NON-ARCHIMEDEAN MONOTONE FUNCTIONS

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Introduction.

In the sequel, \( K \) is a non-archimedean, non-trivially valued field, that is complete under the metric induced by the valuation. The residue class field of \( K \) is denoted by \( \kappa \). \( X \) will always be a closed, non-empty subset of \( K \) without isolated points (except in 2.2, if you want).

Since \( K \) admits no ordering in the usual sense it is not possible to define monotone functions \( X \rightarrow K \) just by taking over the classical definitions. Thus, our procedure will be to try and find statements for functions \( \mathbb{R} \rightarrow \mathbb{R} \) equivalent to monotony, and formulated in terms that are translatable to \( K \). This way we will obtain several definitions of "\( f : X \rightarrow K \) is monotone", that are, although not equivalent, closely related.

The connections between these various definitions and the properties of the non-archimedean monotone functions can be put together to form a little theory which is interesting in its own right, but of which the relations to the other parts of \( p \)-adic analysis are yet not very tight.

1. Monotone functions.

For a function \( f : \mathbb{R} \rightarrow \mathbb{R} \) the following conditions are equivalent:

(a) \( f \) is monotone (in the non-strict sense),

(b) If \( C \subset \mathbb{R} \) is convex then \( f^{-1}(C) \) is convex,

(c) If \( x \) is between \( y \), \( z \) then \( f(x) \) is between \( f(y) \) and \( f(z) \).

Also, the following conditions are equivalent:

(a) \( f \) is strictly monotone,

(b) \( f \) is injective. If \( C \subset \mathbb{R} \) is convex then \( f(C) \) is relatively convex in \( f(\mathbb{R}) \),

(c) If \( f(x) \) is between \( f(y) \) and \( f(z) \) then \( x \) is between \( y \) and \( z \).

Let \( x, y \in K \). Then the smallest ball that contains \( x, y \) is denoted by [\( x, y \)]. \( z \in K \) is between \( x \) and \( y \) if \( z \in [x, y] \). (If \( z \notin [x, y] \), we

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call $x, y$ at the same side of $z$). A subset $C \subset K$ is called convex if $x, y \in C, z \in [x, y]$ implies $z \in C$. Each convex subset of $K$ can be written in at least one of the following forms

\[ \{ x : |x - a| < r \}, \{ x : |x - a| \leq r \} \]

for some $a \in K, r \in (0, \infty)$.

Let $Z \subset Y \subset K$. Then $Z$ is called convex in $Y$ if $Z = C \cap Y$, where $C$ is convex.

With all these definitions we have the following theorem.

**Theorem 1.1.** Let $f : X \to K$. Then the following conditions are equivalent:

1. If $x, y, z \in X$, $x$ is between $y$ and $z$ then $f(x)$ is between $f(y)$ and $f(z)$,
2. If $C \subset K$ is convex, then $f(C)$ is convex in $X$.

We denote the collection of those $f : X \to K$ satisfying (1) or (2) by $M_b(X)$, i.e. $f \in M_b(X)$ if, and only if, for each $x, y, z \in X$,

\[ |x - y| \leq |y - z| \text{ implies } |f(x) - f(y)| \leq |f(y) - f(z)|. \]

Isometries are in $M_b$ (viz. exp), but also non trivial locally constant functions (e.g., choose a center in each ball of radius $r > 0$, and let $f$ be the map assigning to $x \in X$ the center of the ball of radius $r$ to which $x$ belongs. Then $f \in M_b(X)$).

**Theorem 1.2.** Let $f : X \to K$. Then the following conditions are equivalent:

1'. If $x, y, z \in X$, $f(x)$ is between $f(y)$ and $f(z)$ then $x$ is between $y$ and $z$,
2'. If $C \subset X$ is convex in $X$ then $f(C)$ is convex in $f(X)$.

We denote the collection of those $f : X \to K$ satisfying (1') or (2') by $M_s(X)$, i.e. $f \in M_s(X)$ if, and only if, for each $x, y, z \in X$,

\[ |x - y| < |y - z| \text{ implies } |f(x) - f(y)| < |f(y) - f(z)|. \]

The classical situations suggests the question as to whether $M_s(X) \subset M_b(X)$ and also whether $f \in M_b(X)$, $f$ injective implies $f \in M_s(X)$. In general, both statements are false, but we do have the following:

**Theorem 1.3.** $f \in M_s(X)$ implies $f^{-1} \in M_b(f(X))$. $f \in M_b(X)$, $f$ injective implies $f^{-1} \in M_s(f(X))$. If $k$ is finite and $X$ is convex, then an injective $M_b$-function is in $M_s(X)$.
So we are led to define \( M_{ba}(X) := M_b(X) \cap M_s(X) \) as being the more or less natural translation of "the space of the strictly monotone functions".

The following theorem concerns continuity of monotone functions. For a function \( f : X \to K \), we define its oscillation function, \( \omega_f \), in the usual way:

\[
\omega_f(a) := \lim_{n \to \infty} \sup \{|f(x) - f(y)| : |x - a| \leq \frac{1}{n}, |y - a| \leq \frac{1}{n}\}
\]

\[
= \lim_{n \to \infty} \sup \{|f(x) - f(\alpha)| : |x - a| \leq \frac{1}{n}\} \quad (a \in X).
\]

\( f \) is continuous at \( a \) if, and only if, \( \omega_f(a) = 0 \).

**Theorem 1.4.** Let \( f \) be either in \( M_b(X) \) or in \( M_s(X) \). Then

1. \( \omega_f(a) = \inf_{\not\in f(a)} |f(z) - f(a)| \quad (a \in X) \)
2. \( f \) is bounded on compact subsets of \( X \),
3. For each \( a \in X \) we have the following alternative. Either \( f \) is continuous at \( a \), or for each sequence \( x_1, x_2, \ldots \) (\( x_n \neq a \)) converging to \( a \), the sequence \( f(x_1), f(x_2), \ldots \) is bounded and has no convergent subsequence.

Let \( g \in M_b(X) \). If \( Y \subset X \) is spherically complete, then so is \( g(Y) \).

Let \( h \in M_s(X) \). If \( Z \subset h(X) \) is spherically complete, then so is \( h^{-1}(Z) \).

**Proof (sketch).** If \( f \in M_b(X) \cup M_s(X) \), then:

\[
|x - y| < |y - a| \quad \text{implies} \quad |f(x) - f(y)| < |f(y) - f(z)|.
\]

So \( f \) is locally bounded, and (ii) follows. Of (i), only the \( \leq \) part is interesting. Choose \( z \neq a \). If \( |x - a| < |z - a| \), then

\[
|f(x) - f(a)| \leq |f(z) - f(a)| \quad \text{whence} \quad \omega_f(a) \leq |f(z) - f(a)|.
\]

Let \( \lim x_n = a \) (\( x_n \neq a \) for all \( n \)) and \( \lim f(x_n) = \alpha \). Let \( \lim y_n = a \). It suffices to show that \( \lim f(y_n) = \alpha \). Indeed, let \( \varepsilon > 0 \), and choose \( k \) such that \( |f(x_k) - \alpha| < \varepsilon \). Then \( |y_n - a| < |x_k - a| \) for large \( n \), so

\[
|y_n - x_m| < |x_k - x_m|
\]

for large \( m \) depending on \( n \). Hence \( |f(y_n) - f(x_n)| \leq |f(x_n) - f(x_m)| \), so \( (m \to \infty) \) \( |f(y_n) - f(x_n)| < |f(x_k) - f(x_m)| \), so

\[
|f(y_n) - f(x_n)| \leq |f(x_k) - f(x_m)| < \varepsilon, \quad \text{and we have (iii). The rest of the proof is straightforward.}
\]

**Corollary 1.5.** Let \( f : X \to K \) be in \( M_b(X) \cup M_s(X) \).

1. If \( K \) is a local field, then \( f \) is continuous,
2. If \( |X| \) is discrete, then \( f \in M_b(X) \Rightarrow f \) is a homeomorphism \( X \sim f(X) \),
3. The graph of \( f \) is closed in \( K^2 \),
4. If \( f(X) \) has no isolated points, then \( f \) is continuous.
An \( M_b \)-function may be everywhere discontinuous on \( K \) (even when \(|K|\) is discrete).

**THEOREM 1.6.** Let \( B \) be the unit ball of \( K \),

(i) If \( K \) is a local field and \( f \in M_b(B) \cup M_s(B) \), then \( f \) has bounded difference quotients (i.e., there is \( C > 0 \) such that \(|f(x) - f(y)| \leq C|x - y| \) for all \( x \in B \)). If, in addition, \( f(B) \) is convex, then \( f \) is a similarity (i.e., a scalar multiple of an isometry).

(ii) If \( K \) has discrete valuation and \( f \in M_b(B) \) is bounded, then \( f \) has bounded difference quotients. If \( f \in M_{bs}(B) \) and if \( f(B) \) is convex, then \( f \) is a similarity.

2. **Monotone functions having a type.**

In this section, we want to translate the usual classification of (strictly) monotone functions \( R \rightarrow R \) into two types: the increasing and the decreasing functions. The equivalence relation in \( R^r \): \( x \sim y \) if \( x \) and \( y \) are at the same side of \( 0 \), yields \((-\infty, 0)\) and \((0, \infty)\) as equivalence classes. The relation \( \sim \) is compatible with the canonical group homomorphism \( R^* \rightarrow R^*/R^+ \), the latter group being \([-1, 1] \). \( \pi(x) \) (usually called \( sgn(x) \)) assigns +1 to every positive element and -1 to every negative element. A function \( f: R \rightarrow R \) is strictly monotone if there exists \( \sigma: R^*/R^+ \rightarrow R^*/R^+ \) such that for all \( x \neq y \)

\[
\pi(f(x) - f(y)) = \sigma(\pi(x - y)).
\]

If \( \sigma \) is the identity then \( f \) is called increasing; if \( \sigma(1) = -1 \), \( \sigma(-1) = 1 \), \( f \) is called decreasing. Other maps \( \sigma: [-1, 1] \rightarrow [-1, 1] \) cannot occur (i.e., there is no \( f \) such that, for all \( x \neq y \)

\[
\pi(f(x) - f(y)) = \sigma(\pi(x - y)).
\]

This rather weird description of real monotone functions can be used in the non-archimedean case.

For \( x, y \in K^* \) define \( x \sim y \) if \( x, y \) are at the same side of \( 0 \). This means: \( 0 \not\in [x, y] \), or \(|x - y| > |y| \), or \(|xy^{-1} - 1| < 1 \). Thus \( x \sim y \) if, and only if, \( xy^{-1} \in K^+ \) where

\[
K^+ := \{x \in K; |1 - x| < 1 \}. \]

We call the elements of \( K^+ \) the positive element of \( K \).

The relation \( \sim \) is compatible with the canonical homomorphism of (multiplicative) groups

\[
\pi: K^* \rightarrow K^*/K^+ =: \Sigma.
\]

We call \( \Sigma \) the group of signs and \( \pi(x) \) the sign of an element \( x \in K^* \) (\( x \) is
positive if, and only if, \( \pi(x) = 1 \).

If \( K \) is a local field, we can make a group embedding \( \rho : \Sigma \rightarrow K^* \) such that \( \pi \circ \rho \) is the identity on \( \Sigma \). For example, if \( K = \mathbb{Q}_p \), \( \delta \) is a primitive \((p-1)\)th root of unity, then

\[
\pi\left(\sum_{k \geq 0} a_k p^k\right) = a_k p^k \quad (k \in \mathbb{Z}, \ a_k \neq 0)
\]

(Here \( a_n \in \{0, 1, \delta, \ldots, \delta^{p-2}\} \) for each \( n \)).

**Definition 2.1.** Let \( \sigma : \Sigma \rightarrow \Sigma \) be any map. A function \( f : X \rightarrow K \) is monotone of type \( \sigma \) if, for all \( x, y \in X, x \neq y \),

\[
\pi(f(x) - f(y)) = \sigma(\pi(x - y))
\]

(i.e., if \( x - y \in \sigma \in \Sigma \) then \( f(x) - f(y) \in \sigma(\sigma) \)).

We call \( f \) of type \( \beta \in \Sigma \) if \( f \) is of type \( \sigma \) where \( \sigma \) is the multiplication with \( \beta \), i.e.,

\[
f(x) - f(y) \in \beta \quad (x, y \in X, x \neq y).
\]

We call \( f \) increasing if \( f \) is of type \( \sigma \) where \( \sigma \) is the identity, i.e.,

\[
\frac{f(x) - f(y)}{x - y} \text{ is positive (} x \neq y \).
\]

Clearly, if \( f \) is of type \( \beta \), and if \( b \in \beta \), then \( b^{-1} f \) is increasing. First, we look at increasing functions, then we discuss more general types \( \sigma \).

Notice that increasing functions are isometries hence are in \( M_k^+(X) \). If \( f \) is increasing then \( f(x) = x + h(x) \), where \( |h(x) - h(y)| < |x - y| \) \((x, y \in X, x \neq y) \). Such \( h \) we call pseudo-contractions.

**Lemma 2.2.** Let \( X \) be an ultrametric space. Then the following are equivalent

\[(a) \ X \text{ is spherically complete,}
\[(b) \text{ Each pseudocontraction } X \rightarrow X \text{ has a (unique) fixed point.}
\]

**Proof (sketch).** \( (a) \rightarrow (b) \). Let \( \sigma : X \rightarrow X \) be a pseudocontraction. A convex set \( C \subset X \) is called invariant if \( \sigma(C) \subset C \). It is easily proved that the invariant convex subsets of \( X \) form a nest. Let \( C_0 \) be the smallest invariant convex set. If \( a \in C_0 \) and \( \sigma(a) \neq a \) then

\[
B_0 := \{ x \in X ; \ d(x, \sigma(a)) < d(a, \sigma(a)) \}
\]

is invariant, convex, and does not contain \( a \). Hence \( \sigma(a) = a \) for all \( a \in C_0 \), and \( C_0 \) is a singleton. \( (b) \rightarrow (a) \). If \( B_1 \neq B_2 \neq \ldots \) are balls in \( X \) with \( \cap B_n = \emptyset \) then choose \( x_n \in B_n \setminus B_{n+1} \) \((n \in \mathbb{N}) \). The map \( \sigma : X \rightarrow X \) defined by

\[
\sigma(x) = x_{n+1} \quad (x \in B_n \setminus B_{n+1})
\]

is a pseudocontraction without a fixed point.
COROLLARY 2.3. - Let \( X \) be convex, let \( K \) be spherically complete, and let \( f : X \to K \) be increasing. Then \( f(x) \) is convex. If \( f(x) \subset X \), then \( f \) is surjective.

Proof. - Let \( f(x) \subset X \). Choose \( x \in X \). Then \( x \mapsto -f(x) + x + x \) is a pseudocontraction mapping \( X \) into \( X \), hence has a fixed point. So \( f(x) = x \) for some \( x \in X \).

If \( K \) is not spherically complete, we have always increasing \( f : K \to K \) that are not surjective. (Let \( h : K \to K \) be a pseudocontraction without a fixed point. Let \( f(x) = x - h(x) \) \( (x \in K) \), then \( 0 \notin \text{im} f \). The inverse \( f^{-1} : f(K) \to K \) can, of course, not be extended to an increasing function \( K \to K \).

THEOREM 2.4. - Let \( K \) be spherically complete, and let \( f : X \to K \) be increasing. Then \( f \) can be extended to an increasing function \( K \to K \).

Proof. - By Zorn's Lemma, it suffices to extend \( f \) to an increasing function on \( X \cup \{ a \} \), where \( a \notin X \). We are done if we can find \( x \in K \) such that, for all \( x \in X \),

\[
\left| \frac{a - f(x)}{a - x} - 1 \right| < 1
\]

i.e. \( a \in B_{f(x) - (a - x)} |a - x| \) for all \( x \in X \). These balls form a nest.

Let us call a function \( f : X \to K \) positive if \( f(x) \subset K^+ \).

THEOREM 2.5.

(i) If \( f : X \to K \) is increasing then \( f' \) is positive,

(ii) If \( g : X \to K \) is a positive Baire class one function, then \( g \) has an increasing antiderivative,

(iii) If \( g : X \to K \) is continuous and positive, then \( g \) has a \( C^1 \)-antiderivative,

(iv) If \( f \in C^1(X) \) and \( f' \) is positive then \( f = j + h \) where \( j \) is increasing, and \( h \) is locally constant.

EXAMPLES.

1° The exponential function (defined on its natural convergence region) is increasing.

2° Let \( f \in C(Z_p) \), and let \( e_0 = 5_{Z_p} \), for \( n \in \mathbb{N} \),

\[
e_n(x) = \begin{cases} 1 & \text{if } |x - n| < \frac{1}{n} \\ 0 & \text{elsewhere} \end{cases} \quad (x \in \mathbb{Z}_p) .
\]

Then \( e_0, e_1, \ldots \) form an orthonormal base of \( C(Z_p) \), so there exist \( \lambda_0, \lambda_1, \ldots \in \mathbb{Q}_p \) such that \( f = \sum_{n=0}^{\infty} \lambda_n e_n \), uniformly.
Let \( a, b \in \Sigma \). If the set theoretic sum \( a + b := \{ x + y ; x \in a, y \in b \} \) does not contain \( 0 \) then \( a + b \in \Sigma \), notation \( a \oplus b \). It follows that \( a \oplus b \) is defined if, and only if, \( a \neq -b \).

If \( x, y \in a \in \Sigma \) then \( |x| = |y| \). This defines \( |\cdot| \) in a natural way.

We have the following results.

**Theorem 2.6.** Let \( f : K \to K \) be monotone of type \( \sigma : \Sigma \to \Sigma \). Let \( a, b \in \Sigma \),

(i) \( \sigma(-a) = -\sigma(a) \),

(ii) If \( \sigma(a) \oplus \sigma(b) \) is defined then so is \( a \oplus b \) and \( \sigma(a \oplus b) = \sigma(a) \oplus \sigma(b) \),

(iii) \( |a| < |b| \) implies \( |\sigma(a)| < |\sigma(b)| \),

(iv) If \( |b| = 1 \), \( b \) contains an element of the prime field of \( K \) then \( \sigma(ba) = b\sigma(a) \),

(v) \( f \in M_b(K) \),

(vi) \( f \) is either nowhere continuous or uniformly continuous.

**Theorem 2.7.** Let \( f : K \to K \) be monotone of type \( \sigma : \Sigma \to \Sigma \). Then the following conditions are equivalent,

(a) \( \sigma \) is injective,

(b) \( f \in M_b(x) \),

(c) If for some \( a, b \in \Sigma \), \( a \oplus b \) is defined, then so is \( \sigma(a) \oplus \sigma(b) \),

(d) \( |\sigma(a)| < |\sigma(b)| \) implies \( |a| < |b| \) \( (a, b \in \Sigma) \).

**Corollary 2.8.** Let \( K \) be a prime field, and let \( f : K \to K \) be monotone of type \( \sigma : \Sigma \to \Sigma \). Then \( \sigma \) is injective.

(If \( K = Q_p(\sqrt{-1}) \), \( p = 3 \mod 4 \), we can find an example of an \( f : K \to K \) monotone of type \( \sigma \), where \( \sigma \) is not injective).

**Example 2.9.** Let \( K = Q_p \). Then

\[ \{ \sigma : \Sigma \to \Sigma : \text{there is } f : Q_p \to Q_p, \text{ } f \text{ monotone of type } \sigma \} \]
consists of all $\sigma : \Sigma \to \Sigma$ of the form
$$\sigma^i p^n \to \sigma^i \delta^j(n) p(\lambda(n))$$
where $s : \mathbb{Z} \to \{0, 1, 2, \ldots, p - 2\}$ and $\lambda : \mathbb{Z} \to \mathbb{Z}$ is strictly increasing.

3. Functions of bounded variation.

**Lemma 3.1.** Let $f : X \to K$ have bounded difference quotients. Then $f$ is a linear combination of two increasing functions.

**Proof.** Choose $\lambda \in K$,
$$|\lambda| > \sup\{\frac{f(x) - f(y)}{x - y} ; x \neq y\}.$$

Then $\lambda^{-1} f$ is a (pseudo-) contraction, so $g(x) := x + \lambda^{-1} f(x) (x \in X)$ is increasing. If $h(x) := x (x \in X)$, then $\lambda h + \lambda g = f$.

**Corollary 3.2.** Let $X$ be the unit ball of a local field $K$ and let $f : X \to K$. Then the following are equivalent

(a) $f \in \text{BA}(X)$ (i.e., $\sup\{\frac{f(x) - f(y)}{x - y} ; x \neq y\} < \infty$),

(b) $f$ is a linear combination of two increasing functions,

(γ) $f \in \text{I}_g(X)$,

(δ) $f \in \text{I}_b(X)$.

**Proof.** Use 1.6.

**References**
