NON-ARCHIMEDEAN MONOTONE FUNCTIONS

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Introduction.

In the sequel, $K$ is a non-archimedean, non-trivially valued field, that is complete under the metric induced by the valuation. The residue class field of $K$ is denoted by $k$. $X$ will always be a closed, non-empty subset of $K$ without isolated points (except in 2.2, if you want).

Since $K$ admits no ordering in the usual sense it is not possible to define monotone functions $X \rightarrow K$ just by taking over the classical definitions. Thus, our procedure will be to try and find statements for functions $\mathbb{R} \rightarrow \mathbb{R}$ equivalent to monotony, and formulated in terms that are translatable to $K$. This way we will obtain several definitions of "$f : X \rightarrow K$ is monotone", that are, although not equivalent, closely related.

The connections between these various definitions and the properties of the non-archimedean monotone functions can be put together to form a little theory which is interesting in its own right, but of which the relations to the other parts of $p$-adic analysis are yet not very tight.

1. Monotone functions.

For a function $f : \mathbb{R} \rightarrow \mathbb{R}$ the following conditions are equivalent:

(a) $f$ is monotone (in the non-strict sense),
(b) If $C \subset \mathbb{R}$ is convex then $f^{-1}(C)$ is convex,
(c) If $x$ is between $y$ and $z$ then $f(x)$ is between $f(y)$ and $f(z)$.

Also, the following conditions are equivalent:

(a) $f$ is strictly monotone,
(b) $f$ is injective. If $C \subset \mathbb{R}$ is convex then $f(C)$ is relatively convex in $f(\mathbb{R})$,
(c) If $f(x)$ is between $f(y)$ and $f(z)$ then $x$ is between $y$ and $z$.

Let $x, y \in K$. Then the smallest ball that contains $x, y$ is denoted by $[x, y]$. $z \in K$ is between $x$ and $y$ if $z \in [x, y]$. (If $z \notin [x, y]$, we

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call \( x, y \) at the same side of \( z \). A subset \( C \subseteq K \) is called convex if 
\[ x, y \in C, \quad z \in [x, y] \] implies \( z \in C \). Each convex subset of \( K \) can be 
written in at least one of the following forms 
\[ \{ x : |x - a| < r \}, \{ x : |x - a| \leq r \} \]
for some \( a \in K, \quad r \in (0, \infty) \).

Let \( Z \subseteq Y \subseteq K \). Then \( Z \) is called convex in \( Y \) if \( Z = C \cap Y \), where \( C \) is 
convex.

With all these definitions we have the following theorem.

**Theorem 1.1.** Let \( f : X \to K \). Then the following conditions are equivalent:

1. If \( x, y, z \in X \), \( x \) is between \( y \) and \( z \) then \( f(x) \) is between \( f(y) \) and \( f(z) \).
2. If \( C \subseteq K \) is convex, then \( f(C) \) is convex in \( X \).

We denote the collection of those \( f : X \to K \) satisfying (1) or (2) by \( M_b(X) \), i.e., \( f \in M_b(X) \) if, and only if, for each \( x, y, z \in X \),
\[ |x - y| < |y - z| \] implies \( |f(x) - f(y)| < |f(y) - f(z)| \).

Isometries are in \( M_b \) (viz. exp), but also non-trivial locally constant functions (e.g., choose a center in each ball of radius \( r > 0 \), and let \( f \) be the 
map assigning to \( x \in X \) the center of the ball of radius \( r \) to which \( x \) belongs. Then \( f \in M_b(X) \).

**Theorem 1.2.** Let \( f : X \to K \). Then the following conditions are equivalent:

1. If \( x, y, z \in X \), \( f(x) \) is between \( f(y) \) and \( f(z) \) then \( x \) is between \( y \) and \( z \).
2. If \( C \subseteq X \) is convex in \( X \) then \( f(C) \) is convex in \( f(X) \). \( f \) is injective.

We denote the collection of those \( f : X \to K \) satisfying (1') or (2') by \( M_b(X) \), i.e., \( f \in M_b(X) \) if, and only if, for each \( x, y, z \in X \),
\[ |x - y| < |y - z| \] implies \( |f(x) - f(y)| < |f(y) - f(z)| \).

The classical situations suggests the question as to wether \( M_b(X) \subseteq M_s(X) \) and 
also wether \( f \in M_b(X) \), \( f \) injective implies \( f \in M_s(X) \). In general, both 
statements are false, but we do have the following:

**Theorem 1.3.** \( f \in M_s(X) \) implies \( f^{-1} \in M_b(f(X)) \). \( f \in M_s(X) \), \( f \) injective 
implies \( f^{-1} \in M_s(f(X)) \). \( f \in M_b(X) \), \( f \) injective 
implies \( f^{-1} \in M_s(f(X)) \). If \( k \) is finite and \( X \) is convex, then an injective 
\( M_b \)-function is in \( M_s(X) \).
So we are led to define $M_{ba}(X) := M_b(X) \cap M_s(X)$ as being the more or less natural translation of "the space of the strictly monotone functions".

The following theorem concerns continuity of monotone functions. For a function $f : X \to K$, we define its oscillation function, $\omega_f$, in the usual way:

$$\omega_f(a) := \lim_{n \to \infty} \sup \{|f(x) - f(y)| : |x - a| \leq \frac{1}{n}, \quad |y - a| \leq \frac{1}{n}\}$$

$$= \lim_{n \to \infty} \sup \{|f(x) - f(a)| : |x - a| \leq \frac{1}{n}\} \quad (a \in X).$$

$f$ is continuous at $a$ if, and only if, $\omega_f(a) = 0$.

**THEOREM 1.4.** - Let $f$ be either in $M_b(X)$ or in $M_s(X)$. Then

(i) $\omega_f(a) = \inf_{z \neq a} |f(z) - f(a)|$ \quad (a $\in X$)

(ii) $f$ is bounded on compact subsets of $X$,

(iii) For each $a \in X$ we have the following alternative. Either $f$ is continuous at $a$, or for each sequence $x_1, x_2, \ldots$ ($x_n \neq a$) converging to $a$, the sequence $f(x_1), f(x_2), \ldots$ is bounded and has no convergent subsequence.

Let $g \in M^b(X)$. If $Y \subset X$ is spherically complete, then so is $g(Y)$.

Let $h \in M^s(X)$. If $Z \subset h(X)$ is spherically complete, then so is $h^{-1}(Z)$.

**Proof (sketch).** - If $f \in M_b(X) \cup M_s(X)$, then:

$$|x - y| < |y - z| \text{ implies } |f(x) - f(y)| \leq |f(y) - f(z)|$$

So $f$ is locally bounded, and (ii) follows. Of (i), only the $\leq$ part is interesting. Choose $z \neq a$. If $|x - a| < |z - a|$, then

$$|f(x) - f(a)| \leq |f(z) - f(a)| \text{ whence } \omega_f(a) \leq |f(z) - f(a)|.$$

Let $\lim x_n = a$ ($x_n \neq a$ for all $n$) and $\lim f(x_n) = \alpha$. Let $\lim y_n = a$. It suffices to show that $\lim f(y_n) = \alpha$. Indeed, let $\varepsilon > 0$, and choose $k$ such that $|f(x_k) - \alpha| < \varepsilon$. Then $|y_n - a| < |x_k - a|$ for large $n$, so

$$|y_n - x_m| < |x_k - x_m|$$

for large $m$ depending on $m$. Hence $|f(y_n) - f(x_n)| \leq |f(x_n) - f(x_m)|$, so ($m \to \infty$) $|f(y_n) - \alpha| \leq |f(x_k) - \alpha| < \varepsilon$, and we have (iii). The rest of the proof is straightforward.

**COROLLARY 1.5.** - Let $f : X \to K$ be in $M_b(X) \cup M_s(X)$.

(i) If $K$ is a local field, then $f$ is continuous.

(ii) If $|K|$ is discrete, then $f \in M_b(X) \Rightarrow f$ is a homeomorphism $X \to f(X)$, and $f \in M_s(X) \Rightarrow f$ is a closed map.

(iii) The graph of $f$ is closed in $K^2$.

(iv) If $f(X)$ has no isolated points, then $f$ is continuous.
An $M_b$-function may be everywhere discontinuous on $K$ (even when $|K|$ is discrete).

**THEOREM 1.6.** Let $B$ be the unit ball of $K$,

(i) If $K$ is a local field and $f \in M_b(B) \cup M_s(B)$, then $f$ has bounded difference quotients (i.e., there is $C > 0$ such that $|f(x) - f(y)| \leq C|x - y|$ for all $x \in B$). If, in addition, $f(B)$ is convex, then $f$ is a similarity (i.e., a scalar multiple of an isometry).

(ii) If $K$ has discrete valuation and $f \in M_b(B)$ is bounded, then $f$ has bounded difference quotients. If $f \in M_{bs}(B)$ and if $f(B)$ is convex, then $f$ is a similarity.

2. Monotone functions having a type.

In this section, we want to translate the usual classification of (strictly) monotone functions $\mathbb{R} \to \mathbb{R}$ into two types: the increasing and the decreasing functions. The equivalence relation in $\mathbb{R}^r$: $x \sim y$ if $x$ and $y$ are at the same side of 0, yields $(-\infty, 0)$ and $(0, \infty)$ as equivalence classes. The relation $\sim$ is compatible with the canonical group homomorphism $\mathbb{R}^* \to \mathbb{R}^*/\mathbb{R}^+$, the latter group being $\{1, -1\}$. $\pi(x)$ (usually called $\text{sgn}(x)$) assigns $+1$ to every positive element and $-1$ to every negative element. A function $f: \mathbb{R} \to \mathbb{R}$ is strictly monotone if there exists $\sigma: \mathbb{R}^*/\mathbb{R}^+ \to \mathbb{R}^*/\mathbb{R}^+$ such that for all $x \neq y$

$$\pi(f(x) - f(y)) = \sigma(\pi(x - y)) .$$

If $\sigma$ is the identity then $f$ is called increasing; if $\sigma(1) = -1$, $\sigma(-1) = 1$, $f$ is called decreasing. Other maps $\sigma: [-1, 1] \to [-1, 1]$ can not occur (i.e., there is no $f$ such that, for all $x \neq y$,

$$\pi(f(x) - f(y)) = \sigma(\pi(x - y)) .$$

This rather weird description of real monotone functions can be used in the nonarchimedean case.

For $x, y \in K^*$ define $x \sim y$ if $x, y$ are at the same side of 0. This means: $0 \notin [x, y]$, or $|x - y| > |y|$, or $|xy^{-1} - 1| < 1$. Thus $x \sim y$ if, and only if, $xy^{-1} \in K^+$ where

$$K^+ := \{x \in K; \ |1 - x| < 1\} .$$

We call the elements of $K^+$ the positive element of $K$.

The relation $\sim$ is compatible with the canonical homomorphism of (multiplicative) groups

$$\pi: K^* \to K^*/K^+ =: \Sigma .$$

We call $\Sigma$ the group of signs and $\pi(x)$ the sign of an element $x \in K^*$ ( $x$ is
positive if, and only if, $\pi(x) = 1$.

If $K$ is a local field, we can make a group embedding $\rho : \Sigma \hookrightarrow K^*$ such that $\pi \circ \rho$ is the identity on $\Sigma$. For example, if $K = \mathbb{Q}_p$, $\delta$ is a primitive $(p - 1)$th root of unity, then

$$\pi(\sum_{n \geq 0} a_n p^n) = a_k p^k \quad (k \in \mathbb{Z}, \ a_k \neq 0)$$

(Here $a_n \in \{0, 1, 5, \ldots, 6^{p-2}\}$ for each $n$).

**Definition 2.1.** Let $\sigma : \Sigma \rightarrow \Sigma$ be any map. A function $f : X \rightarrow K$ is monotone of type $\sigma$ if, for all $x, y \in X$, $x \neq y$,

$$\pi(f(x) - f(y)) = \sigma(\pi(x - y))$$

(i.e., if $x-y \in \sigma$ then $f(x) - f(y) \in \sigma(a)$).

We call $f$ of type $\beta \in \Sigma$ if $f$ is of type $\sigma$ where $\sigma$ is the multiplication with $\beta$, i.e.

$$\frac{f(x) - f(y)}{x - y} \in \beta \quad (x, y \in X, x \neq y).$$

We call $f$ increasing if $f$ is of type $\sigma$ where $\sigma$ is the identity, i.e.,

$$\frac{f(x) - f(y)}{x - y}$$

is positive $(x \neq y)$.

Clearly, if $f$ is of type $\beta$, and if $b \in \beta$, then $b^{-1} f$ is increasing.

First, we look at increasing functions, then we discuss more general types $\sigma$.

Notice that increasing functions are isometries hence are in $N_\beta(X)$. If $f$ is increasing then $f(x) = x + h(x)$, where $|h(x) - h(y)| < |x - y| (x, y \in X, x \neq y)$.

Such $h$ we call pseudo-contractions.

**Lemma 2.2.** Let $X$ be an ultrametric space. Then the following are equivalent

(a) $X$ is spherically complete,

(b) Each pseudocontraction $X \rightarrow X$ has a (unique) fixed point.

**Proof (sketch).** (a) $\Rightarrow$ (b). Let $\sigma : X \rightarrow X$ be a pseudocontraction. A convex set $C \subset X$ is called invariant if $\sigma(C) \subset C$. It is easily proved that the invariant convex subsets of $X$ form a nest. Let $C_0$ be the smallest invariant convex set. If $a \in C_0$ and $\sigma(a) \neq a$ then

$$B_0 := \{x \in X ; \ a(x, \sigma(a)) < a(a, \sigma(a))\}$$

is invariant, convex, and does not contain $a$. Hence $\sigma(a) = a$ for all $a \in C_0$, and $C_0$ is a singleton. (b) $\Rightarrow$ (a). If $B_1 \neq B_2 \neq \ldots$ are balls in $X$ with $\cap B_n = \emptyset$ then choose $x_n \in B_n \setminus B_{n+1}$ $(n \in \mathbb{N})$. The map $\sigma : X \rightarrow X$ defined by

$$\sigma(x) = x_{n+1} \quad (x \in B_n \setminus B_{n+1})$$

is a pseudocontraction without a fixed point.
COROLLARY 2.3. - Let $X$ be convex, let $K$ be spherically complete, and let $f : X \to K$ be increasing. Then $f(x)$ is convex. If $f(x) \in X$, then $f$ is surjective.

Proof. - Let $f(X) \subset X$. Choose $x \in X$. Then $x \mapsto f(x) + x + \alpha$ is a pseudocontraction mapping $X$ into $X$, hence has a fixed point. So $f(x) = \alpha$ for some $x \in X$.

If $K$ is not spherically complete, we have always increasing $f : K \to K$ that are not surjective. (Let $h : K \to K$ be a pseudocontraction without a fixed point. Let $f(x) = x - h(x)$ ($x \in K$), then $0 \notin f(x)$. The inverse $f^{-1} : f(K) \to K$ can, of course, not be extended to an increasing function $K \to K$.

THEOREM 2.4. - Let $K$ be spherically complete, and let $f : X \to K$ be increasing. Then $f$ can be extended to an increasing function $K \to K$.

Proof. - By Zorn's Lemma, it suffices to extend $f$ to an increasing function on $X \cup \{a\}$, where $a \notin X$. We are done if we can find $a \in K$ such that, for all $x \in X$,

$$\frac{\alpha - f(x)}{a - x} - 1 < 1$$

i.e. $a \in B_f(x)-(a-x)\{a - x\}$ for all $x \in X$. These balls form a nest.

Let us call a function $f : X \to K$ positive if $f(x) \in K^+$.

THEOREM 2.5.

(i) If $f : X \to K$ is increasing then $f'$ is positive,

(ii) If $g : X \to K$ is a positive Baire class one function, then $g$ has an increasing antiderivative,

(iii) If $g : X \to K$ is continuous and positive, then $g$ has a $C^1$-antiderivative,

(iv) If $f \in C^1(X)$ and $f'$ is positive then $f = j + h$ where $j$ is increasing, and $h$ is locally constant.

EXAMPLES.

1° The exponential function (defined on its natural convergence region) is increasing.

2° Let $f \in C(\mathbb{Z}_p)$, and let $e_0 = 0$, for $n \in \mathbb{N}$,

$$e_n(x) = \begin{cases} 1 & \text{if } |x - n| < \frac{1}{n} \\ 0 & \text{elsewhere} \end{cases} (x \in \mathbb{Z}_p).$$

Then $e_0, e_1, \ldots$ form an orthonormal base of $C(\mathbb{Z}_p)$, so there exist $\lambda_0, \lambda_1, \ldots \in \mathbb{Q}_p$ such that $f = \sum_{n=0}^{\infty} \lambda_n e_n$, uniformly.
Let $f$ be increasing if, and only if, for all $n \in \mathbb{N}$,
\[ |\lambda_n - \{n\}| < \{n\} \]
(where, if $n = a_0 + a_1 p + \ldots + a_k p^k$ ($a_i \in \{0, 1, \ldots, p-1\}$ for each $i$, $a_k \neq 0$), then $\{n\}_1 = a_k p^k$).

In other words, $f = \sum \lambda_n \in C(\mathbb{Z}_p)$ is increasing if, and only if, $\lambda_n/\{n\}$ is positive for all $n \in \mathbb{N}$.

Let $\alpha, \beta \in \Sigma$. If the set theoretic sum $\alpha + \beta := \{x + y; x \in \alpha, y \in \beta\}$ does not contain 0 then $\alpha + \beta \in \Sigma$, notation $\alpha \oplus \beta$. It follows that $\alpha \oplus \beta$ is defined if, and only if, $\alpha \neq -\beta$.

If $x, y \in \alpha \in \Sigma$ then $|x| = |y|$. This defines $|\alpha|$ in a natural way.

We have the following results.

**Theorem 2.6.** Let $f : \Sigma \to \Sigma$ be monotone of type $\sigma : \Sigma \to \Sigma$. Let $\alpha, \beta \in \Sigma$,

(i) $\sigma(-\alpha) = -\sigma(\alpha)$,

(ii) If $\sigma(\alpha) \oplus \sigma(\beta)$ is defined then so is $\alpha \oplus \beta$ and $\sigma(\alpha \oplus \beta) = \sigma(\alpha) \oplus \sigma(\beta)$,

(iii) $|\alpha| < |\beta|$ implies $|\sigma(\alpha)| < |\sigma(\beta)|$,

(iv) If $|\beta| = 1$, $\beta$ contains an element of the prime field of $K$ then $\sigma(\beta \alpha) = \beta \sigma(\alpha)$,

(v) $f \in M_b(K)$,

(vi) $f$ is either nowhere continuous or uniformly continuous.

**Theorem 2.7.** Let $f : \Sigma \to \Sigma$ be monotone of type $\sigma : \Sigma \to \Sigma$. Then the following conditions are equivalent,

(a) $\sigma$ is injective,

(b) $f \in M_b(x)$,

(c) If for some $\alpha, \beta \in \Sigma$, $\alpha \oplus \beta$ is defined, then so is $\sigma(\alpha) \oplus \sigma(\beta)$,

(d) $|\sigma(\alpha)| < |\sigma(\beta)|$ implies $|\alpha| < |\beta| (\alpha, \beta \in \Sigma)$.

**Corollary 2.8.** Let $K$ be a prime field, and let $f : K \to K$ be monotone of type $\sigma : \Sigma \to \Sigma$. Then $\sigma$ is injective.

(If $K = \mathbb{Q}_p(\sqrt{p-1})$, $p = 3 \bmod 4$, we can find an example of an $f : K \to K$ monotone of type $\sigma$, where $\sigma$ is not injective).

**Example 2.9.** Let $K = \mathbb{Q}_p$. Then

$\{\sigma : \Sigma \to \Sigma : \text{there is } f : \mathbb{Q}_p \to \mathbb{Q}_p, f \text{ monotone of type } \sigma\}$
consists of all \( \sigma : \Sigma \rightarrow \Sigma \) of the form
\[
\delta_{p}^{i} \rightarrow \delta_{p}^{j} \delta(n) p \lambda(n)
\]
where \( s : \mathbb{Z} \rightarrow \{0, 1, 2, \ldots, p - 2\} \) and \( \lambda : \mathbb{Z} \rightarrow \mathbb{Z} \) is strictly increasing.

3. Functions of bounded variation.

**Lemma 3.1.** Let \( f : X \rightarrow K \) have bounded difference quotients. Then \( f \) is a linear combination of two increasing functions.

**Proof.** Choose \( \lambda \in K \),
\[
|\lambda| > \sup \left[ \frac{|f(x) - f(y)|}{x - y}; x \neq y \right].
\]
Then \( \lambda^{-1} f \) is a (pseudo-) contraction, so \( g(x) := -x + \lambda^{-1} f(x) \) \((x \in X)\) is increasing. If \( h(x) := x \) \((x \in X)\), then \( \lambda h + \lambda g = f \).

**Corollary 3.2.** Let \( X \) be the unit ball of a local field \( K \) and let \( f : X \rightarrow K \). Then the following are equivalent:

(a) \( f \in BA(X) \) \(\text{i.e.} \ \sup \left[ \frac{|f(x) - f(y)|}{x - y}; x \neq y \right] < \infty \),
(b) \( f \) is a linear combination of two increasing functions,

(d) \( f \in \mathbb{I}_{b}(X) \),

(e) \( f \in \mathbb{I}_{b}(X) \).

**Proof.** Use 1.6.

**References**
