \ldots$ form an orthonormal base of $C(\mathbb{Z}_p)$, so there exist $\lambda_0, \lambda_1, \ldots \in \mathbb{Q}_p$ such that $f = \sum_{n=0}^{\infty} \lambda_n e_n$, uniformly.
\( f \) is increasing if, and only if, for all \( n \in \mathbb{N} \),
\[
|\lambda_n - \{n\}| < \{n\}
\]
(where, if \( n = a_0 + a_1 p + \cdots + a_k p^k \) (\( a_i \in \{0, 1, \ldots, p - 1\} \) for each \( i \), \( a_k \neq 0 \)), then \( \{n\}_1 = a_k p^k \).

In other words, \( f = \sum \lambda_n 2^n \in C(\mathbb{Z}_p) \) is increasing if, and only if, \( \lambda_n/\{n\} \) is positive for all \( n \in \mathbb{N} \).

Let \( \alpha, \beta \in \Sigma \). If the set theoretic sum \( \alpha + \beta := \{x + y ; x \in \alpha, y \in \beta\} \) does not contain 0 then \( \alpha + \beta \in \Sigma \), notation \( \alpha \oplus \beta \). It follows that \( \alpha \oplus \beta \) is defined if, and only if, \( \alpha \neq - \beta \).

If \( x, y \in \alpha \in \Sigma \) then \( |x| = |y| \). This defines \( |\alpha| \) in a natural way.

We have the following results.

**Theorem 2.6.** Let \( f : K \to K \) be monotone of type \( \sigma : \Sigma \to \Sigma \). Let \( \alpha, \beta \in \Sigma \),
\[(i) \quad \sigma(- \alpha) = - \sigma(\alpha),
(ii) \quad \text{If } \sigma(\alpha) \oplus \sigma(\beta) \text{ is defined then so is } \alpha \oplus \beta \text{ and } \sigma(\alpha \oplus \beta) = \sigma(\alpha) \oplus \sigma(\beta),
(iii) \quad |\alpha| < |\beta| \text{ implies } |\sigma(\alpha)| < |\sigma(\beta)|,
(iv) \quad \text{If } |\beta| = 1, \beta \text{ contains an element of the prime field of } K \text{ then } \sigma(\beta \alpha) = \beta \sigma(\alpha),
(v) \quad \sigma \in M_b(K),
(vi) \quad f \text{ is either nowhere continuous or uniformly continuous.}

**Theorem 2.7.** Let \( f : K \to K \) be monotone of type \( \sigma : \Sigma \to \Sigma \). Then the following conditions are equivalent,
\[(a) \quad \sigma \text{ is injective},
(b) \quad f \in M_b(\Sigma),
(c) \quad \text{If for some } \alpha, \beta \in \Sigma, \alpha \oplus \beta \text{ is defined, then so is } \sigma(\alpha) \oplus \sigma(\beta),
(d) \quad |\sigma(\alpha)| < |\sigma(\beta)| \text{ implies } |\alpha| < |\beta| (\alpha, \beta \in \Sigma).

**Corollary 2.8.** Let \( k \) be a prime field, and let \( f : K \to K \) be monotone of type \( \sigma : \Sigma \to \Sigma \). Then \( \sigma \) is injective.

(If \( K = \mathbb{Q}_p(\sqrt{-1}) \), \( p = 3 \mod 4 \), we can find an example of an \( f : K \to K \) monotone of type \( \sigma \), where \( \sigma \) is not injective).

**Example 2.9.** Let \( K = \mathbb{Q}_p \). Then
\[
\{\sigma : \Sigma \to \Sigma : \text{there is } f : \mathbb{Q}_p \to \mathbb{Q}_p, f \text{ monotone of type } \sigma\}
consists of all $\sigma : \Sigma \rightarrow \Sigma$ of the form
$$\sigma^p \rightarrow \sigma \sigma^p (n) \lambda(n)$$
where $s : \mathbb{Z} \rightarrow \{0, 1, 2, \ldots, p - 2\}$ and $\lambda : \mathbb{Z} \rightarrow \mathbb{Z}$ is strictly increasing.

3. Functions of bounded variation.

**Lemma 3.1.** Let $f : X \rightarrow K$ have bounded difference quotients. Then $f$ is a linear combination of two increasing functions.

**Proof.** Choose $\lambda \in K$,
$$|\lambda| > \sup \left\{ \frac{|f(x) - f(y)|}{x - y} \; ; \; x \neq y \right\}.$$
Then $\lambda^{-1} f$ is a (pseudo-) contraction, so $g(x) := -x + \lambda^{-1} f(x)$ ($x \in X$) is increasing. If $h(x) := x$ ($x \in X$), then $\lambda h + \lambda g = f$.

**Corollary 3.2.** Let $X$ be the unit ball of a local field $K$ and let $f : X \rightarrow K$. Then the following are equivalent

(a) $f \in BA(X)$ (i.e., $\sup \left\{ \frac{|f(x) - f(y)|}{x - y} \; ; \; x \neq y \right\} < \infty$),

(b) $f$ is a linear combination of two increasing functions,

(c) $f \in L_{\text{loc}}(X)$,

(d) $f \in L_{\text{loc}}(X)$.

**Proof.** Use 1.6.

**References**
